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Optimization problems for nonlinear eigenvalues

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CONTENTS

INTRODUCTION

This thesis is mainly focused on the study of some variational problems and elliptic partial differential equations that takes in to account a possible anisotropy. This kind of equations and functional arise from a generalization of the Euclidean case and are studied by means of symmetrization techniques, shape optimization and properties of Finsler metrics. On these questions Alvino, Bellettini, Ferone, Kawohl, Lions, Novaga, Trombetti (see e.g. [2, 4, 10, 12]) have obtained relevant results and later many authors have successifully continued the study of anisotropic eigenvalue problems (we refer for example to [44, 45, 46, 47, 48, 49, 50]).

In the first Chapter we recall some definitions and properties of rearrangements referring to [16, 31, 78, 83, 94, 102, 107, 109]. We also introduce some notions of Finsler metrics, the definition of the Wulff shape and some "generalized"definitions and properties of perimeter, total variation, coarea formulas, isoperimetric inequalities [6, 4, 33, 103, 112].

In the second chapter we study geometric properties of the eigenvalues of the anisotropic p-Laplacian

$$\mathcal{Q}_p u := \operatorname{div}\left(\frac{1}{p}\nabla_{\xi} F^p(\nabla u)\right), \tag{1}$$

with Dirichlet or Neumann boundary conditions, where F is a suitable norm (see Chapther 1 for details) and $1 < p \leq +\infty$. In this chapter we study some isoperimetric problems, consisting in optimizing a domain dependent functional while keeping its volume fixed. Among the isoperimetric problems, we are interested in those ones linking the shape of domain to the sequence of its eigenvalue. Their study involves different fields of mathematics (spectral theory, partial differential equations, calculus of variations, shape optimization, rearrangement theory). One of the first question on optimization of eigenvalues appeared in the book of Lord Raylegh "The theory of Sound"(1894). He conjectured that the first Dirichlet eigenvalue of Laplacian (*the first frequency of the fixed membrane*) is minimal for the disk. Thirty years later, Faber [61] and Krahn [86] proved this result with the means of rearrangements techniques, in particular by the Pólya-Szegö inequality [102]. Later Krahn [87], proved that for the second eigenvalue the minimizer is the union of two identic balls.

Minimization of the first nontrivial Neumann eigenvalue of the Laplacian (*the frequencies of the free membrane*) among open sets of a given measure is a trivial problem, since the infimum is zero (even among convex sets). But it is bounded away from zero among convex sets with given diameter (Payne and Weinberger [97]). Moreover, contrary to the first Dirichlet eigenvalue, Szegö [106] (in the plane) and Weinberger [115] (in higher dimensions) proved that balls maximize the first notrivial Neumann eigenvalue among the open sets with given volume.

One of the first attempt to solve a problem with an anisotropic function F, is contained in a paper of Wulff [116] dating back to 1901. However, only in 1944 A. Dinghas [54] proved that the set minimizing a generalized perimeter among open set with fixed volume is set homothetic to the unit ball of F^o, the dual norm of F, i.e.

$$\mathcal{W} := \{x \in \mathbb{R}^n : F^o(x) \leq 1\}, \tag{2}$$

that is the so-called Wulff shape, centered in the origin. Moreover, we denote by $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape centered in x_0 with radius r and we put $\mathcal{W}_r := \mathcal{W}_r(x_0)$ if no misunderstanding occurs.

This chapter is divided in four Section. In Section 2.1, by means of Schwarz (or spherical) symmetrization, it is possible to obtain comparison results for solutions to linear elliptic problems:

$$-\mathrm{div}(a(x,u,\nabla u)) = f \quad \text{in } \Omega, \quad u \in H_0^1(\Omega) \tag{3}$$

where

$$a(x,\eta,\xi)\cdot\xi \geq F^2(\xi) \quad \text{a.e.} \quad x\in\Omega, \quad \eta\in\mathbb{R}, \quad \xi\in\mathbb{R}^n. \tag{4}$$

The authors in [4], using convex symmetrization, estimate a solution of (3) in terms of a function v that solves

$$-\Delta v = f^{\#} \quad \text{in } \Omega^{\#}, \quad v\in H_0^1(\Omega^{\#}),$$

where $f^{\#}$ is the spherically decreasing rearrangemetns of f (i.e. the function such that its level sets are balls which have the same measure as the level sets of f) and $\Omega^{\#}$ is the ball centered in the origin such that $|\Omega^{\#}| = |\Omega|$.

In [99], we have considered a lower order term $b(x,\nabla u)$ for (3), that is

$$-\mathrm{div}(a(x,u,\nabla u)) + b(x,\nabla u) = f \quad \text{in } \Omega, \quad u\in H_0^1(\Omega) \tag{5}$$

where a satisfies the ellipticity condition (4) and on b we assume that

$$|b(x,\xi)| \leq B(x)F(\xi)$$

where $B(x)$ is an integrable function. We have used convex symmetrization, to obtain comparison results with solutions of the convexly symmetric problem

$$\begin{cases} -\mathrm{div}(F(\nabla v)\nabla F_\xi(\nabla v)) + \tilde{b}(F^o(x))F(\nabla v)(\nabla F^o_\xi(x)\cdot\nabla F_\xi(\nabla v)) = f^\star \text{ in } \Omega^\star \\ v\in H_0^1(\Omega^\star), \end{cases}$$

where F^o is polar to F, $\tilde{b}$ is an auxiliary function related to B, $f^\star$ is the convex rearrangement of f with respect to F (i.e. the function such that its level sets are Wulff shape which have the same measure as the level sets of f) and $\Omega^\star$ is the set homothetic to the Wulff shape centered at the origin having the same measure as Ω.

We have obtained the following estimates:

$$u^\star \leq v \tag{6}$$
$$\int_\Omega F^q(\nabla u) \leq \int_{\Omega^\star} F^q(\nabla v), \tag{7}$$

where $0 < q \leq 2$ and $u^\star$ is the convex rearrangement of u. The proof is based on some differential inequalities for rearranged functions obtained using Schwarz and Hardy inequalities and the properties of homogeneity and convexity of the function F. Finally we consider the case where $\tilde{b}$ is essentially bounded by a constant β; we compare solutions of (5) with solutions to

$$\begin{cases} -\mathrm{div}(F(\nabla v)\nabla F_\xi(\nabla v)) - \beta F(\nabla v)(\nabla F^o_\xi(x)\cdot\nabla F_\xi(\nabla v)) = f^\star \text{ in } \Omega^\star \\ v\in H_0^1(\Omega^\star) \end{cases}$$

and we obtain same estimates similar to (6) and (7) of the preceding case.

In Section 2.2, our main aim is to study some properties of the Dirichlet eigenvalues of the anisotropic p-Laplacian operator (1). Namely, in [51], we analyze the values λ such that the problem

$$\begin{cases} -\mathcal{Q}_p u = \lambda(p,\Omega)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

admits a nontrivial solution in $W_0^{1,p}(\Omega)$. Let us observe that the operator in (1) reduces to the p-Laplacian when F is the Euclidean norm on $\mathbb{R}^n$ and, for a general norm F, $\mathcal{Q}_p$ is anisotropic and can be highly nonlinear. In literature, several papers are devoted to the study of the smallest eigenvalue of (8), denoted by $\lambda_1(p,\Omega)$, in bounded domains, which has the variational characterization

$$\lambda_1(p,\Omega) = \min_{\varphi \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega F^p(\nabla\varphi)\,dx}{\displaystyle\int_\Omega |\varphi|^p\,dx}.$$

Let Ω be a bounded domain (i.e. an open connected set). It is known (see [12]) that $\lambda_1(p,\Omega)$ is simple, the eigenfunctions have constant sign and it is isolated and the only positive eigenfunctions are the first eigenfunctions. Furthermore, the Faber-Krahn inequality holds:

$$\lambda_1(p,\Omega) \geq \lambda_1(p,\Omega^\star).$$

Many other results are known for $\lambda_1(p,\Omega)$. The interested reader may refer, for example, to [12, 15, 28, 48, 84]. As matter of fact, also different kind of boundary conditions have been considered as, for example, in the papers [44, 52] (Neumann case), [47] (Robin case).

Among the results contained in the quoted papers, we recall that if Ω is a bounded domain, it has been proved in [15] that

$$\lim_{p\to\infty} \lambda_1(p,\Omega)^{\frac{1}{p}} = \frac{1}{\rho_F(\Omega)},$$

where $\rho_F(\Omega)$ is the radius of the bigger Wulff shape contained in Ω, generalizing a well-known result in the Euclidean case contained in [82].

Actually, very few results are known for higher eigenvalues in the anisotropic case. In [69] the existence of a infinite sequence of eigenvalues is proved, obtained by means of a $\min-\max$ characterization. As in the Euclidean case, it is not known if this sequence exhausts all the set of the eigenvalues. Here we will show that the spectrum of $-\mathcal{Q}_p$ is a closed set, that the eigenfunctions are in $C^{1,\alpha}(\Omega)$ and admit a finite number of nodal domains. We recall the reference [92], where many results for the spectrum of the p-Laplacian in the Euclidean case have been summarized.

The core of the result of this Section relies in the study of the second eigenvalue $\lambda_2(p,\Omega)$, $p \in]1,+\infty[$, in bounded open sets, defined as

$$\lambda_2(p,\Omega) := \begin{cases} \min\{\lambda > \lambda_1(p,\Omega) \colon \lambda \text{ is an eigenvalue}\} & \text{if } \lambda_1(p,\Omega) \text{ is simple} \\ \lambda_1(p,\Omega) & \text{otherwise,} \end{cases}$$

and in analyzing its behavior when $p \to \infty$.

First of all, we show that if Ω is a domain, then $\lambda_2(p,\Omega)$ admits exactly two nodal domains. Moreover, for a bounded open set Ω, we prove a sharp lower bound for λ_2, namely the Hong-Krahn-Szego inequality

$$\lambda_2(p,\Omega) \geq \lambda_2(p,\widetilde{\mathcal{W}}),$$

where $\widetilde{\mathcal{W}}$ is the union of two disjoint Wulff shapes, each one of measure $\frac{|\Omega|}{2}$.

In the Euclidean case, such inequality is well-known for $p=2$, and it has been recently studied for any $1<p<+\infty$ in [20].

Finally, we address our attention to the behavior of $\lambda_2(p,\Omega)$ when Ω is a bounded open set and $p\to+\infty$. In particular, we show that

$$\lim_{p\to\infty} \lambda_2(p,\Omega)^{\frac{1}{p}} = \frac{1}{\rho_{2,F}(\Omega)},$$

where $\rho_{2,F}(\Omega)$ is the radius of two disjoint Wulff shapes $\mathcal{W}_1,\mathcal{W}_2$ such that $\mathcal{W}_1\cup\mathcal{W}_2$ is contained in Ω. Furthermore, the normalized eigenfunctions of $\lambda_2(p,\Omega)$ converge to a function u_∞ that is a viscosity solution to the fully nonlinear elliptic problem:

$$\begin{cases} A(u,\nabla u,\nabla^2 u) = \min\{F(\nabla u)-\lambda u, -\mathcal{Q}_\infty u\} = 0 & \text{in } \Omega, \text{if } u>0,\\ B(u,\nabla u,\nabla^2 u) = \max\{-F(\nabla u)-\lambda u, -\mathcal{Q}_\infty u\} = 0 & \text{in } \Omega, \text{if } u<0,\\ -\mathcal{Q}_\infty u = 0 & \text{in } \Omega, \text{if } u=0,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where

$$\mathcal{Q}_\infty u = F^2(\nabla u)(\nabla^2 u\ \nabla_\xi F(\nabla u))\cdot\nabla_\xi F(\nabla u). \tag{10}$$

In the Euclidean case, this kind of result has been proved for bounded domains in [81]. We consider both the nonconnected case and general norm F because our aim is twofold: first, to consider the case of a general Finsler norm F; second, to extend also the results known in the case of domains, to the case of nonconnected sets.

In Section 2.3, we consider the set $\mathscr{F}(\mathbb{R}^n)$ of lower semicontinuous functions, positive in $\mathbb{R}^n\setminus\{0\}$ and positively 1-homogeneous and we denote by $L^p_\omega(\Omega)$ the weighted $L^p(\Omega)$ space, where ω is a log concave function.

In a general anisotropic case and for bounded convex domains Ω of $\mathbb{R}^n$, we prove a sharp lower bound for the optimal constant $\Lambda_{p,\mathcal{F},\omega}(\Omega)$ in the Poincaré-type inequality

$$\inf_{t\in\mathbb{R}} \|u-t\|_{L^p_\omega(\Omega)} \leq \frac{1}{[\Lambda_{p,\mathcal{F},\omega}(\Omega)]^{\frac{1}{p}}}\|\mathcal{F}(\nabla u)\|_{L^p_\omega(\Omega)},$$

with $1<p<+\infty$ and $\mathcal{F}\in\mathscr{F}(\mathbb{R}^n)$.

If $\mathcal{F}$ is the Euclidean norm of $\mathbb{R}^n$ and $\omega=1$, then $\Lambda(p,\Omega)=\Lambda_{p,\mathcal{E},\omega}(\Omega)$ is the first nontrivial eigenvalue of the Neumann $p-$Laplacian:

$$\begin{cases} -\Delta_p u = \Lambda(p,\Omega)|u|^{p-2}u & \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

then for a convex set Ω, it holds that

$$\Lambda(p,\Omega) \geq \left(\frac{\pi_p}{\operatorname{diam}_{\mathcal{E}}(\Omega)}\right)^p,$$

where

$$\pi_p = 2\int_0^{+\infty} \frac{1}{1+\frac{1}{p-1}s^p}ds = 2\pi\frac{(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}, \qquad \mathrm{diam}_{\mathcal{E}}(\Omega) \text{ Euclidean diameter of } \Omega.$$

For other properties of π_p and of generalized trigonometric functions, we refer to [91].

This estimate, proved in the case $p = 2$ in [97] (see also [9]), has been generalized the case $p \neq 2$ in [1, 58, 66, 111] and for $p \to \infty$ in [57, 104]. Moreover the constant $\left(\frac{\pi_p}{\mathrm{diam}_{\mathcal{E}}(\Omega)}\right)^p$ is the optimal constant of the one-dimensional Poincaré-Wirtinger inequality, with $\omega = 1$, on a segment of length $\mathrm{diam}_{\mathcal{E}}(\Omega)$. When $p = 2$ and $\omega = 1$, in [17] an extension of the estimate in the class of suitable non-convex domains has been proved.

Our aim, in [52], is to prove an analogous sharp lower bound for $\Lambda_{p,\mathcal{F},\omega}(\Omega)$, in a general anisotropic case. More precisely, we prove the following inequality in a bounded convex domain $\Omega \subset \mathbb{R}^n$,

$$\Lambda_{p,\mathcal{F},\omega}(\Omega) = \inf_{\substack{u\in W^{1,\infty}(\Omega)\\ \int_\Omega |u|^{p-2}u\omega\,dx=0}} \frac{\displaystyle\int_\Omega \mathcal{F}(\nabla u)^p\omega\,dx}{\displaystyle\int_\Omega |u|^p\omega\,dx} \geq \left(\frac{\pi_p}{\mathrm{diam}_{\mathcal{F}}(\Omega)}\right)^p, \tag{11}$$

where $\mathrm{diam}_{\mathcal{F}}(\Omega) = \sup_{x,y\in\Omega} \mathcal{F}^o(y-x)$, $1 < p < \infty$ and ω is a positive log-concave function defined in Ω. This result has been proved in the case $p = 2$ and $\omega = 1$, when $\mathcal{F}$ is a strongly convex, smooth norm of $\mathbb{R}^n$ in [113] with a completely different method than the one presented here.

In Section 2.4, we study the limiting problem of the anisotropic p-Laplacian eigenvalue with Neumann boundary condition:

$$\begin{cases} -\mathcal{Q}_p u = \Lambda(p,\Omega)|u|^{p-2}u & \text{in } \Omega \\ \nabla_\xi F^p(\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

This problem is related to the Payne-Weinberger inequality (11). In [101], we study the the limit as $p \to \infty$ of eigenvalue problem (12), we consider

$$\begin{cases} A(u,\nabla u,\nabla^2 u) = \min\{F(\nabla u) - \Lambda u, -\mathcal{Q}_\infty u\} = 0 & \text{in } \Omega, \text{if } u > 0, \\ B(u,\nabla u,\nabla^2 u) = \max\{-F(\nabla u) - \Lambda u, -\mathcal{Q}_\infty u\} = 0 & \text{in } \Omega, \text{if } u < 0, \\ -\mathcal{Q}_\infty u = 0 & \text{in } \Omega, \text{if } u = 0, \\ \nabla F(\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where ν is the outer normal to $\partial\Omega$ and $\mathcal{Q}_\infty$ is defined as in (10). In the euclidean case ($F(\cdot) = |\cdot|$) this problem has been treated in [57, 104]. The solutions of (13) have be treated in viscosity sense and we refer to [32] and references therein for viscosity solutions theory and to [74] for Neumann problems condition in viscosity sense.

Let us observe that for $\Lambda = 0$ problem (13) has trivial solutions.

We prove that all nontrivial eigenvalues Λ of (13) are greater or equal than:

$$\Lambda(\infty,\Omega) := \frac{2}{\mathrm{diam}_F(\Omega)}.$$

This result has lots of interesting consequences. The first one is a Szegö-Weinberger inequality for convex sets, i.e. we prove that the Wulff shape $\Omega^\star$, that has the same measure of Ω, maximizes the first ∞-eigenvalue among sets with prescribed measure:

$$\Lambda(\infty,\Omega) \leq \Lambda(\infty,\Omega^\star).$$

Then we prove that the first positive Neumann eigenvalue of (13) is never larger than the first Dirichlet eigenvalue of (9):

$$\Lambda(\infty, \Omega) \leq \lambda(\infty, \Omega),$$

and that the equality holds if and only if Ω is a Wulff shape. Finally we prove two important results regarding the geometric properties of the first nontrivial ∞-eigenfunction. The first one shows that closed nodal domain cannot exist in Ω; the second one says that the first ∞-eigenfunction attains its maximum only on the boundary of Ω.

In the third chapter we are interested in variational problems that are called "non-local". This kind of problems are associated to non-standard Euler-Lagrange equations, in particular we consider equations perturbed with an integral term of the unknown function calculated on the entire domain. This kind of equations and functionals leads to a generalization of Sobolev inequality. On one hand, we introduce the non linearity with the means of a convex function F, on the other hand we add an integral term that represents the non-locality. Therefore, we study the optimal constant $\lambda(\alpha, \Omega)$ in the following Sobolev-Poincaré inequality:

$$\int_\Omega u^2 \, dx \leq \left[\frac{1}{\lambda(\alpha, \Omega)}\right] \left(\int_\Omega (F(\nabla u))^2 \, dx + \alpha \left(\int_\Omega u \, dx\right)^2\right), \quad u \in H_0^1(\Omega).$$

In Section 3.1, we consider the following minimization problem

$$\lambda(\alpha, \Omega) = \inf_{u \in H_0^1(\Omega)} \mathscr{Q}_\alpha(u, \Omega) \tag{14}$$

with

$$\mathscr{Q}_\alpha(u, \Omega) = \frac{\int_\Omega (F(\nabla u))^2 \, dx + \alpha (\int_\Omega u \, dx)^2}{\int_\Omega u^2 \, dx}$$

where α is a real parameter. In this case, the Euler-Lagrange equation associated to problem (14) presents an integral term calculated over all Ω, indeed the minimization problem (14) leads to the following eigenvalue problem

$$\begin{cases} -\operatorname{div}(F(\nabla u) \nabla_\xi F(\nabla u)) + \alpha \int_\Omega u \, dx = \lambda u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{15}$$

In the euclidean case, when $F(\xi) = |\xi|$, problems like the above ones arise, for example, in the study of reaction-diffusion equations describing chemical processes (see [105]). More examples can be found in [19], [29], [43], [70] and [98].

The extension to a general $F(\xi)$ is considered here as it has been made in other contexts to take into account a possible anisotropy of the problem. Typical examples are anisotropic elliptic equations ([4], [12]), anisotropic eigenvalue problems ([47], [48]), anisotropic motion by mean curvature ([10], [11]).

We also observe that, when $\alpha \to +\infty$, problem (14) becomes a twisted problem in the form (see [72] for the euclidean case)

$$\lambda^T(\Omega) = \inf_{u \in H_0^1(\Omega)} \left\{ \frac{\int_\Omega F^2(\nabla u) \, dx}{\int_\Omega u^2 \, dx}, \int_\Omega u \, dx = 0 \right\}.$$

As in [72], we prove the following isoperimetric inequality

$$\lambda^T(\Omega) \geq \lambda^T(\mathcal{W}_1 \cup \mathcal{W}_2),$$

where $\mathcal{W}_1$ and $\mathcal{W}_2$ are two disjoint Wulff shape, each one with measure $|\Omega|/2$.

Now, our principal objective consists in finding an optimal domain Ω which minimizes $\lambda(\alpha,\cdot)$ among all bounded open sets with a given measure. If we denote with κ_n the measure of $\mathcal{W}$, in the local case ($\alpha = 0$) we have a Faber-Krahn type inequality

$$\lambda(0,\Omega) \geq \lambda(0,\Omega^\star) = \frac{\kappa_n^{2/n} j_{n/2-1,1}}{|\Omega|^{2/n}},$$

where $j_{\nu,1}$ is the first positive zero of $J_\nu(z)$, the ordinary Bessel function of order ν, and $\Omega^\star$ is the Wulff shape centered at the origin with the same measure of Ω. Hence, when α vanishes, the optimal domain is a Wulff shape. We show that the non local term affects the minimizer of problem (14) in the sense that, up to a critical value of α, the minimizer is again a Wulff shape, but, if α is big enough, the minimizer becomes the union of two disjoint Wulff shapes of equal radii. This is a consequence of the fact that the problem (14) have an unusual rescaling with respect to the domain. Indeed, we have

$$\lambda(\alpha, t\Omega) = \frac{1}{t^2}\lambda(t^{n+2}\alpha,\Omega),$$

which, for $\alpha = 0$, becomes

$$\lambda(0, t\Omega) = \frac{1}{t^2}\lambda(0,\Omega),$$

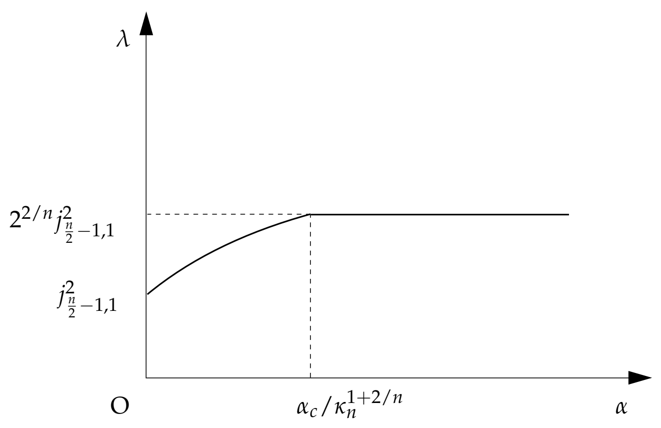
that is the rescaling in the local case. Therefore we show that we have a Faber-Krahn-type inequality only up to a critical value. Above this, we show a saturation phenomenon (see [71] for another example), that is the estimate cannot be improved and the optimal value remains constant. More precisely, in [100] we prove, for every $n \geq 2$, that, for every bounded, open set Ω in $\mathbb{R}^n$ and for every real number α, it holds

$$\lambda(\alpha,\Omega) \geq \begin{cases} \lambda(\alpha,\Omega^\star) & \text{if } \alpha|\Omega|^{1+2/n} \leq \alpha_c, \\ \frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}} & \text{if } \alpha|\Omega|^{1+2/n} \geq \alpha_c, \end{cases}$$

where

$$\alpha_c = \frac{2^{3/n}\kappa_n^{2/n} j^3_{n/2-1,1} J_{n/2-1,1}(2^{1/n} j_{n/2-1,1})}{2^{1/n} j_{n/2-1,1} J_{n/2-1}(2^{1/n} j_{n/2-1,1}) - n J_{n/2}(2^{1/n} j_{n/2-1,1})}.$$

If equality sign holds when $\alpha|\Omega|^{1+2/n} < \alpha_c$ then Ω is a Wulff shape, while if inequality sign holds when $\alpha|\Omega|^{1+2/n} > \alpha_c$ then Ω is the union of two disjoint Wulff shapes of equal measure. In Figure 1 we illustrate the transition between the two minimizers.



The continuous line represents the minimum of $\lambda(\alpha,\Omega)$, among the open bounded sets of measure κ_n, as a function of α.

In Section 3.2, we consider the following one-dimensional problem:

$$\lambda(\alpha,q)=\inf\left\{\mathcal{Q}[u,\alpha],\ u\in H_0^1(-1,1),\ u\not\equiv 0\right\}, \tag{16}$$

where $\alpha\in\mathbb{R}$, $1\le q\le 2$ and

$$\mathcal{Q}[u,\alpha]:=\frac{\displaystyle\int_{-1}^{1}|u'|^2dx+\alpha\left|\int_{-1}^{1}|u|^{q-1}u\,dx\right|^{\frac{2}{q}}}{\displaystyle\int_{-1}^{1}|u|^2dx}.$$

Let us observe that $\lambda(\alpha,q)$ is the optimal value in the inequality

$$\lambda(\alpha,q)\int_{-1}^{1}|u|^2dx\le\int_{-1}^{1}|u'|^2dx+\alpha\left|\int_{-1}^{1}|u|^{q-1}u\,dx\right|^{\frac{2}{q}}.$$

which holds for any $u\in H_0^1(-1,1)$. Moreover, in the local case ($\alpha=0$), this inequality reduces to the classical one-dimensional Poincaré inequality; in particular,

$$\lambda(0,q)=\frac{\pi^2}{4}$$

for any q.

The minimization problem (16) leads, in general, to a nonlinear nonlocal eigenvalue problem. Indeed, supposing $\int_{-1}^{1}y|y|^{q-1}\,dx\ge 0$, the associated Euler-Lagrange equation is

$$\begin{cases}-y''+\alpha\left(\displaystyle\int_{-1}^{1}y|y|^{q-1}\,dx\right)^{\frac{2}{q}-1}|y|^{q-1}=\lambda(\alpha,q)\,y & \text{in }]-1,1[\\ y(-1)=y(1)=0.\end{cases}$$

Our purpose, in [53], is to study some properties of $\lambda(\alpha,q)$. In particular, depending on α and q, we aim to prove symmetry results for the minimizers of (16).

Under this point of view, in the multidimensional case ($N\ge 2$) the problem has been settled out in [19] (when $q=1$) and in [43] (when $q=2$).

We show that the nonlocal term affects the minimizer of problem (16) in the sense that it has constant sign up to a critical value of α and, for α larger than the critical value, it has to change sign, and a saturation effect occurs. For $1\le q\le 2$, we prove that there exists a positive number α_q such that, if $\alpha<\alpha_q$, then $\lambda(\alpha,q)<\pi^2$, and any minimizer y of $\lambda(\alpha,q)$ has constant sign in $]-1,1[$. If $\alpha\ge\alpha_q$, then $\lambda(\alpha,q)=\pi^2$. Moreover, if $\alpha>\alpha_q$, the function $y(x)=\sin\pi x$, $x\in[-1,1]$, is the only minimizer, up to a multiplicative constant, of $\lambda(\alpha,q)$. Hence it is odd, $\int_{-1}^{1}|y(x)|^{q-1}y(x)\,dx=0$, and $\bar{x}=0$ is the only point in $]-1,1[$ such that $y(\bar{x})=0$.

Furthermore, we analyze the behaviour of the minimizers for the critical value $\alpha=\alpha_q$. If $q=1$, we have $\alpha_1=\frac{\pi^2}{2}$. Moreover, if $\alpha=\alpha_1$, there exists a positive minimizer of $\lambda(\alpha_1,1)$, and for any $\bar{x}\in]-1,1[$ there exists a minimizer y of $\lambda(\alpha_1,1)$ which changes sign in $\bar{x}$, non-symmetric and with $\int_{-1}^{1}y(x)\,dx\ne 0$ when $\bar{x}\ne 0$. If $1<q\le 2$ and $\alpha=\alpha_q$, then $\lambda(\alpha_q,q)$ in $[-1,1]$ admits both a positive minimizer and the minimizer $y(x)=\sin\pi x$, up to a multiplicative constant. Hence, any minimizer has constant sign or it is odd.

Let us observe that, for any $\alpha \in \mathbb{R}$, it holds that

$$\lambda(\alpha, q) \leq \Lambda_q = \pi^2,$$

where

$$\Lambda_q := \min \left\{ \frac{\displaystyle\int_{-1}^{1} |u'|^2 dx}{\displaystyle\int_{-1}^{1} |u|^2 dx},\ u \in H_0^1(-1,1),\ \int_{-1}^{1} |u|^{q-1} u\, dx = 0,\ u \not\equiv 0 \right\}. \tag{17}$$

It is known that, when $q \in [1,2]$, then $\Lambda_q = \Lambda_1 = \pi^2$, and the minimizer of (17) is, up to a multiplicative constant, $y(x) = \sin \pi x$, $x \in [-1,1]$ (see for example [34]).

Problems with prescribed averages of u and boundary value conditions have been studied in several papers. We refer the reader, for example, to [13, 27, 34, 55, 56, 75, 95]. In recent literature, also the multidimensional case has been adressed (see, for example [18, 72, 35, 36, 96]).

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1 PRELIMINARIES

1.1 REARRANGEMENTS

Let Ω be a measurable and not negligible subset of $\mathbb{R}^n$. We denote with $|\Omega|$ its n-dimensional Lebesgue measure. Let $\mathbb{R}^{n\times n}$ the space of real matrices, we denote the matrix product between two matrices $A, B \in \mathbb{R}^{n\times n}$ by (AB). Let $\xi, \zeta \in \mathbb{R}^n$, we denote the scalar product between ξ and ζ by $\xi \cdot \zeta$. We recall some definitions and properties of rearrangements (we refer to [31, 65, 85, 83]).

Definition 1.1. *Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the* distribution function *of u as the map $\mu : [0, \infty[\to [0, \infty[$ such that*

$$\mu(t) := |\{x \in \Omega : |u(x)| > t\}|. \tag{18}$$

Such function represents the measure of the level sets of u and satisfies the following properties.

Proposition 1.2. *Let μ defined as in (18), then*

1. *$\mu(\cdot)$ is monotone decreasing;*
2. *$\mu(0) = |\operatorname{supp} u|$;*
3. *$\operatorname{supp} \mu = [0, \operatorname{ess\,sup} |u|]$;*
4. *$\mu(\cdot)$ is right-continuous;*
5. *$\mu(t^-) - \mu(t) = |\{x \in \Omega : |u(x)| = t\}|$.*

Proof. Properties (1), (2) and (3) follows immediately from the definition. To prove (4) and (5), let us observe that

$$\{x \in \Omega : |u(x)| > t\} = \bigcup_{k=1}^{\infty} \left\{x \in \Omega : |u(x)| > t + \frac{1}{k}\right\}$$

and

$$\{x \in \Omega : |u(x)| > t\} = \bigcup_{k=1}^{\infty} \left\{x \in \Omega : |u(x)| > t - \frac{1}{k}\right\}.$$

Therefore

$$\mu(t^+) = \lim_{k\to\infty} \left| \left\{x \in \Omega : |u(x)| > t + \frac{1}{k}\right\} \right| = \mu(t),$$

and

$$\mu(t^-) = \lim_{k\to\infty} \left| \left\{x \in \Omega : |u(x)| > t + \frac{1}{k}\right\} \right| = \mu(t) + |\{x \in \Omega : |u(x)| = t\}|.$$

□

We stress that the distribution function μ is discontinuous only for the value t such that $|\{x \in \Omega : |u(x)| = t\}| \neq 0$.

Definition 1.3. *Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the* decreasing rearrangement *of u as the map $u^* : [0, \infty[\to [0, \infty[$ such that*

$$u^*(s) = \sup\{t > 0 \ : \ \mu(t) > s\}.$$

The decreasing rearrangements is a "generalization"of the inverse of μ in the sense that u^* is the distribution function of μ. Therefore u^* satisfies the following:

1. u^* is monotone decreasing;
2. u^* is right-continuous;
3. $u^*(0) = \operatorname{ess\,sup} u$;
4. $\operatorname{supp} u^* = [0, |\operatorname{supp} u|]$;
5. $u^*(\mu(t)) \leq t$ and $\mu(u^*(s)) \leq s$.

Definition 1.4. *Let u and $v : \Omega \to \mathbb{R}$ two measurable function, we say that u and v are* equimisurable *if they have the same distribution function.*

Proposition 1.5. *The functions $u : \Omega \to \mathbb{R}$ and $u^* : [0, |\Omega|] \to [0, \infty[$ are equimisurable, that is for all $t \geq 0$,*

$$|\{x \in \Omega : |u(x)| > t\}| = |\{s \in [0, |\Omega|] : u^*(s) > t\}| . \tag{19}$$

Proof. By the definition of u^*, it follows that

$$\begin{aligned} &\text{if } u^*(s) > t, \text{then } s < \mu(t); \\ &\text{if } u^*(s) \leq t, \text{then } s \geq \mu(t). \end{aligned}$$

Hence we have

$$\{s \geq 0 : u^*(s) > t\} = [0, \mu(t)[,$$

that gives (19). □

Proposition 1.6. *Let $u \in L^p(\Omega, \mathbb{R})$ and $1 \leq p \leq \infty$, then $u^* \in L^p(0, |\Omega|)$, and*

$$||u||_{L^p(\Omega)} = ||u^*||_{L^p(0,|\Omega|)}. \tag{20}$$

Proof. If $p < \infty$, then, by Cavalieri's principle we have

$$\int_\Omega u^p \, dx = \int_0^{+\infty} \mu(t) d(t^p).$$

Hence (20) follows from Proposition 1.5. If $p = \infty$, the result follows from the definition of rearrangement. □

Corollary 1.7. *Let $u : \Omega \to \mathbb{R}$ be a measurable function, we have*

$$|\{x \in \Omega : |u(x)| \geq t\}| = |\{s \in [0, |\Omega|] : u^*(s) \geq t\}| . \tag{21}$$
$$|\{x \in \Omega : |u(x)| < t\}| = |\{s \in [0, |\Omega|] : u^*(s) < t\}| . \tag{22}$$
$$|\{x \in \Omega : |u(x)| \leq t\}| = |\{s \in [0, |\Omega|] : u^*(s) \leq t\}| . \tag{23}$$

Proof. Since (23) is equivalent to (19) and (21) is equivalent to (22), then it is sufficient to prove that (19) and (21) are equivalent. Indeed, being

$$\lim_{h\to 0^+} |\{x \in \Omega : |u(x)| > t+h\}| = |\{x \in \Omega : |u(x)| > t\}|$$

and

$$\lim_{h\to 0^+} |\{x \in \Omega : |u(x)| > t-h\}| = |\{x \in \Omega : |u(x)| \geq t\}|,$$

we get the thesis. □

Definition 1.8. *We denote by $\Omega^\#$ the ball centered in the origin having the same measure as Ω. Let $u : \Omega \to \mathbb{R}$, we define the* sferically decreasing rearrangement *or* Schwarz symmetrization *of u, as the map $u^\# : \Omega^\# \to [0, \infty[$ such that*

$$u^\#(x) = u^*(\omega_n |x|^n), \quad x \in \Omega^\#,$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$, namely

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

For other details we refer to [83]. Now we state the following Theorem, which says that a function in $W_0^{1,p}(\Omega)$ is also in $W_0^{1,p}(\Omega^\#)$ and that the L^p-norm of the gradient decreases under the effect of rearrangement.

Theorem 1.9. *If $u \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < +\infty$, is a nonnegative function with compact support, then $u^\# W^{1,p}(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |\nabla u^\#|^p \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p \, dx.$$

To introduce the notion of perimeter given by De Giorgi in [42], we define the bounded variation function. For these results we mainly refer to [63, 60, 7, 117]. We denote by $\mathcal{B}(\Omega)$ the σ-algebra of all Borel subsets of Ω.

Definition 1.10. *Let $\nu : \mathcal{B}(\Omega) \to \mathbb{R}^k$ a vector-valued Radon measure on Ω, we define the* total variation *of ν in Ω by*

$$|\nu|(\Omega) = \sup\left\{ \sum_{i=1}^{k} \int_\Omega \varphi(x)\, d\nu_i(x) \; : \; \varphi \in C_0(\Omega, \mathbb{R}^k),\ ||\varphi||_\infty \leq 1 \right\}.$$

Definition 1.11. *Let $u \in L^1(\Omega)$, we say that u is a* function with bounded variation *in Ω, shortly a* BV function, *if there exists a Radon measure λ with values in $\mathbb{R}^n$ such that for any $i = 1, \ldots, n$ and any $\varphi \in C_0^1(\Omega)$*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega \varphi \, d\lambda_i.$$

The measure λ is also called the *measure derivative* of u and is denoted by the symbol Du. By $BV(\Omega)$ we denote the vector space of functions with bounded variations in Ω. This space can be endowed with the norm $||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + |Du|(\Omega)$, thus becoming a Banach space. We mainly use the following characterization of the BV functions.

