STEFANIA BOFFA

Sequences of Refinements of Rough Sets: Logical and Algebraic Aspects

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Sequences of Refinements of Rough Sets: Logical and Algebraic Aspects

Stefania Boffa

University of Insubria
sboffa@uninsubria.it

Abstract. In this thesis, a generalization of the classical *Rough set theory* [85] is developed considering the so-called *sequences of orthopairs* that we define in [20] as special sequences of rough sets.
Mainly, our aim is to introduce some *operations between sequences of orthopairs*, and to discover how to generate them starting from the operations concerning standard rough sets (defined in [32]). Also, we prove several *representation theorems* representing the class of *finite centered Kleene algebras with the interpolation property* [31], and some classes of *finite residuated lattices* (more precisely, we consider *Nelson algebras* [89], *Nelson lattices* [23], *IUML-algebras* [74] and *Kleene lattice with implication* [27]) as sequences of orthopairs.
Moreover, as an application, we show that a sequence of orthopairs can be used to represent *an examiner's opinion on a number of candidates applying for a job*, and we show that opinions of two or more examiners can be combined using operations between sequences of orthopairs in order to get a final decision on each candidate.
Finally, we provide the original *modal logic* SO_n with semantics based on sequences of orthopairs, and we employ it to describe the knowledge of an agent that increases over time, as new information is provided. Modal logic SO_n is characterized by the sequences $(\Box_1, \ldots, \Box_n)$ and $(\bigcirc_1, \ldots, \bigcirc_n)$ of n modal operators corresponding to a sequence $(t_1, \ldots, t_n)$ of consecutive times. Furthermore, the operator $\Box_i$ of $(\Box_1, \ldots, \Box_n)$ represents the knowledge of an agent at time t_i, and it coincides with the *necessity modal operator* of $S5$ logic [29]. On the other hand, the main innovative aspect of modal logic SO_n is the presence of the sequence $(\bigcirc_1, \ldots, \bigcirc_n)$, since $\bigcirc_i$ establishes whether an agent is *interested in knowing* a given fact at time t_i.

Keywords: Rough sets · Orthopairs · Refinements · Many-valued logic · Modal logic

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1 Introduction

We can only see a short distance ahead, but we can see plenty there that needs to be done.

Alan Turing

Rough sets and orthopairs are mathematical tools that are used to deal with vague, imprecise and uncertain information. Rough set theory was introduced by the Polish mathematician Zdzislaw Pawlak in 1980 [85] [84] [86], and successively numerous researchers of several fields have contributed to its development. The rough set approach appears of fundamental importance in many research domains, for example in artificial intelligence and cognitive sciences, especially in the areas of machine learning, knowledge acquisition, decision analysis, knowledge discovery from databases, expert systems, inductive reasoning and pattern recognition [79] [113] [55] [87]. Also, rough set theory has been applied to solve many real-life problems in medicine, pharmacology, engineering, banking, finance, market analysis, environment management, etc. (see [95] [98] [53] for some examples). On the other hand, rough sets are also explored in mathematical logic for their relationship with three-valued logics [92] [104] [34]. Rough set philosophy is founded on the assumption that each object of the universe of discourse is described by some information, some data, or knowledge. Objects characterized by the same data are *indiscernible* in view of the available information about them. In this way, an *indiscernibility relation* between objects is generated, and it is the mathematical basis of rough set theory. The set of all indiscernible objects is named *elementary set*, and we can say that it is the *basic granule of knowledge* about the universe. Indiscernibility relations are equivalence relations, and elementary sets are their equivalence classes. Then, given an equivalence relation R defined on U, the *rough set* of a subset X of the universe U is the pair $(\mathcal{L}_R(X), \mathcal{U}_R(X))$ consisting respectively of the union of all equivalence classes fully contained in X, named *lower approximation* of X with respect to R, and the union of all the equivalence classes that have at least one element in common with X, named *upper approximation* of X with respect to R. Therefore, the rough set $(\mathcal{L}_R(X), \mathcal{U}_R(X))$ is the approximation of X with respect to the relation R. The set $\mathcal{B}_R(X)$ is called the *R-boundary region* of X, and it is the set $\mathcal{U}_R(X) \setminus \mathcal{L}_R(X)$. The objects of $\mathcal{B}_R(X)$ cannot be classified as belonging to X with certainty.