Theorem 1.12. *Let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if*

$$V = \sup\left\{ \int_\Omega u(x)\,\mathrm{div}\,\varphi(x)\,dx \;:\; \varphi \in C_0^1(\Omega,\mathbb{R}^n), ||\varphi||_\infty \le 1 \right\} < \infty. \tag{24}$$

Moreover the supremum V in (24) is equal to the total variation $|Du|(\Omega)$ of Du in Ω.

We recall that a sequence ν_h of Radon measures is said to converge weakly* to a Radon measure ν if $\lim_{h\to\infty}\sum_{i=1}^k \int_\Omega \varphi_i\, d(\nu_h)_i = \sum_{i=1}^k \int_\Omega \varphi_i\, d\nu_i$ for any $\varphi \in C_c(\Omega,\mathbb{R}^k)$. Moreover a sequence u_h in $BV(\Omega)$ converges weakly* to a $BV(\Omega)$ function u if $u_h \to u$ in $L^1(\Omega)$ and Du_h converges to Du weakly* in Ω in the sense of measures. The following approximation theorem states that we can approximate the BV functions by smooth functions in the weak* convergence sense.

Theorem 1.13. *Let $u \in BV(\Omega)$. Then, there exists a sequence $\{u_h\}_{h\in\mathbb{N}} \subset C^\infty(\Omega)\cap BV(\Omega)$ such that*

$$u_h \to u \text{ weakly* in } BV(\Omega), \qquad \lim_{h\to\infty}\int_\Omega |\nabla u_h|\,dx = |Du|(\Omega)$$

Proof. We give the result in the case $\Omega = \mathbb{R}^n$. The general case is proved using the same methods, only with some extra technicalities. Let ρ be a positive, radially symmetric function with compact support in B_1, such that $\int_{\mathbb{R}^n}\rho\,dx = 1$. For all $\varepsilon > 0$, we set $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(\frac{x}{\varepsilon})$ and

$$u_\varepsilon(x) = (u*\rho_\varepsilon)(x) = \int_{\mathbb{R}^n}\rho_\varepsilon(x-y)u(y)\,dy.$$

By the properties of mollified functions, we have that $u_\varepsilon \to u$ in $L^1(\mathbb{R}^n)$. Moreover, for any $x \in \mathbb{R}^n$

$$\frac{\partial u_\varepsilon}{\partial x_i}(x) = \int_{\mathbb{R}^n} u(y)\frac{\partial}{\partial x_i}\rho_\varepsilon(x-y)\,dy = \int_{\mathbb{R}^n} u(y)\frac{\partial}{\partial y_i}\rho_\varepsilon(x-y)\,dy = -\int_{\mathbb{R}^n}\rho_\varepsilon(x-y)\,dD_iu(y), \tag{25}$$

for $i = 1,...,n$. Now we fix $\varphi \in C_0(\mathbb{R}^n)$, with $||\varphi||_\infty \le 1$ and, from (25), we get

$$\begin{aligned}\int_{\mathbb{R}^n}\varphi\cdot\nabla u_\varepsilon\,dx &= -\sum_{i=1}^n\int_{\mathbb{R}^n}\varphi_i(x)\,dx\int_{\mathbb{R}^n}\rho_\varepsilon(x-y)dD_iu(y)\\ &= -\sum_{i=1}^n\int_{\mathbb{R}^n}dD_iu(y)\int_{\mathbb{R}^n}\rho_\varepsilon(x-y)\varphi_i(x)\,dx = -\sum_{i=1}^n\int_{\mathbb{R}^n}(\varphi_i*\rho_\varepsilon)(y)\,dD_iu(y)\end{aligned}$$

Since also $||\varphi*\rho_\varepsilon||_\infty \le 1$, we can take the supremum over all such functions φ to obtain

$$\int_{\mathbb{R}^n}|\nabla u_\varepsilon|\,dx \le |Du|(\mathbb{R}^n). \tag{26}$$

Therefore the measures $\nabla u_\varepsilon\,dx$ converge weakly* to the measure Du and thus, by the lower semicontinuity of the total variation, we have

$$|Du|(\mathbb{R}^n) \le \liminf_{\varepsilon\to 0}\int_{\mathbb{R}^n}|\nabla u_\varepsilon|\,dx.$$

This inequality, together with (26), concludes the proof. □

Now we give the definition of perimeter.

Definition 1.14. *Let E be a measurable subset of $\mathbb{R}^n$ and Ω an open set. The* perimeter *of E in Ω is defined by the quantity*

$$P(E;\Omega) = \sup\left\{ \int_E \operatorname{div}\varphi\, dx \;:\; \varphi \in C_0^1(\Omega;\mathbb{R}^n),\ ||\varphi||_\infty \le 1 \right\}.$$

If $P(E;\Omega) < \infty$, we say that E is a set of finite perimeter *in Ω.*

We recall that by the use of Theorem 1.13 we have the following approximation result for sets of finite perimeter. We shall write simply $P(E)$ to denote the perimeter of E in $\mathbb{R}^n$.

Theorem 1.15. *Let E be a set of finite perimeter in $\mathbb{R}^n$ with $|E| < \infty$. Then, there exists a sequence of bounded open sets E_h with C^∞ boundaries, such that $\chi_{E_h} \to \chi_E$ in $L^1(\mathbb{R}^n)$ and $P(E_h) \to P(E)$.*

Finally we state the isoperimetric inequality (we refer to [73, 108]).

Theorem 1.16. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter with finite measure. Then,*

$$P(E) \ge n\omega_n^{\frac{1}{n}} |E|^{1-\frac{1}{n}}.$$

Proof. By Theorem 1.13 we can approximate χ_E by a sequence of function such that

$$u_h(x) \to \chi_E(x) \text{ a.e } x \in \mathbb{R}^n, \quad \int_{\mathbb{R}^n} |\nabla u_h|\, dx \to |D\chi_E|(\mathbb{R}^n) = P(E).$$

Then follows immediately by the classical Sobolev imbedding Theorem

$$\int_{\mathbb{R}^n} |\nabla u_h|\, dx \ge n\omega_n^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} |u_h|^{\frac{n}{n-1}}\, dx \right)^{\frac{n}{n-1}}.$$

□

1.2 CONVEX SYMMETRIZATION

Throughout this thesis we will consider a convex even 1-homogeneous function (see also [2, 4, 33, 103])

$$\xi \in \mathbb{R}^n \mapsto F(\xi) \in [0, +\infty[,$$

that is a convex function such that

$$F(t\xi) = |t| F(\xi), \quad t \in \mathbb{R},\ \xi \in \mathbb{R}^n, \tag{27}$$

and such that

$$a|\xi| \le F(\xi), \quad \xi \in \mathbb{R}^n, \tag{28}$$

for some constant $0 < a$. Under this hypothesis it is easy to see that there exists $b \ge a$ such that

$$F(\xi) \le b|\xi|, \quad \xi \in \mathbb{R}^n.$$

Moreover, we assume that

$$\nabla^2_{\xi}[F^p](\xi) \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}, \tag{29}$$

with $1 < p < +\infty$.

The hypothesis (29) on F assures that the operator

$$\mathcal{Q}_p[u] := \operatorname{div}\left(\frac{1}{p}\nabla_{\xi}[F^p](\nabla u)\right)$$

is elliptic, hence there exists a positive constant γ such that

$$\frac{1}{p}\sum_{i,j=1}^{n} \nabla^2_{\xi_i\xi_j}[F^p](\eta)\xi_i\xi_j \geq \gamma|\eta|^{p-2}|\xi|^2,$$

for some positive constant γ, for any $\eta \in \mathbb{R}^n \setminus \{0\}$ and for any $\xi \in \mathbb{R}^n$.

Remark 1.17. *We stress that for $p \geq 2$ the condition*

$$\nabla^2_{\xi}[F^2](\xi) \textit{ is positive definite in } \mathbb{R}^n \setminus \{0\},$$

implies (29).

The polar function $F^o \colon \mathbb{R}^n \to [0, +\infty[$ of F is defined as

$$F^o(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{F(\xi)}.$$

It is easy to verify that also F^o is a convex function which satisfies properties (27) and (28). Furthermore,

$$F(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{F^o(\xi)}.$$

From the above property it holds that

$$\xi \cdot \eta \leq F(\xi)F^o(\eta), \qquad \forall \xi, \eta \in \mathbb{R}^n. \tag{30}$$

The set

$$\mathcal{W} = \{\xi \in \mathbb{R}^n \colon F^o(\xi) < 1\}$$

is the so-called Wulff shape centered at the origin. We put

$$\kappa_n = |\mathcal{W}|,$$

where $|\mathcal{W}|$ denotes the Lebesgue measure of $\mathcal{W}$. More generally, we denote with $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape centered at x_0 with measure $\kappa_n r^n$, and $\mathcal{W}_r(0) = \mathcal{W}_r$.

The following properties of F and F^o hold true (see for example [11]):

$$\nabla_{\xi}F(\xi) \cdot \xi = F(\xi), \quad \nabla_{\xi}F^o(\xi) \cdot \xi = F^o(\xi), \tag{31}$$

$$F(\nabla_{\xi}F^o(\xi)) = F^o(\nabla_{\xi}F(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \tag{32}$$

$$F^o(\xi)\nabla_{\xi}F(\nabla_{\xi}F^o(\xi)) = F(\xi)\nabla_{\xi}F^o(\nabla_{\xi}F(\xi)) = \xi \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \tag{33}$$

$$\sum_{j=1}^{n} \nabla^2_{\xi_i\xi_j}F(\xi)\xi_j = 0, \quad \forall\, i = 1, \ldots, n. \tag{34}$$

Definition 1.18. *Let $u \in BV(\Omega)$, we define the* total variation of u with respect to F *as*

$$\int_\Omega |Du|_F = \sup\left\{\int_\Omega u \operatorname{div}\sigma\, dx \;:\; \sigma \in C_0^1(\Omega;\mathbb{R}^n), F^o(\sigma) \le 1\right\}$$

and the perimeter of a set E with respect to F*:*

$$P_F(E;\Omega) = \int_\Omega |D\chi_E|_F = \sup\left\{\int_E \operatorname{div}\sigma\, dx \;:\; \sigma \in C_0^1(\Omega;\mathbb{R}^n), F^o(\sigma) \le 1\right\}$$

These definition yields to the following co-area formula

$$\int_\Omega |Du|_F = \int_0^\infty P_F(\{u > s\};\Omega)\, ds \quad \forall u \in BV(\Omega)$$

and to the equality

$$P_F(E;\Omega) = \int_{\Omega\cap\partial^* E} F(\nu^E)\, d\mathbb{H}^{n-1}(x),$$

where $\partial^* E$ is the reduced boundary of E and ν^E is the outer normal to E (see also [6]).

Definition 1.19. *We denote by $\Omega^\star$ the Wulff shape centered in the origin having the same measure as Ω. Let $u : \Omega \to \mathbb{R}$, we define the* (decreasing) convex rearrangement *of u (see [4]) as the map $u^\star : \Omega^\star \to [0,\infty[$, such that*

$$u^\star(x) = u^*(\kappa_n(F^o(x))^n). \tag{35}$$

By definition it holds

$$||u||_{L^p(\Omega)} = ||u^\star||_{L^p(\Omega^\star)}, \quad \text{for } 1 \le p \le +\infty.$$

Furthermore, when u coincides with its convex rearrangement, we have (see [4])

$$\nabla u^\star(x) = u^{*'}(\kappa_n(F^o(x))^n)n\kappa_n(F^o(x))^{n-1}\nabla_\xi F^o(x); \tag{36}$$

$$F(\nabla u^\star(x)) = -u^{*'}(\kappa_n(F^o(x))^n)n\kappa_n(F^o(x))^{n-1}; \tag{37}$$

$$\nabla_\xi F(\nabla u^\star(x)) = \frac{x}{F^o(x)}. \tag{38}$$

Now, we recall here a result about a Pólya-Szegö principle related to H (we refer to [4], [22]) in the equality case (see [59], [64] for further details).

Proposition 1.20. *Let $u \in W_0^{1,p}(\Omega)$, $p \ge 1$. Then $u^\star \in W_0^{1,p}(\Omega)$*

$$\int_\Omega (F(\nabla u))^2\, dx \ge \int_{\Omega^\star} (F(\nabla u^\star))^2\, ds. \tag{39}$$

Furthermore, if u satisfies the equality in (39), then, for a.e. $t \in [0, \operatorname{ess\,sup} u]$, the set $\{x \in \Omega : u(x) > t\}$ is equivalent to a Wulff shape.

Proposition 1.21. *Let E be a subset of $\mathbb{R}^n$. Then $P_F(E;\Omega)$ is finite if and only if the usual perimeter $P(E;\Omega)$ is finite. Moreover we have*

$$\alpha P(E;\Omega) \le P_F(E;\Omega) \le \beta P(E;\Omega).$$

Proof. By (27) and (28), we have

$$\frac{1}{\beta}|\xi| \leq F^o(\xi) \leq \frac{1}{\alpha}|\xi|, \quad \forall \xi \in \mathbb{R}^n.$$

and hence the result follows. □

To show an isoperimetric inequality which estimate from below the perimeter with respect to a gauge function F of a set E, we give the following approximation results.

Proposition 1.22. *Let $u \in BV(\Omega)$. A sequence $\{u_h\}_{h\in\mathbb{N}} \subseteq C^\infty(\Omega)$ exists, such that:*

$$\lim_{h\to\infty} \int_\Omega |u_h - u| = 0$$

and

$$\lim_{h\to\infty} \int_\Omega |Du_h|_F = \int_\Omega |Du|_F.$$

Proof. By mollifying u, we define a sequence $\{u_h\}_{h\in\mathbb{N}}$ with the required properties following the proof of Theorem 1.17 of [76]. □

Proposition 1.23. *Let E be a set of finite perimeter in Ω. There exists a sequence $\{E_h\}_{h\in\mathbb{N}}$ of C^∞ sets such that:*

$$\lim_{h\to\infty} \int_\Omega |\chi_{E_h} - \chi_E| = 0$$

and

$$\lim_{h\to\infty} \int_\Omega |D\chi_{E_h}|_F = P_F(E;\Omega)$$

Proof. We find the proof in [4]. We mollify the function χ_E as in Proposition 1.22, hence we find a sequence $\{f_h\}_{h\in\mathbb{N}} \subset C^\infty(\Omega)$ such that:

$$\lim_{h\to\infty} \int_\Omega |f_h - \chi_E| = 0$$

and

$$\lim_{h\to\infty} \int_\Omega |Df_h|_F = P_F(E;\Omega).$$

By the coarea formula we have

$$\int_\Omega |u_h|_F = \int_0^1 P_F(\{u_h > s\};\Omega) ds.$$

Sard's theorem implies that the sets $E_s^{(h)} = \{u_h > s\}$ have C^∞ boundary for almost every $s \in (0,1)$ and We consider only such levels s. Let us fix $\varepsilon \in]0, \frac{1}{4}[$ and $h = h(\varepsilon)$ such that:

$$\int_\Omega |f_h - \chi_E| < \varepsilon.$$

Arguing as in [93, Lemma 2, p.299], we get

$$\int_\Omega |\chi_E - \chi_{E_s^{(h)}}| < \varepsilon^{\frac{1}{2}}, \tag{40}$$

for every $s \in [\varepsilon^{\frac{1}{2}}, 1-\varepsilon^{\frac{1}{2}}]$. On the other hand, for every h there exists $s_h \in (\varepsilon^{\frac{1}{2}}, 1-\varepsilon^{\frac{1}{2}})$ such that:

$$(1-2\varepsilon^{\frac{1}{2}}) P_F(E^{(h)}_{s_h};\Omega) \leq \int_0^1 P_F(E^{(h)}_t;\Omega)\, dt. \tag{41}$$

Moreover we have

$$P_F(E;\Omega) = \lim_{h\to\infty} \int_\Omega |Df_h|_F = \lim_{h\to\infty} \int_0^1 P_F(E^{(h)}_t;\Omega)\, dt. \tag{42}$$

By (40), it follows that $\chi_{E^{(h)}_{s_h}} \to \chi_E$ in $L^1(\Omega)$ and by (41) and (42) we have

$$\limsup_{\varepsilon\to 0} P_F(E^{(h)}_{s_h};\Omega) \leq P_F(E;\Omega).$$

Since P_F is lower semicontinuous, the result follows. □

We observe that if $u \in W^{1,1}(\Omega)$ then

$$\int_\Omega |Du|_F = \int_\Omega F(\nabla u)\, dx$$

and it holds

$$-\frac{d}{dt}\int_{u>t} |\nabla u|_F\, dx = P_F(\{u>t\};\Omega). \tag{43}$$

Another important result that has been generalized to the anisotropic case is the isoperimetric inequality [41, 67, 68].

Proposition 1.24. *If E is a set of finite perimeter in $\mathbb{R}^n$, then:*

$$P_F(E;\mathbb{R}^n) \geq n\kappa_n^{\frac{1}{n}} |E|^{1-1/n}. \tag{44}$$

Proof. If E is a smooth set, then in [26] is proved the following

$$P_F(E;\mathbb{R}^n) = \int_{\mathbb{R}^n} |D\chi_E|_F = \int_E F(\nu^E)\, d\sigma \geq n\kappa_n^{\frac{1}{n}} |E|^{1-1/n}, \tag{45}$$

where ν^E is the outer normal to E. Then by Proposition 1.23 and (45), the result follows. □

Now, we recall the useful definitions of anisotropic distance, diameter and inradius. We define the anisotropic distance function (or F-distance) to $\partial\Omega$ as

$$d_F(x) := \inf_{y\in\partial\Omega} F^o(x-y), \quad x \in \overline{\Omega},$$

and the anisotropic inradius as

$$\rho_F := \max\{d_F(x),\ x \in \overline{\Omega}\}.$$

We denote the diameter diam_F of Ω with respect to the norm F on $\mathbb{R}^n$ as

$$\mathrm{diam}_F(\Omega) := \sup_{x,y\in\overline{\Omega}} F^o(x-y). \tag{46}$$

It will be useful in the sequel an anisotropic version of the isodiametric inequality.

Proposition 1.25. *Let Ω be a convex set in $\mathbb{R}^n$. Then*

$$|\Omega| \leq \frac{\kappa_n}{2^n} \operatorname{diam}_F(\Omega)^n. \tag{47}$$

The equality sign holds if and only if Ω is equivalent to a Wulff shape.

Proof. We want prove that

$$\frac{\operatorname{diam}_F(\Omega)^n}{|\Omega|} \geq \frac{2^n}{\kappa_n} = \frac{\operatorname{diam}_F(\mathcal{W})^n}{|\mathcal{W}|}.$$

We argue similarly as in [24, Th 11.2.1]. Firstly, we observe that from definitions, it follows that Ω has the same anisotropic diameter of its convex envelope, but it has a lower or equal volume. Hence, if we denote by Ω^C the convex envelope of Ω, we have that

$$\frac{\operatorname{diam}_F(\Omega)^n}{|\Omega|} \geq \frac{\operatorname{diam}_F(\Omega^C)^n}{|\Omega^C|}. \tag{48}$$

Therefore, we can suppose that Ω is a convex set and we prove that the minimum of the right hand side of (48) is reached by a Wulff shape.

Let us suppose that $\operatorname{diam}_F \Omega \leq 1$, we denote by Ω' the set that is symmetric to Ω with respect to the origin and put $B := (\Omega + \Omega')/2$. The function $|t\Omega + (1-t)\Omega'|^{1/n}$, $0 \leq t \leq 1$, is concave so that $|\Omega| = |\Omega'| \leq |B|$ and the equality sign holds only if Ω is homothetic to Ω', i.e. if Ω has a center of symmetry. Let us call a and b the point that realize the diameter of B: $F^o(a-b) = \operatorname{diam}_F B$. Now, $a = x + x'/2$, $b = y + y'/2$, where $x, y \in \Omega$ and $x', y' \in \Omega'$, hence:

$$\begin{aligned} F^o(a-b) = \frac{1}{2} F^o(x + x' - y - y') &\leq \frac{1}{2}\left(F^o(x-y) + F^o(x'-y')\right) \\ &\leq \frac{1}{2}\operatorname{diam}_F \Omega + \frac{1}{2}\operatorname{diam}_F \Omega' \end{aligned}$$

and therefore $\operatorname{diam}_F B \leq 1$. Now, it is sufficient to assume that Ω has a center of symmetry. But then $\operatorname{diam}_F(\Omega) \leq 1$ implies that Ω is contained in Wulff shape of unit diameter, i.e. $|\Omega| \leq \kappa_n/2^n$. This in turn implies (47). □

Finally we observe that, in general, F and F^o are not rotational invariant. Anyway, let us consider $A \in SO(n)$ and define

$$F_A(x) = F(Ax).$$

Since $A^T = A^{-1}$, then

$$(F_A)^o(\xi) = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x \cdot \xi}{F_A(x)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{A^T y \cdot \xi}{F(y)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{y \cdot A\xi}{F(y)} = (F^o)_A(\xi).$$

Moreover, we also have

$$\operatorname{diam}_{F_A}(A^T\Omega) = \sup_{x,y \in A^T\Omega} (F^o)_A(y-x) = \sup_{\bar{x},\bar{y} \in \Omega} F^o(\bar{y} - \bar{x}) = \operatorname{diam}_F(\Omega).$$

2 ANISOTROPIC LAPLACIAN EIGENVALUE PROBLEMS

In this chapther we analyze some properties of the eigenvalues of the anisotropic p-Laplacian operator

$$-\mathcal{Q}_p u := -\operatorname{div}\left(F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)\right), \tag{49}$$

where F is a suitable smooth norm of $\mathbb{R}^n$ and $p \in]1, +\infty]$. We provide sharp estimates for eigenvalues of $\mathcal{Q}_p u$ with both Dirichlet and Neumann boundary condition.

2.1 CONVEX SYMMETRIZATION FOR ANISOTROPIC ELLIPTIC EQUATIONS WITH A LOWER ORDER TERM

2.1.1 Preliminary results

In this Section, we estimate the solution of the eigenvalue problem of the anisotropic Laplacian with a lower order term, when Dirichlet boundary condition holds. We obtain comparison results with solutions of the convexly symmetric problem:

$$\begin{cases} -\operatorname{div}(F(\nabla v)\nabla F(\nabla v)) - \tilde{b}(F^o(x))(\nabla F^o(x) \cdot \nabla F(\nabla v))F(\nabla v) = f^\star \text{ in } \Omega^\star \\ v = 0 \text{ in } \partial\Omega^\star \end{cases} \tag{50}$$

where $\tilde{b}$ is a suitable auxiliary function (see further for details). Firstly, we give three Lemmas, that are basic for our treatment.

Lemma 2.1. *If u is any member of $H_0^1(\Omega)$, then*

$$\frac{1}{n^2\kappa_n^{\frac{2}{n}}}\mu(t)^{\frac{2}{n}-2}\left[-\mu'(t)\right]\left[-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u)\right] \geq 1 \tag{51}$$

for a.e. t such that $0 < t < \operatorname{ess\,sup}|u|$.

Proof. For $h > 0$, Schwarz inequality gives

$$\frac{1}{h}\int_{t<|u|\leq t+h} F(\nabla u) \leq \frac{1}{h}\left(\int_{t<|u|\leq t+h} dx\right)^{\frac{1}{2}}\left(\int_{t<|u|\leq t+h} F^2(\nabla u)\right)^{\frac{1}{2}}$$

and

$$\frac{1}{h}\int_{t<|u|\leq t+h} F(\nabla u) \leq \left(\frac{1}{h}(\mu(t) - \mu(t+h)\right)^{\frac{1}{2}}\left(\frac{1}{h}\int_{t<|u|\leq t+h} F^2(\nabla u)\right)^{\frac{1}{2}}.$$

Therefore, as $h \to 0^+$, we obtain

$$-\frac{d}{dt}\int_{|u|>t} F(\nabla u) \leq (-\mu'(t))^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u)\right)^{\frac{1}{2}}.$$

By (44) and (43), we have

$$n\kappa_n^{1/n}\mu(t)^{1-\frac{1}{n}} \le \sqrt{-\mu'(t)}\left(-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u)\right)^{\frac{1}{2}}.$$

Then squaring and dividing by $n^2\kappa_n^{2/n}\mu(t)^{2-\frac{2}{n}}$, we obtain (51). □

Lemma 2.2.

$$\int_E |f| \le \int_0^{|E|} f^*(s)\, ds$$

for any measurable set E.

This Lemma is a special case of a theorem by Hardy and Littlewood (*see* [78], Theorem 378).

Lemma 2.3. *If φ is bounded and*

$$\varphi(t) \le \int_t^{+\infty} K(s)\varphi(s)\, ds + \psi(t)$$

for a.e. $t > 0$, then

$$\varphi(t) \le \int_t^{+\infty} \exp\left(\int_t^s K(r)\, dr\right)(-d\psi(s))$$

for a.e. $t > 0$. Here K is any nonnegative integrable function, ψ has bounded variation and vanishes at $+\infty$.

Lemma 2.3 is a generalization of Gronwall's lemma.

2.1.2 Main result

In this section we discuss our main result. It consists in showing that a solution to (5) can be compared in term of a solution to (50), where the function $\tilde{b}$ is known as a pseudo rearrangement of $B(x)$. It can be defined as

$$\tilde{b}\left(\left(\frac{s}{\kappa_n}\right)^{\frac{1}{n}}\right) = \left(\frac{d}{ds}\int_{|u|>u^*(s)} B^2(x)\right)^{\frac{1}{2}}, \tag{52}$$

We refer to [5] and [107] for further details.

Theorem 2.4. *Let $u \in H_0^1(\Omega)$ be a solution to the problem*

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + b(x,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{53}$$

where $a(x,\eta,\xi) \equiv \{a_i(x,\eta,\xi)\}_{i=1,\dots,n}$ are Carathéodory functions satisfying

$$a(x,\eta,\xi)\cdot\xi \ge F^2(\xi) \quad a.e. \quad x\in\Omega, \quad \eta\in\mathbb{R}, \quad \xi\in\mathbb{R}^n. \tag{54}$$

and $b(x,\xi)$ is such that:

$$|b(x,\xi)| \le B(x)F(\xi), \tag{55}$$

where $B \in L^k(\Omega)$, *with* $k > n$. *We assume further that* $f \in L^{\frac{2n}{n+2}}(\Omega)$ *if* $n \geq 3$; $f \in L^p(\Omega)$, $p > 1$, *if* $n = 2$;
$F : \mathbb{R}^n \to [0, \infty[$ *is a convex function satisfying* (27)-(28).
Then

$$u^\star \leq v \tag{56}$$

$$\int_\Omega F^q(\nabla u) \leq \int_{\Omega^\star} F^q(\nabla v) \tag{57}$$

with $0 < q \leq 2$, *and*

$$v(x) = \int_{F^o(x)}^{\left(\frac{|\Omega|}{\kappa_n}\right)^{1/n}} \frac{1}{t^{n-1}} dt \int_0^t \exp\left(\int_t^r \tilde{b}(r')dr'\right) f^*(\kappa_n r^n) r^{n-1} dr. \tag{58}$$

where $\tilde{b}$ *is defined as in* (52).

Remark 2.5. *The function in* (58) *is convexly symmetric, in the sense that* $v(x) = v^\star(x)$. *Indeed the function*

$$v^*(s) = \int_s^{|\Omega|} \frac{1}{n^2 \kappa_n^{2/n}} t^{\frac{2}{n}-2} dt \int_0^t \exp\left(\int_{\left(\frac{r}{\kappa_n}\right)^{1/n}}^{\left(\frac{t}{\kappa_n}\right)^{1/n}} \tilde{b}(r')dr'\right) f^*(r) dr$$

is decresing and $v(x) = v^*(\kappa_n (F^o(x))^n)$. *We observe that* $v(x)$ *is a solution in* $H_0^1(\Omega^\star)$ *to the problem*

$$\begin{cases} -div(F(\nabla v)\nabla F(\nabla v)) - \tilde{b}(F^o(x))(\nabla F^o(x) \cdot \nabla F(\nabla v))F(\nabla v) = f^\star \text{ in } \Omega^\star \\ v = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega^\star. \end{cases} \tag{59}$$

In fact, if we define $\rho = F^o(x)$ *and we look for a solution such that* $v(\rho) = v(F^o(x))$, *we obtain*

$$\nabla v = v'(\rho)\nabla_\xi F^o(x), \tag{60}$$

$$F(\nabla v) = -v'(\rho)F(\nabla_\xi F^o(x)) = -v'(\rho), \tag{61}$$

$$\nabla_\xi F(\nabla v) = \nabla_\xi F(v'(\rho)\nabla_\xi F^o(x)) = \nabla_\xi F(\nabla_\xi F^o(x)) = \frac{x}{F^o(x)}. \tag{62}$$

A direct computation gives

$$\begin{aligned} -div(F(\nabla v)\nabla_\xi F(\nabla v)) - \tilde{b}(F^o(x))F(\nabla v)\nabla_\xi F^o(x) \cdot \nabla_\xi F(\nabla v) \\ = -v''(\rho) - \frac{n-1}{\rho} v'(\rho) + \tilde{b}(F^o(x))v'(\rho). \end{aligned}$$

Using (58), *we can write:*

$$v(\rho) = \int_\rho^{\left(\frac{|\Omega|}{\kappa_n}\right)^{1/n}} \frac{1}{t^{n-1}} dt \int_0^t \exp\left(\int_\rho^t g(r')dr'\right) f^*(\kappa_n r^n) r^{n-1} dr \tag{63}$$

and we have:

$$-v''(\rho) - \frac{n-1}{\rho} v'(\rho) + \tilde{b}(F^o(x))v'(\rho) = f^\star(\rho). \tag{64}$$

Collecting (63) *and* (64) *we obtain that the function in* (58) *solves* (59).

Remark 2.6. *We can compute $\int_{\Omega^\star} F^q(\nabla v)$. By (61) we have*

$$[F(\nabla v(x))]^q = [v'(\rho)]^q = \left[-\frac{1}{\rho^{n-1}} \int_0^\rho \exp\left(\int_r^\rho \tilde{b}(r')dr' \right) f^*(\kappa_n r^n) r^{n-1} dr \right]^q$$

where $\rho = F^o(x)$. An integration by the substitution $s = \kappa_n r^n$ gives

$$[F(\nabla v(x))]^q = \left[-\frac{1}{n\kappa_n \rho^{n-1}} \int_0^{\kappa_n \rho^n} \exp\left(\int_{\left(\frac{s}{\kappa_n}\right)^{1/n}}^{\rho} \tilde{b}(r')dr' \right) f^*(s) ds \right]^q,$$

therefore, by an integration on $\Omega^\star$, we have

$$\int_{\Omega^\star} [F(\nabla v(x))]^q$$
$$= \int_0^{|\Omega|} \left[-\frac{1}{n\kappa_n \rho^{n-1}} \int_0^{\kappa_n \rho^n} \exp\left(\int_{\left(\frac{s}{\kappa_n}\right)^{1/n}}^{\rho} \tilde{b}(r')dr' \right) f^*(s) ds \right]^q d\rho.$$

Hence, by the sustitution $\tau = \kappa_n \rho^n$, we have

$$\int_{\Omega^\star} [F(\nabla v(x))]^q$$
$$= \int_0^{|\Omega|} \left[-\frac{1}{n\kappa_n^{1/n}} \tau^{\frac{1}{n}-1} \int_0^{\tau} \exp\left(\int_{\left(\frac{s}{\kappa_n}\right)^{1/n}}^{\left(\frac{\tau}{\kappa_n}\right)^{1/n}} \tilde{b}(r')dr' \right) f^*(s) ds \right]^q d\tau.$$

Theorem 2.7. *Let $u \in H_0^1(\Omega)$ be a solution to problem (53) under the assumption (54). Furthermore we suppose that (55) holds with*

$$||B||_{L^\infty(\Omega)} = \beta \leq \infty;$$

$f \in L^{\frac{2n}{n+2}}(\Omega)$ if $n \geq 3$; $f \in L^p(\Omega)$, $p > 1$, if $n = 2$; $F : \mathbb{R}^n \to [0, \infty[$ is a convex function satisfying (27)-(28).
Then (56) and (57) holds with

$$v(x) = \int_{F^o(x)}^{\left(\frac{|\Omega|}{\kappa_n}\right)^{1/n}} \frac{1}{t^{n-1}} dt \int_0^t e^{\beta(r-t)} f^*(\kappa_n r^n) r^{n-1} dr. \tag{65}$$

Remark 2.8. *The function $v(x)$ in (65) is a solution in $H_0^1(\Omega^\star)$ to the problem*

$$\begin{cases} -\mathrm{div}(F(\nabla v)\nabla_\xi F(\nabla v)) - \beta F(\nabla v)\nabla_\xi F^o(x) \cdot \nabla_\xi F(\nabla v) = f^\star & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star. \end{cases}$$

The proof of Theorem 2.7 is similar to that of Theorem 2.4 and it can be obtained from it considering the function $B(x)$ as a constant.

2.1.3 Proof of main Theorem

Let us start by proving a preliminary result about the function $\tilde{b}$ (*see* [107]).

Lemma 2.9. *If $\tilde{b}$ is defined by (52), then*

$$\left(-\frac{d}{dt} \int_{|u|>t} B^2(x) \right)^{\frac{1}{2}} = \sqrt{-\mu'(t)}\, \tilde{b}\left(\left(\frac{\mu(t)}{\kappa_n} \right)^{\frac{1}{n}} \right) \tag{66}$$

and

$$-\frac{d}{dt}\int_{|u|>t} B(x)F(\nabla u) \le \left(-\frac{d}{dt}\int_0^{\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac{1}{n}}} \tilde{b}(r)dr\right)\left(-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u)\right) \tag{67}$$

for almost every $t \in [0, \operatorname{ess\,sup}_\Omega |u|]$.