In this dissertation, we focus on *orthopairs* generated by an equivalence relation. They are equivalent to rough sets and are defined as follows. Let R be an equivalence relation on U, and let X be a subset of U, the *orthopair* of X determined by R is the pair $(\mathcal{L}_R(X), \mathcal{E}_R(X))$, where $\mathcal{L}_R(X)$ is the lower approximation and $\mathcal{E}_R(X)$, called *impossibility domain* or *exterior region* of X with respect to R, is the union of equivalence classes of R with no elements in common with X [32]. Orthopairs and rough sets are obtained from one another; indeed, the impossibility domain coincides with the complement of the upper approximation with respect to the universe. A pair (A, B) of disjoint subsets of a universe U can be viewed as the orthopair of a subset of U generated by an equivalence relation on U; in this case, we can say that (A, B) is an orthopair on U. We can view any orthopair (A, B) on the universe U as a three-valued function $f : U \mapsto \{0, \frac{1}{2}, 1\}$ such that, let $x \in U$, $f(x) = 1$ if $x \in A$, $f(x) = 0$ if $x \in B$ and $f(x) = \frac{1}{2}$ otherwise. Conversely, the three-valued function $f : U \mapsto \{0, \frac{1}{2}, 1\}$ determines the orthopair (A, B) on U, where $A = \{x \in U | f(x) = 1\}$ and $B = \{x \in U | f(x) = 0\}$. Several kinds of operations between rough sets have been considered [34]. They correspond to connectives in three-valued logics. Logical approaches to some of these connectives have been given, such as Łukasiewicz, Nilpotent Minimum, Nelson and Gödel connectives [83] [9] [13] [4].

Several authors generalized the definitions of rough sets and orthopairs by considering binary relations that are not equivalence relations, since the latter are not usually suitable to describe the real-world relationships between elements [111] [97]. We consider orthopairs generated by a tolerance relation, that is a reflexive and symmetric binary relation [96]. Given a tolerance relation R defined on U and an element x of U, by *tolerance class* of x with respect to R, we mean the set of elements of U indiscernible to x with respect to R. The set of all tolerance classes of R is a covering of U, that is a set of subsets of U whose union is U. Moreover, if R is an equivalence relation, then the set of all equivalence classes is a partition of U (a partition is a set of subsets of U that are pairwise disjoint and whose union is U). Therefore, we can define rough sets and orthopairs determined by a covering (or a partition) instead of a tolerance relation (or an equivalence relation).

In this thesis, we focus on *sequences of orthopairs* generated by refinement sequences of coverings [20] [19]. A *refinement sequence* of a universe U is a finite sequence $(C_1, \ldots, C_n)$ of coverings of U such that C_i is finer than C_j (each block of C_i is included at least in a block of C_j) for each $j \leq i$. Clearly, for each subset X of U, the refinement sequence $(C_1, \ldots, C_n)$ generates the sequence

$$((\mathcal{L}_1(X), \mathcal{E}_1(X)), \ldots, (\mathcal{L}_n(X), \mathcal{E}_n(X))),$$

where $(\mathcal{L}_i(X), \mathcal{E}_i(X))$ is the orthopair of X determined by C_i. Furthermore, we deal with sequences of *partial coverings*. These are coverings that do not fully cover the universe, and they are suitable for describing situations in which some information is lost during the refinement process [39]. Refinement sequences of partial coverings are obtained starting from *incomplete information tables*, that are tables where a set of objects is described by a set of attributes, but some information is lost or not available [67]. It is interesting to notice that when $(C_1, \ldots, C_n)$ consists of all partitions of U, the pair $(U, (C_1, \ldots, C_n))$ is an *Aumann structure*, that is a mathematical structure used by economists and game theorists to represent the knowledge [6] [7]. Refinement sequences can be represented as partially ordered sets. Hence, sequences of orthopairs generated by refinement sequences can be represented as pairs of upward closed subsets of such partially ordered sets. By using this correspondence, we give a concrete representation of some finite algebraic structures related with Kleene algebras. *Kleene algebras* form a subclass of *De Morgan algebras*. The latter were introduced by Moisil [76], and successively, they were explored by several authors, in particular, by Kalman [64] (under the name of *distributive i-lattices*), and by Bialynicki-Birula and Rasiowa, which called them *quasi-Boolean algebras* [12]. The notation that is still used was introduced by Monteiro [77]. We are interested in the family of *finite centered Kleene algebras with the interpolation property*, studied by the Argentinian mathematician Roberto Cignoli. In particular, in [31], he proved that centered Kleene algebras with the interpolation property are represented by *bounded distributive lattices* [88]. By Birkhoff representation, each bounded distributive lattice is characterized as a set of upsets of a partially ordered set with set intersection and union [14]. In this thesis, we prove that each finite centered Kleene algebra with the interpolation property is isomorphic to the set of sequences of orthopairs generated by a refinement sequence with operations obtained extending the *Kleene operations* between orthopairs (see [34]) to the sequences of orthopairs. We obtain a similar result for some other finite structures that are *residuated lattices* [104], and having as reduct a centered Kleene algebras with the interpolation property. More exactly, we show that some subclasses of *Nelson algebras*, *Nelson lattices* and *IUML-algebras* are represented as sequences of orthopairs in which the residuated operations are respectively obtain by extending *Nelson implication*, *Łukasiewicz conjunction and implication*, and *Sobociński conjunction and implication* between orthopairs (listed in [34]) to sequences of orthopairs. In Table 1 each structure is associated with its orthopaired operations.

Structures	*Operations between orthopairs*
Nelson algebras	Kleene conjunction and Nelson implication
Nelson lattices	Łukasiewicz conjunction and implication
IUML-algebras	Sobociński conjunction and implication

Table 1: Structures and Operations between orthopairs

Nelson algebras were introduced by Rasiowa [89], under the name of N-lattices, as the algebraic counterparts of the constructive logic with strong negation considered by Nelson and Markov [90] [22]. The centered Nelson algebras with the interpolation property are represented by Heyting algebras [11]. Nelson lattices are involutive residuated lattices, and are equationally equivalent to centered Nelson algebras [23]. IUML-algebras are the algebraic models of the logic IUML, which is a substructural fuzzy logic that is an axiomatic extension of the multiplicative additive intuitionistic linear logic MAILL [74]. IUML-algebras can also be defined as *bounded odd Sugihara monoids*, where a Sugihara monoid is the equivalent algebraic semantics for the relevance logic RM^t of R-mingle as formulated with Ackermann constants. In [49], a dual categorical equivalence is shown between IUML-algebras and suitable topological spaces defined starting from Kleene spaces. In this dissertation we focus only on finite IUML-algebras, and we refer to [3] and [74].

Moreover, we investigate the relationship between sequences of orthopairs and some finite lattices with implication. The latter are more general than Nelson lattices and form a subclass of *algebras with implication*, (DLI-algebras for short) [28]. We find a pair of operations that allows us to consider sequences of orthopairs as Kleene lattices with implication, but they coincide with no pair of three-valued operations. Consequently, we can introduce new operations between orthopairs, and so between rough sets.

On the other hand, some three-valued algebraic structures have been represented as rough sets generated by one covering [61] [62] [63] [4] [40]. Our results are more general, since many-valued algebraic structures correspond to sequences of rough sets determined by a sequence of coverings.

An important application of rough set theory is to partition a given universe into three pairwise disjoint regions: the *acceptance region* (i.e. the lower approximation), the *rejection region* (i.e. the impossibility domain), and the *uncertain region* (i.e. the boundary region). This classification is at the basis of the *three-way decision theory* [107], which allows us to make a decision on each object by considering the region to which it belongs. In this framework, we use a sequence of orthopairs to represent an examiner's opinion on a number of candidates applying for a job. Moreover, we show that the opinions of two or more examiners can be combined using operations between sequences of orthopairs in order to get a final decision on each candidate. On the other hand, we also show that sequences of orthopairs are identified as *decision trees* with three outcomes. Decision trees are graphical models widely used in machine learning for describing sequential decision problems [48].

Rough sets can be interpreted as the *necessity* and *possibility* operators in modal logic $S5$ [82] [8]. Moreover, the relationships between modal logic and many generalizations of rough set theory have been examined by several authors [70] [110]. In Chapter 5, we present a new modal logic, named SO_n logic, with semantics based on sequences of orthopairs. Modal logic SO_n is characterized by two families of modal operators, $(\Box_1, \dots, \Box_n)$ and $(\bigcirc_1, \dots, \bigcirc_n)$, which are semantically interpreted through the Kripke frame $(U, (R_1, \dots, R_n))$, where $(R_1, \dots, R_n)$ is a sequence of equivalence relations defined on the domain U, such that $R_j(u) \subseteq R_i(u)$, for each $i \leq j$ and $u \in U$.