Proof. Let $p(t)$ and $q(s)$ be the integrals of $B(x)$ over $\{|u| > t\}$ and $\{|u| > u^*(s)\}$ respectively, hence $p'(t) = q'(\mu(t))\mu'(t)$ for almost every $t \in [0, \operatorname{ess\,sup}_\Omega u]$. So equality (66) is proved.
By Hölder inequality, we have

$$-\frac{d}{dt}\int_{|u|>t} B(x)F(\nabla u) \le \left(-\frac{d}{dt}\int_{|u|>t} B(x)\right)^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u)\right)^{\frac{1}{2}},$$

by (66) we obtain

$$-\frac{d}{dt}\int_{|u|>t} B(x)F(\nabla u) \le \sqrt{-\mu'(t)}\;\tilde{b}\left(\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac{1}{n}}\right)\left(-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u)\right)^{\frac{1}{2}},$$

hence, by Lemma 2.1 ,

$$\begin{aligned}-\frac{d}{dt}&\int_{|u|>t} B(x)F(\nabla u)\\ &\le -\mu'(t)\frac{\mu(t)^{\frac{1}{n}-1}}{n\kappa_n^{1/n}}\;\tilde{b}\left(\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac{1}{n}}\right)\left(-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u)\right),\end{aligned}$$

that is equal to the right-hand side of (67). □

Proof of Theorem 2.4. Suppose u is a weak solution of problem (53), then

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi + \int_\Omega b(x,\nabla u)\varphi = \int_\Omega f\varphi, \qquad \forall\varphi \in H_0^1(\Omega). \tag{68}$$

For $h > 0$, $t > 0$, let φ be the following test function

$$\varphi_h(x) = \begin{cases} h, & \text{if } |u| > t+h \\ |u|-t, & \text{if } t < |u| \le t+h \\ 0, & \text{if } |u| \le t,\end{cases}$$

then

$$\nabla_i\varphi_h(x) = \begin{cases} 0, & \text{if } |u| > t+h \\ \nabla_i u, & \text{if } t < |u| \le t+h \\ 0, & \text{if } |u| \le t.\end{cases}$$

Inserting this test function in (68), we have

$$\begin{aligned}\int_{t<|u|\le t+h} a(x,u,\nabla u)\cdot\nabla u &+ \int_{|u|>t+h} b(x,\nabla u)h\\ &= \int_{|u|>t+h} fh + \int_{t<|u|\le t+h} (f - b(x,\nabla u))(|u|-t)\operatorname{sign} u.\end{aligned}$$

The last term is smaller than $\int_{t<|u|\le t+h}(f-b(x,\nabla u))(|u|-t)$ and, by hypothesis (54) and (55), we have

$$\int_{t<|u|\le t+h} F^2(\nabla u) - h\int_{|u|>t+h} B(x)F(\nabla u) \le \int_{|u|>t+h} fh + \int_{t<|u|\le t+h} (f-b(x,\nabla u))(|u|-t). \tag{69}$$

Dividing each term by h, as $h\to 0^+$, (69) becomes

$$-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u) - \int_{|u|>t} B(x)F(\nabla u) \le \int_{|u|>t} f,$$

and, by Lemma 2.2,

$$-\frac{d}{dt}\int_{|u|>t} F^2(\nabla u) - \int_{|u|>t} B(x)F(\nabla u) \le \int_0^{\mu(t)} f^*(s)ds. \tag{70}$$

Now, we can write

$$\int_{|u|>t} B(x)F(\nabla u) = \int_t^{+\infty}\left(-\frac{d}{ds}\int_{|u|>s} B(x)F(\nabla u)\right)ds$$

and hence, by Lemma 2.9, we have

$$\begin{aligned}\int_{|u|>t} & B(x)F(\nabla u)\\ &\le \int_t^{+\infty}\left(-\frac{d}{dt}\int_0^{\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac1n}} \tilde b(r)dr\right)\left(-\frac{d}{ds}\int_{|u|>s}F^2(\nabla u)\right)ds.\end{aligned} \tag{71}$$

Inserting (71) in (70) we obtain

$$\begin{aligned}&-\frac{d}{dt}\int_{|u|>t}F^2(\nabla u)\\ &\le \int_t^{+\infty}\left(-\frac{d}{dt}\int_0^{\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac1n}} \tilde b(r)dr\right)\left(-\frac{d}{ds}\int_{|u|>s}F^2(\nabla u)\right)ds + \int_0^{\mu(t)} f^*(s)ds.\end{aligned}$$

Now we can use Lemma 2.3 with $\varphi(t) = -\frac{d}{dt}\int_{|u|>t}F^2(\nabla u)$. We have

$$-\frac{d}{dt}\int_{|u|>t}F^2(\nabla u) \le \int_t^{+\infty}\exp\left(\int_t^s -\frac{d}{dr}\int_0^{\left(\frac{\mu(r)}{\kappa_n}\right)^{\frac1n}}\tilde b(r')dr'\right)\left[-d\psi(s)ds\right],$$

where $\psi(s) = \int_0^{\mu(s)} f^*(\xi)d\xi$.
Using the substitution $\rho=\mu(s)$ and $\sigma=\mu(r)$, we obtain

$$-\frac{d}{dt}\int_{|u|>t}F^2(\nabla u) \le \int_0^{\mu(t)}\exp\left(\int_{\left(\frac{\sigma}{\kappa_n}\right)^{\frac1n}}^{\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac1n}}\tilde b(\rho)d\rho\right) f^*(\sigma)d\sigma. \tag{72}$$

Inequality (72) and Lemma 2.1 give

$$1\le \frac{1}{n^2\kappa_n^{2/n}}\mu(t)^{\frac2n-2}(-\mu'(t))\int_0^{\mu(t)}\exp\left(\int_{\left(\frac{\sigma}{\kappa_n}\right)^{\frac1n}}^{\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac1n}}\tilde b(\rho)d\rho\right) f^*(\sigma)d\sigma.$$

for a.e. $t \in [0, \operatorname{ess\,sup}|u|]$, then integration of both sides with respect to t over the interval $[0, u^*(s)]$ yields

$$u^*(s) \leq \int_s^{|\Omega|} dt \frac{1}{n^2 \kappa_n^{2/n}} t^{\frac{2}{n}-2} \int_0^t \exp\left(\int_{\left(\frac{\sigma}{\kappa_n}\right)^{\frac{1}{n}}}^{\left(\frac{t}{\kappa_n}\right)^{\frac{1}{n}}} \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma \tag{73}$$

From formula (58), we learn that $v^*(s)$ is the right-hand side of (73), so (56) is satisfied. In order to prove (57), we observe that Hölder inequality gives

$$\frac{1}{h} \int_{t<|u|\leq t+h} F^q(\nabla u) \leq \left(\frac{1}{h} \int_{t<|u|\leq t+h} dx \right)^{1-\frac{q}{2}} \left(\frac{1}{h} \int_{t<|u|\leq t+h} F^2(\nabla u) \right)^{\frac{q}{2}}$$

and hence, for $t \to 0^+$,

$$-\frac{d}{dt} \int_{|u|>t} F^q(\nabla u) \leq (-\mu'(t))^{1-\frac{q}{2}} \left(-\frac{d}{dt} \int_{|u|>t} F^2(\nabla u) \right)^{\frac{q}{2}}, \tag{74}$$

provided that $0 < q \leq 2$. Lemma 2.1 gives

$$\left[-\frac{d}{dt} \int_{|u|>t} F^2(\nabla u) \right]^{\frac{1}{2}} \leq \frac{1}{n \kappa_n^{1/n}} \mu(t)^{\frac{1}{n}-1} (-\mu'(t))^{\frac{1}{2}} \left[-\frac{d}{dt} \int_{|u|>t} F^2(\nabla u) \right],$$

hence by inequality (72)

$$\begin{aligned} &\left[-\frac{d}{dt} \int_{|u|>t} F^2(\nabla u) \right]^{\frac{1}{2}} \\ &\leq \frac{1}{n \kappa_n^{1/n}} \mu(t)^{\frac{1}{n}-1} (-\mu'(t))^{\frac{1}{2}} \int_0^{\mu(t)} \exp\left(\int_{\left(\frac{\sigma}{\kappa_n}\right)^{\frac{1}{n}}}^{\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac{1}{n}}} \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma. \end{aligned} \tag{75}$$

Coupling (75) with (74)

$$\begin{aligned} &-\frac{d}{dt} \int_{|u|>t} F^q(\nabla u) \\ &\quad \leq (-\mu'(t))^{1-q/2} \left[\left(-\frac{d}{dt} \int_{|u|>t} F^2(\nabla u) \right)^{\frac{1}{2}} \right]^q \\ &\quad \leq (-\mu'(t))^{1-q/2} \\ &\qquad \left[\frac{1}{n \kappa_n^{(1/n)}} \mu(t)^{\frac{1}{n}-1} (-\mu'(t))^{\frac{1}{2}} \int_0^{\mu(t)} \exp\left(\int_{\left(\frac{\sigma}{\kappa_n}\right)^{\frac{1}{n}}}^{\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac{1}{n}}} \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma \right]^q. \end{aligned}$$

Consequently

$$\begin{aligned} &\int_\Omega F^q(\nabla u) \\ &\quad \leq \int_0^{|\Omega|} -\mu'(t) \\ &\qquad \left[\frac{1}{n \kappa_n^{(1/n)}} \mu(t)^{\frac{1}{n}-1} \int_0^{\mu(t)} \exp\left(\int_{\left(\frac{\sigma}{\kappa_n}\right)^{\frac{1}{n}}}^{\left(\frac{\mu(t)}{\kappa_n}\right)^{\frac{1}{n}}} \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma \right]^q dt, \end{aligned}$$

and hence, by the substitution $\tau = \mu(t)$,

$$\int_\Omega F^q(\nabla u)$$
$$\leq \int_0^{|\Omega|} \left[\frac{1}{n\kappa_n^{(1/n)}} \tau^{\frac{1}{n}-1} \int_0^\tau \exp\left(\int_{\left(\frac{\sigma}{\kappa_n}\right)^{\frac{1}{n}}}^{\left(\frac{\tau}{\kappa_n}\right)^{\frac{1}{n}}} \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma \right]^q d\tau$$
$$= \int_{\Omega^\star} F^q(\nabla v),$$

so the theorem is proved.

2.2 ON THE SECOND DIRICHLET EIGENVALUE OF SOME NONLINEAR ANISOTROPIC ELLIPTIC OPERATORS

2.2.1 The Dirichlet eigenvalue problem for $-\mathcal{Q}_p$

In this Section, we study the second eigenvalue $\lambda_2(p,\Omega)$ of the anisotropic p-Laplacian operator (49) with Dirichlet condition:

$$\begin{cases} -\mathcal{Q}_\text{p}\, u = \lambda(p,\Omega)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We provide a lower bound of $\lambda_2(p,\Omega)$ among bounded open sets of given measure, showing the validity of a Hong-Krahn-Szego type inequality. Furthermore, we investigate the limit problem as $p \to +\infty$. Firstly, we recall the following

Definition 2.10. *A domain of $\mathbb{R}^n$ is a connected open set.*

Here we state the eigenvalue problem for $\mathcal{Q}_\text{p}$. Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, $1 < p < +\infty$, and consider the problem

$$\begin{cases} -\mathcal{Q}_\text{p}\, u = \lambda|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{76}$$

Definition 2.11. *We say that $u \in W_0^{1,p}(\Omega)$, $u \neq 0$, is an eigenfunction of (76), if*

$$\int_\Omega F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)\cdot\nabla\varphi\, dx = \lambda\int_\Omega |u|^{p-2}u\varphi\, dx \tag{77}$$

for all $\varphi \in W_0^{1,p}(\Omega)$. The corresponding real number λ is called an eigenvalue of (76).

Obviously, if u is an eigenfunction associated to λ, then

$$\lambda = \frac{\displaystyle\int_\Omega F^p(\nabla u)\, dx}{\displaystyle\int_\Omega |u|^p\, dx} > 0.$$

The first eigenvalue

Among the eigenvalues of (76), the smallest one, denoted here by $\lambda_1(p,\Omega)$, has the following well-known variational characterization:

$$\lambda_1(p,\Omega) = \min_{\varphi\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega F^p(\nabla\varphi)\, dx}{\displaystyle\int_\Omega |\varphi|^p\, dx}. \tag{78}$$

In the following theorems its main properties are recalled.

Theorem 2.12. *If Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, there exists a function $u_1 \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ which achieves the minimum in (78), and satisfies the problem (76) with $\lambda = \lambda_1(p,\Omega)$. Moreover, if Ω is connected, then $\lambda_1(p,\Omega)$ is simple, that is the corresponding eigenfunctions are unique up to a multiplicative constant, and the first eigenfunctions have constant sign in Ω.*

Proof. The proof can be immediately adapted from the case of Ω connected and we refer the reader, for example, to [89, 12]. □

Theorem 2.13. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$. Let $u \in W_0^{1,p}(\Omega)$ be an eigenfunction of (76) associated to an eigenvalue λ. If u does not change sign in Ω, then there exists a connected component Ω_0 of Ω such that $\lambda = \lambda_1(p, \Omega_0)$ and u is a first eigenfunction in Ω_0. In particular, if Ω is connected then $\lambda = \lambda_1(p, \Omega)$ and a constant sign eigenfunction is a first eigenfunction.*

Proof. If Ω is connected, a proof can be found in [89, 47]. Otherwise, if $u \geq 0$ in Ω disconnected, by the maximum principle u must be either positive or identically zero in each connected component of Ω. Hence there exists a connected component Ω_0 such that u coincides in Ω_0 with a positive eigenfunction relative to λ. By the previous case, $\lambda = \lambda_1(p, \Omega_0)$ and the proof is completed. □

Here we list some other useful and interesting properties that can be proved in a similar way than the Euclidean case.

Proposition 2.14. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, the following properties hold.*

1. *For $t > 0$ it holds $\lambda_1(p, t\Omega) = t^{-p}\lambda_1(p, \Omega)$.*
2. *If $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, then $\lambda_1(p, \Omega_1) \geq \lambda_1(p, \Omega_2)$.*
3. *For all $1 < p < s < +\infty$ we have $p[\lambda_1(p, \Omega)]^{1/p} < s[\lambda_1(s, \Omega)]^{1/s}$.*

Proof. The first two properties are immediate from (78). As regards the third property, the inequality derives from the Hölder inequality, similarly as in [90]. Indeed, taking $\phi = |\psi|^{\frac{s}{p}-1}\psi$, $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\psi \geq 0$, we have by (27) that

$$[\lambda_1(p, \Omega)]^{\frac{1}{p}} \leq \frac{s}{p}\left(\frac{\displaystyle\int_\Omega |\psi|^{s-p}F^p(\nabla\psi)dx}{\displaystyle\int_\Omega |\psi|^s dx}\right)^{\frac{1}{p}} \leq \frac{s}{p}\left(\frac{\displaystyle\int_\Omega F^s(\nabla\psi)dx}{\displaystyle\int_\Omega |\psi|^s dx}\right)^{\frac{1}{s}}$$

By minimizing with respect to ψ, we get the thesis. □

In addition, the Faber-Krahn inequality for $\lambda_1(p, \Omega)$ holds.

Theorem 2.15. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, then*

$$|\Omega|^{p/N}\lambda_1(p, \Omega) \geq \kappa_N^{p/N}\lambda_1(p, \mathcal{W}). \tag{79}$$

Moreover, equality sign in (79) holds if Ω is homothetic to the Wulff shape.

The proof of this inequality, contained in [12], is based on a symmetrization technique introduced in [4] (see [59, 64] for the equality cases).

Using the previous result we can prove the following property of $\lambda_1(p, \Omega)$.

Proposition 2.16. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. The first eigenvalue of (76), $\lambda_1(p, \Omega)$, is isolated.*

Proof. We argue similarly as in [92]. For completeness we give the proof. For convenience we write λ_1 instead of $\lambda_1(p, \Omega)$. Let $\lambda_k \neq \lambda_1$ a sequence of eigenvalues such that

$$\lim_{k \to +\infty} \lambda_k = \lambda_1$$

Let u_k be a normalized eigenfunction associated to λ_k that is,

$$\lambda_k = \int_\Omega F^p(\nabla u_k)\,dx \quad \text{and} \quad \int_\Omega |u_k|^p\,dx = 1 \tag{80}$$

By (80), there exists a function $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence

$$u_k \to u \quad \text{in } L^p(\Omega) \quad \nabla u_k \rightharpoonup \nabla u \text{ weakly in } L^p(\Omega).$$

By the strong convergence of u_k in $L^p(\Omega)$ and, recalling that F is convex, by weak lower semicontinuity, it follows that

$$\int_\Omega |u|^p dx = 1 \quad \text{and} \quad \int_\Omega F^p(\nabla u)\,dx \le \lim_{k\to\infty} \lambda_k = \lambda_1.$$

Hence, u is a first eigenfunction. On the other hand, being u_k not a first eigenfunction, by Theorem 2.13 it has to change sign. Hence, the sets $\Omega_k^+ = \{u_k > 0\}$ and $\Omega_k^- = \{u_k < 0\}$ are nonempty and, as a consequence of the Faber-Krahn inequality and of Theorem 2.13, it follows that

$$\lambda_k = \lambda_1(p, \Omega_k^+) \ge \frac{C_{n,F}}{|\Omega_k^+|^{\frac{p}{n}}}, \qquad \lambda_k = \lambda_1(p, \Omega_k^-) \ge \frac{C_{n,F}}{|\Omega_k^-|^{\frac{p}{n}}}.$$

This implies that both $|\Omega_k^+|$ and $|\Omega_k^-|$ cannot vanish as $k \to +\infty$ and finally, that u_k converges to a function u which changes sign in Ω. This is in contradiction with the characterization of the first eigenfunctions, and the proof is completed. □

Higher eigenvalues

First of all, we recall the following result (see [69, Theorem 1.4.1] and the references therein), which assures the existence of infinite eigenvalues of $-\mathcal{Q}_p$. We use the following notation. Let $\mathbb{S}^{n-1}$ be the unit Euclidean sphere in $\mathbb{R}^n$, and

$$M = \{u \in W_0^{1,p}(\Omega) \colon \int_\Omega |u|^p dx = 1\}. \tag{81}$$

Moreover, let $\mathcal{C}_n$ be the class of all odd and continuous mappings from $\mathbb{S}^{n-1}$ to M. Then, for any fixed $f \in \mathcal{C}_n$, we have $f : \omega \in \mathbb{S}^{n-1} \mapsto f_\omega \in M$.

Proposition 2.17. *Let Ω be a bounded open set of $\mathbb{R}^n$, for any $k \in \mathbb{N}$, the value*

$$\tilde\lambda_k(p,\Omega) = \inf_{f\in\mathcal{C}_n} \max_{\omega\in\mathbb{S}^{n-1}} \int_\Omega F^p(\nabla f_\omega) dx$$

is an eigenvalue of $-\mathcal{Q}_p$. Moreover,

$$0 < \tilde\lambda_1(p,\Omega) = \lambda_1(p,\Omega) \le \tilde\lambda_2(p,\Omega) \le \ldots \le \tilde\lambda_k(p,\Omega) \le \tilde\lambda_{k+1}(p,\Omega) \le \ldots,$$

and

$$\tilde\lambda_k(p,\Omega) \to \infty \text{ as } k \to \infty.$$

Hence, we have at least a sequence of eigenvalues of $-\mathcal{Q}_\mathrm{p}$. Furthermore, the following proposition holds.

Proposition 2.18. *Let Ω be a bounded open set of $\mathbb{R}^n$. The spectrum of $-\mathcal{Q}_\mathrm{p}$ is a closed set.*

Proof. Let λ_k be a sequence of eigenvalues converging to $\mu < +\infty$ and let u_k be the corresponding normalized eigenfunctions, that is such that $\|u_k\|_{L^p(\Omega)} = 1$. We have to show that μ is an eigenvalue of $-\mathcal{Q}_{\mathrm{p}}$.

We have that

$$\int_\Omega F^{p-1}(\nabla u_k)\nabla_\xi F(\nabla u_k) \cdot \nabla \varphi dx = \lambda_k \int_\Omega |u_k|^{p-2} u_k \varphi \, dx \tag{82}$$

for any test function $\varphi \in W_0^{1,p}(\Omega)$. Since

$$\lambda_k = \int_\Omega F^p(\nabla u_k)\, dx,$$

and being λ_k a convergent sequence, up to a subsequence we have that there exists a function $u \in W_0^{1,p}(\Omega)$ such that $u_k \to u$ strongly in $L^p(\Omega)$ and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. Our aim is to prove that u is an eigenfunction relative to λ.

Choosing $\varphi = u_k - u$ as test function in the equation solved by u_k, we have

$$\begin{aligned}&\int_\Omega (F^{p-1}(\nabla u_k)\nabla_\xi F(\nabla u_k) - F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)) \cdot \nabla(u_k - u) dx \\ &= \lambda_k \int_\Omega |u_k|^{p-2} u_k (u_k - u)\ dx - \int_\Omega F^{p-1}(\nabla u)\nabla_\xi F(\nabla u) \cdot \nabla(u_k - u) dx.\end{aligned}$$

By the strong convergence of u_k and the weak one of ∇u_k, the right-hand side of the above identity goes to zero as k diverges. Hence

$$\lim_{k\to\infty} \int_\Omega (F^{p-1}(\nabla u_k)\nabla_\xi F(\nabla u_k) - F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)) \cdot \nabla(u_k - u) dx = 0$$

By nowadays standard arguments, this limit implies the strong convergence of the gradient, hence we can pass to the limit under the integral sign in (82) to obtain

$$\int_\Omega F^{p-1}(\nabla u) F_\xi(\nabla u) \cdot \nabla \varphi \, dx = \lambda \int_\Omega |u|^{p-2} u \varphi \, dx$$

This shows that λ is an eigenvalue and the proof is completed. □

Finally, we list some properties of the eigenfunctions, well-known in the Euclidean case (see for example [92, 3]). Recall that a nodal domain of an eigenfunction u is a connected component of $\{u > 0\}$ or $\{u < 0\}$.

Proposition 2.19. *Let $p > 1$, and let Ω be a bounded open set in $\mathbb{R}^n$. Then the following facts hold.*

(i) Any eigenfunction of $-\mathcal{Q}_{\mathrm{p}}$ has only a finite number of nodal domains.

(ii) Let λ be an eigenvalue of $-\mathcal{Q}_{\mathrm{p}}$, and u be a corresponding eigenfunction. The following estimate holds:

$$\|u\|_{L^\infty(\Omega)} \le C_{n,p,F} \lambda^{\frac{n}{p}} \|u\|_{L^1(\Omega)} \tag{83}$$

where $C_{n,p,F}$ is a constant depending only on n, p and F.

(iii) All the eigenfunctions of (1) are in $C^{1,\alpha}(\Omega)$, for some $\alpha \in (0,1)$.

Proof. Let λ be an eigenvalue of $-\mathcal{Q}_p$, and u a corresponding eigenfunction.

In order to prove (i), let us denote by Ω_j^+ a connected component of the set $\Omega^+ := \{u > 0\}$. Being $\lambda = \lambda_1(\Omega_j^+)$, then by (79)

$$|\Omega_j^+| \geq C_{n,p,F} \lambda^{-\frac{n}{p}}.$$

Then, the thesis follows observing that

$$|\Omega| \geq \sum_j |\Omega_j^+| \geq C_{n,p,F} \lambda^{-\frac{n}{p}} \sum_j 1.$$

In order to prove (ii), let $k > 0$, and choose $\varphi(x) = \max\{u(x) - k, 0\}$ as test function in (77). Then

$$\int_{A_k} F^p(\nabla u)\, dx = \lambda \int_{A_k} |u|^{p-2} u(u-k)\, dx \tag{84}$$

where $A_k = \{x \in \Omega \colon u(x) > k\}$. Being $k|A_k| \leq ||u||_{L^1(\Omega)}$, then $|A_k| \to 0$ as $k \to \infty$. By the inequality $a^{p-1} \leq 2^{p-1}(a-k)^{p-1} + 2^{p-1}k^{p-1}$, we have

$$\int_{A_k} |u|^{p-2} u(u-k)\, dx \leq 2^{p-1} \int_{A_k} (u-k)^p\, dx + 2^{p-1} k^{p-1} \int_{A_k} (u-k)\, dx. \tag{85}$$

By Poincaré inequality and property (28), then (84) and (85) give that

$$(1 - \lambda C_{n,p,F} |A_k|^{p/n}) \int_{A_k} (u-k)^p\, dx \leq \lambda |A_k|^{p/n} C_{n,p,F} k^{p-1} \int_{A_k} (u-k)\, dx.$$

By choosing k sufficiently large, the Hölder inequality implies

$$\int_{A_k} (u-k)\, dx \leq \tilde{C}_{n,p,F} \lambda^{\frac{1}{p-1}} k |A_k|^{1+\frac{p}{n(p-1)}}.$$

This estimate allows to apply [88, Lemma 5.1, p. 71] in order to get the boundedness of $\operatorname{ess\,sup} u$. Similar argument gives that $\operatorname{ess\,inf} u$ is bounded.

Since (83) holds, by standard elliptic regularity theory (see e.g. [88]) the eigenfunction is $C^{1,\alpha}(\Omega)$. ☐

2.2.2 The second Dirichlet eigenvalue of $-\mathcal{Q}_p$

If Ω is a bounded domain, Proposition 2.16 assures that the first eigenvalue $\lambda_1(p, \Omega)$ of (1) is isolated. This suggests the following definition.

Definition 2.20. *Let Ω be a bounded open set of $\mathbb{R}^n$. Then the second eigenvalue of $-\mathcal{Q}_p$ is*

$$\lambda_2(p,\Omega) := \begin{cases} \min\{\lambda > \lambda_1(p,\Omega) \colon \lambda \textit{ is an eigenvalue}\} & \textit{if } \lambda_1(p,\Omega) \textit{ is simple} \\ \lambda_1(p,\Omega) & \textit{otherwise.} \end{cases}$$

Remark 2.21. *If Ω is connected, by theorems 2.12 and 2.13 we deduce the following characterization of the second eigenvalue:*

$$\lambda_2(p,\Omega) = \min\{\lambda : \lambda \textit{ admits a sign-changing eigenfunction}\}. \tag{86}$$

We point out that in [69] it is proved that in a bounded open set it holds

$$\lambda_2(p,\Omega) = \tilde{\lambda}_2(p,\Omega) = \inf_{\gamma\in\Gamma_\Omega(u_1,-u_1)} \max_{u\in\gamma([0,1])} \int_\Omega F^p(\nabla u(x))\,dx \tag{87}$$

where $\tilde{\lambda}_2(p,\Omega)$ is given in Proposition 2.17, and

$$\Gamma_\Omega(u,v) = \{\gamma : [0,1] \to M \colon \gamma \text{ is continuous and } \gamma(0) = u,\ \gamma(1) = v\},$$

with M as in (81). As immediate consequence of (87) we get

Proposition 2.22. *If $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, then $\lambda_2(p,\Omega_1) \geq \lambda_2(p,\Omega_2)$.*

By adapting the method contained in [37, 38], it is possible to prove the following result.

Proposition 2.23. *Let Ω be a bounded domain in $\mathbb{R}^n$. The eigenfunctions associated to $\lambda_2(p,\Omega)$ admit exactly two nodal domains.*

Proof. We will proceed as in the proof of [38, Th. 2.1]. In such a case, $\lambda_2(p,\Omega)$ is characterized as in (86). Then any eigenfunction u_2 has to change sign, and it admits at least two nodal domains $\Omega_1 \subset \Omega^+$ and $\Omega_2 \subset \Omega^-$. Let us assume, by contradiction, the existence of a third nodal domain Ω_3 and let us suppose, without loss of generality, that $\Omega_3 \subset \Omega^+$.
Claim. There exists a connected open set $\widetilde{\Omega}_2$, with $\Omega_2 \subset \widetilde{\Omega}_2 \subset \Omega$ such that $\widetilde{\Omega}_2 \cap \Omega_1 = \emptyset$ or $\widetilde{\Omega}_2 \cap \Omega_3 = \emptyset$.

The proof of the claim follows line by line as in [38, Th. 2.1]. One of the main tool is the Hopf maximum principle, that for the operator $-\mathcal{Q}_p$ is proved for example in [39, Th. 2.1].

Now, without loss of generality, we assume that $\widetilde{\Omega}_2$ is disjoint of Ω_1 and from this fact a contradiction is derived.

By the fact that u_2 does not change sign on the nodal domains and by Proposition 2.22, we have that $\lambda_1(p,\Omega_1) = \lambda_2(p,\Omega)$ and that $\lambda_1(p,\widetilde{\Omega}_2) < \lambda_1(p,\Omega_2) = \lambda_2(p,\Omega)$. Now, we may construct the disjoint sets $\widetilde{\widetilde{\Omega}}_2$ and $\widetilde{\Omega}_1$ such that $\Omega_2 \subset \widetilde{\widetilde{\Omega}}_2 \subset \widetilde{\Omega}_2$ and $\Omega_1 \subset \widetilde{\Omega}_1$, in order to have

$$\lambda_1(p,\widetilde{\Omega}_1) < \lambda_2(p,\Omega), \qquad \lambda_1(p,\widetilde{\widetilde{\Omega}}_2) < \lambda_2(p,\Omega).$$

Now let v_1 and v_2 be the extension by zero outside $\widetilde{\Omega}_1$ and $\widetilde{\widetilde{\Omega}}_2$, respectively, of the positive normalized eigenfunctions associated to $\lambda_1(p,\widetilde{\Omega}_1)$ and $\lambda(p,\widetilde{\widetilde{\Omega}}_2)$. Hence we easily verify that the function $v = v_1 - v_2$ belongs to $W_0^{1,p}(\Omega)$, it changes sign and satisfies

$$\frac{\int_\Omega F^p(\nabla v_+)\,dx}{\int_\Omega v_+^p\,dx} < \lambda_2(p,\Omega), \qquad \frac{\int_\Omega F^p(\nabla v_-)\,dx}{\int_\Omega v_-^p\,dx} < \lambda_2(p,\Omega).$$

The final aim is to construct a path $\gamma([0,1])$ such

$$\max_{u\in\gamma([0,1])} \int_\Omega F^p(\nabla u(x))\,dx < \lambda_2(p,\Omega),$$

obtaining a contradiction from (87). The construction of this path follows adapting the method contained in [37, 38].

□

Remark 2.24. *In order to better understand the behavior of $\lambda_1(p,\Omega)$ and $\lambda_2(p,\Omega)$ on disconnected sets, a meaningful model is given when*

$$\Omega = \mathcal{W}_{r_1} \cup \mathcal{W}_{r_2}, \text{ with } r_1, r_2 > 0 \text{ and } \mathcal{W}_{r_1} \cap \mathcal{W}_{r_2} = \emptyset.$$

We distinguish two cases.

CASE $r_1 < r_2$. *We have*

$$\lambda_1(p,\Omega) = \lambda_1(p,\mathcal{W}_{r_2}).$$

Hence $\lambda_1(p,\Omega)$ is simple, and any eigenfunction is identically zero on $\mathcal{W}_1$ and has constant sign in $\mathcal{W}_2$. Moreover,

$$\lambda_2(p,\Omega) = \min\{\lambda_1(p,\mathcal{W}_{r_1}), \lambda_2(p,\mathcal{W}_{r_2})\}.$$

Hence, if r_1 is not too small, then the second eigenvalue is $\lambda_1(p,\mathcal{W}_{r_1})$, and the second eigenfunctions of Ω coincide with the first eigenfunctions of $\mathcal{W}_{r_1}$, that do not change sign in $\mathcal{W}_{r_1}$, and vanish on $\mathcal{W}_{r_2}$.

CASE $r_1 = r_2$. *We have*

$$\lambda_1(p,\Omega) = \lambda_1(p,\mathcal{W}_{r_i}), \quad i = 1,2.$$

The first eigenvalue $\lambda_1(p,\Omega)$ is not simple: choosing, for example, the function $U = u_1\chi_{\mathcal{W}_{r_1}} - u_2\chi_{\mathcal{W}_{r_2}}$, where u_i, $i=1,2$, is the first normalized eigenfunction of $\lambda_1(p,\mathcal{W}_{r_i})$, and $V = u_1\chi_{\mathcal{W}_{r_1}}$, then U and V are two nonproportional eigenfunctions relative to $\lambda_1(p,\Omega)$. Hence, in this case, by definition,

$$\lambda_2(p,\Omega) = \lambda_1(p,\Omega) = \lambda_1\left(p,\mathcal{W}_{r_i}\right).$$

In order to prove the Hong-Krahn-Szego inequality, we need the following key lemma.

Proposition 2.25. *Let Ω be an open bounded set of $\mathbb{R}^n$. Then there exists two disjoint domains Ω_1, Ω_2 of Ω such that*

$$\lambda_2(p,\Omega) = \max\{\lambda_1(p,\Omega_1), \lambda_1(p,\Omega_2)\}.$$

Proof. Let $u_2 \in W_0^{1,p}(\Omega)$ be a second normalized eigenfunction. First of all, suppose that u_2 changes sign in Ω. Then, consider two nodal domains $\Omega_1 \subseteq \Omega_+$ and $\Omega_2 \subseteq \Omega_-$. By definition, Ω_1 and Ω_2 are connected sets. The restriction of u_2 to Ω_1 is, by Theorem 2.13, a first eigenfunction for Ω_1 and hence $\lambda_2(p,\Omega) = \lambda_1(p,\Omega_1)$. Analogously for Ω_2, hence

$$\lambda_2(p,\Omega) = \lambda_1(p,\Omega_1) = \lambda_1(p,\Omega_2),$$

and the proof of the proposition is completed, in the case u_2 changes sign.

In the case that u_2 has constant sign in Ω, for example $u_2 \geq 0$, then by Theorem 2.13 Ω must be disconnected. If $\lambda_1(p,\Omega)$ is simple, by definition $\lambda_2(p,\Omega) > \lambda_1(p,\Omega)$. Otherwise, $\lambda_1(p,\Omega) = \lambda_2(p,\Omega)$. Hence in both cases, we can consider a first nonnegative normalized eigenfunction u_1 not proportional to u_2.

Observe that in any connected component of Ω, by the Harnack inequality, u_i, $i=1,2$, must be positive or identically zero. Hence we can choose two disjoint connected open sets Ω_1 and Ω_2, contained respectively in $\{x \in \Omega \colon u_1(x) > 0\}$ and $\{x \in \Omega \colon u_2(x) > 0\}$. Then, u_1 and u_2 are first Dirichlet eigenfunctions in Ω_1 and Ω_2, respectively, and

$$\lambda_1(p,\Omega) = \lambda_1(p,\Omega_1) \leq \lambda_2(p,\Omega), \qquad \lambda_2(p,\Omega) = \lambda_1(p,\Omega_2),$$

and the proof is completed. □

Now we are in position to prove the Hong-Krahn-Szego inequality for $\lambda_2(p,\Omega)$.

Theorem 2.26. *Let Ω be a bounded open set of $\mathbb{R}^n$. Then*

$$\lambda_2(p,\Omega) \geq \lambda_2(p,\widetilde{\mathcal{W}}), \tag{88}$$

where $\widetilde{\mathcal{W}}$ is the union of two disjoint Wulff shapes, each one of measure $\frac{|\Omega|}{2}$. Moreover equality sign in (88) occurs if Ω is the disjoint union of two Wulff shapes of the same measure.

Proof. Let Ω_1 and Ω_2 given by Proposition 2.25. By the Faber-Krahn inequality we have

$$\lambda_2(p,\Omega) = \max\{\lambda_1(p,\Omega_1),\lambda_1(p,\Omega_2)\} \geq \max\{\lambda_1(p,\mathcal{W}_{r_1}),\lambda_1(p,\mathcal{W}_{r_2})\}$$

with $|\mathcal{W}_{r_i}| = |\Omega_i|$. By the rescaling property of $\lambda_1(p,\cdot)$, and observing that, being Ω_1 and Ω_2 disjoint subsets of Ω, $|\Omega_1|+|\Omega_2| \leq |\Omega|$, we have that

$$\begin{aligned}\max\{\lambda_1(p,\mathcal{W}_{r_1}),\lambda_1(p,\mathcal{W}_{r_2})\} = \lambda_1(p,\mathcal{W})\kappa_n^{\frac{p}{n}} \max\left\{|\Omega_1|^{-\frac{p}{n}}, |\Omega_2|^{-\frac{p}{n}}\right\} \geq \\ \geq \lambda_1(p,\mathcal{W})\kappa_n^{\frac{p}{n}}\left(\frac{|\Omega|}{2}\right)^{-\frac{p}{n}} = \lambda_1(p,\widetilde{\mathcal{W}}).\end{aligned}$$

□

2.2.3 The limit case $p \to \infty$

In this section we derive some information on $\lambda_2(p,\Omega)$ as p goes to infinity. First of all we recall some known result about the limit of the first eigenvalue. Let us consider a bounded open set Ω.

The anisotropic distance of $x \in \overline{\Omega}$ to the boundary of Ω is the function

$$d_F(x) = \inf_{y\in\partial\Omega} F^o(x-y), \quad x \in \overline{\Omega}.$$

We stress that when $F = |\cdot|$ then $d_F = d_{\mathcal{E}}$, the Euclidean distance function from the boundary.

It is not difficult to prove that d_F is a uniform Lipschitz function in $\overline{\Omega}$ and

$$F(\nabla d_F(x)) = 1 \quad \text{a.e. in } \Omega.$$

Obviously, $d_F \in W_0^{1,\infty}(\Omega)$. Let us consider the quantity

$$\rho_F = \max\{d_F(x),\ x \in \overline{\Omega}\}.$$

If Ω is connected, ρ_F is called the anisotropic inradius of Ω. If not, ρ_F is the maximum of the inradii of the connected components of Ω.

For further properties of the anisotropic distance function we refer the reader to [33].

Remark 2.27. *It is easy to prove (see also [82, 15]) that the distance function satisfies*

$$\frac{1}{\rho_F(\Omega)} = \frac{1}{\|d_F\|_{L^\infty(\Omega)}} = \min_{\varphi\in W_0^{1,\infty}(\Omega)\setminus\{0\}} \frac{\|F(\nabla\varphi)\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}}. \tag{89}$$

Indeed it is sufficient to observe that if $\varphi \in C_0^1(\Omega)\cap C(\overline{\Omega})$, then $\varphi \in C_0^1(\Omega_i)\cap C(\overline{\Omega}_i)$, for any connected component Ω_i of Ω. Then for a.e. $x \in \Omega_i$, for $y \in \partial\Omega_i$ which achieves $F^o(x-y) = d_F(x)$, it holds

$$\begin{aligned}|\varphi(x)| = |\varphi(x)-\varphi(y)| = |\nabla\varphi(\xi)\cdot x - y| \leq \\ \leq F(\nabla\varphi(\xi))\,F^o(x-y) \leq \|F(\nabla\varphi)\|_{L^\infty(\Omega)} d_F(x).\end{aligned}$$

Passing to the supremum and by density we get (89).

The following result holds (see [15, 82]).

Theorem 2.28. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $\lambda_1(p,\Omega)$ be the first eigenvalue of (76). Then*

$$\lim_{p\to\infty} \lambda_1(p,\Omega)^{\frac{1}{p}} = \frac{1}{\rho_F(\Omega)}.$$

Now let us define

$$\lambda_1(\infty,\Omega) = \frac{1}{\rho_F(\Omega)}.$$

The value $\lambda_1(\infty,\Omega)$ is related to the so-called anisotropic infinity Laplacian operator defined in [15], that is

$$\mathcal{Q}_\infty u = F^2(\nabla u)(\nabla^2 u\, \nabla F(\nabla u)) \cdot \nabla F(\nabla u).$$

Indeed, in [15] the following result is proved.