Modal logic SO_n can also be viewed as an epistemic logic. More precisely, SO_n can represent the knowledge of an agent that increases over time, as new information is provided. Epistemic logic is the logic of knowledge and belief [59]. Epistemic modal logic provides models to formalize and describe the process of accumulating knowledge by individual knowers and groups of knowers by using modal logic [16] [46] [60]. Its applications include addressing numerous complex problems in philosophy, artificial intelligence, economics, linguistics and in other fields [99] [58]. Therefore, the sequences $(\Box_1, \dots, \Box_n)$ and $(\bigcirc_1, \dots, \bigcirc_n)$ correspond to a sequence $(t_1, \dots, t_n)$ of consecutive instants of time. The operator $\Box_i$

of $(\Box_1, \ldots, \Box_n)$ represents the knowledge of an agent at time t_i, and it coincides with the *necessity modal operator* of S5 logic [57]. The main innovative aspect of our logic is the presence of $(\bigcirc_1, \ldots, \bigcirc_n)$, since its element $\bigcirc_i$ establishes whether the agent is *interested in knowing* the truth or falsity of the sentences at time t_i.

Contents of the thesis

We conclude this introductory chapter by briefly describing the contents of the following chapters.

Chapter 2 reviews the basic notions and the notation that we will use throughout the thesis along with some simple preliminary results. Specially, we will focus on rough set theory, partial order theory and lattice theory.

In *Chapter 3*, we introduce the definition of *refinement sequences of partial coverings* as special sequences of coverings representing situations where new information is gradually provided on ever smaller sets of objects. We provide examples of environments in which refinement sequences arise; in detail, we obtain refinement sequences starting from incomplete information tables and formal contexts. Some families of sequences are defined considering how much the blocks of their coverings overlap. We identify refinement sequences as partially ordered sets. Moreover, the notion of *sequences of orthopairs* is introduced in order to generalize the rough set theory. We represent each sequence of orthopairs as a pair of disjoint upsets of a partially ordered set, or equivalently, as a labelled poset. Finally, we view sequences of orthopairs as decision trees with only three outcomes.

Preliminary versions of this chapter appeared in [1] [19] [20] [2].

In *Chapter 4*, we equip sets of sequences of orthopairs with some operations in order to obtain finite many-valued algebraic structures. Furthermore, we prove theorems wherewith to represent such structures as sequences of orthopairs. We show that, when sequences of orthopairs are generated by one covering, our operations coincide with some operations between orthopairs listed in [34]. Also, we discover how to generate operations between sequences of orthopairs starting from those concerning individual orthopairs. Finally, we use a sequence of orthopairs to represent an examiner's opinion on a number of candidates applying for a job. Moreover, we show that opinions of two or more examiners can be combined using our operations in order to get a final decision on each candidate.

Some results shown in this chapter can be found in [1] [19] [20] [2].

In *Chapter 5*, we recall some basic notions of modal logic and the existing connections between modal logic and rough sets. Then, we develop the original modal logic SO_n, defining its language, introducing its Kripke models, and providing its axiomatization. Moreover, we investigate the properties of our logic system, such as the consistency, the soundness and the completeness with respect to Kripke's semantics. We explore the relationships between modal logic SO_n and sequences of orthopairs. We consider the operations between orthopairs and between sequences of orthopairs from the logical point of view. Eventually, we employ modal logic SO_n to represent the knowledge of an agent that increases over time, as new information is provided.

We conclude this dissertation with *Chapter 6*, in which we briefly summarize the results that we have obtained, and we discuss their potential further developments along with new research objectives.

2 Preliminaries

That language is an instrument of human reason, and not merely a medium for the expression of thought, is a truth generally admitted.

George Boole

In this chapter, we introduce the basic notions and the notation that we will use throughout the thesis along with some simple preliminary results. Briefly, in Section 2.1, we recall the main definitions of rough set theory. In Section 2.2, we list several operations between orthopairs that are found in [34]; moreover, we show the connection between these operations and three-valued connectives. Finally, Section 2.3 focuses on some important contents of partial order theory and lattice theory.

2.1 Rough sets and orthopairs

Rough set theory, developed by Pawlak [85] [84], is a mathematical tool used to deal with imprecise and vague information of datasets, and it finds numerous applications in several areas of science, such as, for instance chemistry [66], medicine [102], marketing [52], social network [18], [41], etc. Rough sets provide approximations of sets with respect to equivalence relations.

Definition 1 (Equivalence relation). *An equivalence relation R of U is a subset on $U \times U$ such that*

1. $(x, x) \in R$ *(reflexivity),*
2. *if* $(x, y) \in R$*, then* $(y, x) \in R$ *(symmetry),*
3. *if* $(x, y) \in R$ *and* $(y, z) \in R$*, then* $(x, z) \in R$ *(transitivity),*

for each $x, y, z \in U$.

Moreover, let $x \in U$, we set $R(x) = \{y \in U \mid (x, y) \in R\}$, and we call $R(x)$ equivalence class *of x with respect to R.*

Definition 2 (Rough set). *Let R be an equivalence relation on U, and let $X \subseteq U$. Then, the* rough set *of X determined by R is the pair $(\mathcal{L}_R(X), \mathcal{U}_R(X))$, where*

$$\mathcal{L}_R(X) = \{x \in U \mid R(x) \subseteq X\} \text{ and}$$
$$\mathcal{U}_R(X) = \{x \in U \mid R(x) \cap X \neq \emptyset\}.$$

$\mathcal{L}_R(X)$ and $\mathcal{U}_R(X)$ are respectively called lower approximation *and* upper approximation *of X with respect to R.*

We write $(\mathcal{L}(X), \mathcal{U}(X))$ instead of $(\mathcal{L}_R(X), \mathcal{U}_R(X))$, when R is clear from the context.

Also, we call the *R-boundary region* of X the set $\mathcal{B}_R(X) = \mathcal{U}_R(X) \setminus \mathcal{L}_R(X)$.

Remark 1. Let R be an equivalence relation on U, and let $X \subseteq U$. Then,

$$\mathcal{L}_R(X) \subseteq X \subseteq \mathcal{U}_R(X) \quad \text{and} \quad \mathcal{U}_R(X) = \mathcal{L}_R(X) \cup \mathcal{B}_R(X).$$

Definition 3 (Orthopair). *Let R be an equivalence relation on U, and let $X \subseteq U$. Then, the* orthopair *of X determined by R is the pair $(\mathcal{L}_R(X), \mathcal{E}_R(X))$, where*

$\mathcal{L}_R(X)$ is the lower approximation given in Definition 2, and
$\mathcal{E}_R(X) = \{x \in U \mid R(x) \cap X = \emptyset\}$.

$\mathcal{E}_R(X)$ is called impossibility domain *or* exterior domain *of X. We write $(\mathcal{L}(X), \mathcal{E}(X))$ instead of $(\mathcal{L}_R(X), \mathcal{E}_R(X))$, when R is clear from the context.*

Remark 2. Let R be an equivalence relation on U, and let $X \subseteq U$. Then,

$$\mathcal{L}_R(X) \cap \mathcal{E}_R(X) = \emptyset \quad \text{and} \quad \mathcal{E}_R(X) = U \setminus \mathcal{U}_R(X).$$

The lower and upper approximations, the R-boundary region and the impossibility domain are depicted in Figure 1. The blocks, that cover the universe U (the largest rectangle), represent the equivalence classes with respect to an equivalence relation R on U. Moreover, if X is represented by the oval shape, then $\mathcal{L}(X)$ is the union of green blocks, $\mathcal{U}(X)$ is the union of green and white blocks, $\mathcal{B}(X)$ is the union of white blocks, and $\mathcal{E}(X)$ is the union of red blocks.

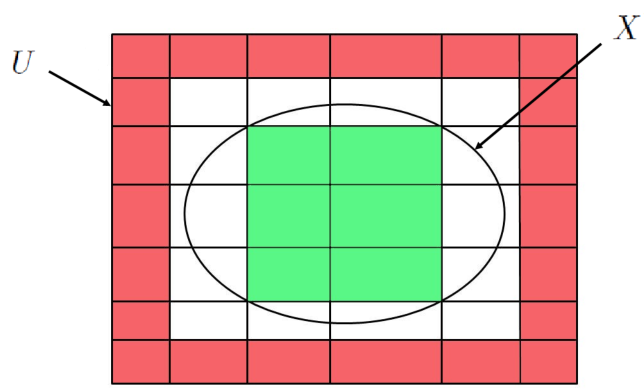

Fig. 1: Graphic representation of $\mathcal{L}(X)$, $\mathcal{U}(X)$, $\mathcal{B}(X)$ and $\mathcal{E}(X)$

In Rough set theory, given an equivalence relation R on the universe U, the pair (U, R) is called *Pawlak space*.