Theorem 2.29. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then, there exists a positive solution $u_\infty \in W_0^{1,\infty}(\Omega) \cap C(\bar\Omega)$ which satisfies, in the viscosity sense, the following problem:*

$$\begin{cases} \min\{F(\nabla u) - \lambda u, -\mathcal{Q}_\infty u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{90}$$

with $\lambda = \lambda_1(\infty,\Omega)$. Moreover, any positive solution $v \in W_0^{1,\infty}(\Omega)$ to (90) with $\lambda = \lambda_1(\infty,\Omega)$ satisfies

$$\frac{\|F(\nabla v)\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\Omega)}} = \min_{\varphi\in W_0^{1,\infty}(\Omega)\setminus\{0\}} \frac{\|F(\nabla \varphi)\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}} = \lambda_1(\infty,\Omega) = \frac{1}{\rho_F(\Omega)}.$$

Finally, if problem (90) admits a positive viscosity solution in Ω, then $\lambda = \lambda_1(\infty,\Omega)$.

Proposition 2.30. *Theorem 2.28 holds also when Ω is a bounded open set of $\mathbb{R}^n$.*

Proof. Suppose that Ω is not connected, and consider a connected component Ω_0 of Ω with anisotropic inradius $\rho_F(\Omega)$. By the monotonicity property of $\lambda_1(p,\Omega)$ given in Proposition 2.14, we have

$$\lambda_1(p,\Omega) \le \lambda_1(p,\Omega_0).$$

Then up to a subsequence, passing to the limit as $p \to +\infty$ and using Theorem 2.28 we have

$$\tilde\lambda = \lim_{p_j\to\infty} \lambda_1(p_j,\Omega)^{\frac{1}{p_j}} \le \frac{1}{\rho_F(\Omega)}. \tag{91}$$

In order to prove that $\tilde\lambda = \rho_F(\Omega)^{-1}$, let u_{p_j} the first nonnegative normalized eigenfunction associated to $\lambda_1(p_j,\Omega)$. Reasoning as in [15], the sequence u_{p_j} converges to a function u_∞ in $C^0(\Omega)$ which is a viscosity solution of (90) associated to $\tilde\lambda$. Then by the maximum principle contained in [8, Lemma 3.2], in each connected component of Ω, u_∞ is either positive or identically zero. Denoting by $\tilde\Omega$ a connected component of $\{u_\infty > 0\}$, by the uniform convergence, for p_j large, also u_{p_j} is positive in $\tilde\Omega$. Then by Theorem 2.13 we have

$$\lambda_1(p_j,\tilde\Omega) = \lambda_1(p_j,\Omega), \quad \text{and then} \quad \frac{1}{\rho_F(\tilde\Omega)} = \tilde\lambda.$$

By (91) and by definition of ρ_F, $\tilde\lambda \le \rho_F(\Omega)^{-1} \le \rho_F(\tilde\Omega)^{-1} = \tilde\lambda$; then necessarily $\tilde\lambda = \rho_F(\Omega)^{-1}$. $\square$

In order to define the eigenvalue problem for $\mathcal{Q}_\infty$, let us consider the following operator

$$\mathcal{A}_\lambda(s,\xi,X)=\begin{cases}\min\{F(\xi)-\lambda s, -F^2(\xi)(X\,\nabla F(\xi))\cdot\nabla F(\xi)\} & \text{if } s>0,\\ -F^2(\xi)(X\,\nabla F(\xi))\cdot\nabla F(\xi) & \text{if } s=0,\\ \max\{-F(\xi)-\lambda s, -F^2(\xi)(X\,\nabla F(\xi))\cdot\nabla F(\xi)\} & \text{if } s<0,\end{cases}$$

with $(s,\xi,X)\in\mathbb{R}\times\mathbb{R}^n\times S^{n\times n}$, where $S^{n\times n}$ denotes the space of real, symmetric matrices of order n. Clearly $\mathcal{A}_\lambda$ is not continuous in $s=0$.

For completeness we recall the definition of viscosity solution for the operator $\mathcal{A}_\lambda$.

Definition 2.31. *Let $\Omega\subset\mathbb{R}^n$ a bounded open set. A function $u\in C(\Omega)$ is a viscosity subsolution (resp. supersolution) of $\mathcal{A}_\lambda(x,u,\nabla u)=0$ if*

$$\mathcal{A}_\lambda(\phi(x),\nabla\phi(x),\nabla^2\phi(x))\le 0\quad (\textit{resp. } \mathcal{A}_\lambda(\phi(x),\nabla\phi(x),\nabla^2\phi(x))\ge 0),$$

for every $\phi\in C^2(\Omega)$ such that $u-\phi$ has a local maximum (resp. minimum) zero at x. A function $u\in C(\Omega)$ is a viscosity solution of $\mathcal{A}_\lambda=0$ if it is both a viscosity subsolution and a viscosity supersolution and in this case the number λ is called an eigenvalue for $\mathcal{Q}_\infty$.

Definition 2.32. *We say that $u\in C(\bar{\Omega})$, $u|_{\partial\Omega}=0$, $u\not\equiv 0$ is an eigenfunction for the anisotropic $\infty-$Laplacian if there exists $\lambda\in\mathbb{R}$ such that*

$$\mathcal{A}_\lambda(u,\nabla u,\nabla^2 u)=0\quad \textit{in } \Omega \tag{92}$$

in the viscosity sense. Such value λ will be called an eigenvalue for the anisotropic $\infty-$Laplacian.

In order to define the second eigenvalue for $\mathcal{Q}_\infty$ we introduce the following number:

$$\rho_{2,F}(\Omega)=\sup\{\rho>0\colon \text{there are two disjoint Wulff shapes } \mathcal{W}_1,\mathcal{W}_2\subset\Omega \text{ of radius } \rho\},$$

and let us define

$$\lambda_2(\infty,\Omega)=\frac{1}{\rho_{2,F}(\Omega)}.$$

Clearly

$$\lambda_1(\infty,\Omega)\le\lambda_2(\infty,\Omega).$$

Remark 2.33. *It is easy to construct open sets Ω such that $\lambda_1(\infty,\Omega)=\lambda_2(\infty,\Omega)$. For example, this holds when Ω coincides with the union of two disjoint Wulff shapes with same measure, or their convex envelope.*

Remark 2.34. *A simple example of $\rho_{2,F}(\Omega)$ is given when Ω is the union of two disjoint Wulff shapes, $\Omega=\mathcal{W}_{r_1}\cup W_{r_2}$, with $r_2\le r_1$. In this case, $\lambda_1(\infty,\Omega)=\frac{1}{r_1}$ and, if r_2 is not too small, then $\lambda_2(\infty,\Omega)=\frac{1}{r_2}$.*

Theorem 2.35. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set and let $\lambda_2(p,\Omega)$ be the second Dirichlet eigenvalue of $-\mathcal{Q}_p$ in Ω. Then*

$$\lim_{p\to\infty}\lambda_2(p,\Omega)^{\frac{1}{p}}=\lambda_2(\infty,\Omega)=\frac{1}{\rho_{2,F}(\Omega)}.$$

Moreover $\lambda_2(\infty,\Omega)$ is an eigenvalue of $\mathcal{Q}_\infty$, that is $\lambda_2(\infty,\Omega)$ is an eigenvalue for the anisotropic infinity Laplacian in the sense of Definition 2.32.

Proof. First we observe that $\lambda_2(p,\Omega)^{\frac{1}{p}}$ is bounded from above with respect to p. More precisely we have

$$\lambda_1(\infty,\Omega) \leq \limsup_{p\to\infty} \lambda_2(p,\Omega)^{\frac{1}{p}} \leq \lambda_2(\infty,\Omega). \tag{93}$$

Indeed if we consider two disjoint Wulff shapes $\mathcal{W}_1$ and $\mathcal{W}_2$ of radius $\rho_{2,F}(\Omega)$, clearly $\mathcal{W}_1 \cup \mathcal{W}_2 \subset \Omega$ and then by monotonicity property (Proposition 2.22) of $\lambda_2(p,\Omega)$ we have

$$\lambda_1(p,\Omega)^{\frac{1}{p}} \leq \lambda_2(p,\Omega)^{\frac{1}{p}} \leq \lambda_2(p,\mathcal{W}_1 \cup \mathcal{W}_2)^{\frac{1}{p}} = \lambda_1(p,\mathcal{W}_1)^{\frac{1}{p}},$$

where last equality follows from Remark 2.24. Then passing to the limit as $p \to \infty$ in the right hand side, by Theorem 2.28 we have (93). Hence there exists a sequence p_j such that $p_j \to +\infty$ as $j \to \infty$, and

$$\frac{1}{\rho_F(\Omega)} = \lambda_1(\infty,\Omega) \leq \lim_{j\to\infty} \lambda_2(p_j,\Omega)^{\frac{1}{p_j}} = \overline{\lambda} \leq \lambda_2(\infty,\Omega) = \frac{1}{\rho_{2,F}(\Omega)}. \tag{94}$$

In order to conclude the proof we have to show that $\overline{\lambda}$ is an eigenvalue for $\mathcal{Q}_\infty$ and that $\overline{\lambda} = \lambda_2(\infty,\Omega)$.

Let us consider $u_j \in W_0^{1,p}(\Omega)$ eigenfunction of $\lambda_2(p_j,\Omega)$ such that $\|u_j\|_{L^{p_j}(\Omega)} = 1$. Then by standard arguments u_j, converges, up to a subsequence of p_j, uniformly to a function $u \in W_0^{1,\infty}(\Omega) \cap C(\bar{\Omega})$. The function u is a viscosity solution of (92) with $\lambda = \bar{\lambda}$. Indeed, let $x_0 \in \Omega$. If $u(x_0) > 0$, being u continuous, it is positive in a sufficiently small ball centered at x_0. Then it is possible to proceed exactly as in [15] in order to obtain that, in the viscosity sense,

$$\min\{F(\nabla u(x_0)) - \overline{\lambda} u(x_0), -\mathcal{Q}_\infty u(x_0)\} = 0.$$

Similarly, if $u(x_0) < 0$ then

$$\max\{-F(\nabla u(x_0)) - \overline{\lambda} u(x_0), -\mathcal{Q}_\infty u(x_0)\} = 0.$$

It remains to consider the case $u(x_0) = 0$. We will show that u is a subsolution of (92).

Let φ a $C^2(\Omega)$ function such that $u - \varphi$ has a strict maximum point at x_0. By the definition of $\mathcal{A}_{\bar{\lambda}}$, we have to show that $-\mathcal{Q}_\infty \varphi(x_0) \leq 0$.

For any j, let x_j be a maximum point of $u_j - \varphi$, so that $x_j \to x_0$ as $j \to \infty$. Such sequence exists by the uniform convergence of u_j. By [15, Lemma 2.3] u_j verifies in the viscosity sense $-\mathcal{Q}_p u_j = \lambda_2(p_j,\Omega)|u_j|^{p_j-2}u_j$. Then

$$\begin{aligned}
&- \mathcal{Q}_p \varphi_j(x_j) = \\
&= -(p_j-2)F^{p_j-4}(\nabla\varphi(x_j))\mathcal{Q}_\infty\varphi(x_j) - F^{p_j-2}(\nabla\varphi(x_j))\mathcal{Q}_2\varphi(x_j) \leq \\
&\qquad\qquad\qquad \leq \lambda_2(p_j,\Omega)|u_j(x_j)|^{p_j-2}u_j(x_j);
\end{aligned}$$

If $\nabla\varphi(x_0) \neq 0$, then dividing the above inequality by $(p_j-2)F^{p_j-4}(\nabla\varphi)$ we have

$$-\mathcal{Q}_\infty\varphi(x_j) \leq \frac{F^2(\nabla\varphi(x_j))\mathcal{Q}_2\varphi(x_j)}{p_j-2} + \left(\frac{\lambda_2(p_j,\Omega)^{\frac{1}{p_j-4}}|u_j(x_j)|}{F(\nabla\varphi(x_j))}\right)^{p_j-4} \frac{u_j(x_j)^3}{p_j-2} =: \ell_j.$$

Passing to the limit as $j \to \infty$, recalling that $\varphi \in C^2(\Omega)$, $F \in C^2(\mathbb{R}^n \setminus \{0\})$, $\lambda_2(p_j,\Omega)^{\frac{1}{p_j}} \to \bar{\lambda}$, $\nabla\varphi(x_0) \neq 0$ and $u_j(x_j) \to 0$ we get

$$-\mathcal{Q}_\infty\varphi(x_0) \leq 0.$$

Finally, we note that if $\nabla\varphi(x_0) = 0$, the above inequality is trivially true. Hence, we can conclude that u is a viscosity subsolution.

The proof that u is also a viscosity supersolution can be done by repeating the same argument than before, considering $-u$.

Last step of the proof of the Theorem consists in showing that $\bar{\lambda} = \lambda_2(\infty, \Omega)$. We distinguish two cases.

Case 1: The function u changes sign in Ω.

Let us consider the following sets

$$\Omega^+ = \{x \in \Omega \colon u(x) > 0\} \qquad \Omega^- = \{x \in \Omega \colon u(x) < 0\}.$$

Being $u \in C^0(\Omega)$ then Ω^+, Ω^- are two disjoint open sets of $\mathbb{R}^n$ and $|\Omega^+| > 0$ and $|\Omega^-| > 0$.

By Theorem 2.29 we have

$$\overline{\lambda} = \lambda_1(\infty, \Omega^+) \quad \text{and} \quad \overline{\lambda} = \lambda_1(\infty, \Omega^-).$$

Then by definition of $\rho_{2,F}$ we get

$$\rho_F(\Omega^+) = \rho_F(\Omega^-) = \frac{1}{\overline{\lambda}} \leq \rho_{2,F}(\Omega),$$

that implies, by (94) that

$$\overline{\lambda} = \lambda_2(\infty, \Omega).$$

Case 2: The function u does not change sign in Ω.

We first observe that in this case Ω cannot be connected. Indeed since u_j converges to u in $C^0(\overline{\Omega})$, for sufficiently large p we have that there exist second eigenfunctions relative to $\lambda_2(p, \Omega)$ with constant sign in Ω and this cannot happen if Ω is connected.

Then in this case, we have to replace the sequence u_j (and then the function u) in order to find two disjoint connected open subsets Ω_1, Ω_2 of Ω, such that

$$\lambda_1(\infty, \Omega) = \lambda_1(\infty, \Omega_1) \tag{95}$$

and

$$\overline{\lambda} = \lambda_1(\infty, \Omega_2). \tag{96}$$

Once we prove that such subsets exist, by (94) and the definition of $\rho_{2,F}$ we obtain

$$\rho_F(\Omega_2) = \frac{1}{\overline{\lambda}} \leq \rho_{2,F}(\Omega) \leq \rho_F(\Omega) = \rho_F(\Omega_1),$$

that implies, again by (94),

$$\overline{\lambda} = \lambda_2(\infty, \Omega).$$

In order to prove (95) and (96), we consider $u_{1,\infty}$, an eigenfunction associated to $\lambda_1(\infty, \Omega)$, obtained as limit in $C^0(\Omega)$ of a sequence $u_{1,p}$ of first normalized eigenfunctions associated to $\lambda_1(p, \Omega)$, and consider a connected component of Ω, say Ω_1, where $u_{1,\infty} > 0$ and such that $\lambda_1(\infty, \Omega) = \lambda_1(\infty, \Omega_1)$. The argument of the proof of Proposition 2.30 gives that such $u_{1,\infty}$ and Ω_1 exist. Then, let $u_{2,p} \geq 0$ be a normalized eigenfunction associated to $\lambda_2(p, \Omega)$ such that for any p sufficiently large, $\operatorname{supp}(u_{2,p}) \cap \Omega_1 = \emptyset$.

The existence of such a sequence is guaranteed from this three observations:

- if $u_{2,p}$ changes sign for a divergent sequence of p's, then we come back to the case 1;
- by the maximum principle, in each connected component of Ω $u_{2,p}$ is either positive or identically zero;
- the condition $\text{supp}(u_{2,p}) \cap \Omega_1 = \emptyset$ depends from the fact that $u_{2,p}$ can be chosen not proportional to $u_{1,p}$.

Hence, there exists Ω_2 connected component of Ω disjoint from Ω_1, such that $u_{2,p}$ converges to $u_{2,\infty}$ (up to a subsequence) in $C^0(\Omega_2)$, and where $u_{2,\infty} > 0$. By Theorem 2.29, (96) holds. □

Theorem 2.36. *Given Ω bounded open set of $\mathbb{R}^n$, let $\lambda > \lambda_1(\infty, \Omega)$ be an eigenvalue for $\mathcal{Q}_\infty$. Then $\lambda \geq \lambda_2(\infty, \Omega)$ and $\lambda_2(\infty, \Omega)$ is the second eigenvalue of $\mathcal{Q}_\infty$, in the sense that there are no eigenvalues of $\mathcal{Q}_\infty$ between $\lambda_1(\infty, \Omega)$ and $\lambda_2(\infty, \Omega)$.*

Proof. Let u_λ be an eigenfunction corresponding to λ. We distinguish two cases.
Case 1: The function u_λ changes sign in Ω.

Let us consider the following sets

$$\Omega^+ = \{x \in \Omega \colon u_\lambda(x) > 0\} \qquad \Omega^- = \{x \in \Omega \colon u_\lambda(x) < 0\}.$$

Being $u_\lambda \in C^0(\Omega)$ then Ω^+, Ω^- are two disjoint open sets of $\mathbb{R}^n$ and $|\Omega^+| > 0$ and $|\Omega^-| > 0$.

By Theorem 2.29 we have

$$\lambda = \lambda_1(\infty, \Omega^+) \quad \text{and} \quad \lambda = \lambda_1(\infty, \Omega^-).$$

Then by definition of $\rho_{2,F}$ we get

$$\rho_F(\Omega^+) = \rho_F(\Omega^-) = \frac{1}{\lambda} \leq \rho_{2,F}(\Omega),$$

that implies, by (94) that

$$\lambda \geq \lambda_2(\infty, \Omega).$$

Case 2: The function u_λ does not change sign in Ω.

By Theorem 2.29 Ω cannot be connected being $\lambda > \lambda_1(\infty, \Omega)$.

In this case, again by Theorem 2.29 we can find two disjoint connected open subsets Ω_1, Ω_2 of Ω, such that

$$\lambda_1(\infty, \Omega) = \lambda_1(\infty, \Omega_1)$$

and

$$\lambda = \lambda_1(\infty, \Omega_2).$$

Being $\lambda > \lambda_1(\infty, \Omega)$, we obtain

$$\rho_F(\Omega_2) = \frac{1}{\lambda} < \rho_F(\Omega) = \rho_F(\Omega_1),$$

that by the definition of $\rho_{2,F}$ implies,

$$\lambda \geq \lambda_2(\infty, \Omega).$$

□

Remark 2.37. *We observe that if Ω is a bounded open set and $\widetilde{\mathcal{W}}$ is the union of two disjoint Wulff shapes with the same measure $|\Omega|/2$, it holds that*

$$\rho_{2,F}(\Omega) \leq \rho_{2,F}(\widetilde{\mathcal{W}}),$$

that is,

$$\lambda_2(\infty,\Omega) \geq \lambda_2(\infty,\widetilde{\mathcal{W}}),$$

that is the Hong-Krahn-Szego inequality for the second eigenvalue of $-\mathcal{Q}_\infty$.

2.3 A SHARP WEIGHTED ANISOTROPIC POINCARÉ INEQUALITY FOR CONVEX DOMAINS

2.3.1 Definition and statement of the problem

In this Section we prove, in a general anisotropic case, an optimal lower bound for the best constant in a class of weighted anisotropic Poincaré inequalities. We prove a sharp lower bound for the optimal constant $\Lambda_{p,\mathcal{F},\omega}(\Omega)$ in the Poincaré-type inequality

$$\inf_{t\in\mathbb{R}} \|u-t\|_{L^p_\omega(\Omega)} \leq \frac{1}{[\Lambda_{p,\mathcal{F},\omega}(\Omega)]^{\frac{1}{p}}} \|\mathcal{F}(\nabla u)\|_{L^p_\omega(\Omega)},$$

with $1 < p < +\infty$, Ω is a bounded convex domain of $\mathbb{R}^n$, $\mathcal{F} \in \mathscr{F}(\mathbb{R}^n)$, where $\mathscr{F}(\mathbb{R}^n)$ is the set of lower semicontinuous functions, positive in $\mathbb{R}^n \setminus \{0\}$ and positively 1-homogeneous; moreover, let ω be a log-concave function. A function

$$\xi \in \mathbb{R}^n \mapsto \mathcal{F}(\xi) \in [0,+\infty[$$

belongs to the set $\mathscr{F}(\mathbb{R}^n)$ if it verifies the following assumptions:

1. $\mathcal{F}$ is positively 1-homogeneous, that is
$$\text{if } \xi \in \mathbb{R}^n \text{ and } t \geq 0, \text{ then } \mathcal{F}(t\xi) = t\mathcal{F}(\xi);$$
2. if $\xi \in \mathbb{R}^n \setminus \{0\}$, then $\mathcal{F}(\xi) > 0$;
3. $\mathcal{F}$ is lower semi-continuous.

If $\mathcal{F} \in \mathscr{F}(\mathbb{R}^n)$, properties (1), (2), (3) give that there exists a positive constant a such that

$$a|\xi| \leq \mathcal{F}(\xi), \quad \xi \in \mathbb{R}^n.$$

The polar function $\mathcal{F}^o \colon \mathbb{R}^n \to [0,+\infty[$ of $\mathcal{F} \in \mathscr{F}(\mathbb{R}^n)$ is defined as

$$\mathcal{F}^o(\eta) = \sup_{\xi\neq 0} \frac{\xi\cdot\eta}{\mathcal{F}(\xi)}.$$

The function $\mathcal{F}^o$ belongs to $\mathscr{F}(\mathbb{R}^n)$. Moreover it is convex on $\mathbb{R}^n$, and then continuous. If $\mathcal{F}$ is convex, it holds that

$$\mathcal{F}(\xi) = (\mathcal{F}^o)^o(\xi) = \sup_{\eta\neq 0} \frac{\xi\cdot\eta}{\mathcal{F}^o(\eta)}.$$

If $\mathcal{F}$ is convex and $\mathcal{F}(\xi) = \mathcal{F}(-\xi)$ for all $\xi \in \mathbb{R}^n$, then $\mathcal{F}$ is a norm on $\mathbb{R}^n$, and the same holds for $\mathcal{F}^o$.

We recall that if $\mathcal{F}$ is a smooth norm of $\mathbb{R}^n$ such that $\nabla^2(\mathcal{F}^2)$ is positive definite on $\mathbb{R}^n \setminus \{0\}$, then $\mathcal{F}$ is called a Finsler norm on $\mathbb{R}^n$.

If $\mathcal{F} \in \mathscr{F}(\mathbb{R}^n)$, by definition we have

$$\xi\cdot\eta \leq \mathcal{F}(\xi)\mathcal{F}^o(\eta), \qquad \forall \xi,\eta \in \mathbb{R}^n. \tag{97}$$

Remark 2.38. *Let $\mathcal{F} \in \mathscr{F}(\mathbb{R}^n)$, and consider the convex envelope of $\mathcal{F}$, that is the largest convex function $\overline{\mathcal{F}}$ such that $\overline{\mathcal{F}} \leq \mathcal{F}$. It holds that $\overline{\mathcal{F}}$ and $\mathcal{F}$ have the same polar function:*

$$(\overline{\mathcal{F}})^o = \mathcal{F}^o \quad \text{in } \mathbb{R}^n.$$

Indeed, being $\overline{\mathcal{F}} \leq \mathcal{F}$, by definition it holds that $(\overline{\mathcal{F}})^o \geq \mathcal{F}^o$. To show the reverse inequality, it is enough to prove that $(\mathcal{F}^o)^o \leq \mathcal{F}$. Then, being $\overline{\mathcal{F}}$ the convex envelope of $\mathcal{F}$, it must be $(\mathcal{F}^o)^o \leq \overline{\mathcal{F}}$, that implies $(\overline{\mathcal{F}})^o \leq \mathcal{F}^o$. Denoting by $G(x) = (\mathcal{F}^o)^o(x)$, for any x there exists $\overline{v}_x$ such that

$$G(x) = \frac{x \cdot \overline{v}_x}{\mathcal{F}^o(\overline{v}_x)}, \quad \text{and} \quad x \cdot \overline{v}_x \leq \mathcal{F}^o(\overline{v}_x)\mathcal{F}(x), \quad \text{that implies} \quad G(x) \leq \mathcal{F}(x).$$

Let $\mathcal{F} \in \mathscr{F}(\mathbb{R}^n)$, and consider a bounded convex domain Ω of $\mathbb{R}^n$. Throughout the chapther $D_{\mathcal{F}}(\Omega) \in]0, +\infty[$ will be

$$D_{\mathcal{F}}(\Omega) = \sup_{x,y \in \Omega} \mathcal{F}^o(y - x).$$

We explicitly observe that since $\mathcal{F}^o$ is not necessarily even, in general $\mathcal{F}^o(y-x) \neq \mathcal{F}^o(x-y)$. When $\mathcal{F}$ is a norm, then $D_{\mathcal{F}}(\Omega)$ is the so called anisotropic diameter of Ω with respect to $\mathcal{F}^o$. In particular, if $\mathcal{F} = \mathcal{E}$ is the Euclidean norm in $\mathbb{R}^n$, then $\mathcal{E}^o = \mathcal{E}$ and $D_{\mathcal{E}}(\Omega)$ is the standard Euclidean diameter of Ω. We refer the reader, for example, to [30, 62] for remarkable examples of convex not even functions in $\mathscr{F}(\mathbb{R}^n)$. On the other hand, in [112] some results on isoperimetric and optimal Hardy-Sobolev inequalities for a general function $\mathcal{F} \in \mathscr{F}(\mathbb{R}^n)$ have been proved, by using a generalizazion of the so called convex symmetrization introduced in [4] (see also [47, 48, 49]).

Remark 2.39. *In general $\mathcal{F}$ and $\mathcal{F}^o$ are not rotational invariant. Anyway, if $A \in SO(n)$, defining*

$$\mathcal{F}_A(x) = \mathcal{F}(Ax), \tag{98}$$

and being $A^T = A^{-1}$, then $\mathcal{F}_A \in \mathscr{F}(\mathbb{R}^n)$ and

$$(\mathcal{F}_A)^o(\xi) = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x \cdot \xi}{\mathcal{F}_A(x)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{A^T y \cdot \xi}{\mathcal{F}(y)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{y \cdot A\xi}{\mathcal{F}(y)} = (\mathcal{F}^o)_A(\xi).$$

Moreover,

$$D_{\mathcal{F}_A}(A^T\Omega) = \sup_{x,y \in A^T\Omega} (\mathcal{F}^o)_A(y-x) = \sup_{\bar{x},\bar{y} \in \Omega} \mathcal{F}^o(\bar{y} - \bar{x}) = D_{\mathcal{F}}(\Omega). \tag{99}$$

2.3.2 Proof of the Payne-Weinberger inequality

To state and prove Theorem 2.42, the following Wirtinger-type inequality, contained in [66] is needed.

Proposition 2.40. *Let f be a positive log-concave function defined on $[0, L]$ and $p > 1$, then*

$$\inf \left\{ \frac{\displaystyle\int_0^L |u'|^p f \, dx}{\displaystyle\int_0^L |u|^p f \, dx}, \; u \in W^{1,p}(0, L), \; \int_0^L |u|^{p-2} u f dx = 0 \right\} \geq \frac{\pi_p^p}{L^p}.$$

The proof of the main result is based on a slicing method introduced in [97] in the Laplacian case. The key ingredient is the following Lemma. For a proof, we refer the reader, for example, to [97, 9, 66].

Lemma 2.41. *Let Ω be a convex set in $\mathbb{R}^n$ having (Euclidean) diameter $D_{\mathcal{E}}(\Omega)$, let ω be a positive log-concave function on Ω, and let u be any function such that $\int_\Omega |u|^{p-2}u\omega\,dx = 0$. Then, for all positive ε, there exists a decomposition of the set Ω in mutually disjoint convex sets Ω_i $(i = 1, \dots, k)$ such that*

$$\bigcup_{i=1}^{k} \overline{\Omega} = \overline{\Omega}$$
$$\int_{\Omega_i} |u|^{p-2}u\omega\,dx = 0$$

and for each i there exists a rectangular system of coordinates such that

$$\Omega_i \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon 0 \le x_1 \le d_i, |x_l| \le \varepsilon, l = 2, \dots, n\},$$

where $d_i \le D_{\mathcal{E}}(\Omega)$, $i = 1, \dots, k$.

Our aim is to prove an analogous sharp lower bound for $\Lambda_{p,\mathcal{F},\omega}(\Omega)$, in a general anisotropic case. More precisely, our main result is:

Theorem 2.42. *Let $\mathcal{F} \in \mathscr{F}(\mathbb{R}^n)$, $\mathcal{F}^o$ be its polar function. Let us consider a bounded convex domain $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, and take a positive* log-concave *function ω defined in Ω. Then, given*

$$\Lambda_{p,\mathcal{F},\omega}(\Omega) = \inf_{\substack{u\in W^{1,\infty}(\Omega)\\ \int_\Omega |u|^{p-2}u\omega\,dx=0}} \frac{\displaystyle\int_\Omega \mathcal{F}(\nabla u)^p\omega\,dx}{\displaystyle\int_\Omega |u|^p\omega\,dx},$$

it holds that

$$\Lambda_{p,\mathcal{F},\omega}(\Omega) \ge \left(\frac{\pi_p}{\operatorname{diam}_{\mathcal{F}}(\Omega)}\right)^p, \tag{100}$$

where $\operatorname{diam}_{\mathcal{F}}(\Omega) = \sup_{x,y\in\Omega} \mathcal{F}^o(y-x)$ *and*

$$\pi_p = 2\int_0^{+\infty} \frac{1}{1+\frac{1}{p-1}s^p}ds = 2\pi\frac{(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}, \qquad \operatorname{diam}_{\mathcal{E}}(\Omega) \text{ Euclidean diameter of } \Omega.$$

This result has been proved in the case $p = 2$ and $\omega = 1$, when $\mathcal{F}$ is a strongly convex, smooth norm of $\mathbb{R}^n$ in [113] with a completely different method than the one presented here.

Proof. By density, it is sufficient to consider a smooth function u with uniformly continuous first derivatives and $\int_\Omega |u|^{p-2}u\omega\,dx = 0$.

Hence, we can decompose the set Ω in k convex domains Ω_i as in Lemma 2.41. In order to prove (100), we will show that for any $i \in \{1, \dots, k\}$ it holds that

$$\int_{\Omega_i} F^p(\nabla u)\omega\,dx \ge \frac{\pi_p^p}{D_{\mathcal{F}}(\Omega)^p}\int_{\Omega_i} |u|^p\omega\,dx. \tag{101}$$

By Lemma 2.41, for each fixed $i \in \{1, \dots, k\}$, there exists a rotation $A_i \in SO(n)$ such that

$$A_i\Omega_i \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_1 \le d_i,\ |x_l| \le \varepsilon,\ l = 2, \dots, n\}.$$

By changing the variable $y = A_i x$, recalling the notation (98) and using (99) it holds that

$$\int_{\Omega_i} \mathcal{F}^p(\nabla u(x))\, \omega(x)\, dx = \int_{A_i\Omega_i} \mathcal{F}_{A_i^T}(\nabla u(A_i^T y))^p\, \omega(A_i^T y)\, dy; \qquad D_{\mathcal{F}}(\Omega) = D_{\mathcal{F}_{A_i^T}}(A_i\Omega).$$

We deduce that it is not restrictive to suppose that for any $i \in \{1, \dots, n\}$ A_i is the identity matrix, and the decomposition holds with respect to the x_1-axis.

Now we may argue as in [66]. For any $t \in [0, d_i]$ let us denote by $v(t) = u(t, 0, \dots, 0)$, and $f_i(t) = g_i(t)\omega(t, 0, \dots, 0)$, where $g_i(t)$ will be the $(n-1)$ volume of the intersection of Ω_i with the hyperplane $x_1 = t$. By Brunn-Minkowski inequality g_i, and then f_i, is a log-concave function in $[0, d_i]$. Since u, u_{x_1} and ω are uniformly continuous in Ω there exists a modulus of continuity $\eta(\cdot)$ with $\eta(\varepsilon) \searrow 0$ for $\varepsilon \to 0$, indipendent of the decomposition of Ω and such that

$$\left| \int_{\Omega_i} |u_{x_1}|^p \omega\, dx - \int_0^{d_i} |v'|^p f_i\, dt \right| \le \eta(\varepsilon)|\Omega_i|, \qquad \left| \int_{\Omega_i} |u|^p \omega\, dx - \int_0^{d_i} |v|^p f_i\, dt \right| \le \eta(\varepsilon)|\Omega_i|,$$

and

$$\left| \int_0^{d_i} |v|^{p-2} v f_i\, dt \right| \le \eta(\varepsilon)|\Omega_i|.$$

Now, by property (97) we deduce that for any vector $\eta \in \mathbb{R}^n$

$$|\nabla u \cdot \eta| \le \mathcal{F}(\nabla u) \max\{\mathcal{F}^o(\eta), \mathcal{F}^o(-\eta)\}.$$

Then choosing $\eta = e_1$ and denoting by $M = \max\{\mathcal{F}^o(e_1), \mathcal{F}^o(-e_1)\}$, Proposition 2.40 gives

$$\begin{aligned} \int_{\Omega_i} \mathcal{F}^p(\nabla u)\omega\, dx &\ge \frac{1}{M^p} \int_{\Omega_i} |u_{x_1}|^p \omega\, dx \ge \frac{1}{M^p} \int_0^{d_i} |v'|^p f_i\, dt - \frac{\eta(\varepsilon)|\Omega_i|}{M^p} \\ &\ge \frac{\pi_p}{d_i^p M^p} \int_0^{d_i} |v|^p f_i\, dt + C\eta(\varepsilon)|\Omega_i| \ge \frac{\pi_p^p}{d_i^p M^p} \int_{\Omega_i} |u|^p \omega\, dx + C\eta(\varepsilon)|\Omega_i|, \end{aligned}$$

where C is a constant which does not depend on ε. Being $d_i \le D_{\mathcal{E}}(\Omega)$, and then $d_i M \le D_{\mathcal{F}}(\Omega)$, by letting ε to zero we get (101). Hence, by summing over i we get the thesis. □

Remark 2.43. *In order to prove an estimate for $\Lambda_{p,\mathcal{F},\omega}$, we could use directly property (97) with $v = \frac{\nabla u}{|\nabla u|}$, and the Payne-Weinberger inequality in the Euclidean case, obtaining that*

$$\int_{\Omega} \mathcal{F}^p(\nabla u)\omega\, dx \ge \int_{\Omega} \frac{|\nabla u|^p}{\mathcal{F}^o(v)^p} \omega\, dx \ge \frac{\pi_p^p}{D_{\mathcal{E}}(\Omega)^p \mathcal{F}^o(v_m)^p} \int_{\Omega} |u|^p \omega\, dx,$$

where $\mathcal{F}^o(v_m) = \max_{|\nu|=1} \mathcal{F}^o(\nu)$. However, we have a worst estimate than (100) because $D_{\mathcal{E}}(\Omega) \cdot \mathcal{F}^o(v_m)$ is, in general, strictly larger than $D_{\mathcal{F}}(\Omega)$, as shown in the following example.

Example 2.44. *Let* $\mathcal{F}(x,y) = \sqrt{a^2x^2 + b^2y^2}$*, with* $a < b$*. Then* $\mathcal{F}$ *is a even, smooth norm with* $\mathcal{F}^o(x,y) = \sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}$ *and the Wulff shapes* $\{\mathcal{F}^o(x,y) < R\}$*,* $R > 0$*, are ellipses. Clearly we have:*

$$D_{\mathcal{E}}(\Omega) = 2b \quad \textit{and} \quad D_{\mathcal{F}}(\Omega) = 2$$

Let us compute $\mathcal{F}^o(v_m)$*. We have:*

$$\max_{|v|=1} \mathcal{F}^o(v) = \max_{\vartheta \in [0,2\pi]} \sqrt{\frac{(\cos\vartheta)^2}{a^2} + \frac{(\sin\vartheta)^2}{b^2}} = \mathcal{F}^o(0,\pm 1) = \frac{1}{a}.$$

Then $D_{\mathcal{E}}(\Omega) \cdot \mathcal{F}^o(v_m) = 2\dfrac{b}{a} > 2.$

2.4 THE ANISOTROPIC ∞-LAPLACIAN EIGENVALUE PROBLEM WITH NEUMANN BOUNDARY CONDITIONS

2.4.1 The limiting problem

Throughout this section, we denote by $||\cdot||_p^p$ the main norm of functions in L^p-space, i.e. $||f||_p^p = \frac{1}{|\Omega|}\int_\Omega |f|^p\,dx$ for all $f \in L^p(\Omega)$. We study the minimum problem

$$\Lambda_1(p,\Omega) = \min\left\{\frac{\int_\Omega F^p(\nabla u)\,dx}{\int_\Omega |u|^p\,dx} : u \in W^{1,p}(\Omega), \int_\Omega u|u|^{p-2}\,dx = 0\right\}. \tag{102}$$

Let us consider a minimizer u_p of (102) such that $||u_p||_p = 1$ and $\mathcal{Q}_p$ the operator defined in (1). Then, for every $p > 1$, u_p solves the Neumann eigenvalue problem:

$$\begin{cases} -\mathcal{Q}_p u_p = \Lambda_1(p,\Omega)|u_p|^{p-2}u_p & \text{in } \Omega \\ \nabla_\xi F^p(\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν is the euclidean outer normal to $\partial\Omega$.