Remark 3. Let U be a universe, we denote the power set of U (i.e. the set of all subsets of U) with 2^U. Then, the structure $(2^U, \cap, \cup, \neg, \emptyset, U)$ is a *Boolean algebra* [105], where $\cap$, $\cup$ and $\neg$ are the usual set-theoretic operators. On the other hand, lower and upper approximations can be defined as unary operators on 2^U satisfying some properties [72], and so they are also named *approximation operators*. Thus, given an equivalence relation R on U, the system

$(2^U, \cap, \cup, \neg, \mathcal{L}_R, \mathcal{U}_R, \emptyset, U)$, called *Pawlak rough set algebra*, is a topological algebra [91], which is an extension of the Boolean algebra $(2^U, \cap, \cup, \neg, \emptyset, U)$. This means that we can regard Rough set theory as an extension of Set theory with the additional approximation operators [109].

We can observe that equivalence relations are equivalent to partitions that are defined as follows.

Definition 4 (Partition). *By* partition P *of the universe* U*, we mean a set* $\{b_1, \ldots, b_n\}$ *such that*

1. $b_1, \ldots, b_n \subseteq U$,
2. $b_i \cap b_j = \emptyset$*, for each* $i \neq j$,
3. $b_1 \cup \ldots \cup b_n = U$.

Therefore, a partition of U is a set of subsets of U that are pairwise disjoint and whose union is U.

Remark 4. The equivalence relation R of U determines the partition P_R of U made of all equivalence classes of R, namely

$$P_R = \{R(x) \mid x \in U\};$$

vice-versa, the partition P of U generates the equivalence relation R_P on U such that, let $x, y \in U$,

$$x \; R_P \; y \text{ if and only if } x \text{ and } y \text{ belong to the same element of } P.$$

We call blocks both equivalence classes and elements of partitions.

By Remark 4, it follows that rough sets and orthopairs can also be defined starting from partitions. Therefore, the following definition is equivalent to Definition 2 and Definition 3.

Definition 5 (Rough set and Orthopair). *Let* P *be a partition of* U*, and let* $X \subseteq U$*. The* rough set *and the* orthopair *of* X *determined by* P *are respectively the pairs* $(\mathcal{L}_P(X), \mathcal{U}_P(X))$ *and* $(\mathcal{L}_P(X), \mathcal{E}_P(X))$*, where*

$\mathcal{L}_P(X) = \cup\{b \in P \mid b \subseteq X\}$,
$\mathcal{U}_P(X) = \cup\{b \in P \mid b \cap X \neq \emptyset\}$*, and*
$\mathcal{E}_P(X) = \cup\{b \in P \mid b \cap X = \emptyset\}$.

Several authors generalize the classical definitions of rough sets and orthopairs, by considering binary relations that are not equivalence relations, since the latter are not usually suitable to describe the real-world relationships between elements (e.g. [111] [97]).

In this thesis, we consider orthopairs generated by tolerance relations [96] [71], or equivalently by coverings [33] [36].

Definition 6 (Tolerance relation). *A* tolerance relation R *on* U *is a subset of* $U \times U$ *such that*

1. $(x, x) \in R$ *(reflexivity),*
2. *if* $(x, y) \in R$*, then* $(y, x) \in R$ *(symmetry),*

for each $x, y, z \in U$*.*

Moreover, let $x \in U$*, we set* $R(x) = \{y \in U \mid (x, y) \in R\}$ *and we call* $R(x)$ *the* tolerance class *of* x *with respect to* R*.*

Trivially, an equivalence relation is also a tolerance relation. Moreover, tolerance relations generate coverings that are defined as follows.

Definition 7 (Covering). *By* covering C *of the universe* U*, we mean a set* $\{b_1, \ldots, b_n\}$ *such that*

1. $b_1, \ldots, b_n \subseteq U$*,*
2. $b_1 \cup \ldots \cup b_n = U$*.*

We can say that a partition is a covering that satisfies the additional property to have blocks pairwise disjoint.

2.2 Operations between orthopairs

In this section, we focus on some operations between orthopairs corresponding to three-valued connectives; moreover, here, by orthopair on U, we mean any pair of disjoint subsets of U, which may not even be the approximation of a subset of U with respect to a relation on U (see Definition 3).

The relationship between orthopairs and three-valued logics is based on the idea expressed in the following observation.

Remark 5. The orthopair (A, B) on the universe U generates the three-valued function $f_{(A,B)} : U \mapsto \{0, \frac{1}{2}, 1\}$ such that, let $x \in U$,

$$f_{(A,B)}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in B, \\ \frac{1}{2} & \text{if } x \in U \setminus (A \cup B). \end{cases}$$

Conversely, the three-valued function $f : U \mapsto \{0, \frac{1}{2}, 1\}$ determines the orthopair (A_f, B_f) on U, where

$$A_f = \{x \in U \mid f(x) = 1\} \text{ and } B_f = \{x \in U \mid f(x) = 0\}.$$

The most simple operations between orthopairs are defined as follows.

Definition 8. *Let* (A, B) *and* (C, D) *be two orthopairs on the universe* U*, we set*

$(A, B) \wedge_{\mathcal{K}} (C, D) = (A \cap C, B \cup D)$ *and*
$(A, B) \vee_{\mathcal{K}} (C, D) = (A \cup C, B \cap D)$*.*

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

Table 2: Kleene conjunction

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

Table 3: Kleene disjunction

Theorem 1 states that $\wedge_{\mathcal{K}}$ and $\vee_{\mathcal{K}}$ are respectively obtained from the *Kleene conjunction* and the *Kleene disjunction* on $\{0, \frac{1}{2}, 1\}$. The latter are defined by Table 2 and Table 3, respectively.

Also, we notice that $\wedge$ and $\vee$ are the minimum and the maximum on $\{0, \frac{1}{2}, 1\}$, respectively.

Theorem 1. *Let (A, B) and (C, D) be orthopairs on U. Then,*

$$(A, B) \wedge_{\mathcal{K}} (C, D) = (E, F) \quad \textit{and} \quad (A, B) \vee_{\mathcal{K}} (C, D) = (G, H),$$

where

$E = \{x \in U \mid f_{(A,B)}(x) \wedge f_{(C,D)}(x) = 1\}$,
$F = \{x \in U \mid f_{(A,B)}(x) \wedge f_{(C,D)}(x) = 0\}$,
$G = \{x \in U \mid f_{(A,B)}(x) \vee f_{(C,D)}(x) = 1\}$ *and*
$H = \{x \in U \mid f_{(A,B)}(x) \vee f_{(C,D)}(x) = 0\}$.

Proof. Let $x \in U$. By Remark 5, $x \in A \cap C$ if and only if $f_{(A,B)}(x) = 1$ and $f_{(C,D)}(x) = 1$, namely $f_{(A,B)}(x) \wedge f_{(C,D)}(x) = 1$ (see Table 2). Similarly, we can prove that $x \in B \cup D$ if and only if $f_{(A,B)}(x) \wedge f_{(C,D)}(x) = 0$. By Remark 5 and starting from Table 3, we can prove that

$x \in A \cup C$ if and only if $f_{(A,B)}(x) \vee f_{(C,D)}(x) = 1$, and
$x \in B \cap D$ if and only if $f_{(A,B)}(x) \vee f_{(C,D)}(x) = 0$.

The next operations between orthopairs are equivalent to some three-valued connectives belonging to the families of conjunctions and implications on $\{0, \frac{1}{2}, 1\}$. Now, we recall the definitions of conjunction and implication that are based on some intuitive properties in scope of modelling incomplete information.

Definition 9 (Conjunction). *A* conjunction *on $\{0, \frac{1}{2}, 1\}$ is a map*

$$* : \left\{0, \frac{1}{2}, 1\right\} \times \left\{0, \frac{1}{2}, 1\right\} \mapsto \left\{0, \frac{1}{2}, 1\right\}$$

satisfying the following properties: let $x, y, z \in \{0, \frac{1}{2}, 1\}$,

1. *if $x \leq y$, then $x * z \leq y * z$,*
2. *if $x \leq y$, then $z * x \leq z * y$,*
3. $0 * 0 = 0 * 1 = 1 * 0 = 0$ *and* $1 * 1 = 1$.

$\circledast_{\mathcal{L}}$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

Table 4: Łukasiewicz conjunction

$\circledast_{\mathcal{S}}$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	0	1	1

Table 5: Sobociński conjunction

Example 1. Among the conjunctions listed in [34], we only consider the Kleene conjunction, the *Łukasiewicz conjunction* and the *Sobociński conjunction [100]*. The latter two are defined by Table 4 and Table 5.

Definition 10 (Implication). *An* implication *on* $\{0, \frac{1}{2}, 1\}$ *is a map*

$$\rightarrow: \left\{0, \frac{1}{2}, 1\right\} \times \left\{0, \frac{1}{2}, 1\right\} \mapsto \left\{0, \frac{1}{2}, 1\right\}$$

satisfying the following properties: let $x, y \in \{0, \frac{1}{2}, 1\}$,

1. *if* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$,
2. *if* $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$,
3. $0 \rightarrow 0 = 1 \rightarrow 1 = 1$ *and* $1 \rightarrow 0 = 0$.