Definition 2.45. *Let $u \in W^{1,p}(\Omega)$. We say that u is a weak solution of (12) if it holds the following inequality:*

$$\int_\Omega F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)\cdot\nabla\varphi\,dx = \Lambda\int_\Omega |u|^{p-2}u\varphi\,dx \tag{103}$$

for all $\varphi \in W^{1,p}(\Omega)$. The corresponding real number Λ is called an eigenvalue of (12).

We analyze the Neumann eigenvalue problem (12) with the means of viscosity solutions and we use the following notation

$$G_p(u,\nabla u,\nabla^2 u) := -(p-2)F^{p-4}(\nabla u)\mathcal{Q}_\infty u - F^{p-2}(\nabla u)\Delta_F(\nabla u) - \Lambda_1(p,\Omega)|u|^{p-2}u$$

where $\Delta_F(\nabla u) = \operatorname{div}(F(\nabla u)\nabla_\xi F(\nabla u))$ is the anisotropic Laplacian. Following for instance [74], we define the viscosity (sub- and super-) solutions to the following Neumann eigenvalue problem

$$\begin{cases} G_p(u,\nabla u,\nabla^2 u) = 0 & \text{in } \Omega \\ \nabla_\xi F^p(\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{104}$$

Definition 2.46. *A lower semicontinuous function u is a viscosity supersolution (subsolution) to (104) if for every $\phi \in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict minimum (maximum) at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have that:*

if $x_0 \in \Omega$, we require

$$G_p(\phi(x_0),\nabla\phi(x_0),\nabla^2\phi(x_0)) \geq 0 \tag{105}$$
$$(G_p(\phi(x_0),\nabla\phi(x_0),\nabla^2\phi(x_0)) \leq 0) \tag{106}$$

and if $x_0 \in \Omega$, then the inequality holds

$$\max\{G_p(\phi(x_0),\nabla\phi(x_0),\nabla^2\phi(x_0)),\nabla_\xi F^p(\nabla\phi(x_0))\cdot\nu\} \geq 0 \tag{107}$$
$$(\min\{G_p(\phi(x_0),\nabla\phi(x_0),\nabla^2\phi(x_0)),\nabla_\xi F^p(\nabla u\phi(x_0))\cdot\nu\} \leq 0) \tag{108}$$

Definition 2.47. *A continuous function u is a viscosity solution to (104) if and only if it is both a viscosity supersolution and a viscosity subsolution to (104).*

Now we prove that a weak solution to the Neumann anisotropic $p-$Laplacian problem (12) is also a viscosity solution to (104).

Lemma 2.48. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to*

$$\begin{cases} -\mathcal{Q}_p u = \Lambda_1(p,\Omega)|u|^{p-2}u & \text{in } \Omega \\ \nabla_\xi F^p(\nabla u)\cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

then u is a viscosity solution to

$$\begin{cases} G_p(u,\nabla u,\nabla^2 u) = 0 & \text{in } \Omega \\ \nabla_\xi F^p(\nabla u)\cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. In [15, Lemma 2.3] it is proved that every weak solution to $-\mathcal{Q}_p u = \Lambda_1(p,\Omega)|u|^{p-2}u$ is a viscosity solution to $G_p(u,\nabla u,\nabla^2 u) = 0$ in Ω. It remains to show that the Neumann boundary condition is satisfied in the viscosity sense, as defined in (107) - (108). We firstly prove that u is a supersolution. Hence, let $x_0 \in \partial\Omega$, $\phi \in C^2(\overline{\Omega})$ such that $u(x_0) = \phi(x_0)$ and $\phi(x) < u(x)$ when $x \neq x_0$. By contradiction we assume that

$$\max\{G_p(\phi(x_0),\nabla\phi(x_0),\nabla^2\phi(x_0)),\nabla_\xi F^p(\nabla\phi(x_0))\cdot\nu\} < 0. \tag{109}$$

Therefore, there exists $r > 0$ such that (109) holds for all $x \in \overline{\Omega}\cap W_r(x_0)$. We set $m := \inf_{\overline{\Omega}\cap\partial W_r(x_0)}(u-\phi) > 0$ and by $\psi(x) := \phi(x) + \frac{m}{2}$. If we take $(\psi - u)^+$ as test function in (103), we have both

$$\int_{\{\psi>u\}} F^{p-1}(\nabla\psi)\nabla_\xi F(\nabla\psi)\nabla(\psi-u)\,dx < \Lambda_1(p,\Omega)\int_{\{\psi>u\}}|\phi|^{p-2}\phi(\psi-u)\,dx$$

and

$$\int_{\{\psi>u\}} F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)\nabla(\psi-u)\,dx = \Lambda_1(p,\Omega)\int_{\{\psi>u\}}|u|^{p-2}u(\psi-u)\,dx.$$

If we subtract these last two relation each other, by the convexity of F^p, we have

$$\begin{aligned} 0 &\le \int_{\{\psi>u\}} \left(F^{p-1}(\nabla\psi)\nabla_\xi F(\nabla\psi) - F^{p-1}(\nabla u)\nabla_\xi F(\nabla u)\right)\nabla(\psi-u)\,dx \\ &< \Lambda_1(p,\Omega)\int_{\{\psi>u\}}\left(|\phi|^{p-2}\phi - |u|^{p-2}u\right)(\psi-u)\,dx < 0. \end{aligned}$$

This is absurd and hence conclude the proof. □

The eigenvalue problem (13) arises as an asymptotic limit of the nonlinear eigenvalue problem (12). Indeed, on covex sets, the first nontrivial eigenfunction of the Neumann eigenvalue problem (12) converges to a viscosity solution of (13) and the limiting eigenvalue of (12) as $p \to \infty$ is the first nontrivial eigenvalue of the limit problem (13). Moreover this eigenvalue is closely related to the geometry of the considered domain Ω and, to give a geometric characterization, we define

$$\Lambda_1(\infty,\Omega) := \frac{2}{\mathrm{diam}_F(\Omega)}, \tag{110}$$

where the anisotropic diameter is defined as in (46). Moreover, in the convex set Ω, we define the anisotropic distance between two points $x,y \in \Omega$ as

$$d_F(x,y) = F^o(x-y)$$

and the anisotropic distance between a point $x \in \Omega$ and a set $E \subset \Omega$ as

$$d_F(x,E) = \inf_{y\in E} F^o(x-y).$$

In the following Lemma we prove that (110) is the first nontrivial Neumann eigenvalue of (13).

Lemma 2.49. *Let Ω be a bounded open connected set in $\mathbb{R}^n$ with Lipschitz boundary, then*

$$\lim_{p\to\infty} \Lambda_1(p,\Omega)^{\frac{1}{p}} = \Lambda_1(\infty,\Omega)$$

Proof. We will proceed by adapting the proof of [57, Lem. 1]. We divide the proof in two steps.

Step 1. $\limsup_{p\to\infty} \Lambda_1(p,\Omega)^{\frac{1}{p}} \le \frac{2}{\mathrm{diam}_F(\Omega)}$.

We fix $x_0 \in \Omega$ and $c_p \in \mathbb{R}$ such that $w(x) := d_F(x,x_0) - c_p$ is an admissible test function in (102), that is $\int_\Omega |w|^{p-2} w\, dx = 0$. Recalling that $F(\nabla d_F(x,x_0)) = 1$ for all $x \in \mathbb{R}^n \backslash 0$, we get

$$\Lambda_1(p,\Omega)^{\frac{1}{p}} \le \frac{1}{\left(\frac{1}{|\Omega|}\int_\Omega |\, d_F(x,x_0) - c_p|^p\, dx\right)^{\frac{1}{p}}}.$$

Since $0 \le c \le \mathrm{diam}_F(\Omega)$, then there exists a constant c such that, up to a subsequence, $c_p \to c$ and $0 \le c \le \mathrm{diam}_F(\Omega)$. Therefore we have that

$$\liminf_{p\to\infty}\left(\frac{1}{|\Omega|}\int_\Omega |\, d_F(x,x_0) - c_p|^p\, dx\right)^{\frac{1}{p}} = \sup_{x\in\Omega} |\, d_F(x,x_0) - c| \ge \frac{\sup_{x\in\Omega}\, d_F(x,x_0)}{2}$$

for all $x_0 \in \Omega$, hence

$$\liminf_{p\to\infty} \Lambda_1(p,\Omega)^{-\frac{1}{p}} \ge \frac{\mathrm{diam}_F(\Omega)}{2}.$$

Step 2. $\limsup_{p\to\infty} \Lambda_1(p,\Omega)^{\frac{1}{p}} \ge \frac{2}{\mathrm{diam}_F(\Omega)}$.

The minimum u_p of (102) is such that

$$\left(\frac{1}{|\Omega|}\int_\Omega F^p(\nabla u_p)\, dx\right)^{\frac{1}{p}} = \Lambda_1(p,\Omega)^{\frac{1}{p}}$$

Let us fix m such that $n < m < p$, then, by Hölder inequality we have

$$\left(\frac{1}{|\Omega|}\int_\Omega F^m(\nabla u_p)\, dx\right)^{\frac{1}{m}} \le \Lambda_1(p,\Omega)^{\frac{1}{p}}.$$

Hence $\{u_p\}_{p\geq m}$ is uniformly bounded in $W^{1,m}(\Omega)$ and therefore weakly converges in $W^{1,m}(\Omega)$ to a function $u_\infty \in C_c(\Omega)$. By lower semicontinuity of $\int_\Omega F(\cdot)$ and by Hölder inequality, we have

$$\begin{aligned}
\frac{||F(u_\infty)||_m}{||u_\infty||_m} &\leq \limsup_{p\to\infty} \frac{\left(\frac{1}{|\Omega|}\int_\Omega F^m(\nabla u_p)\,dx\right)^{\frac{1}{m}}}{\left(\frac{1}{|\Omega|}\int_\Omega |u_p|^m\,dx\right)^{\frac{1}{m}}} \leq \\
&\leq \limsup_{p\to\infty} \frac{\left(\frac{1}{|\Omega|}\int_\Omega F^p(\nabla u_p)\,dx\right)^{\frac{1}{p}}}{\left(\frac{1}{|\Omega|}\int_\Omega |u_p|^m\,dx\right)^{\frac{1}{m}}} = \\
&= \limsup_{p\to\infty} \Lambda_1(p,\Omega)^{\frac{1}{p}} \frac{\left(\frac{1}{|\Omega|}\int_\Omega |u_p|^p\,dx\right)^{\frac{1}{p}}}{\left(\frac{1}{|\Omega|}\int_\Omega |u_p|^m\,dx\right)^{\frac{1}{m}}} = \limsup_{p\to\infty} \Lambda_1(p,\Omega)^{\frac{1}{p}} \frac{||u_\infty||_\infty}{||u_\infty||_m}.
\end{aligned}$$

Sending $m \to \infty$, we get

$$\frac{||F(u_\infty)||_\infty}{||u_\infty||_\infty} \leq \limsup_{p\to\infty} \Lambda_1(p,\Omega)^{\frac{1}{p}}.$$

Now we show that condition $\int_\Omega |u_p|^{p-2}u_p\,dx = 0$ leads to

$$\sup u_\infty = -\inf u_\infty u_\infty. \tag{111}$$

Indeed, we have

$$\begin{aligned}
0 &\leq \left| ||(u_\infty)^+||_{p-1} - ||(u_\infty)^-||_{p-1} \right| = \\
&= \left| ||(u_\infty)^+||_{p-1} - ||(u_p)^+||_{p-1} + ||(u_p)^-||_{p-1} - ||(u_\infty)^-||_{p-1} \right| \leq \\
&\leq \left| ||(u_\infty)^+||_{p-1} - ||(u_p)^+||_{p-1} \right| + \left| ||(u_\infty)^-||_{p-1} - ||(u_p)^-||_{p-1} \right| \leq \\
&\leq ||(u_\infty)^+ - (u_p)^+||_{p-1} + ||(u_\infty)^- - (u_p)^-||_{p-1}.
\end{aligned}$$

Letting $p \to \infty$, we obtain (111). Now, let us fix $x, y \in \Omega$ and let us define $v(t) = u_\infty(tx + (1-t)y)$. Using the scalar product property (30), we get

$$\begin{aligned}
|u_\infty(x) - u_\infty(y)| = |v(1) - v(0)| = \left|\int_0^1 v'(t)dt\right| &= \left|\int_0^1 \nabla u_\infty(tx+(1-t)y)\cdot(x-y)dt\right| \leq \\
&\leq \int_0^1 F(\nabla u_\infty(tx+(1-t)y))F^o(x-y)dt \leq \\
&\leq ||F(\nabla u_\infty)||_\infty \int_0^1 F^o(x-y)dt \leq ||F(\nabla u_\infty)||_\infty d_F(x,y).
\end{aligned}$$

Hence we conclude by observing that

$$\begin{aligned}
2||u||_\infty = \sup u_\infty - \inf u_\infty &\leq |u_\infty(x) - u_\infty(y)| \leq \\
&\leq ||F(\nabla u_\infty)||_\infty d_F(x,y) \leq ||F(\nabla u_\infty)||_\infty \operatorname{diam}_F(\Omega).
\end{aligned}$$

□

We also treat the eigenvalue problem (13) in viscosity sense, hence now we recall the definition of viscosity supersolutions and viscosity subsolutions to this problem.

Definition 2.50. *An upper semicontinuous function u is a viscosity subsolution to (13) if whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that*

$$u(x_0) = \phi(x_0), \text{ and } u(x) < \phi(x) \text{ if } x \neq x_0,$$

then

$$\begin{aligned} &A(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \leq 0 \quad \text{if } u(x_0) > 0 \\ &B(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \leq 0 \quad \text{if } u(x_0) < 0 \\ &-\mathcal{Q}_\infty\phi(x_0) \leq 0 \quad \text{if } u(x_0) = 0 \end{aligned}$$

while if $x_0 \in \partial\Omega$ and $\phi \in C^2(\overline{\Omega})$ are such that

$$u(x_0) = \phi(x_0), \text{ and } u(x) < \phi(x) \text{ if } x \neq x_0,$$

then

$$\begin{aligned} &\min\{A(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \nabla_\xi F(\nabla\phi(x_0)) \cdot \nu\} \leq 0 \quad \text{if } u(x_0) > 0 \\ &\min\{B(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \nabla_\xi F(\nabla\phi(x_0)) \cdot \nu\} \leq 0 \quad \text{if } u(x_0) < 0 \\ &\min\{-\mathcal{Q}_\infty\phi(x_0), \nabla_\xi F(\nabla\phi(x_0)) \cdot \nu\} \leq 0 \quad \text{if } u(x_0) = 0 \end{aligned}$$

Definition 2.51. *A lower semicontinuous function u is a viscosity supersolution to (13) if whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that*

$$u(x_0) = \phi(x_0), \text{ and } u(x) > \phi(x) \text{ if } x \neq x_0,$$

then

$$A(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \geq 0 \quad \text{if } u(x_0) > 0 \tag{112}$$
$$B(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \geq 0 \quad \text{if } u(x_0) < 0 \tag{113}$$
$$-\mathcal{Q}_\infty\phi(x_0) \geq 0 \quad \text{if } u(x_0) = 0 \tag{114}$$

while if $x_0 \in \partial\Omega$ and $\phi \in C^2(\overline{\Omega})$ are such that

$$u(x_0) = \phi(x_0), \text{ and } u(x) > \phi(x) \text{ if } x \neq x_0,$$

then

$$\max\{A(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \nabla_\xi F(\nabla\phi(x_0)) \cdot \nu\} \geq 0 \quad \text{if } u(x_0) > 0 \tag{115}$$
$$\max\{B(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \nabla_\xi F(\nabla\phi(x_0)) \cdot \nu\} \geq 0 \quad \text{if } u(x_0) < 0 \tag{116}$$
$$\max\{-\mathcal{Q}_\infty\phi(x_0), \nabla_\xi F(\nabla\phi(x_0)) \cdot \nu\} \geq 0 \quad \text{if } u(x_0) = 0 \tag{117}$$

Definition 2.52. *A continuous function u is a viscosity solution to (13) if and only if it is both a viscosity subsolution and a viscosity supersolution to (13).*

Definition 2.53. *We say that a function $u \in C(\overline{\Omega})$ is an eigenfunction of (13) if there exists $\Lambda \in \mathbb{R}$ such that u solves (13) in viscosity sense. The number Λ is called an $\infty-$eigenvalue.*

Theorem 2.54. *Let Ω be an open bounded connected set $\mathbb{R}^n$. If u_∞ and $\Lambda_1(\infty, \Omega)$ are defined as in Lemma 2.49 above, then u_∞ satisfies (13) in viscosity sense with $\Lambda = \Lambda_1(\infty, \Omega)$.*

Proof. In Lemma 2.49 we have proved that there exists a subsequence u_{p_i} uniformly converging to u_∞ in Ω. To prove that u_∞ is a viscosity supersolution to (13) in Ω, we fix $x_0 \in \Omega$, $\phi \in C^2(\Omega)$ such that $\phi(x_0) = u_\infty(x_0)$ and $\phi(x) < u_\infty(x)$ for $x \in \Omega\backslash\{x_0\}$.

There exists $r > 0$ such that $u_{p_i} \to u_\infty$ uniformly in the Wulff shape $W_r(x_0)$, therefore it can be proved that $u_{p_i} - \phi$ has a local minimum in x_i such that $\lim_{i\to\infty} x_i = x_0$. By Lemma 2.49 again, we observe that u_{p_i} is a viscosity solution to (104) and in particular is a viscosity supersolution. Choosing $\psi(x) = \phi(x) - \phi(x_i) + u_{p_i}(x_i)$ as test function in (103), we obtain that (105) holds, therefore

$$\begin{aligned}-(p_i-2)F^{p_i-4}(\nabla\phi(x_i))\mathcal{Q}_\infty\phi(x_i) - F^{p_i-2}(\nabla\phi(x_i))\Delta_F(\phi(x_i)) \geq \\ \Lambda_1(p_i,\Omega)|u_{p_i}(x_i)|^{p_i-2}u_{p_i}(x_i).\end{aligned} \tag{118}$$

Hence three cases can occur.

Case 1: $u_\infty(x_0) > 0$. If p_i is sufficiently large then also $\phi(x_i) > 0$ and $\nabla\phi(x_i) \neq 0$ otherwise we reach a contradiction in (118). Dividing by $(p_i-2)F^{p_i-4}(\nabla\phi(x_i))$ both members of (118), we have

$$-\mathcal{Q}_\infty\phi(x_i) - \frac{\Delta_F(\phi(x_i))}{p_i-2} \geq \left(\frac{\Lambda_1(p_i,\Omega)^{\frac{1}{p_i}}u_{p_i}(x_i)}{F(\nabla\phi(x_i))}\right)^{p_i-4}\frac{\Lambda_1(p_i,\Omega)^{\frac{4}{p_i}}u^3_{p_i}(x_i)}{p_i-2}. \tag{119}$$

Sending $p_i \to \infty$, we obtain the necessary condition

$$\frac{\Lambda_1(\infty,\Omega)\phi(x_0)}{F(\nabla\phi(x_0))} < 1. \tag{120}$$

Taking into account (120) and sending $p_i \to \infty$ in (119), we obtain

$$-\mathcal{Q}_\infty\phi(x_0) \geq 0. \tag{121}$$

Inequalities (120) and (121) must hold together, and therefore we have

$$\min\{F(\nabla\phi(x_0)) - \Lambda_1(\infty,\Omega)\phi(x_0),\ -\mathcal{Q}_\infty\phi(x_0)\} \geq 0.$$

Case 2: $u_\infty(x_0) < 0$. Let us observe that, by definition, also $\phi(x_0) < 0$. We have to show that

$$\max\{-F(\nabla\phi(x_0)) - \Lambda_1(\infty,\Omega)\phi(x_0),\ -\mathcal{Q}_\infty\phi(x_0)\} \geq 0.$$

If $-F(\nabla\phi(x_0)) - \Lambda_1(\infty,\Omega)\phi(x_0) \geq 0$, the proof is terminated. Therefore we assume $-F(\nabla\phi(x_0)) - \Lambda_1(\infty,\Omega)\phi(x_0) < 0$, that is

$$0 > \frac{\Lambda_1(\infty,\Omega)\phi(x_0)}{F(\nabla\phi(x_0))} > -1.$$

Now let us observe that also in this case, if p_i is sufficiently large, then $\nabla\phi(x_i) \neq 0$. Therefore

$$0 > \lim_{p_i\to\infty}\Lambda_1(p_i,\Omega)^{\frac{1}{p_i}}\lim_{x_i\to x_0}\frac{u_{p_i}(x_i)}{F(\nabla\phi(x_i))} > -1$$

and hence, if p is sufficiently large, by continuity of ϕ, this inequality holds

$$0 > \frac{\Lambda_1(p_i,\Omega)^{\frac{1}{p_i}}u_{p_i}(x_i)}{F(\nabla\phi(x_i))} > -1. \tag{122}$$

Dividing again by $(p_i-2)F(\nabla\phi(x_i))^{p_i-4}$ both members of (118), we have

$$-\mathcal{Q}_\infty\phi(x_i)-\frac{\Delta_F(\phi(x_i))}{p_i-2}\geq-\frac{\Lambda_1(p_i,\Omega)^{\frac{4}{p_i}}u_{p_i}^3(x_i)}{p_i-2}\left(-\frac{\Lambda_1(p_i,\Omega)^{\frac{1}{p_i}}u_{p_i}(x_i)}{F(\nabla\phi(x_i))}\right)^{p_i-4}. \quad (123)$$

Taking into account (122) and sending $p_i\to\infty$ in (123), we obtain

$$-\mathcal{Q}_\infty\phi(x_0)\geq 0.$$

that ends the proof in the case 2.

Case 3: $u_\infty(x_0)=0$. If $\nabla\phi(x_0)=0$ then, by definition, $-\mathcal{Q}_\infty\phi(x_0)=0$ and $A(\phi(x_0),\nabla\phi(x_0),\nabla^2\phi(x_0))$. On the other hand, if $\nabla\phi(x_0)\neq 0$ we have that $\lim_{i\to\infty}\frac{\Lambda_1(p_i,\Omega)^{\frac{1}{p_i}}\phi(x_i)}{F(\nabla\phi(x_i))}=0$. Then, again dividing by $(p_i-2)F^{p_i-4}(\nabla\phi(x_i))$ both members of (118) and sending $p_i\to\infty$ in (119), we obtain

$$-\mathcal{Q}_\infty\phi(x_0)\geq 0.$$

Finally we prove that u_∞ satisfies also the boundary condition in viscosity sense. We assume that $x_0\in\partial\Omega$, $\phi\in C^2(\overline{\Omega})$ is such that $\phi(x_0)=u_\infty(x_0)$ and $\phi(x)<u_\infty(x)$ in $\overline{\Omega}\backslash\{0\}$. Using again the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i}-\phi$ has a minimum point $x_i\in\overline{\Omega}$, with $\lim_{i\to\infty}x_i=x_0$.

When $x_i\in\Omega$ for infinitely many i, arguing as before, we get

$$\begin{aligned}\min\{F(\nabla\phi(x_0))-\Lambda_1(\infty,\Omega)\phi(x_0),\ -\mathcal{Q}_\infty\phi(x_0)\}\geq 0,&\quad \text{if } u_\infty(x_0)>0,\\ \max\{-F(\nabla\phi(x_0))-\Lambda_1(\infty,\Omega)\phi(x_0),\ -\mathcal{Q}_\infty\phi(x_0)\}\geq 0,&\quad \text{if } u_\infty(x_0)<0,\\ -\mathcal{Q}_\infty\phi(x_0)\geq 0,&\quad \text{if } u_\infty(x_0)=0.\end{aligned}$$

When $x_i\in\partial\Omega$, since u_{p_i} is a viscosity solution to (104), for infinitely many i we have

$$\max\{G_p(\phi(x_i),\nabla\phi(x_i),\nabla^2\phi(x_i)),\nabla_\xi F^p(\nabla\phi(x_i))\cdot\nu\}\geq 0.$$

If $G_p(\phi(x_i),\nabla\phi(x_i),\nabla^2\phi(x_i))\geq 0$, we argue again as before, otherwise we have that $\nabla_\xi F^p(\nabla\phi(x_i))\cdot\nu\geq 0$, i.e. $F^{p-1}(\nabla\phi(x_i))\nabla_\xi F(\nabla\phi(x_i))\cdot\nu\geq 0$. This implies $\nabla_\xi F(\nabla\phi(x_i))\cdot\nu\geq 0$ and passing to the limit for $i\to\infty$ we have $\nabla_\xi F(\nabla\phi(x_0))\cdot\nu\geq 0$, that concludes the proof. Arguing in the same way we can prove that u_∞ is a viscosity subsolution to (13) in Ω. □

2.4.2 Proof of the Main Result

In this Section we will use some comparison result for viscosity solutions. Let us observe that uniqueness and comparison theorems for elliptic equations of second order (see for example [32]) of the form $G(x,u,\nabla u,\nabla^2 u)=0$ require that the function $G(x,r,p,X)$ has to satisfy a fundamental monotonicity condition:

$$G(x,r,p,X)\leq G(x,s,p,Y)\quad\text{whenever } r\leq s \text{ and } Y\leq X,$$

for all $x\in\mathbb{R}^n$, $r,s\in\mathbb{R}$, $p\in\mathbb{R}^n$, $X,Y\in S^n$, where S^n is the set of symmetric $n\times n$ matrices. The equation

$$\begin{cases}A(u,\nabla u,\nabla^2 u)=\min\{F(\nabla u)-\Lambda u,-\mathcal{Q}_\infty u\}=0 & \text{in } \Omega,\ \text{if } u>0,\\ B(u,\nabla u,\nabla^2 u)=\max\{-F(\nabla u)-\Lambda u,-\mathcal{Q}_\infty u\}=0 & \text{in } \Omega,\ \text{if } u<0,\\ -\mathcal{Q}_\infty u=0 & \text{in } \Omega,\ \text{if } u=0\end{cases}$$

does not satisfy this monotonicity condition.

So, for $\varepsilon > 0$ small enough, in the sequel we will use a comparison result for lower semicontinuous functions u that has a strictly positive minimum m in an open bounded set. It is easily seen that if u is a viscosity supersolution to the first equation of (13), then it is also a viscosity supersolution to

$$\min\{F(\nabla u) - \varepsilon, -\mathcal{Q}_\infty u\} = 0, \tag{124}$$

with $\varepsilon = \Lambda m$.

To state the main Theorem, we give two preliminary results. We can argue as in [57, Lem. 3, Lem.4, Prop. 1]. For completeness we give the proof.

Lemma 2.55. *Let Ω be a smooth open bounded convex set in $\mathbb{R}^n$, let $\Lambda > 0$ be an eigenvalue for problem (13) that admits a nontrivial eigenfunction u.*

(1) If Ω_1 is an open connected subset Ω such that $u \geq m$ in $\overline{\Omega}_1$ for some positive constant m, then $u > m$ in Ω_1.

(2) The eigenfunction u changes sign.

Proof. To prove *(1)*, we fix $x_0 \in \Omega_1$ and we prove that $u(x_0) > m$. Firstly, let us observe that u is a viscosity supersolution and that $u \not\equiv m$ for any $W_R(x_0) \subset \Omega_1$. Otherwise $F(\nabla u) - \Lambda u < 0$ (in viscosity sense), that contradicts (112). Therefore, there exists $x_1 \in W_{\frac{R}{4}}(x_0)$ such that $u(x_1) > m$. For $\varepsilon > 0$ small enough, there exists $r \leq d_F(x_0, x_1)$ such that $u > m + \varepsilon$ on $\partial W_r(x_1)$. Therefore the function

$$v(x) = m + \frac{\varepsilon}{\frac{R}{2} - r}\left(\frac{R}{2} - F^o(x - x_1)\right) \qquad \text{in } W_{\frac{R}{2}}(x_1)\backslash W_r(x_1),$$

by using (34), satisfies

$$-\mathcal{Q}_\infty v = 0 \qquad \text{in } W_{\frac{R}{2}}(x_1)\backslash W_r(x_1).$$

Hence v is a solution and in particular a viscosity subsolution to $-\mathcal{Q}_\infty v = 0$, and therefore v is a viscosity subsolution to (124). Furtherly, u is a viscosity supersolution to (124) with $\varepsilon = \Lambda m$ and

$$u \geq v \qquad \text{in } \partial W_{\frac{R}{2}}(x_1)\backslash \partial W_r(x_1).$$

The comparison principle in [80] implies $u \geq v > m$ in $W_{\frac{R}{2}}(x_1)\backslash W_r(x_1)$. Therefore $u(x_0) > m$ and this conclude the proof of *(1)*.

To prove *(2)*, we observe that the solution u to (13) is a nontrivial solution, so we can assume that it is positive somewhere, at most changing sign. We have to prove that the minimum m of u in $\overline{\Omega}$ is negative. By contradiction we assume $m \geq 0$ and two cases occur.

Case 1: $m > 0$. By *(1)*, the minimum cannot be obtained in Ω.

Case 2: $m = 0$. Since $u \not\equiv 0$, if the minimum is reached in Ω, then there would exists a point $x_0 \in \Omega$ and a Wulff shape $W_R(x_0) \subset \Omega$ such that $u(x_0) = 0$ and $\max_{W_{\frac{R}{4}}(x_0)} u > 0$. Now let $x_1 \in W_{\frac{R}{4}}(x_0)$ such that $u(x_1) > 0$. The continuity of u implies that there exists $r \leq d_F(x_0, x_1)$ such that $u > \frac{u(x_1)}{2}$ on $\partial W_r(x_1)$. Therefore the function

$$v(x) = \frac{u(x_1)}{R - 2r}\left(\frac{R}{2} - F^o(x - x_1)\right) \qquad \text{in } W_{\frac{R}{2}}(x_1)\backslash W_r(x_1),$$

is such that

$$-\mathcal{Q}_\infty v = 0 \qquad \text{in } W_{\frac{R}{2}}(x_1)\backslash W_r(x_1).$$

Hence v is a solution and in particular a viscosity subsolution to $-\mathcal{Q}_\infty v = 0$, therefore v is a viscosity subsolution to (124). Furtherly, u is a viscosity supersolution to (124) with $\varepsilon = \Lambda m$ and

$$u \geq v \qquad \text{in } \partial W_{\frac{R}{2}}(x_1)\backslash \partial W_r(x_1).$$

The comparison principle in [80] implies $u \geq v > 0$ in $W_{\frac{R}{2}}(x_1)\backslash W_r(x_1)$, and therefore $u(x_0) > 0$.

We have proved that there exists a nonnegative minimum point $x_0 \in \partial\Omega$. We shall prove that u does not satisfies the boundary condition (115)-(117) for viscosity supersolutions. Indeed there certainly exists $\bar{x} \in \Omega$ and $r > 0$ such that the Wulff shape $W_r(x_0)$ is inner tangential to $\partial\Omega$ at x_0 and $\partial W_r(\bar{x}) \cap \partial\Omega = \{x_0\}$. Then the function

$$v(x) = u(\bar{x}) - \left(\frac{u(\bar{x}) - u(x_0)}{r}\right) F^o(x - \bar{x}) \qquad \text{in } W_r(\bar{x})\backslash\{\bar{x}\}$$

is such that

$$-\mathcal{Q}_\infty v = 0 \qquad \text{in } W_r(\bar{x})\backslash\{\bar{x}\}.$$

Hence v is a solution and in particular is a viscosity subsolution to $-\mathcal{Q}_\infty v = 0$, therefore v is a viscosity subsolution to (124). Furtherly, u is a viscosity supersolution to (124) with $\varepsilon = \Lambda m$ and

$$u \geq v \qquad \text{in } \partial W_r(\bar{x}) \cup \{\bar{x}\}.$$

The comparison principle in [80] implies $u \geq v > 0$ in $\overline{W_r(\bar{x})}$. Therefore the function

$$\phi(x) = u(\bar{x}) - (u(\bar{x}) - u(x_0)) \left(\frac{F^o(x - \bar{x})}{r}\right)^{\frac{1}{2}}$$

is such that $\phi \in C^2(\overline{\Omega}\backslash\{\bar{x}\})$,

$$\begin{aligned} \phi < v \leq u \qquad & \text{in } W_r(\bar{x})\backslash\{\bar{x}\}, \\ \phi(x) < u(x_0) \leq u(x) \qquad & \text{in } \Omega\backslash W_r(\bar{x}) \end{aligned}$$

and

$$u(x_0) = \phi(x_0).$$

Hence ϕ gives a contradiction with the boundary condition for viscosity supersolution. Indeed we have that $-\mathcal{Q}_\infty \phi(x_0) = \frac{1}{8r^4\sqrt{r}}(u(\bar{x}) - u(x_0))^3 < 0$ that is in contradiction with (117) if $u(x_0) = 0$. Otherwise, if $u(x_0) > 0$, we have that

$$A(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) = \min\{F(\nabla\phi(x_0)) - \Lambda\phi(x_0), -\mathcal{Q}_\infty\phi(x_0)\} < 0.$$

Furthermore,

$$\nabla_\xi F(\nabla\phi(x_0)) \cdot \nu = -\frac{x_0 - \bar{x}}{F^o(x_0 - \bar{x})} \cdot \nu < 0,$$

and hence

$$\max\{A(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \nabla_\xi F(\nabla\phi(x_0)) \cdot \nu\} < 0$$

that contradicts (115). □

Now we prove that $\Lambda_\infty(\Omega)$ as defined in (110) is the first nontrivial eigenvalue.

Proposition 2.56. *Let Ω be a smooth open bounded convex set in $\mathbb{R}^n$. If for some $\Lambda > 0$ the eigenvalue problem (13) admits a nontrivial eigenfunction u, then $\Lambda \geq \Lambda_1(\infty, \Omega)$.*

Proof. Let us denote by $\Omega_+ = \{x \in \Omega \,:\, u(x) > 0\}$ and $\Omega_- = \{x \in \Omega \,:\, u(x) < 0\}$. By Lemma 2.55, they are both nonempty. Now we call $\bar{u}$ the normalized function of u such that

$$\max_{\overline{\Omega}} \bar{u} = \frac{1}{\Lambda}.$$

The fact that $\Lambda\bar{u} \leq 1$ and that u is a viscosity subsolution to (13) imply that $\bar{u}$ is also a viscosity subsolution to

$$\min\{F(\nabla\bar{u}) - 1, -\mathcal{Q}_\infty \bar{u}\} = 0 \qquad \text{in } \Omega_+.$$

For all $x_0 \in \Omega \backslash \Omega_+$, $\varepsilon > 0$ and $\gamma > 0$, we consider the function

$$g_{\varepsilon,\gamma}(x) = (1+\varepsilon)F^o(x - x_0) - \gamma(F^o(x - x_0))^2.$$

It belongs to $C^2(\Omega \backslash W_\rho(x_0))$ for every $\rho > 0$ and, if γ is small enough compared with ε, it verifies

$$\min\{F(\nabla g_{\varepsilon,\gamma}) - 1, -\mathcal{Q}_\infty g_{\varepsilon,\gamma}\} \geq 0 \qquad \text{in } \Omega_+.$$

Hence, the comparison principle in [80] hence implies that

$$m = \inf_{x\in\Omega_+} (g_{\varepsilon,\gamma}(x) - u(x)) = \inf_{x\in\partial\Omega_+} (g_{\varepsilon,\gamma}(x) - u(x)). \tag{125}$$

We show now that the minimum is reached on Ω. By (125) this means that we want to prove that

$$m = \inf_{x\in\Omega_+} (g_{\varepsilon,\gamma}(x) - u(x)) = \inf_{x\in\partial\Omega_+\cap\Omega} (g_{\varepsilon,\gamma}(x) - u(x)) \geq 0. \tag{126}$$

We assume that there exists $\bar{x} \in \partial\Omega \cap \partial\Omega_+$ such that $g_{\varepsilon,\gamma}(\bar{x}) - u(\bar{x}) = m$. We get $g_{\varepsilon,\gamma}(x) - m$ as test function in (2.50), then, by construction for every $x \in \partial\Omega \cap \partial\Omega_+$ and $\gamma < \frac{\varepsilon}{2\,\mathrm{diam}_F(\Omega)}$, it results that

$$\begin{aligned} F(\nabla g_{\varepsilon,\gamma}(x)) &= 1 + \varepsilon - 2\gamma F^o(x - x_0) > 1, \\ \nabla F(\nabla g_{\varepsilon,\gamma}(x)) \cdot \nu &= \frac{x - x_0}{F^o(x - x_0)} \cdot \nu > 0, \\ -\mathcal{Q}_\infty g_{\varepsilon,\gamma}(x) &= 2\gamma F^2(\nabla g_{\varepsilon,\gamma}(x)) > 0 \end{aligned}$$

which gives a contradiction to (2.50).

Hence (126) implies that

$$g_{\varepsilon,\gamma}(x) \geq u(x) \quad \forall\, x \in \overline{\Omega^+}\,, \forall\, x_0 \in \overline{\Omega^-}.$$

Sending ε and γ go to zero we have that

$$F^o(x - x_0) \geq u(x) \quad \forall\, x \in \overline{\Omega^+}\,, \forall\, x_0 \in \overline{\Omega^-},$$

therefore

$$d_F^+ = \sup_{x \in \overline{\Omega_+}} d_F(x, \{u = 0\}) \geq \frac{1}{\Lambda}.$$

Arguing in the same way we obtain

$$d_F^- = \sup_{x \in \overline{\Omega_-}} d_F(x, \{u = 0\}) \geq \frac{1}{\Lambda}.$$

Finally

$$\mathrm{diam}_F(\Omega) \geq d_F^+ + d_F^- \geq \frac{2}{\Lambda},$$

which concludes the proof of our proposition. □

In conclusion, Theorem 2.54 and Proposition 2.56 leads to the main result.

Theorem 2.57. *Let Ω be a smooth open bounded convex set in $\mathbb{R}^n$. Then a necessary condition for existence of nonconstant continuous solutions to (13) is*

$$\Lambda \geq \Lambda_1(\infty, \Omega) = \frac{2}{\mathrm{diam}_F(\Omega)}.$$

Problem (13) admits a Lipschitz solution when $\Lambda = \frac{2}{\mathrm{diam}_F(\Omega)}$.

One of most interesting consequences of this result is that, with the use of the isodiametric inequality (47), we can state an anisotropic version of a Szegö-Weinberger inequality.

Theorem 2.58. *The Wulff shape f $\Omega^\star$ maximizes the first nontrivial Neumann ∞-eigenvalue among smooth open bounded convex sets Ω of fixed volume:*

$$\Lambda_1(\infty, \Omega) \leq \Lambda_1(\infty, \Omega^\star).$$

2.4.3 Geometric properties of the first ∞-eigenvalue

A consequence of the main Theorem 2.57 is in showing that the the first nontrivial Neumann ∞-eigenvalue $\Lambda_1(\infty, \Omega)$ is never large than the first Dirichlet ∞-eigenvalue $\lambda_1(\infty, \Omega)$. To prove this result, we first recall two preliminary Lemmas from [21, Lem. A.1, Lem. 2.2].

Lemma 2.59. *Let $\ell > 0$ and $g : [-\ell, \ell] \to \mathbb{R}^+$ defined by*

$$g(s) = \omega_{n-1} |\ell - s|^{n-1}.$$

Then, the problem

$$\eta := \inf_{v \in W^{1,p}((-\ell,\ell)\setminus\{0\})} \left\{ \frac{\displaystyle\int_{-\ell}^{\ell} |v'|^p g \, ds}{\displaystyle\int_{-\ell}^{\ell} |v|^p g \, ds} \; : \; \int_{-\ell}^{\ell} |v|^{p-2} v g \, ds = 0 \right\}.$$

admits a solution. Any optimizer f is a weak solution of

$$\begin{cases} -(g|f'|^{p-2}f')' = \eta g |f|^{p-2} f, & \text{in } (-\ell, \ell), \\ f'(-\ell) = f'(\ell) = 0. \end{cases}$$

Moreover, f vanishes at $x = 0$ only and thus is also a weak solution of

$$\begin{cases} -(g|f'|^{p-2}f')' = \eta g|f|^{p-2}f, & \text{in } (0,\ell), \\ f'(0) = f'(\ell) = 0. \end{cases}$$

Lemma 2.60. *Let Ω be an open convex set, and let $x_0 \in \partial\Omega$. Then*

$$(x - x_0) \cdot \nu(x) \leq 0, \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where ν is the outer unit normal to $\partial\Omega$ at the point x.

Now we give an important spectral Theorem that extends the result in [21, Theorem 3.1] to the anisotropic case.

Proposition 2.61. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set $1 < p < \infty$. Then we have*

$$\Lambda_1(p,\Omega) < \lambda_1(p,W)\left(\frac{\mathrm{diam}_F(W)}{\mathrm{diam}_F(\Omega)}\right)^p \tag{127}$$

where W is any n-dimensional Wulff shape.

Equality sign in (127) is never achieved but the inequality is sharp. More precisely, there exists a sequence $\{\Omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ of convex sets such that:

- $\mathrm{diam}_F(\Omega_k) = d > 0$ *for every $k \in \mathbb{N}$;*
- *Ω_k converges to a segment of anisotropic lenght (that is the diameter) d in the Hausdorff topology;*
- *it holds*

$$\lim_{k\to\infty} \Lambda_1(p,\Omega_k) = \lambda_1(p,W_{\frac{d}{2}}) \tag{128}$$

where $W_{\frac{d}{2}}$ is an n-dimensional Wulff shape of anisotropic radius $\frac{d}{2}$.

Proof. We split the proof into two parts: at the first we prove (127), then we construct the sequence $\{\Omega_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ verifying (128).

Step 1. Without loss of generality, since (127) is in scaling invariant form, we have only to prove that

$$\Lambda_1(p,\Omega) < \lambda_1(p,W)$$

where W is the Wulff shape centered in the origin such that $\mathrm{diam}_F(\Omega) = \mathrm{diam}_F(W)$. Let us take $u \in C^{1,\alpha}(\overline{W}) \cap C^\infty(W\backslash\{0\})$ the first Dirichlet eigenfunction for the Wulff shape W such that it is positive and normalized by the condition $||u||_{L^p(W)} = 1$. This function u is convexly symmetric in the sense of (35), i.e. $u(x) = u^\star(x)$, and solves (see for examples [45])

$$\begin{cases} -\mathcal{Q}_p u = \lambda_1(p,W)u^{p-1} & \text{in } W, \\ u = 0 & \text{on } \partial W. \end{cases} \tag{129}$$

Now, we have two points $x_0, x_1 \in \partial\Omega$ such that $F^o(x_0 - x_1) = \mathrm{diam}_F(\Omega)$ and we define the sets

$$\Omega_i = \left\{ x \in \Omega \ : \ F^o(x - x_i) < \frac{\mathrm{diam}_F(\Omega)}{2} \right\}, \quad i = 0, 1,$$

which are mutually disjoint. Then we consider the $W^{1,p}(\Omega)$ function

$$\varphi(x) = u(x-x_0)\chi_{\Omega_0}(x) - cu(x-x_0)\chi_{\Omega_1}(x)$$

where $c = \frac{\int_{\Omega_0} u(x-x_0)^{p-1}\,dx}{\int_{\Omega_1} u(x-x_1)^{p-1}\,dx}$, so that $\int_\Omega |\varphi|^{p-2}\varphi\,dx = 0$. By using this function in the Raylegh quotient, we have

$$\Lambda_1(p,\Omega) = \min_{u\in W^{1,p}(\Omega)} \frac{\int_\Omega F^p(\nabla u)\,dx}{\int_\Omega |u|^p\,dx}$$
$$\leq \frac{\int_{\Omega_0} F^p(\nabla u(x-x_0))\,dx + \int_{\Omega_1} F^p(\nabla u(x-x_0))\,dx}{\int_{\Omega_0} |u(x-x_0)|^p\,dx + \int_{\Omega_1} |u(x-x_0)|^p\,dx}$$

Now we prove that this inequality is strict. In fact, by contradiction, if φ achieves the minimum $\Lambda_1(p,\Omega)$ of the Raylegh quotient, then φ solves $-\mathcal{Q}_p u = \Lambda_1(p,\Omega)|u|^{p-2}u$ in Ω, in the weak sense. Let us take $y_0 \in \partial\Omega_0 \cap \Omega$, by picking a Wulff shape $W_\rho(y_0)$ with radius ρ sufficiently small so that $W_\rho(y_0) \subset \Omega\backslash\Omega_1$, we would obtain that φ is a nonnegative solution to the equation above in $W_\rho(y_0)$. Then, by Harnack's inequality (see [110]) we obtain

$$0 < \max_{W_\rho(x_0)} \varphi \leq \min_{W_\rho(x_0)} \varphi = 0,$$

that is absurd. Hence

$$\Lambda_1(p,\Omega) < \frac{\int_{\Omega_0} F^p(\nabla u(x-x_0))\,dx + \int_{\Omega_1} F^p(\nabla u(x-x_0))\,dx}{\int_{\Omega_0} |u(x-x_0)|^p\,dx + \int_{\Omega_1} |u(x-x_0)|^p\,dx}$$

Let us observe that $u(x-x_i) = 0$ on $\partial\Omega_i \cap \Omega$ and therefore, by an integration by parts, by (31) and by (129), we have

$$\begin{aligned}
\int_{\Omega_0} F(\nabla u(x-x_0))^p\,dx &= \int_{\Omega_0} F^{p-1}(\nabla u(x-x_0))\nabla_\xi F(\nabla u(x-x_0))\nabla u(x-x_0)\,dx = \\
&= \int_{\partial\Omega\cap\partial\Omega_0} F^{p-1}(\nabla u(x-x_0))\nabla_\xi F(\nabla u(x-x_0))\cdot\nu\; u(x-x_0)\,dx \\
&\quad - \int_{\Omega_0} \operatorname{div}(F^{p-1}(\nabla u(x-x_0))\nabla_\xi F(\nabla u(x-x_0))u(x-x_0)\,dx \\
&= \int_{\partial\Omega\cap\partial\Omega_0} F^{p-1}(\nabla u(x-x_0))\nabla_\xi F(\nabla u(x-x_0))\cdot\nu\; u(x-x_0)\,dx \\
&\quad + \lambda_1(p,W)\int_{\Omega_0} u^p(x-x_0)\,dx.
\end{aligned}$$

Since u is a convexly symmetric function, i.e. it coincides with its convex rearrangement (35), by (36)-(37)-(38) we have $\nabla_\xi F(\nabla u(x-x_0)) = \frac{x-x_0}{F^o(x-x_0)}$ and hence

$$\nabla_\xi F(\nabla u(x-x_0))\cdot\nu = \frac{1}{F^o(x-x_0)}(x-x_0)\cdot\nu$$

that is negative by Lemma 2.60. An analogous computation holds on Ω_1. Finally we obtain

$$\Lambda_1(p,\Omega) < \lambda_1(p,W)\frac{\int_{\Omega_0} |u(x-x_0)|^p\,dx + c^p\int_{\Omega_1} |u(x-x_1)|^p\,dx}{\int_{\Omega_0} |u(x-x_0)|^p\,dx + c^p\int_{\Omega_1} |u(x-x_1)|^p\,dx} = \lambda_1(p,W)$$

Step 2. Let $W_{\frac{d}{2}}$ a Wulff shape of radius $\frac{d}{2}$. Now we construct a sequence of convex sets $\{\Omega_k\}_{k\in\mathbb{N}}$, with $\mathrm{diam}_F(\Omega_k) = d$ and such that

$$\lambda_1(p, W_{\frac{d}{2}}) \leq \liminf_{k\to\infty} \Lambda_1(p, \Omega_k).$$

As observed in (99), the diameter is invariant by rotation. Hence we can suppose that there exists a rotation $A \in SO(n)$ such that the anisotropic diameter is on the x_1 axis. Moreover we observe that, by the change of variables $y = Ax$ and using (98), we have

$$\int_\Omega F^p(\nabla u(x))\, dx = \int_{A\Omega} F_A^p(\nabla u(A^T y))\, dy.$$

Therefore we can suppose that A is the identity matrix. By the properties of F, we observe that when we fix the direction e_1 of the x_1 axis, there exists a positive constant γ such that $\alpha \leq \gamma \leq \beta$ and

$$F^o(\xi) = \gamma|\xi| \quad \text{and} \quad F(\xi) = \frac{1}{\gamma}|\xi|, \ \ \forall \xi \in \mathrm{Span}\{e_1\}. \tag{130}$$

Let $s \in \mathbb{R}$ and $k \in \mathbb{N}\setminus\{0\}$, we denote by

$$C_k^-(s) = \{(x_1, x') \in \mathbb{R}\times\mathbb{R}^{n-1} \ : \ (x_1 - s)_- > k\,|x'|\}$$

and

$$C_k^+(s) = \{(x_1, x') \in \mathbb{R}\times\mathbb{R}^{n-1} \ : \ (x_1 - s)_+ > k\,|x'|\}$$

the left and right circular infinite cone in $\mathbb{R}^n$ whose axis is the x_1-axis, having vertex in $(s,0) \in \mathbb{R}\times\mathbb{R}^{n-1}$, and whose opening angle is $\alpha = 2\arctan\frac{1}{k}$. We set $\frac{d}{2\gamma} = \ell$

$$\Omega_k = C_k^-(\ell) \cap C_k^+(-\ell).$$

Let us observe that for k big enough, the points that realize the anisotropic diameter of Ω_k are $(-\ell, 0)$ and $(\ell, 0) \in \mathbb{R}\times\mathbb{R}^{n-1}$. They have anisotropic distance that is $F^o(\ell+\ell, 0) = 2\gamma\ell = d$. Whenever $u \in W^{1,p}(\Omega_k)$, then $v(x_1, x') = u\left(x_1, \frac{x'}{k}\right)$ belong to $W^{1,p}(\Omega_1)$ and we have

$$\begin{aligned}
&\int_{\Omega_1} F^p\left(\frac{\partial v}{\partial x_1}, k\nabla_{x'} v\right)\, dx = k^{n-1}\int_{\Omega_k} F^p(\nabla u)\, dx,\\
&\int_{\Omega_1} |v|^p = k^{n-1}\int_{\Omega_k} |u|^p\, dx,\\
&\int_{\Omega_1} |v|^{p-2}v\, dx = k^{n-1}\int_{\Omega_k} |u|^{p-2}u\, dx = 0.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\Lambda_1(p, \Omega_k) &= \min_{u\in W^{1,p}(\Omega_k)\setminus\{0\}} \left\{ \frac{\int_{\Omega_k} F^p(\nabla u)\, dx}{\int_{\Omega_k} |u|^p\, dx} \ : \ \int_{\Omega_k} |u|^{p-2}u\, dx \right\}\\
&= \min_{v\in W^{1,p}(\Omega_1)\setminus\{0\}} \left\{ \frac{\int_{\Omega_1} F^p(\frac{\partial v}{\partial x_1}, k\nabla_{x'} v)\, dx}{\int_{\Omega_1} |v|^p\, dx} \ : \ \int_{\Omega_1} |v|^{p-2}u\, dx \right\} := \gamma_k(\Omega_1).
\end{aligned}$$

Now we denote by u_k a function which minimizes the Raylegh quotient defining $\Lambda_1(p,\Omega_k)$ and and by $v_k(x_1,x')=u_k\left(x_1,\frac{x'}{k}\right)$ the corresponding function which minimizes the functional defining $\gamma_k(\Omega_1)$. Without loss of generality we can assume that $||v_k||_{L^p(\Omega_1)}=1$. Inequality (127) implies that

$$\int_{\Omega_1} F^p\left(\frac{\partial v}{\partial x_1},k\nabla_{x'}v\right)\,dx\le C_{n,p,d} \tag{131}$$

for all $k\in\mathbb{N}\backslash\{0\}$, then there exists $w\in W^{1,p}(\Omega_1)\backslash\{0\}$ so that $v_k\rightharpoonup w$ in $W^{1,p}(\Omega_1)$ and strongly in $L^p(\Omega_1)$. So we have that $\int_{\Omega_1}|w|^{p-2}w\,dx=0$ and the bound (131) implies that for every given $k_0\in\mathbb{N}\backslash\{0\}$, we have

$$\begin{aligned}k_0^p\alpha^p\int_{\Omega_1}|\nabla_{x'}w|^p\,dx&\le\alpha^p\int_{\Omega_1}\left(\left|\frac{\partial w}{\partial x_1}\right|^2+k_0^2|\nabla_{x'}w|^2\right)^{\frac{p}{2}}\,dx\\ &\le\int_{\Omega_1}F^p\left(\frac{\partial w}{\partial x_1},k_0\nabla_{x'}w\right)\,dx\le\liminf_{k\to\infty}\int_{\Omega}F^p\left(\frac{\partial v_k}{\partial x_1},k_0\nabla_{x'}v_k\right)\,dx\le C_{n,p,d}\end{aligned}$$

which gives $\nabla_{x'}w=0$ by the arbitrariness of k_0. Thus w does not depend on the x' variables and with an abuse of notation, we will write $w=w(x_1)$. For all $t\in[-\ell,\ell]$ we denote by Γ_t the section of Ω_1 which is ortogonal to the x_1 axis at $x_1=t$ and we set $g(t)=\mathcal{H}^{n-1}(\Gamma_t)$. Also using (130), we get

$$\begin{aligned}\liminf_{k\to\infty}\gamma_k(\Omega_1)&=\liminf_{k\to\infty}\frac{\int_{\Omega_1}F^p(\frac{\partial v_k}{\partial x_1},k\nabla_{x'}v_k)\,dx}{\int_{\Omega_1}|v_k|^p\,dx}\\ &\ge\frac{\int_{\Omega_1}F^p(w',0,\dots,0)\,dx}{\int_{\Omega_1}|v|^p\,dx}=\frac{1}{\gamma^p}\frac{\int_{-\ell}^{\ell}|w'(t)|^pg(t)\,dt}{\int_{-\ell}^{\ell}|w(t)|^pg(t)\,dt}\\ &\ge\frac{1}{\gamma^p}\min_{\varphi\in W^{1,p}(-\ell,\ell)}\left\{\frac{\displaystyle\int_{-\ell}^{\ell}|\varphi'(t)|^pg(t)\,dt}{\displaystyle\int_{-\ell}^{\ell}|\varphi(t)|^pg(t)\,dt},\ \int_{-\ell}^{\ell}|\varphi(t)|^{p-2}\varphi(t)g(t)\,dt=0\right\}\end{aligned}$$

Let us denote by η the previous minimal value, then, by Lemma 2.59, a minimizer f exists and it is a solution to the following boundary value problem

$$\begin{cases}-(g(t)|f'(t)|^{p-2}f'(t))'=\eta g(t)|f(t)|^{p-2}f(t), \text{ in } (-\ell,\ell)\\ f'(-\ell)=f'(\ell)=0.\end{cases}$$

Still by 2.59, we have that $f(0)=0$ and hence solves

$$\begin{cases}-(g(t)|f'(t)|^{p-2}f'(t))'=\eta g(t)|f(t)|^{p-2}f(t), \text{ in } (0,\ell)\\ f'(0)=f'(\ell)=0.\end{cases}$$

Finally, by remainding that $g(t)=\omega_{n-1}(\ell-t)^{n-1}$ for $t\in(-\ell,\ell)$, if we set $h(r)=f(\ell-r)$, then this solves

$$\begin{cases}-(r^{n-1}|h'(r)|^{p-2}h'(r))'=\eta r^{n-1}|h(r)|^{p-2}h(r), \text{ in } (0,\ell)\\ h'(0)=h'(\ell)=0.\end{cases}$$

which means that the function $H(x)=h(F^o(x))$ is a Dirichlet eigenfunction of $\mathcal{Q}_p$ of on n-dimensional Wulff shape of anisotropic radius ℓ, namely W_ℓ. Hence $\eta\ge\lambda_1(p,W_\ell)=\lambda_1\left(p,W_{\frac{d}{2\gamma}}\right)$ and we get

$$\liminf_{k\to\infty}\Lambda_1(p,\Omega_k)=\liminf_{k\to\infty}\gamma_k(\Omega_1)\ge\frac{1}{\gamma^p}\eta\ge\frac{1}{\gamma^p}\lambda_1\left(p,W_{\frac{d}{2\gamma}}\right)=\lambda_1\left(p,W_{\frac{d}{2}}\right).$$

This concludes the proof. □

From Proposition 2.61 follows the following.

Proposition 2.62. *Let Ω be an open bounded convex set in $\mathbb{R}^n$, then*

$$\Lambda_1(p,\Omega)) < \lambda_1(p,\Omega) \tag{132}$$

Proof. The proof follows by combining (127), the Faber-Krahn inequality [44, Th. 6.1] and the isodiametric inequality (47). □

Now we are in position to give the following Theorem.

Theorem 2.63. *Let Ω be an open convex set in $\mathbb{R}^n$, then the first positive Neumann eigenvalue $\Lambda_1(\infty,\Omega)$ is never larger than the first Dirichlet eigenvalue $\lambda_1(\infty,\Omega)$. Moreover $\Lambda_1(\infty,\Omega) = \lambda_1(\infty,\Omega^\star)$ if and only if Ω is a Wulff shape.*

Proof. By convergence result in [15, Lemma 3.1] for Dirichlet eigenvalues and in Lemma 2.49 for Neumann eigenvalues, the proof follows by getting $p \to \infty$ in (132). The second assertion follows immediately by definitions of $\lambda_1(\infty,\Omega)$ and $\Lambda_1(\infty,\Omega)$. □

Moreover we observe that the main Theorem 2.57 has two other important consequences regarding the geometric properties of the eigenfunction. The first one show that closed nodal domain cannot exist in Ω.

Theorem 2.64. *For convex Ω any Neumann eigenfunctions associated with $\Lambda_1(\infty,\Omega)$ cannot have a closed nodal domain inside Ω.*

Proof. By contradiction, we assume that it exists a closed nodal line inside Ω. Since a Neumann eigenfunction u for the ∞-Laplacian is continuous, this implies that it exists an open subset $\Omega' \subset \Omega$ such that $u > 0$ in Ω' and $u = 0$ in $\partial\Omega'$. Let us observe that u is also a Dirichlet eigenfunction on $\Omega'$of the anisotropic ∞-Laplacian problem, hence, recalling [15, eq. (3.2)], we get

$$\frac{2}{\operatorname{diam}_F(\Omega)} = \Lambda_1(\infty,\Omega) = \lambda_1(\infty,\Omega') = \frac{1}{i_F(\Omega')} \geq \frac{2}{\operatorname{diam}_F(\Omega)}$$

where $i_F(\Omega')$ is the anisotropic inradius of Ω'. The last inequality is strict for all sets other than Wulff sets. This proves the corollary. □

Finally we give a result related to the hot-spot conjecture (see [25]), that says that a first nontrivial Neumann eigenfunction for the linear Laplace operator on a convex domain should attain its maximum or minimum on the boundary of this domain.

Theorem 2.65. *If Ω is convex and smooth, then any first nontrivial Neumann eigenfunction, i.e. any viscosity solution to (13) for $\Lambda = \Lambda_\infty$ attains both its maximum and minimum only on the boundary $\partial\Omega$. Moreover the extrema of u are located at points that have maximal anisotropic distance in $\overline{\Omega}$.*

Proof. If we consider $\overline{x}$ and $\underline{x}$, respectively, the maximum and the minimum point of u, we obtain that

$$d_F(\overline{x},\Omega_-) \geq \frac{1}{\Lambda} \quad \text{and} \quad d_F(\underline{x},\Omega_+) \geq \frac{1}{\Lambda}$$

so that $\operatorname{diam}_F(\Omega) \geq F^o(\overline{x}-\underline{x}) \geq \frac{2}{\Lambda}$. Since $\Lambda = \Lambda_\infty$, equality holds and the maximum and the minimum of u are attained in boundary points which have farthest anisotropic distance from each other. □

3 NONLOCAL PROBLEMS

In this chapter we treats some nonlinear nonlocal anisotropic eigenvalue problems. In the first section we determine the shape that minimizes, among domains with given measure, the first eigenvalue of the anisotropic laplacian perturbed by an integral of the unknown function. Using also some properties related to the associated "twisted"problem, we show that, this problem displays a *saturation* phenomenon: the first eigenvalue increases with the weight up to a critical value and then remains constant. With the intent to give a generalization of some of this and related results, in Section 2, we firstly we analyze the behaviour of the euclidean case in one dimension of the Laplacian.

3.1 A NONLOCAL ANISOTROPIC EIGENVALUE PROBLEM

3.1.1 The first eigenvalue of the nonlocal problem

Now we recall some known results about the anisotropic local ($\alpha = 0$) eigenvalue problem.

Theorem 3.1. *Let Ω be an open bounded set, then*

$$\lambda(0,\Omega) \geq \lambda(0,\Omega^\star) = \frac{\kappa_n^{2/n} j_{n/2-1,1}}{|\Omega|^{2/n}}. \tag{133}$$

The details of the proof can be found in [12, Th. 3.3]. The computation of the first eigenvalue on $\Omega^\star$ comes from the fact that the first eigenfunction $u(x) = u^\star(x)$ in $\mathcal{W}_R$ satisfies (see also [99]):

$$\begin{cases} \frac{d^2}{d\rho^2} u^*(\kappa_n \rho^n) + \frac{n-1}{\rho}\frac{d}{d\rho} u^*(\kappa_n\rho^n) + \lambda u^*(\kappa_n\rho^n) = 0 & \text{in } \mathcal{W}_R \\ u^*(\kappa_n\rho^n) = 0 & \text{on } \partial\mathcal{W}_R, \end{cases}$$

where $\rho = F^o(x)$ and R is the radius of the set $\Omega^\star$, which is a Wulff shape.

As a consequence of these and other related Theorems, we have:

Proposition 3.2. *Let Ω be the union of two disjoint Wulff shapes of radii $R_1, R_2 \geq 0$.*

(a) If $R_1 < R_2$, then the first eigenvalue $\lambda(0,\Omega)$ coincides with the first eigenvalue on the larger Wulff shape. Hence, any associated eigenfunction is simple and identically zero on the smaller set and it does not change sign on the larger one.

(b) If Ω is the union of two disjoint Wulff shapes of equal radii, then the first eigenvalue $\lambda(0,\Omega)$ is $\frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$. It is not simple and there exists an associated eigenfunction with zero average.

In this Section we collect some properties of problem (14), which will be fundamental in the proof of the main theorem.

Proposition 3.3. *Let Ω be an open bounded set, then the problem (14) admits a solution $\forall \alpha \in \mathbb{R}$.*

Proof. The direct methods in the Calculus of Variation provide an existence proof for a minimizer of (14). In a bounded domain Ω, the existence of a first eigenfunction (and of the first eigenvalue) is established via a minimizing sequence u_k for the Raylegh quotient. By homogeneity, it is possible the normalization and, using the Rellich-Kondrachov imbedding theorem [9, Th. IX.16], we find a minimizer by the lower semicontinuity [77, Th. 4.5] of the functional. □

Remark 3.4. *Let us note that if $u \in H_0^1(\Omega)$ is a minimizer of problem (14), then it satisfies the associated Euler-Lagrange equation, that we can write as $L_\alpha u = \lambda u$, where*

$$L_\alpha u := -\mathrm{div}(F(\nabla u)\nabla F(\nabla u)) + \alpha \int_\Omega u \, dx. \tag{134}$$

Proposition 3.5. *Let Ω be a bounded open set which is union of two disjoint Wulff shapes $\mathcal{W}_{R_1}(x_1)$ and $\mathcal{W}_{R_2}(x_2)$, with $R_1, R_2 \geq 0$, and let L_α be the operator as in (134). Then:*

(a) if a real number λ is an eigenvalue of $L_\alpha u = \lambda u$ for some nonzero α, either there exists no other real value of α for which λ is an eigenvalue of L_α or λ is an eigenvalue of the local problem ($\alpha = 0$); in the last case λ is an eigenvalue of L_α for all real α.

(b) λ is an eigenvalue of $L_\alpha u = \lambda u$ for all α if and only if it is an eigenvalue of the local problem having an eigenfunction with zero average in Ω.

Proof. We set $\mathcal{W}_i = \mathcal{W}_{R_i}(x_i)$, $i = 1, 2$. We assume that λ is an eigenvalue for two distinct parameters α_1 and α_2 and that u and v are the corresponding eigenfunctions. If we denote $u_i := u|_{\mathcal{W}_i}$ and $v_i := v|_{\mathcal{W}_i}$, $i = 1, 2$, then the functions u_i satisfy

$$-\mathrm{div}(F(\nabla u_i)\nabla F(\nabla u_i)) + \alpha_1 \left(\int_\Omega u \, \mathrm{d}x \right) = \lambda u_i \quad \text{on } \mathcal{W}_i, \text{ for } i = 1, 2; \tag{135}$$

and the functions v_i satisfy

$$-\mathrm{div}(F(\nabla v_i)\nabla F(\nabla v_i)) + \alpha_2 \left(\int_\Omega v \, \mathrm{d}x \right) = \lambda v_i \quad \text{on } \mathcal{W}_i, \text{ for } i = 1, 2. \tag{136}$$

We observe that $u_i(x) = u_i^*(\kappa_n(F^o(x - x_i))^n)$ and $v_i(x) = v_i^*(\kappa_n(F^o(x - x_i))^n)$, $i = 1, 2$. This means that, by (36), (37) and (38), we have

$$\begin{aligned} \int_{\mathcal{W}_i} & F(\nabla u_i)\nabla F(\nabla u_i)\nabla v_i \, \mathrm{d}x \\ &= \int_{\mathcal{W}_i} -u_i^{*'}(\kappa_n(F^o(x - x_i))^n) n\kappa_n(F^o(x - x_i))^{n-1} \frac{x - x_i}{F^o(x - x_i)} \cdot \\ &\qquad \cdot v_i^{*'}(\kappa_n(F^o(x - x_i))^n) n\kappa_n(F^o(x - x_i))^{n-1} \nabla F^o(x - x_i) \, \mathrm{d}x \\ &= \int_{\mathcal{W}_i} F(\nabla v_i)\nabla F(\nabla v_i)\nabla u_i \, \mathrm{d}x. \end{aligned} \tag{137}$$

for $i = 1, 2$. Now, we multiply the first equations of (135) and (136) respectively by v_1 and u_1, the second ones by v_2 and u_2 and then we integrate the first equations on $\mathcal{W}_1$ and the second ones on $\mathcal{W}_2$. By subtracting each one the equations integrated on $\mathcal{W}_1$ and using (137), we get

$$\alpha_1 \int_{\mathcal{W}_1} v_1 \, \mathrm{d}x \int_\Omega u \, \mathrm{d}x - \alpha_2 \int_{\mathcal{W}_1} u_1 \, \mathrm{d}x \int_\Omega v \, \mathrm{d}x = 0, \tag{138}$$

in the same way we get also

$$\alpha_1 \int_{\mathcal{W}_2} v_2 \, dx \int_{\Omega} u \, dx - \alpha_2 \int_{\mathcal{W}_2} u_2 \, dx \int_{\Omega} v \, dx = 0. \tag{139}$$

Hence, the sum of (138) and (139) leads to

$$(\alpha_1 - \alpha_2) \int_{\Omega} u \, dx \int_{\Omega} v \, dx = 0. \tag{140}$$

The result (a) follows because, if α_1 and α_2 are distinct, either u_1 or u_2 must have zero average, and hence satisfy the local equation. Finally, if (140) is valid for all α_1, α_2, there is at least one eigenfunction with zero average and also (b) is proved. □

Proposition 3.6. *Let Ω be a connected bounded open set and $\alpha \leq 0$. Then the first eigenvalue of (14) is simple and the corresponding eigenfunction has constant sign in all Ω.*

Proof. For any $u \in H_0^1(\Omega)$ we have $\mathcal{Q}_\alpha(u, \Omega) \geq \mathcal{Q}_\alpha(|u|, \Omega)$ with equality if and only if $u = |u|$ or $u = -|u|$. From now on, without loss of generality, we can assume that $u \geq 0$. Let us observe that if u is a minimizer of (14), then it satisfies (134) with $\alpha \int_\Omega u \leq 0$. Therefore u is strictly positive in Ω by a weak Harnack inequality (see [110, Th. 1.2]). Now, we give a proof of simplicity following the arguments of [12] and [14]. Let u and v be two positive eigenfunctions, then we can find a real constant c such that u and cv have the same integral:

$$\int_{\Omega} u \, dx = \int_{\Omega} c \, v \, dx. \tag{141}$$

We call w the function cv, which is again an eigenfunction and we set

$$\varphi = \left(\frac{u^2 + w^2}{2} \right)^{1/2}$$

which is an admissible function. A short calculation yields

$$\nabla \varphi = \frac{\sqrt{2}}{2} \frac{u \nabla u + w \nabla w}{(u^2 + w^2)^{1/2}}$$

and hence, by homogeneity, we have

$$\begin{aligned} F^2(\nabla \varphi) &= \frac{u^2 + w^2}{2} F^2 \left(\frac{u \nabla u + w \nabla w}{u^2 + w^2} \right) \\ &= \frac{u^2 + w^2}{2} F^2 \left(\frac{u^2 \nabla \log u + w^2 \nabla \log w}{u^2 + w^2} \right). \end{aligned}$$

Because of the convexity of $F(\xi)$ and the fact that $u^2/(u^2 + w^2)$ and $w^2/(u^2 + w^2)$ add up to 1, we can use Jensen's inequality to obtain

$$\begin{aligned} F^2(\nabla \varphi) &\leq \frac{u^2 + w^2}{2} \left[\frac{u^2}{u^2 + w^2} F^2(\nabla \log u) + \frac{w^2}{u^2 + w^2} F^2(\nabla \log w) \right] \\ &= \frac{1}{2} F^2(\nabla u) + \frac{1}{2} F^2(\nabla w). \end{aligned} \tag{142}$$

On the other hand, we have

$$\left(\int_{\Omega} \varphi \, dx \right)^2 \geq \left(\int_{\Omega} \left(\frac{u}{2} + \frac{w}{2} \right) \, dx \right)^2 = \frac{1}{2} \left(\int_{\Omega} u \, dx \right)^2 + \frac{1}{2} \left(\int_{\Omega} w \, dx \right)^2 \tag{143}$$

Hence, definition (14) and inequalities (142)-(143) yield the following inequality chain

$$\begin{aligned}\lambda(\alpha,\Omega) &\le \frac{\int_\Omega F^2(\nabla\varphi)\,dx + \alpha\left(\int_\Omega \varphi\,dx\right)^2}{\int_\Omega \varphi^2\,dx}\\ &\le \frac{\frac12\int_\Omega F^2(\nabla u)\,dx + \frac12\int_\Omega F^2(\nabla w)\,dx + \frac{\alpha}{2}\left(\int_\Omega u\,dx\right)^2 + \frac{\alpha}{2}\left(\int_\Omega w\,dx\right)^2}{\frac12\int_\Omega u^2\,dx + \frac12\int_\Omega w^2\,dx}\\ &= \lambda(\alpha,\Omega).\end{aligned} \tag{144}$$

Therefore, inequalities in (144) hold as equalities. This implies that $F^2(\nabla\varphi) = \frac12 F^2(\nabla u) + \frac12 F^2(\nabla w)$ almost everywhere. By (142), the strict convexity of the level sets of H gives that $\nabla \log u = \nabla \log w$ a.e.. This proves that u and w are constant multiples of each other and, in view of (141), we have $u = w$. Therefore u and v are proportional. □

Proposition 3.7. *Let Ω be a bounded open set, then:*

(a) the first eigenvalue of (14) $\lambda(\alpha,\Omega)$ is Lipschitz continuous and non-decreasing with respect to α (increasing when the eigenfunction relative to $\lambda(\alpha,\Omega)$ has nonzero average);

(b) for nonnegative values of α, the first eigenvalue of (14) $\lambda(\alpha,\Omega)$ satisfies

$$\lambda(\alpha,\Omega) \ge \frac{\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}; \tag{145}$$

(c) for nonnegative values of α, if Ω is the union of two disjoint Wulff shapes of equal radii, the first eigenvalue of (14) $\lambda(\alpha,\Omega)$ is equal to $\frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$.

Proof.

(a) By simple computation we have the following inequalities

$$\mathcal{Q}_\alpha(u,\Omega) \le \mathcal{Q}_{\alpha+\varepsilon}(u,\Omega) \le \mathcal{Q}_\alpha(u,\Omega) + |\Omega|\varepsilon \quad \forall\, \varepsilon > 0.$$

Taking the minimum over all $u \in H_0^1(\Omega)$, we obtain

$$\lambda(\alpha,\Omega) \le \lambda(\alpha+\varepsilon,\Omega) \le \lambda(\alpha,\Omega) + |\Omega|\varepsilon \quad \forall\, \varepsilon > 0,$$

and, in view of Proposition 3.5 *(a)-(b)*, the claim follows.

(b) By monotonicity of $\lambda(\alpha,\Omega)$ with respect to α, we have that $\lambda(\alpha,\Omega) \ge \lambda(0,\Omega)$; then, by (133), we obtain the (145).

(c) By Proposition 3.2 *(b)*, if Ω is the union of two disjoint Wulff shapes of equal radii, $\frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$ is the first eigenvalue of the local problem and it admits an eigenfunction with zero average. This implies that, by Proposition 3.5 *(b)*, $\frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$ is an eigenvalue of L_α for all α.

□

3.1.2 On the First Twisted Dirichlet Eigenvalue

In this Section we prove a Raylegh-Faber-Krahn type equation for the twisted eigenvalue problem

$$\lambda^T(\Omega) = \inf_{\substack{u\in H_0^1(\Omega)\\ u\not\equiv 0}} \mathscr{Q}^T(u,\Omega), \tag{146}$$

where

$$\mathscr{Q}^T(u,\Omega) = \left\{ \frac{\int_\Omega F^2(\nabla u)\, dx}{\int_\Omega u^2\, dx},\ \int_\Omega u\, dx = 0 \right\}.$$

Let us denote by

$$\Omega_+ = \{x \in \Omega, u(x) > 0\} \quad \text{and} \quad \Omega_- = \{x \in \Omega, u(x) < 0\}, \tag{147}$$

and by $\mathcal{W}_+$ and $\mathcal{W}_-$ the Wulff shapes such that $|\mathcal{W}_\pm| = |\Omega_\pm|$.

Lemma 3.8.

$$\lambda^T(\Omega) \geq \lambda^T(\mathcal{W}_+ \cup \mathcal{W}_-)$$

Proof. Let us denote with $u_+^\star$ (resp. $u_-^\star$) the decreasing convex rearrangement of $u|_{\Omega_+}$ (resp. $u|_{\Omega_-}$). The Pólya-Szegö principle (39) and properties of convex rearrangements provide

$$\lambda^T(\Omega) \geq \frac{\int_{\mathcal{W}_+} F^2(\nabla u_+^\star)\, ds + \int_{\mathcal{W}_-} F^2(\nabla u_-^\star)\, ds}{\int_{\mathcal{W}_+} (u_+^\star)^2\, ds + \int_{\mathcal{W}_-} (u_-^\star)^2\, ds} \tag{148}$$

and

$$\int_{\mathcal{W}_+} u_+^\star\, ds - \int_{\mathcal{W}_-} u_-^\star\, ds = \int_{\Omega^+} u\, dx + \int_{\Omega^-} u\, dx = \int_\Omega u\, dx = 0. \tag{149}$$

In view of (148) and (149), we have the following inequality:

$$\lambda^T(\Omega) \geq \lambda^* := \inf_{\substack{(f,g)\in H_0^1(\mathcal{W}_+)\times H_0^1(\mathcal{W}_-)\\ \int_{\mathcal{W}_+} f\, ds = \int_{\mathcal{W}_-} g\, ds}} \frac{\int_{\mathcal{W}_+} F^2(\nabla f)\, ds + \int_{\mathcal{W}_-} F^2(\nabla g)\, ds}{\int_{\mathcal{W}_+} f^2\, ds + \int_{\mathcal{W}_-} g^2\, ds}.$$

Using classical methods of calculus of variations, we can prove that this infimum is attained in (f,g). Now, following the ideas of [72, Sect. 3], the function

$$w = \begin{cases} f & \text{in } \mathcal{W}_+ \\ -g & \text{in } \mathcal{W}_- \end{cases}$$

satisfies

$$\begin{cases} -\operatorname{div}(F(\nabla w)\nabla F(\nabla w)) = \lambda^* w - \frac{1}{|\Omega|}\int_{\mathcal{W}_+\cup\mathcal{W}_-} \operatorname{div}(F(\nabla w)\nabla F(\nabla w))\, dx & \text{in } \mathcal{W}_+ \cup \mathcal{W}_- \\ w = 0 \quad \text{on } \partial(\mathcal{W}_+ \cup \mathcal{W}_-). \end{cases} \tag{150}$$

This shows that λ^* is an eigenvalue of the twisted problem (146) on $\mathcal{W}_+ \cup \mathcal{W}_-$ and therefore, $\lambda^T(\Omega) \geq \lambda^* \geq \lambda^T(\mathcal{W}_+ \cup \mathcal{W}_-)$. □

Throughout this Section, we investigate the first eigenvalue when Ω is the union of two disjoint Wulff shapes, of radii $R_1 \le R_2$. Without loss of generality, we assume that the volume of Ω is such that

$$R_1^n + R_2^n = 1$$

and we denote by $\theta(R_1, R_2)$, the first positive root of equation

$$R_1^n \frac{J_{\frac{n}{2}+1}(\theta\, R_1)}{J_{\frac{n}{2}-1}(\theta\, R_1)} + R_2^n \frac{J_{\frac{n}{2}+1}(\theta\, R_2)}{J_{\frac{n}{2}-1}(\theta\, R_2)} = 0 \tag{151}$$

Now we recall a result given in [72, Prop. 3.2].

Proposition 3.9. *There exists a constant $c_n < 1$, depending on the dimension n, such that*

(a) if $R_1/R_2 < c_n$, then $\lambda^T(\mathcal{W}_{R_1} \cup \mathcal{W}_{R_2}) = \left(\frac{j_{\frac{n}{2},1}}{R_2}\right)^2$;

(b) if $R_1/R_2 \ge c_n$, then $\lambda^T(\mathcal{W}_{R_1} \cup \mathcal{W}_{R_2}) = \theta^2(R_1, R_2)$.

Moreover, if we set $\theta^* = 2^{1/n} j_{\frac{n}{2}-1,1}$, we obtain the following

Proposition 3.10. *The first positive root equation of (151) $\theta(R_1, R_2)$ satisfies*

$$\theta(R_1, R_2) \ge \theta^*,$$

for all R_1, $R_2 \ge 0$.

This result is proved in [72, Lemma 3.3] when Ω has the same measure as the unit ball, but it can be obtained for all sets of finite measure. Now, we show the following isoperimetric inequality.

Theorem 3.11. *Let Ω be any bounded open set in $\mathbb{R}^n$. Then*

$$\lambda^T(\Omega) \ge \lambda^T(\mathcal{W}_1 \cup \mathcal{W}_2)$$

where $\mathcal{W}_1$ and $\mathcal{W}_2$ are two disjoint Wulff shapes of measure $|\Omega|/2$. Equality holds if and only if $\Omega = \mathcal{W}_1 \cup \mathcal{W}_2$.

Proof. Thanks to Lemma 3.8, it remains to prove that the union of two disjoint Wulff shapes with the same measure gives the lowest possible value of $\lambda^T(\cdot)$ among unions of disjoint Wulff shapes with given measure $|\Omega|$. Hence, we compute the first twisted eigenvalue of the union Ω of the Wulff shapes $\mathcal{W}_{R_1}(x_1)$ and $\mathcal{W}_{R_2}(x_2)$, with $R_1 \le R_2$. If we consider the eigenfunction that is zero on the smaller Wulff shape and coincides with the first eigenfunction on the larger one, we trivially have $\lambda_1^T(\mathcal{W}_{R_1}(x_1) \cup \mathcal{W}_{R_2}(x_2)) = \lambda_1^T(\mathcal{W}_{R_2}(x_2))$.

Now we study the case in which the eigenfunction u does not vanish on any of the two Wulff shapes. We denote by u_1 and u_2 the functions that express u respectively on $\mathcal{W}_{R_1}(x_1)$ and $\mathcal{W}_{R_2}(x_2)$. The proof of Lemma 3.8 shows that we can study only functions dependent on the radius of the Wulff shape in which are defined. Therefore, in an abuse of notation, we consider functions such that $u_j(x) = u_j(F^o(x - x_j))$, for $j = 1, 2$, and hence, instead of (150), we can solve equivalently (see [99])

$$\begin{cases} u_j''(\rho) + \frac{n-1}{\rho} u_j'(\rho) + \lambda^T u_j(\rho) = c, \ 0 < \rho < R_j \\ u_j'(0) = 0, \ u_j(R_j) = 0, \end{cases} \tag{152}$$

for $j=1,2$, where $c=\frac{1}{|\Omega|}\int_\Omega \operatorname{div}(F(\nabla u)\nabla_\xi F(\nabla u))\,\mathrm{d}x$. Therefore, the solution u of (152) can be written in the form:

$$u=\begin{cases} u_1=c_1\left((F^o(x-x_1))^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,F^o(x-x_1)\right)\right.\\ \qquad\qquad\left.-R_1^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,R_1\right)\right) \text{ in } \mathcal{W}_{R_1}(x_1)\\ u_2=-c_2\left((F^o(x-x_2))^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,F^o(x-x_2)\right)\right.\\ \qquad\qquad\left.-R_2^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,R_2\right)\right) \text{ in } \mathcal{W}_{R_2}(x_2)\end{cases} \tag{153}$$

Now we express the coupling condition $\int_\Omega u\,\mathrm{d}x=0$ as

$$0=\int_{\mathcal{W}_{R_1}} u_1\,\mathrm{d}x+\int_{\mathcal{W}_{R_2}} u_2\,\mathrm{d}x,$$

and hence we obtain

$$\begin{aligned}0=c_1&\left(\gamma_n\int_0^{R_1} J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,\rho\right)\rho^{\frac{n}{2}}\,\mathrm{d}\rho-\kappa_n R_1^{\frac{n}{2}+1}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,R_1\right)\right)\\ &-c_2\left(\gamma_n\int_0^{R_2} J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,\rho\right)\rho^{\frac{n}{2}}\,\mathrm{d}\rho-\kappa_n R_2^{\frac{n}{2}+1}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,R_2\right)\right).\end{aligned}$$

We use classical properties of Bessel functions [114], namely

$$\int_0^R J_{\frac{n}{2}-1}(kr)\,r^{\frac{n}{2}}dr=\frac{1}{k}R^{\frac{n}{2}}J_{\frac{n}{2}}(kr)\quad\text{and}\quad \frac{n}{kr}J_{\frac{n}{2}}(kr)-J_{\frac{n}{2}-1}(kr)=J_{\frac{n}{2}+1}(kr),$$

together with $\gamma_n=n\kappa_n$, where γ_n is the generalized perimeter of $\mathcal{W}$, to get

$$c_1R_1^{\frac{n}{2}+1}J_{\frac{n}{2}+1}\left(\sqrt{\lambda^T}\,R_1\right)-c_2R_2^{\frac{n}{2}+1}J_{\frac{n}{2}+1}\left(\sqrt{\lambda^T}\,R_2\right)=0.$$

Hence it is possible to take

$$c_1=R_2^{\frac{n}{2}+1}J_{\frac{n}{2}+1}\left(\sqrt{\lambda^T}\,R_2\right)\quad\text{and}\quad c_2=R_1^{\frac{n}{2}+1}J_{\frac{n}{2}+1}\left(\sqrt{\lambda^T}\,R_1\right) \tag{154}$$

in (153).

Now we want that the constant c in (152) is the same for $j=1$ and for $j=2$. This automatically implies that this constant c coincides with the average of the anisotropic laplacian computed on u. Since

$$\begin{aligned}\operatorname{div}(F(\nabla u_1)\nabla_\xi F(\nabla u_1))&=-c_1\lambda^T(F^o(x-x_1))^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,F^o(x-x_1)\right)\\ \operatorname{div}(F(\nabla u_2)\nabla_\xi F(\nabla u_2))&=c_2\lambda^T(F^o(x-x_2))^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,F^o(x-x_2)\right),\end{aligned}$$

we have

$$\begin{aligned}c&=\operatorname{div}(F(\nabla u_1)\nabla_\xi F(\nabla u_1))+\lambda^Tu_1=-c_1\lambda^TR_1^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,R_1\right)\\ c&=\operatorname{div}(F(\nabla u_2)\nabla_\xi F(\nabla u_2))+\lambda^Tu_1=c_2\lambda^TR_2^{1-\frac{n}{2}}J_{\frac{n}{2}-1}\left(\sqrt{\lambda^T}\,R_2\right).\end{aligned}$$

Comparing this two relations and taking in account (154), if we set $\lambda^T(\mathcal{W}_{R_1}(x_1)\cup\mathcal{W}_{R_2}(x_2))=\theta^2$, the condition $-c+c=0$ gives the equation (151). Now we observe that, in the case that $R_1/R_2<c_n$, by Proposition 3.9 *(a)* and by the inequality $j_{\frac{n}{2},1}>2^{1/n}j_{\frac{n}{2}-1,1}$ [72, Cor. A.2], we have

$$\lambda^T(\mathcal{W}_{R_1}(x_1)\cup\mathcal{W}_{R_2}(x_2))\geq\left(\frac{j_{\frac{n}{2},1}}{R_2}\right)^2\geq\left(j_{\frac{n}{2},1}\right)^2>\left(2^{1/n}j_{\frac{n}{2}-1,1}\right)^2=\theta^{*2}.$$

If $R_1/R_2 \geq c_n$, by Proposition 3.9 *(b)* and Proposition 3.10, we have

$$\lambda^T(\mathcal{W}_{R_1}(x_1) \cup \mathcal{W}_{R_2}(x_2)) = (\theta(R_1, R_2))^2 \geq \theta^{*2}.$$

Therefore, in both case, we obtain that $\lambda^T(\mathcal{W}_{R_1}(x_1) \cup \mathcal{W}_{R_2}(x_2)) \geq \theta^{*2}$ and since θ^* is the value of $\lambda^T(\Omega)$ computed on two Wulff shapes with the same measure, this conclude the proof. □

3.1.3 The Nonlocal Problem

The aim of this Section is to prove Theorem 3.16. We start by showing some preliminary results.

Theorem 3.12. *Let Ω be an open bounded set in $\mathbb{R}^n$, then there exists a positive value of α such that the corresponding first eigenvalue $\lambda(\alpha, \Omega)$ is greater or equal than $\frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$.*

Proof. We first observe that $\lambda(\alpha, \Omega)$ is bounded, indeed

$$\lim_{\alpha \to +\infty} \lambda(\alpha, \Omega) \leq \min_{\substack{u \in H_0^1(\Omega) \\ u \not\equiv 0}} \left\{ \frac{\int_\Omega (F(\nabla u))^2 \, dx}{\int_\Omega u^2 \, dx}, \int_\Omega u \, dx = 0 \right\} = \lambda^T(\Omega).$$

Compactness arguments show that there exists a sequence of eigenfunctions u_α, $\alpha \to +\infty$, with norm in $L^2(\Omega)$ equal to 1, weakly converging in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to a function u. Obviously $\int_\Omega u_\alpha \, dx \to \int_\Omega u \, dx = 0$, as $\alpha \to +\infty$ (this limit exists by compactness) and hence, by the lower semicontinuity [77, Th. 4.5]

$$\lim_{\alpha \to +\infty} \lambda(\alpha, \Omega) \geq \inf_{\substack{u \in H_0^1(\Omega) \\ u \not\equiv 0}} \left\{ \frac{\int_\Omega (F(\nabla u))^2 \, dx}{\int_\Omega u^2 \, dx}, \int_\Omega u \, dx = 0 \right\} = \lambda^T(\Omega).$$

In Theorem 3.11 we have proved that the last term is greater or equal than the first eigenvalue on two disjoint Wulff shapes of equal radii. Therefore, by Proposition 3.7 *(c)*, the result follows. □

Proposition 3.13. *If $\alpha > 0$ and Ω is a bounded, open set in $\mathbb{R}^n$ which is not union of two disjoint Wulff shapes. Then there exist $\mathcal{W}_{R_1}$ and $\mathcal{W}_{R_2}$ disjoint such that $|\mathcal{W}_{R_1} \cup \mathcal{W}_{R_2}| = |\Omega|$ and*

$$\lambda(\alpha, \Omega) > \lambda(\alpha, \mathcal{W}_{R_1} \cup \mathcal{W}_{R_2}) = \min_{\substack{A = \mathcal{W}_{R_1} \cup \mathcal{W}_{R_2} \\ |A| = |\Omega|}} \lambda(\alpha, A).$$

Proof. Let u be an eigenfunction of (14), $\Omega_\pm$ be defined as in (147), $u_\pm = u|_{\Omega_\pm}$ and $\Omega^\star_\pm$ be the Wulff shapes with the same measure as $\Omega_\pm$. Using (39), it is easy to show that

$$\lambda(\alpha, \Omega) \geq \min_{\substack{A = \mathcal{W}_{R_1} \cup \mathcal{W}_{R_2} \\ |A| = |\Omega|}} \lambda(\alpha, A). \tag{155}$$

Indeed, we have

$$
\begin{aligned}
\lambda(\alpha,\Omega) &= \frac{\int_\Omega F^2(\nabla u)\,\mathrm{d}x + \alpha\left(\int_\Omega u\,\mathrm{d}x\right)^2}{\int_\Omega u^2\,\mathrm{d}x} \\
&\geq \frac{\int_{\Omega_+^\star} F^2(\nabla(u_+)^\star)\,\mathrm{d}s + \int_{\Omega_-^\star} F^2(\nabla(u_-)^\star)\,\mathrm{d}s + \alpha\left(\int_{\Omega_+^\star}(u_+)^\star\,\mathrm{d}s - \int_{\Omega_-^\star}(u_-)^\star\,\mathrm{d}s\right)^2}{\int_{\Omega_+^\star}(u_+)^{\star 2}\,\mathrm{d}s + \int_{\Omega_-^\star}(u_-)^{\star 2}\,\mathrm{d}s} \\
&\geq \min_{(f,g)\in H_1^0(\Omega_+^\star)\times H_1^0(\Omega_+^\star)} \frac{\int_{\Omega_+^\star} F^2(\nabla f)\,\mathrm{d}s + \int_{\Omega_-^\star} F^2(\nabla g)\,\mathrm{d}s + \alpha\left(\int_{\Omega_+^\star} f\,\mathrm{d}s - \int_{\Omega_-^\star} g\,\mathrm{d}s\right)^2}{\int_{\Omega_+^\star} f^2\,\mathrm{d}s + \int_{\Omega_-^\star} g^2\,\mathrm{d}s} \\
&= \lambda(\alpha,\Omega_+^\star\cup\Omega_-^\star) \\
&\geq \inf_{\substack{A=\mathcal{W}_{R_1}\cup\mathcal{W}_{R_2}\\ |A|=|\Omega|}} \lambda(\alpha,A)
\end{aligned}
\tag{156}
$$

Let us prove that, actually, the inequality (155) is strict. Suppose, by contradiction that (155) holds as an equality. In particular, from (156) we have

$$
\lambda(\alpha,\Omega) = \lambda(\alpha,\Omega_+^\star\cup\Omega_-^\star),
$$

hence, by Theorem 1.20, we deduce that $\Omega_+^\star$ and $\Omega_-^\star$ are Wulff shapes. Then, we may have two cases:

(i) $\Omega = \Omega_+^\star\cup\Omega_-^\star$,

(ii) $|\Omega_+| + |\Omega_-| < |\Omega|$.

In the first case, we have immediately a contradiction because, by hypothesis, Ω is not a union of two Wulff shapes.
In the second case, we observe that eigenfunction u vanishes on a set of positive measure and, by (15), it has zero average. Using the strict monotonicity of the Dirichlet eigenvalue with respect to homotheties, we again reach a contradiction since $\lambda(\alpha,\Omega_+^\star\cup\Omega_-^\star) > \inf_{\substack{A=\mathcal{W}_{R_1}\cup\mathcal{W}_{R_2}\\ |A|=|\Omega|}} \lambda(\alpha,A)$. Therefore, we have

$$
\lambda(\alpha,\Omega) > \inf_{\substack{A=\mathcal{W}_{R_1}\cup\mathcal{W}_{R_2}\\ |A|=|\Omega|}} \lambda(\alpha,A).
$$

Finally, the compactness of family of disjoint pair of Wulff shapes and the continuity of $\lambda(\alpha,\Omega)$ with respect to uniform convergence of the domains gives the existence of the set $A = \mathcal{W}_{R_1}\cup\mathcal{W}_{R_2}$ (see e.g. [23], [79]) . □

Remark 3.14. *Before showing the next result, let us observe that when Ω reduces to the union of two Wulff shapes $\mathcal{W}_{R_1}\cup\mathcal{W}_{R_2}$, then problem (15) becomes*

$$
\begin{cases}
-\mathrm{div}(F(\nabla u)\nabla_\xi F(\nabla u)) + \alpha\left(\int_{\mathcal{W}_{R_1}} u\,dx + \int_{\mathcal{W}_{R_2}} v\,dx\right) = \lambda u & \text{in } \mathcal{W}_{R_1} \\
-\mathrm{div}(F(\nabla v)\nabla_\xi F(\nabla v)) + \alpha\left(\int_{\mathcal{W}_{R_1}} u\,dx + \int_{\mathcal{W}_{R_2}} v\,dx\right) = \lambda v & \text{in } \mathcal{W}_{R_2} \\
u = 0 \quad \text{on } \partial\mathcal{W}_{R_1}, \qquad v = 0 \quad \text{on } \partial\mathcal{W}_{R_2}.
\end{cases}
\tag{157}
$$

Proposition 3.15. *Let Ω be the union of two disjoint Wulff shapes $\mathcal{W}_{R_1}$ and $\mathcal{W}_{R_2}$ such that $\kappa_n(R_1^n + R_2^n) = |\Omega|$. Then, for every $\eta \in \left(\frac{\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}, \frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}} \right)$ and for every $R_1, R_2 \geq 0$, there exists a unique value of α, denoted by α_η, given by*

$$\frac{1}{\alpha_\eta} = \frac{\kappa_n(R_1^n + R_2^n)}{\eta} - \frac{n\kappa_n}{\eta^{3/2}} \left[R_1^{n-1} \frac{J_{n/2}(\sqrt{\eta}R_1)}{J_{n/2-1}(\sqrt{\eta}R_1)} + R_2^{n-1} \frac{J_{n/2}(\sqrt{\eta}R_2)}{J_{n/2-1}(\sqrt{\eta}R_2)} \right], \tag{158}$$

such that $\eta = \lambda(\alpha_\eta, \mathcal{W}_{R_1} \cup \mathcal{W}_{R_2})$.

Proof. In view of Proposition 3.13, problem (15) reduces to (157). Then, we easily verify that the functions

$$u = R_2^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{\eta}R_2\right) \left[(F^o(x))^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{\eta}\, F^o(x)\right) - R_1^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{\eta}\, R_1\right) \right]$$

and

$$v = R_1^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{\eta}R_1\right) \left[(F^o(x))^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{\eta}\, F^o(x)\right) - R_2^{1-\frac{n}{2}} J_{\frac{n}{2}-1}\left(\sqrt{\eta}\, R_2\right) \right].$$

solve problem (157). Now, we show that, for all R_1, R_2 and η as in the hypothesis, there exists at least one value of α such that problem (157) admits a non trivial solution. Indeed, the eigenvalue $\lambda(\alpha, \mathcal{W}_{R_1} \cup \mathcal{W}_{R_2})$ is clearly unbounded from below as $\alpha \to -\infty$ and, by Theorem 3.12, is larger than $\frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$ as $\alpha \to +\infty$. Hence the continuity and the monotonicity of $\lambda(\alpha, \mathcal{W}_{R_1} \cup \mathcal{W}_{R_2})$ with respect to α implies that when $\alpha = \alpha_\eta$, then η is the first eigenvalue of problem (157). This value of α is unique, indeed, arguing by contradiction, if for some η, there exists another value $\alpha \neq \alpha_\eta$ such that η is the first eigenvalue of problem (157), then by Proposition 3.5, η is also an eigenvalue of the local problem and the corresponding eigenfunction has zero average in $\mathcal{W}_{R_1} \cup \mathcal{W}_{R_2}$. Hence, if these Wulff shapes have the same measure, then the eigenvalue η is, by Proposition 3.7 *(c)*, equal to $\frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$ and this contradicts the fact that $\eta < \frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$. Otherwise, if the sets do not have the same measure, by Proposition 3.2 *(a)*, the first eigenfunction is identically zero on the smaller set and it does not change sign on the larger one; this is in contradiction with the fact that the eigenfunction is not trivial and has zero average. □

Theorem 3.16. *For every $n \geq 2$, there exists a positive value*

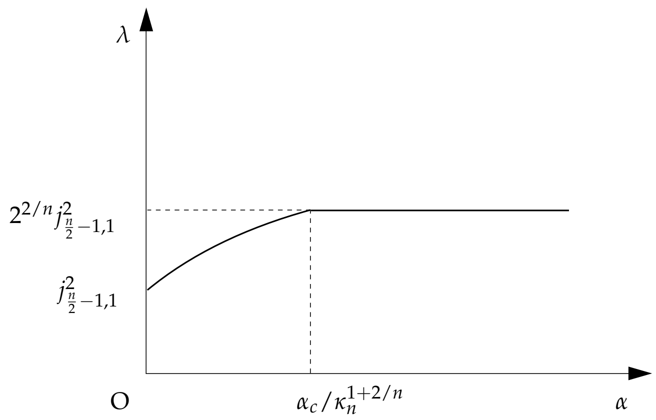
$$\alpha_c = \frac{2^{3/n}\kappa_n^{2/n} j^3_{n/2-1,1} J_{n/2-1,1}(2^{1/n} j_{n/2-1,1})}{2^{1/n} j_{n/2-1,1} J_{n/2-1}(2^{1/n} j_{n/2-1,1}) - n J_{n/2}(2^{1/n} j_{n/2-1,1})}$$

such that, for every bounded, open set Ω in $\mathbb{R}^n$ and for every real number α, it holds

$$\lambda(\alpha, \Omega) \geq \begin{cases} \lambda(\alpha, \Omega^\star) & \text{if } \alpha|\Omega|^{1+2/n} \leq \alpha_c, \\ \frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}} & \text{if } \alpha|\Omega|^{1+2/n} \geq \alpha_c. \end{cases}$$

If equality sign holds when $\alpha|\Omega|^{1+2/n} < \alpha_c$ then Ω is a Wulff shape, while if inequality sign holds when $\alpha|\Omega|^{1+2/n} > \alpha_c$ then Ω is the union of two disjoint Wulff shapes of equal measure.

In Figure 1 we illustrate the transition between the two minimizers.



The continuous line represents the minimum of $\lambda(\alpha, \Omega)$, among the open bounded sets of measure κ_n, as a function of α.

Proof. Let us firstly analize the case of nonpositive α. Let u be an eigenfunction, by (39) we have

$$\lambda(\alpha, \Omega) = \mathcal{Q}_\alpha(u, \Omega) \geq \mathcal{Q}_\alpha(|u|, \Omega) \geq \mathcal{Q}_\alpha(u^\star, \Omega^\star) \geq \lambda(\alpha, \Omega^\star).$$

By Proposition 3.6, we can say that u is positive; therefore Ω coincides with the set $\{x \in \Omega : u(x) > 0\}$ that, by Theorem 1.20, is equivalent to a Wulff shape. Therefore the equality case is proved.

Now, we study the case of positive value of α. In view of Proposition 3.13 we can restrict our study to the case of two disjoint Wulff shapes, of radii R_1 and R_2, whose union has the same measure of Ω. By Proposition 3.7 *(b)-(c)*, the first eigenvalue is greater than $\frac{\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$ and is lower than the first eigenvalue computed on two Wulff shapes with the same measure, that is $\frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$. Hence, we can restrict our study to the eigenvalues in the range $\left(\frac{\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}, \frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}\right)$. Now, let us observe that if Ω is a Wulff shape and $\alpha = \alpha_c|\Omega|^{-1-2/n}$, then, from (158) with $R_2 = 0$, $\lambda(\alpha, \Omega^\star) = \frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}$. Therefore $\alpha = \alpha_c|\Omega|^{-1-2/n}$ is a critical value of α because the first eigenvalue on $\Omega^\star$ coincides with the first eigenvalue on the union of two disjoint Wulff shapes of equal radii.

We firstly analyze the subcritical cases ($0 < \alpha < \alpha_c|\Omega|^{-1-2/n}$). Thanks to Proposition 3.13, it remains to prove that if Ω is union of two non negligible disjoint Wulff shapes, then $\lambda(\alpha, \Omega) > \lambda(\alpha, \Omega^\star)$. Therefore, by Proposition 3.15, this is equivalent to say that, for any $\eta \in \left(\frac{\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}, \frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}\right)$, α_η attains its maximum if and only if R_1 or R_2 vanishes, with the constraint $\kappa_n(R_1^n + R_2^n) = |\Omega|$. This is proved in [19, Prop. 3.4] with κ_n instead of ω_n using Bessel function properties.

The continuity of $\lambda(\alpha, \Omega)$ with respect to α for subcritical values yields $\lambda(\alpha_c, \Omega) \geq \lambda(\alpha_c, \Omega^\star)$. Hence, for supercritical values ($\alpha > \alpha_c|\Omega|^{-1-2/n}$), using the monotonicity of $\lambda(\alpha, \Omega)$ with respect to α, we have

$$\lambda(\alpha, \Omega) \geq \lambda(\alpha_c, \Omega) \geq \lambda(\alpha_c, \Omega^\star) = \frac{2^{2/n}\kappa_n^{2/n} j^2_{n/2-1,1}}{|\Omega|^{2/n}}. \tag{159}$$

If the inequalities in (159) hold as equalities, then Ω is the union of two disjoint Wulff shapes of same measure. Indeed, by Proposition 3.7 *(a)*, the first inequality is strict only when the eigenfunction relative to $\lambda(\alpha, \Omega)$ has nonzero average, that is when the two Wulff shapes have different radii. Hence also the equality case follows and this conclude the proof of the Theorem 3.16. □

3.2 A SATURATION PHENOMENON FOR A NONLINEAR NONLOCAL EIGENVALUE PROBLEM

3.2.1 Some properties of the first eigenvalue

We consider the following problem:

$$\lambda(\alpha,q) = \inf\left\{ \mathcal{Q}[u,\alpha],\ u \in H_0^1(-1,1),\ u \not\equiv 0 \right\}, \tag{160}$$

where $\alpha \in \mathbb{R}$, $1 \le q \le 2$ and

$$\mathcal{Q}[u,\alpha] := \frac{\displaystyle\int_{-1}^{1} |u'|^2 dx + \alpha \left| \int_{-1}^{1} |u|^{q-1} u\, dx \right|^{\frac{2}{q}}}{\displaystyle\int_{-1}^{1} |u|^2 dx}.$$

Let us observe that, for any $\alpha \in \mathbb{R}$, it holds that

$$\lambda(\alpha,q) \le \Lambda_q = \pi^2, \tag{161}$$

where

$$\Lambda_q := \min\left\{ \frac{\displaystyle\int_{-1}^{1} |u'|^2 dx}{\displaystyle\int_{-1}^{1} |u|^2 dx},\ u \in H_0^1(-1,1),\ \int_{-1}^{1} |u|^{q-1} u\, dx = 0,\ u \not\equiv 0 \right\}. \tag{162}$$

It is known that, when $q \in [1,2]$, then $\Lambda_q = \Lambda_1 = \pi^2$, and the minimizer of (162) is, up to a multiplicative constant, $y(x) = \sin \pi x$, $x \in [-1,1]$ (see for example [34]). Let us observe that if y is a minimizer in (160), then is not restrictive to suppose that $\int_{-1}^{1} |y|^{q-1} y\, dx \ge 0$. From now on, we assume that this condition is satisfied by the minimizers.

Proposition 3.17. *Let $1 \le q \le 2$ and $\alpha \in \mathbb{R}$. The following properties of $\lambda(\alpha,q)$ hold.*

(a) Problem (160) admits a minimizer.

(b) Any minimizer y of (160) satisfies the following boundary value problem

$$\begin{cases} -y'' + \alpha\gamma |y|^{q-1} = \lambda(\alpha,q)\, y & \text{in }]-1,1[\\ y(-1) = y(1) = 0, \end{cases} \tag{163}$$

where

$$\gamma = \begin{cases} 0 & \text{if both } q = 2 \text{ and } \displaystyle\int_{-1}^{1} y|y|\, dx = 0, \\ \left(\displaystyle\int_{-1}^{1} y|y|^{q-1}\, dx \right)^{\frac{2}{q}-1} & \text{otherwise.} \end{cases}$$

Moreover, $y \in C^2(-1,1)$.

(c) For any $q \in [1,2]$, the function $\lambda(\cdot,q)$ is Lipschitz continuous and non-decreasing with respect to $\alpha \in \mathbb{R}$.

(d) *If $\alpha \le 0$, the minimizers of (160) do not change sign in $]-1,1[$. In addition,*

$$\lim_{\alpha\to-\infty} \lambda(\alpha,q) = -\infty.$$

(e) *As $\alpha \to +\infty$, we have that*

$$\lambda(\alpha,q) \to \Lambda_q = \pi^2.$$

Proof. The existence of a minimizer follows immediately by the standard methods of Calculus of Variations. Furthermore, any minimizer satisfies (163). In particular, let us explicitly observe that if $1 \le q \le 2$ and there exists a minimizer y of $\lambda(\alpha,q)$ such that $\int_{-1}^{1} |y|^{q-1} y\, dx = 0$, then it holds that $\gamma = 0$ in (163). Indeed, in such a case y is a minimizer also of the problem (162), whose Euler-Lagrange equation is (see [34])

$$\begin{cases} -y'' = \lambda(\alpha,q)\, y & \text{in }]-1,1[, \\ \\ y(-1) = y(1) = 0. \end{cases}$$

From (163) immediately follows that any minimizer y is $C^2(-1,1)$. Hence *(a)-(b)* have been proved.

In order to get property *(c)*, we stress that for all $\varepsilon > 0$, by Hölder inequality, it holds

$$\mathcal{Q}[u,\alpha+\varepsilon] \le \mathcal{Q}[u,\alpha] + \varepsilon \frac{\left(\displaystyle\int_{-1}^{1} |u|^q\, dx\right)^{2/q}}{\displaystyle\int_{-1}^{1} u^2\, dx} \le \mathcal{Q}[u,\alpha] + 2^{\frac{2-q}{q}} \varepsilon, \quad \forall\, \varepsilon > 0.$$

Therefore the following chain of inequalities

$$\mathcal{Q}[u,\alpha] \le \mathcal{Q}[u,\alpha+\varepsilon] \le \mathcal{Q}[u,\alpha] + 2^{\frac{2-q}{q}} \varepsilon, \quad \forall\, \varepsilon > 0,$$

implies, taking the minimum as $u \in H_0^1(-1,1)$, that

$$\lambda(\alpha,q) \le \lambda(\alpha+\varepsilon,q) \le \lambda(\alpha,q) + 2^{\frac{2-q}{q}} \varepsilon, \quad \forall\, \varepsilon > 0,$$

that proves *(c)*.

If $\alpha < 0$, then

$$\mathcal{Q}[u,\alpha] \ge \mathcal{Q}[|u|,\alpha],$$

with equality if and only if $u \ge 0$ or $u \le 0$. Hence any minimizer has constant sign in $]-1,1[$. Finally, it is clear from the definition that $\lim\limits_{\alpha\to-\infty} \lambda(\alpha,q) = -\infty$. Indeed, by fixing a positive test function φ we get

$$\lambda(\alpha,q) \le \mathcal{Q}[\varphi,\alpha].$$

Being $\varphi > 0$ in $]-1,1[$, then $\mathcal{Q}[\varphi,\alpha] \to -\infty$ as $\alpha \to -\infty$, and the proof of *(d)* is completed.

In order to show *(e)*, we recall that $\lambda(\alpha,q) \le \Lambda_q = \pi^2$.

Let $\alpha_k \ge 0$, $k_n \in \mathbb{N}$, be a positively divergent sequence. For any k, consider a minimizer $y_k \in W_0^{1,2}$ of (160) such that $\|y_k\|_{L^2} = 1$. We have that

$$\lambda(\alpha_k,q) = \int_{-1}^{1} |y_k'|^2\, dx + \alpha_k \left(\int_{-1}^{1} y_k |y_k|^{q-1}\, dx\right)^{\frac{2}{q}} \le \Lambda_q.$$

Then y_k converges (up to a subsequence) to a function $y \in H_0^1$, strongly in L^2 and weakly in H_0^1. Moreover $\|y\|_{L^2} = 1$ and

$$\left(\int_{-1}^{1} y_k |y_k|^{q-1} \, dx \right)^{\frac{2}{q}} \le \frac{\Lambda_q}{\alpha_k} \to 0 \quad \text{as } k \to +\infty$$

which gives that $\int_\Omega y|y|^{q-1} \, dx = 0$. On the other hand the weak convergence in H_0^1 implies that

$$\int_{-1}^{1} |y'|^2 \, dx \le \liminf_{k\to\infty} \int_{-1}^{1} |y_k'|^2 \, dx. \tag{164}$$

By definitions of Λ_q and $\lambda(\alpha, q)$, and by (164) we have

$$\begin{aligned} \Lambda_q \le \int_{-1}^{1} |y'|^2 \, dx &\le \liminf_{k\to\infty} \left[\int_{-1}^{1} |y_k'|^2 \, dx + \alpha_k \left(\int_{-1}^{1} y_k |y_k|^{q-1} \, dx \right)^{\frac{2}{q}} \right] \\ &\le \lim_{k\to\infty} \lambda(\alpha_k, q) \le \Lambda_q. \end{aligned}$$

and the result follows. □

Remark 3.18. *If $\lambda(\alpha, q) = 0$, then*

$$-\alpha = \min_{w \in H_0^1(-1,1)} \frac{\displaystyle\int_{-1}^{1} |w'|^2 dx}{\left(\displaystyle\int_{-1}^{1} |w|^q \, dx \right)^{2/q}}. \tag{165}$$

Indeed, if $\lambda(\alpha, q) = 0$ then necessarily $\alpha < 0$ and the minimizers of (160) have constant sign. Let $y \ge 0$ be a minimizer of (160), by definition we have

$$0 = \lambda(\alpha, q) = \frac{\displaystyle\int_{-1}^{1} |y'|^2 \, dx + \alpha \left(\int_{-1}^{1} y^q \, dx \right)^{\frac{2}{q}}}{\displaystyle\int_{-1}^{1} \bar{u}^2 \, dx}$$

and hence

$$-\alpha = \frac{\displaystyle\int_{-1}^{1} |y'|^2 dx}{\left(\displaystyle\int_{-1}^{1} y^q \, dx \right)^{2/q}}. \tag{166}$$

If we denote by v a minimizer of problem (165), we have

$$0 \le \int_{-1}^{1} |v'|^2 \, dx + \alpha \left(\int_{-1}^{1} |v|^q \, dx \right)^{2/q}$$

and therefore

$$-\alpha \le \frac{\displaystyle\int_{-1}^{1} |v'|^2 dx}{\left(\displaystyle\int_{-1}^{1} |v|^q \, dx \right)^{2/q}} = \min_{w \in H_0^1(-1,1)} \frac{\displaystyle\int_{-1}^{1} |w'|^2 dx}{\left(\displaystyle\int_{-1}^{1} |w|^q \, dx \right)^{2/q}}.$$

From (166) the result follows.

3.2.2 Changing-sign minimizers

We first analyze the behavior of the minimizers of (160), by assuming that they have to change sign in $]-1,1[$. In this case, by Proposition 3.17 *(d)*, we may suppose that $\alpha > 0$. Moreover, due the homogeneity of the problem, in all the section we will assume also that

$$\max_{[-1,1]} y(x) = 1, \quad \min_{[-1,1]} y(x) = -\bar{m}, \quad \bar{m} \in]0,1].$$

It is always possible to reduce to this condition, by multiplying the solution by a constant if necessary.
We split the list of the main properties in two propositions.

Proposition 3.19. *Let $1 \le q \le 2$ and suppose that, for some $\alpha > 0$, $\lambda(\alpha, q)$ admits a minimizer y that changes sign in $[-1,1]$. Then the following properties hold.*

(a1) *The minimizer y has in $]-1,1[$ exactly one maximum point, η_M, with $y(\eta_M) = 1$, and exactly one minimum point, $\eta_{\bar{m}}$, with $y(\eta_{\bar{m}}) = -\bar{m}$.*

(b1) *If $y_+ \ge 0$ and $y_- \le 0$ are, respectively, the positive and negative part of y, then y_+ and y_- are, respectively, symmetric about $x = \eta_M$ and $x = \eta_{\bar{m}}$.*

(c1) *There exists a unique zero of y in $]-1,1[$.*

(d1) *In the minimum value $\bar{m}$ of y, it holds that*

$$\lambda(\alpha, q) = H(\bar{m}, q)^2,$$

where $H(m,q)$, $(m,q) \in [0,1] \times [1,2]$, is the function defined as

$$H(m,q) := \int_{-m}^{1} \frac{dy}{\sqrt{1 - z(m,q)(1 - |y|^{q-1}y) - y^2}} =$$

$$= \int_0^1 \frac{dy}{\sqrt{1 - z(m,q)(1 - y^q) - y^2}} + \int_0^1 \frac{m\,dy}{\sqrt{1 - z(m,q)(1 + m^q y^q) - m^2 y^2}}$$

and $z(m,q) = \frac{1-m^2}{1+m^q}$.

Proposition 3.20. *Let us suppose that, for some $\alpha > 0$, $\lambda(\alpha, q)$ admits a minimizer y that changes sign in $[-1,1]$. Then the following properties holds.*

(a2) *If $1 \le q \le 2$, then*

$$\lambda(\alpha, q) = \lambda^T = \pi^2.$$

(b2) *If $1 < q \le 2$, then*

$$\int_{-1}^{1} |y|^{q-1} y \, dx = 0. \tag{167}$$

(c2) *If $1 \le q \le 2$ and (167) holds, then $y(x) = C \sin \pi x$, with $C \in \mathbb{R} \setminus \{0\}$. Hence the only point in $]-1,1[$ where y vanishes is $\bar{x} = 0$.*

Proof of Proposition 3.19. First of all, if y is a minimizer of (160) which changes sign, let us consider $\eta_M, \eta_{\bar m}$ in $]-1,1[$ such that $y(\eta_M) = 1 = \max_{[-1,1]} y$, and $y(\eta_{\bar m}) = -\bar m = \min_{[-1,1]} y$, with $\bar m \in]0,1]$. For the sake of simplicity, we will write $\lambda = \lambda(\alpha, q)$. Multiplying the equation in (163) by y' and integrating we get

$$\frac{y'^2}{2} + \lambda \frac{y^2}{2} = \frac{\alpha\gamma}{q}|y|^{q-1}y + c \qquad \text{in }]-1,1[, \tag{168}$$

for a suitable constant c. Being $y'(\eta_M) = 0$ and $y(\eta_M) = 1$, we have

$$c = \frac{\lambda}{2} - \frac{\alpha}{q}\gamma. \tag{169}$$

Moreover, $y'(\eta_{\bar m}) = 0$ and $y(\eta_{\bar m})$ give also that

$$c = \lambda \frac{\bar m^2}{2} + \frac{\alpha}{q}\bar m^q \gamma. \tag{170}$$

Joining (169) and (170), we obtain

$$\begin{cases} \gamma = \frac{q\lambda}{2\alpha} z(\bar m, q) \\ c = \frac{\lambda}{2} t(\bar m, q) \end{cases} \tag{171}$$

where

$$z(m,q) = \frac{1-m^2}{1+m^q} \quad \text{and} \quad t(m,q) = \frac{m^2+m^q}{1+m^q} = 1 - z(m,q).$$

Then (168) can be written as

$$\frac{y'^2}{2} + \lambda\frac{y^2}{2} = \frac{\lambda}{2} z(\bar m, q)|y|^{q-1}y + \frac{\lambda}{2}t(\bar m, q) \qquad \text{in }]-1,1[.$$

Therefore we have

$$(y')^2 = \lambda[1 - z(\bar m, q)(1 - |y|^{q-1}y) - y^2] \qquad \text{in }]-1,1[. \tag{172}$$

First of all, it is easy to see that the number of zeros of y has to be finite. Let

$$-1 = \zeta_1 < \ldots < \zeta_j < \zeta_{j+1} < \ldots < \zeta_n = 1$$

be the zeros of y.

As observed in [34], it is easy to show that

$$y'(x) = 0 \iff y(x) = -\bar m \text{ or } y(x) = 1. \tag{173}$$

This implies that y has no other local minima or maxima in $]-1,1[$, and in any interval $]\zeta_j, \zeta_{j+1}[$ where $y > 0$ there is a unique maximum point, and in any interval $]\zeta_j, \zeta_{j+1}[$ where $y < 0$ there is a unique minimum point.

To prove (173), let

$$g(Y) = 1 - z(\bar m, q)(1 - |Y|^{q-1}Y) - Y^2, \quad Y \in [-\bar m, 1].$$

So we have

$$(y')^2 = \lambda\, g(y). \tag{174}$$

Observe that $g(-\bar m) = g(1) = 0$. Being $q \le 2$, it holds that $g'(\bar Y) = 0$ implies $g(\bar Y) > 0$. Hence, g does not vanish in $]-\bar m, 1[$. By (174), it holds that $y'(x) \neq 0$ if $y(x) \neq 1$ and $y(x) \neq -\bar m$.

Now, adapting an argument contained in [40, Lemma 2.6], the following three claims below allow to complete the proof of *(a1)*, *(b1)* and *(c1)*.

CLAIM 1: in any interval $]\zeta_j, \zeta_{j+1}[$ given by two subsequent zeros of y and in which $y = y^+ > 0$, has the same length; in any of such intervals, y^+ is symmetric about $x = \frac{\zeta_j+\zeta_{j+1}}{2}$;

CLAIM 2: in any interval $]\zeta_j, \zeta_{j+1}[$ given by two subsequent zeros of y and in which $y = y^- < 0$ has the same length; in any of such intervals, y^- is symmetric about $x = \frac{\zeta_j+\zeta_{j+1}}{2}$;

CLAIM 3: there is a unique zero of y in $]-1, 1[$.

Then, y admits a unique maximum point and a unique minimum point in $]-1,1[$, and the positive and negative parts are symmetric with respect to $x = \eta_M$ and $x = \eta_{\bar{m}}$, respectively.

In order to get claims 1 and 2, To fix the ideas, let us assume that $y > 0$ in $]\zeta_{2k-1}, \zeta_{2k}[$, and $y < 0$ in $]\zeta_{2k}, \zeta_{2k+1}[$. If this is not the case, the procedure is analogous.

Let us consider $y = y_+ > 0$ in $]\zeta_{2k-1}, \zeta_{2k}[$, and denote by η_{2k-1} the unique maximum point in such interval. Then by (172), integrating between ζ_{2k-1} and η_{2k-1} we have

$$\int_0^1 \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 - y^q) - y^2}} = (\eta_{2k-1} - \zeta_{2k-1})\sqrt{\lambda}.$$

Similarly, integrating between η_{2k-1} and ζ_{2k}, it holds

$$\int_0^1 \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 - y^q) - y^2}} = (\zeta_{2k} - \eta_{2k-1})\sqrt{\lambda}.$$

Hence

$$\zeta_{2k} - \zeta_{2k-1} = \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 - y^q) - y^2}}, \quad \text{and} \quad \eta_{2k-1} = \frac{\zeta_{2k-1} + \zeta_{2k}}{2}.$$

Similarly, consider that in $]\zeta_{2k}, \zeta_{2k+1}[$ it holds $y = y_- < 0$, and $y(\eta_{2k}) = -\bar{m}$. By (172), integrating between ζ_{2k} and η_{2k} we have

$$\int_0^{\bar{m}} \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 + y^q) - y^2}} = (\eta_{2k} - \zeta_{2k})\sqrt{\lambda},$$

and then between η_{2k} and ζ_{2k+1}, it holds

$$\int_0^{\bar{m}} \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 + y^q) - y^2}} = (\zeta_{2k+1} - \eta_{2k})\sqrt{\lambda}.$$

Hence

$$\zeta_{2k+1} - \zeta_{2k} = \frac{1}{\sqrt{\lambda}} \int_0^{\bar{m}} \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 + y^q) - y^2}}, \quad \text{and} \quad \eta_{2k} = \frac{\zeta_{2k} + \zeta_{2k+1}}{2}.$$

Resuming, we have that any interval given by two subsequent zeros of y and in which $y = y_+ > 0$, has the same length. Similarly, any interval given by two subsequent zeros of y and in which $y = y_- < 0$, has the same length.
Now, again from (172), if $x \in]\zeta_{2k-1}, \eta_{2k-1}[$ it holds

$$\int_0^{y(x)} \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 - y^q) - y^2}} = (x - \zeta_{2k-1})\sqrt{\lambda}, \tag{175}$$

and, if $t \in]\eta_{2k-1}, \zeta_{2k}[$, then

$$-\int_0^{y(t)} \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 - y^q) - y^2}} = (t - \zeta_{2k})\sqrt{\lambda}.$$

On the other hand, by choosing $t = \zeta_{2k-1} + \zeta_{2k} - x \in]\eta_{2k-1}, \zeta_{2k}[$ it holds

$$-(x - \zeta_{2k-1})\sqrt{\lambda} = (t - \zeta_{2k})\sqrt{\lambda} = -\int_0^{y(t)} \frac{dy}{\sqrt{1 - z(\bar{m}, q)(1 - y^q) - y^2}}.$$

From (175) we deduce that $y(x) = y(t)$, hence y is symmetric about $x = \eta_{2k-1}$ in the interval $]\zeta_{2k-1}, \zeta_{2k}[$.
In the same way, y is symmetric about $x = \eta_{2k}$ in the interval $]\zeta_{2k}, \zeta_{2k+1}[$.
Now we show that the number of the zeros of y is odd. Let us observe that

$$A_+ := \int_{\zeta_{2k-1}}^{\zeta_{2k}} y^q dx \geq \int_{\zeta_{2k}}^{\zeta_{2k+1}} (-y)^q dx =: A_- .$$

Indeed, multiplying (172) by $|y(x)|^{2q}$ and using the symmetry properties of y we have

$$A_+ = \frac{2}{\sqrt{\lambda}} \int_{\zeta_{2k-1}}^{\eta_{2k-1}} \frac{y^q}{\sqrt{1 - z(\bar{m}, q)(1 - y^q) - y^2}} y' dx = \\ = \frac{2}{\sqrt{\lambda}} \int_0^1 \frac{y^q}{\sqrt{1 - z(\bar{m}, q)(1 - y^q) - y^2}} dy$$

and

$$A_- = \frac{2}{\sqrt{\lambda}} \int_{\eta_{2k}}^{\zeta_{2k+1}} \frac{(-y)^q}{\sqrt{1 - z(\bar{m}, q)(1 + |y|^q) - y^2}} y' dx = \\ = \frac{2}{\sqrt{\lambda}} \int_0^1 \frac{\bar{m}^{q+1} y^q}{\sqrt{1 - z(\bar{m}, q)(1 + \bar{m}^q y^q) - \bar{m}^2 y^2}} dy.$$

If y has an even number n of zeros, two cases may occur.
Case 1: $\bar{m} = 1$. Then $z(1, q) = 0$, and by (171) $\gamma = 0$. On the other hand, $A_+ = A_-$, and being n even, then $\gamma = A_+^{\frac{2}{q}-1}$ and this is absurd.
Case 2: $\bar{m} < 1$. Let us consider the function $\tilde{y} \in H_0^1(-1, 1)$ defined as

$$\tilde{y}(x) = \begin{cases} y(x) & \text{if } x \in [\zeta_0, \zeta_{n-1}] \\ -y(x) & \text{if } x \in [\zeta_{n-1}, 1]. \end{cases}$$

We have that

$$\begin{gathered} \int_{-1}^1 (\tilde{y}')^2 dx = \int_{-1}^1 (y')^2 dx, \ \int_{-1}^1 \tilde{y}^2 dx = \int_{-1}^1 y^2 dx, \\ \left| \int_{-1}^1 |\tilde{y}|^{q-1} \tilde{y}\, dx \right| < \int_{-1}^1 |y|^{q-1} y\, dx. \end{gathered} \tag{176}$$

The first two equalities in (176) are obvious. To show last inequality, we recall that $y(x)$ is positive in $]-1, \zeta_2[$, hence if it has an even number of zeros, it is positive in $]\zeta_{n-1}, 1[$. Hence it is sufficient to observe that $A_+ > A_-$ and

$$\int_{-1}^1 |y|^{q-1} y\, dx = \frac{n}{2} A_+ - \frac{n-2}{2} A_-, \qquad \int_{-1}^1 |\tilde{y}|^{q-1} \tilde{y}\, dx = \frac{n-2}{2} A_+ - \frac{n}{2} A_-.$$

Then, (176) implies that $\mathcal{Q}[\tilde{y},\alpha] < \mathcal{Q}[y,\alpha]$ and this contradicts the minimality of y. So, the number n of the zeros of y is odd.
Finally, we conclude that $n = 3$ (Claim 3). If not, by considering the function $w(x) = y\left(\frac{2(x+1)}{n-1} - 1\right)$, $x \in [-1,1]$, we obtain that

$$\mathcal{Q}[w,\alpha] = \frac{\left(\dfrac{2}{n-1}\right)^2 \displaystyle\int_{-1}^{1} |y'|^2 dx + \alpha \left| \int_{-1}^{1} |y|^{q-1} y\, dx \right|^{\frac{2}{q}}}{\displaystyle\int_{-1}^{1} |y|^2 dx} < \mathcal{Q}[y,\alpha],$$

that is absurd. Hence, the solution y has only one zero in $]-1,1[$, and also $(c1)$ is proved.

Now denote by η_M and $\eta_{\bar{m}}$, respectively, the unique maximum and minimum point of y. It is not restrictive to suppose $\eta_M < \eta_{\bar{m}}$. They are such that $\eta_M - \eta_{\bar{m}} = 1$, with $y' < 0$ in $]\eta_M, \eta_{\bar{m}}[$. Then

$$\sqrt{\lambda(\alpha,q)} = \frac{-y'}{\sqrt{1 - z(\bar{m},q)(1 - |y|^{q-1}y) - y^2}} \qquad \text{in }]\eta_M, \eta_{\bar{m}}[.$$

Integrating between η_M and $\eta_{\bar{m}}$, we have

$$\lambda(\alpha,q) = \left[\int_{-\bar{m}}^{1} \frac{dy}{\sqrt{1 - z(\bar{m},q)(1 - |y|^{q-1}y) - y^2}} \right]^2 = H(\bar{m},q),$$

and the proof of the Proposition is completed. □

Remark 3.21. *We stress that properties $(a1) - (a3)$ can be also proved by using a symmetrization argument, by rearranging the functions y^+ and y^- and using the Pólya-Szégo inequality and the properties of rearrangements (see also, for example, [19] and [43]). For the convenience of the reader, we prefer to give an elementary proof without using the symmetrization technique.*

Our aim now is to study the function H defined in Proposition 3.19.

Proposition 3.22. *For any $m \in [0,1]$ and $q \in [1,2]$ it holds that*

$$H(m,q) \geq H(m,1) = \pi.$$

Moreover, if $m < 1$ and $q > 1$, then

$$H(m,q) > \pi,$$

while

$$H(m,1) = \pi, \quad \forall m \in [0,1].$$

Hence if $H(m,q) = \pi$ and $1 < q \leq 2$, then necessarily $m = 1$.

Remark 3.23. *Let us explicitly observe that in the case $\alpha \leq 0$, it holds that $\lambda(\alpha,q) \leq \frac{\pi^2}{4} < H(m,q)^2$ for any $m \in [0,1]$ and $q \in [1,2]$.*

Remark 3.24. *The proof of Proposition 3.22 is based on the study of the integrand function that defines $H(m,q)$, that is*

$$h(m,q,y) := \frac{1}{\sqrt{1 - z(m,q)(1 - y^q) - y^2}} + \frac{m}{\sqrt{1 - z(m,q)(1 + m^q y^q) - m^2 y^2}}.$$

Let us explicitly observe that if $m = 1$, then $z(1, q) = 0$ and

$$h(1, q, y) = \frac{2}{\sqrt{1 - y^2}},$$

that is constant in q. Moreover, if $y = 0$, then

$$h(m, q, 0) = \frac{1 + m}{\sqrt{1 - z(m, q)}}$$

that is strictly increasing in $q \in [1, 2]$. Furthermore, simple computations yield

$$\begin{cases} H(1, q) = \pi, \quad H(0, q) = \dfrac{\pi}{2 - q} \geq \pi \quad (H(0, 2) = +\infty), \\ H(m, 1) = \pi, \quad H(m, 2) = \dfrac{\pi}{2}\sqrt{\dfrac{1 + m^2}{2}}\left(\dfrac{1}{m} + 1\right) \geq \pi. \end{cases}$$

To prove Proposition 3.22, it is sufficient to show that h is monotone in q.

Lemma 3.25. *For any fixed $y \in [0, 1[$ and $m \in]0, 1[$, the function $h(m, \cdot, y)$ is strictly increasing as $q \in [1, 2]$.*

Proof. From the preceding observations, we may assume $m \in]0, 1[$ and $y \in]0, 1[$. Differentiating in q, we have that

$$\begin{aligned} \partial_q h = &- \frac{1}{2F_I^3}\left[- (1 - y^q)\partial_q z + z\, y^q \log y \right] + \\ &- \frac{m}{2F_{II}^3}\left[- (1 + m^q y^q)\partial_q z - z\, m^q y^q (\log m + \log y) \right], \end{aligned}$$

where

$$F_I(m, q, y) := \sqrt{1 - z(m, q)(1 - y^q) - y^2} \leq \sqrt{1 - y^2}, \tag{177}$$

and

$$F_{II}(m, q, y) := \sqrt{1 - z(m, q)(1 + m^q y^q) - m^2 y^2} \geq m\sqrt{1 - y^2}. \tag{178}$$

Being

$$z = \frac{1 - m^2}{1 + m^q}, \quad \partial_q z = -\frac{1 - m^2}{(1 + m^q)^2} m^q \log m,$$

we have that

$$\begin{aligned} \partial_q h = \frac{1}{2}\frac{1 - m^2}{(1 + m^q)^2}\Bigg\{ &\overbrace{\left[- (1 - y^q) m^q \log m - y^q (1 + m^q) \log y \right]}^{h_1(m,q,y)} \frac{1}{F_I^3} + \\ &+ \underbrace{\left[- (1 + m^q y^q) \log m + (1 + m^q) y^q (\log m + \log y) \right]}_{h_2(m,q,y)} \frac{m^{q+1}}{F_{II}^3} \Bigg\}. \end{aligned}$$

Let us observe that $h_1(m, q, y) \geq 0$. Hence, in the set A of (q, m, y) such that $h_2(m, q, y)$ is nonnegative, we have that $\partial_q h(q, m, y) \geq 0$. Moreover, $h_1(q, m, y)$ cannot vanish $(y < 1)$, then $\partial_q h > 0$ in A.

Hence, let us consider the set B where

$$h_2 = (y^q - 1)\log m + (1+m^q)y^q \log y \le 0$$

(observe that in general A and B are nonempty). By (177) and (178) we have that

$$\partial_q h \ge \frac{1}{2}\frac{1-m^2}{(1+m^q)^2}\Big\{\Big[-(1-y^q)m^q\log m - y^q(1+m^q)\log y\Big]\frac{1}{(1-y^2)^{\frac{3}{2}}}+$$

$$+\Big[(y^q-1)\log m + (1+m^q)y^q\log y\Big]\frac{m^{q-2}}{(1-y^2)^{\frac{3}{2}}}\Big\}.$$

Hence, to show that $\partial_q h > 0$ also in the set B it is sufficient to prove that

$$g(m,q,y) := \Big[-(1-y^q)m^q\log m - y^q(1+m^q)\log y\Big]+$$

$$+\Big[(y^q-1)\log m + (1+m^q)y^q\log y\Big]m^{q-2} > 0 \quad (179)$$

when $m \in]0,1[$, $q \in [1,2]$ and $y \in]0,1[$.

Claim 1. *For any $q \in [1,2]$ and $m \in]0,1[$, the function $g(m,q,\cdot)$ is strictly decreasing for $y \in]0,1[$.*

To prove the Claim 1, we differentiate g with respect to y, obtaining

$$\partial_y g = \Big[qy^{q-1}m^q\log m - qy^{q-1}(1+m^q)\log y - y^{q-1}(1+m^q)\Big]+$$

$$+\Big[qy^{q-1}\log m + (1+m^q)(qy^{q-1}\log y + y^{q-1})\Big]m^{q-2} =$$

$$= y^{q-1}\Big[q(m^q+m^{q-2})\log m + q(1+m^q)(m^{q-2}-1)\log y + (1+m^q)(m^{q-2}-1)\Big].$$

Then $\partial_y g < 0$ if and only if

$$(1+m^q)(m^{q-2}-1)(q\log y + 1) < -q(m^q+m^{q-2})\log m.$$

The above inequality is true, as we will show that (recall that $0<m<1$ and $1 \le q \le 2$)

$$\log y < -\frac{1}{q} + \frac{(m^q+m^{q-2})\log m}{(1+m^q)(1-m^{q-2})} =: -\frac{1}{q} + \ell(m,q). \quad (180)$$

If the the right-hand side of (180) is nonnegative, then for any $y \in]0,1[$ the inequality (180) holds.

Claim 2. *For any $q \in [1,2]$ and $m \in]0,1[$, $\ell(m,q) > \frac{1}{q}$.*

We will show that

$$\ell(m,q) > 1 \ge \frac{1}{q}.$$

We have

$$\ell(m,q) = \frac{(m^q+m^{q-2})}{(1+m^q)(m^{q-2}-1)}\log\frac{1}{m} > 1$$

if and only if

$$\Lambda(m,q) = (m^q + m^{q-2})\log\frac{1}{m} - (1+m^q)(m^{q-2}-1) =$$
$$= (m^q + m^{q-2})\log\frac{1}{m} + 1 + m^q - m^{q-2} - m^{2q-2} =$$
$$= m^q\left(\log\frac{1}{m}+1\right) + m^{q-2}\left(\log\frac{1}{m}-1\right) + 1 - m^{2q-2} > 0.$$

Then for $m \in]0,1[$ we have

$$\Lambda(m,q) = m^q\left(\log\frac{1}{m}+1\right) + m^{q-2}\left(\log\frac{1}{m}-1\right) + 1 - m^{2q-2}$$
$$\geq m^q\left(\log\frac{1}{m}+1\right) + m^{q-2}\left(\log\frac{1}{m}-1\right) =$$
$$= m^{q-2}\left(m^2\left(\log\frac{1}{m}+1\right) + \log\frac{1}{m} - 1\right) > 0,$$

and the Claim 2, and then the Claim 1, are proved. To conclude the proof of (179), it is sufficient to observe that

$$g(m,q,y) > g(m,q,1) = 0$$

when $m \in]0,1[$, $q \in [1,2]$ and $y \in]0,1[$.

The Claim 1 gives that $\partial_q h(m,q,y) > 0$ when $m \in]0,1[$, $q \in [1,2]$ and $y \in]0,1[$, and this conclude the proof.

□

Proof of Proposition 3.22. Using Lemma 3.25 and Remark 3.24, it holds that

$$H(m,q) \geq H(m,1) = \pi$$

for $1 \leq q \leq 2$. In particular, if $q \in]1,2]$, $m \in [0,1[$ and $y \in]0,1[$ then

$$h(m,q,y) > h(m,1,y),$$

hence for any $m \in [0,1[$ and $q \in]1,2]$ it holds

$$H(m,q) > H(m,1) = \pi.$$

□

Now we are in position to prove Proposition 3.20.

Proof of Proposition 3.20. Let y be a minimizer of $\lambda(\alpha,q)$ that changes sign in $[-1,1]$, with $\max_{x\in[-1,1]} y(x) = 1$. By *(d1)* of Proposition 3.19 and Proposition 3.22, the eigenvalue $\lambda(\alpha,q)$ has to satisfy the inequality

$$\lambda(\alpha,q) \geq \pi^2.$$

Hence, by (161) it follows that

$$\lambda(\alpha,q) = \pi^2,$$

that is property *(a2)*. Assuming also $1 < q \le 2$, if $-\bar{m}$ is the minimum value of y, again by Proposition 3.22 and *(d1)* of Proposition 3.19, $\lambda(\alpha, q) = \pi^2$ if and only if $\bar{m} = 1$. Hence, $z(1, q) = 0$ and the first identity of (171) gives that

$$\int_{-1}^{1} y|y|^{q-1} dx = 0.$$

and hence *(b2)* follows.

To prove *(c2)*, let us explicitly observe that, when (167) holds, y solves

$$\begin{cases} y'' + \pi^2 y = 0 & \text{in }]-1,1[\\ y(-1) = y(1) = 0. \end{cases}$$

Hence $y(x) = C \sin \pi x$, with $C \in \mathbb{R} \setminus \{0\}$. □

3.2.3 Proof of the main results

Now we are in position to prove the first main result.

Our main result is stated in Theorem 3.26 below. In particular, the nonlocal term affects the minimizer of problem (160) in the sense that it has constant sign up to a critical value of α and, for α larger than the critical value, it has to change sign, and a saturation effect occurs.

Theorem 3.26. *Let $1 \le q \le 2$. There exists a positive number α_q such that:*

1. *if $\alpha < \alpha_q$, then*

   $$\lambda(\alpha, q) < \pi^2,$$

   and any minimizer y of $\lambda(\alpha, q)$ has constant sign in $]-1,1[$.

2. *If $\alpha \ge \alpha_q$, then*

   $$\lambda(\alpha, q) = \pi^2.$$

   Moreover, if $\alpha > \alpha_q$, the function $y(x) = \sin \pi x$, $x \in [-1,1]$, is the only minimizer, up to a multiplicative constant, of $\lambda(\alpha, q)$. Hence it is odd, $\int_{-1}^{1} |y(x)|^{q-1} y(x)\, dx = 0$, and $\bar{x} = 0$ is the only point in $]-1,1[$ such that $y(\bar{x}) = 0$.

Some additional informations are given in the next result.

Theorem 3.27. *The following facts hold.*

1. *For $q = 1$, then $\alpha_1 = \frac{\pi^2}{2}$. Moreover, if $\alpha = \alpha_1$, there exists a positive minimizer of $\lambda(\alpha_1, 1)$, and for any $\bar{x} \in]-1,1[$ there exists a minimizer y of $\lambda(\alpha_1, 1)$ which changes sign in $\bar{x}$, non-symmetric and with $\int_{-1}^{1} y(x)\, dx \ne 0$ when $\bar{x} \ne 0$.*

2. *If $1 < q \le 2$ and $\alpha = \alpha_q$, then $\lambda(\alpha_q, q)$ in $[-1,1]$ admits both a positive minimizer and the minimizer $y(x) = \sin \pi x$, up to a multiplicative constant. Hence, any minimizer has constant sign or it is odd.*

3. *If $q = 2$, then $\alpha_2 = \frac{3}{4}\pi^2$.*

Proof of Theorem 3.26 and Theorem 3.27. We begin the proof with the following claim.

Claim. *There exists a positive value of α such that the minimum problem*

$$\lambda(\alpha,q)=\min_{u\in H_0^1([-1,1])}\frac{\displaystyle\int_{-1}^{1}|u'|^2\,dx+\alpha\left|\int_{-1}^{1}u|u|^{q-1}\,dx\right|^{\frac{2}{q}}}{\displaystyle\int_{-1}^{1}u^2\,dx}$$

admits an eigenfunction y that satisfies $\int_{-1}^{1}y|y|^{q-1}\,dx=0$ In such a case, $\lambda(\alpha,q)=\pi^2$ and, up to a multiplicative constant, $y=\sin\pi x$.

To prove the claim, let us consider the case $1<q\le 2$. By contradiction, we suppose that for any $k\in\mathbb{N}$, there exists a divergent sequence α_k, and a corresponding sequence of eigenfunctions $\{y_k\}_{k\in\mathbb{N}}$ relative to $\lambda(\alpha_k,q)$ such that $\int_{-1}^{1}y_k|y_k|^{q-1}dx>0$ and $\|y_k\|_{L^2(-1,1)}=1$. By Proposition 3.20, these eigenfunctions do not change sign and, as we have already observed, $\lambda(\alpha_k,q)\le\pi^2$. It holds that

$$\int_{-1}^{1}|y_k'|^2\,dx+\alpha_k\left(\int_{-1}^{1}|y_k|^q\,dx\right)^{\frac{2}{q}}\le\pi^2. \tag{181}$$

Hence, y_k converges (up to a subsequence) to a function $y\in W_0^{1,2}(-1,1)$, strongly in $L^2(-1,1)$ and weakly in $H_0^1(-1,1)$. Moreover $\|y\|_{L^2(-1,1)}=1$ and y is not identically zero. Hence $\|y\|_{L^q(-1,1)}>0$. Therefore, letting $\alpha_k\to+\infty$ in (181) we have a contradiction and the claim is proved.

Now, we recall that for any $1\le q\le 2$, $\lambda(\alpha,q)$ is a nondecreasing Lipschitz function in α, and for α sufficiently large, $\lambda(\alpha,q)=\pi^2$. Hence, using the Claim 1, we can define

$$\alpha_q=\min\{\alpha\in\mathbb{R}\colon\lambda(\alpha,q)=\pi^2\}=\sup\{\alpha\in\mathbb{R}\colon\lambda(\alpha,q)<\pi^2\}.$$

Obviously, $\alpha_q>0$. If $\alpha<\alpha_q$, then the minimizers corresponding to $\lambda(\alpha,q)$ has constant sign, otherwise $\lambda(\alpha,q)=\pi^2$. If $\alpha>\alpha_q$, then any minimizer y corresponding to α is such that $\int_{-1}^{1}|y|^{q-1}y\,dx=0$. Indeed, if we assume, by contradiction, that there exist $\bar\alpha>\alpha_q$ and $\bar y$ such that $\int_{-1}^{1}|\bar y|^{q-1}\bar y\,dx>0$, $\|y\|_{L^2}=1$ and $\mathcal{Q}[\bar\alpha,\bar y]=\lambda(\bar\alpha,q)$, then

$$\begin{aligned}\mathcal{Q}[\bar\alpha-\varepsilon,\bar y]&=\mathcal{Q}[\bar\alpha,\bar y]-\varepsilon\left(\int_{-1}^{1}|\bar y|^{q-1}\bar y\,dx\right)^{\frac{q}{2}}\\&=\lambda(\bar\alpha,q)-\varepsilon\left(\int_{-1}^{1}|\bar y|^{q-1}\bar y\,dx\right)^{\frac{q}{2}}<\lambda(\bar\alpha,q).\end{aligned}$$

Hence, for ε sufficiently small, $\pi^2=\lambda(\alpha_q,q)\le\lambda(\bar\alpha-\varepsilon,q)<\lambda(\bar\alpha,q)$ and this is absurd. Finally, by (c2) of Proposition 3.20, the proof of Theorem 3.26 is completed. It is not difficult to see, by means of approximating sequences, that $\lambda(\alpha_q,q)$ admits both a nonnegative minimizer and a minimizer with vanishing q-average, that gives the thesis of Theorem 3.27, in the case $1<q\le 2$. To conclude the proof of Theorem 3.27, we have to study the behavior of the solutions when $q=1$ and $q=2$. Despite its simplicity, the case $q=1$ has a peculiar behavior. Let us recall that $\lambda(0,1)=\frac{\pi^2}{4}$, and, being $\lambda(\alpha,1)$ is Lipschitz, it assumes all the values in the interval $\left]\frac{\pi^2}{4},\pi^2\right]$ as α varies in $]0,+\infty[$.

Suppose that $\frac{\pi^2}{4}<\lambda(\alpha,1)<\pi^2$. Then $0<\alpha<\alpha_1$, the corresponding minimizer y has constant sign in $]-1,1[$, and it is a solution of

$$\begin{cases}y''+\lambda y=\alpha\gamma & \text{in }]-1,1[\\ y(-1)=y(1)=0,\end{cases}$$

where $\lambda = \lambda(\alpha, 1)$ and $\gamma = \int_{-1}^{1} y(x)dx > 0$. Hence

$$y(x) = \frac{\gamma\alpha}{\lambda}\left(1 - \frac{\cos(\sqrt{\lambda}x)}{\cos\sqrt{\lambda}}\right), \quad x \in [-1, 1].$$

Integrating both sides in $[-1, 1]$, we get

$$\alpha = \frac{\lambda\sqrt{\lambda}}{2\sqrt{\lambda} - 2\tan\sqrt{\lambda}},$$

and, letting $\lambda \to \pi^2$,

$$\alpha_1 = \frac{\pi^2}{2}.$$

Finally, in the critical case $\alpha = \alpha_1 = \frac{\pi^2}{2}$, an immediate computation shows that the functions

$$y_A(x) = \frac{A}{2}\left(1 + \cos\left(\pi x\right)\right) - \sqrt{1 - A}\sin\left(\pi x\right),$$

with $A \in [0, 1]$ have average $\gamma = A$ and y_A are minimizers of $\lambda(\alpha_1, 1) = \pi^2$. Moreover, when A varies in $[0, 1[$, the root of y_A in $]-1, 1[$ varies continuously in $[0, 1[$.

It remains to consider the case $q = 2$. If $\alpha = \alpha_2$, the corresponding positive minimizer y is a solution of

$$\begin{cases} y'' + \pi^2 y = \alpha_2 y & \text{in }]-1,1[\\ y(-1) = y(1) = 0. \end{cases}$$

The positivity of the eigenfunction guarantees that

$$\alpha_2 - \pi^2 = \lambda(0, 2) = \frac{\pi^2}{4},$$

hence $\alpha_2 = \frac{3}{4}\pi^2$. □

Remark 3.28. *When $1 < q < 2$, it is possible to obtain the following lower bound on α_q:*

$$\alpha_q \geq \frac{3}{2^{1+\frac{2}{q}}}\pi^2. \tag{182}$$

To get the estimate (182), by choosing $u(x) = \cos\frac{\pi}{2}x$ as test function we get

$$\pi^2 = \lambda(\alpha_q, q) \leq \mathcal{Q}[u, \alpha_q] = \frac{\pi^2}{4} + \alpha_q\left(\int_{-1}^{1} u^q dx\right)^{2/q} \leq \frac{\pi^2}{4} + \alpha_q 2^{\frac{2}{q}-1}.$$

Remark 3.29. *If the interval of integration is $]a, b[$ instead of $]-1, 1[$, then*

$$\lambda(\alpha, q;]a, b[) = \left(\frac{2}{b-a}\right)^2 \cdot \lambda\left(\left(\frac{b-a}{2}\right)^{1+\frac{2}{q}}\alpha,\, q\right).$$

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