Example 2. Among the implications listed in [34], we consider the *Nelson implication*, the *Łukasiewicz implication* and the *Sobociński implication*. They are defined by the following tables, respectively.

$\Rightarrow_{\mathcal{N}}$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	1	1	1
1	0	$\frac{1}{2}$	1

Table 6: Nelson implication

$\Rightarrow_{\mathcal{L}}$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

Table 7: Łukasiewicz implication

$\Rightarrow_{\mathcal{S}}$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	0	0	1

Table 8: Sobociński implication

Now, we regard two multiplications between orthopairs defined as follows.

Definition 11. *Let (A, B) and (C, D) be orthopairs on U, we set*

1. $(A, B) *_{\mathcal{L}} (C, D) = (A \cap C, (U \setminus (A \cup C)) \cup B \cup D)$,
2. $(A, B) *_{\mathcal{S}} (C, D) = ((A \setminus D) \cup (C \setminus B), B \cup D)$.

We can prove that $*_{\mathcal{L}}$ and $*_{\mathcal{S}}$ are respectively equivalent to the three-valued conjunctions $\circledast_{\mathcal{L}}$ and $\circledast_{\mathcal{S}}$. More precisely, the following theorem holds.

Theorem 2. *Let (A, B) and (C, D) be orthopairs on U. Then,*

$$(A, B) *_{\mathcal{L}} (C, D) = (E, F) \text{ and } (A, B) *_{\mathcal{S}} (C, D) = (G, H),$$

where

$E = \{x \in U \mid f_{(A,B)}(x) \circledast_{\mathcal{L}} f_{(C,D)}(x) = 1\}$,
$F = \{x \in U \mid f_{(A,B)}(x) \circledast_{\mathcal{L}} f_{(C,D)}(x) = 0\}$,
$G = \{x \in U \mid f_{(A,B)}(x) \circledast_{\mathcal{S}} f_{(C,D)}(x) = 1\}$ *and*
$H = \{x \in U \mid f_{(A,B)}(x) \circledast_{\mathcal{S}} f_{(C,D)}(x) = 0\}$.

Proof. The proof is similar to that of Theorem 1.

Finally, we consider the following implications between orthopairs.

Definition 12. *Let (A, B) and (C, D) be orthopairs on U, then*

1. $(A, B) \rightarrow_{\mathcal{N}} (C, D) = ((U \setminus A) \cup C, A \cap D)$,
2. $(A, B) \rightarrow_{\mathcal{L}} (C, D) = (((U \setminus A) \cup C) \cap (B \cup (U \setminus D)), A \cap D)$,
3. $(A, B) \rightarrow_{\mathcal{S}} (C, D) = (B \cup C, U \setminus [(((U \setminus A) \cup C) \cap (A \cup (U \setminus D))])$.

The previous implications are respectively obtained from the three-valued implications $\Rightarrow_{\mathcal{N}}$, $\Rightarrow_{\mathcal{L}}$ and $\Rightarrow_{\mathcal{S}}$. More precisely, the following theorem holds.

Theorem 3. *Let (A, B), (C, D) and (E, F) be orthopairs on U. Then,*

$(A, B) \rightarrow_{\mathcal{N}} (C, D) = (E, F)$, *where*
$E = \{x \in U \mid f_{(A,B)}(x) \Rightarrow_{\mathcal{N}} f_{(C,D)}(x) = 1\}$ *and*
$F = \{x \in U \mid f_{(A,B)}(x) \Rightarrow_{\mathcal{N}} f_{(C,D)}(x) = 0\}$.

$(A, B) \rightarrow_{\mathcal{L}} (C, D) = (G, H)$, *where*
$G = \{x \in U \mid f_{(A,B)}(x) \Rightarrow_{\mathcal{L}} f_{(C,D)}(x) = 1\}$ *and*
$H = \{x \in U \mid f_{(A,B)}(x) \Rightarrow_{\mathcal{L}} f_{(C,D)}(x) = 0\}$,

$(A, B) \rightarrow_{\mathcal{S}} (C, D) = (I, J)$,
$I = \{x \in U \mid f_{(A,B)}(x) \Rightarrow_{\mathcal{S}} f_{(C,D)}(x) = 1\}$ *and*
$J = \{x \in U \mid f_{(A,B)}(x) \Rightarrow_{\mathcal{S}} f_{(C,D)}(x) = 0\}$.

Proof. The proof is similar to that of Theorem 1.

On the other hand, there is an equivalent way to describe the relationship between the three-valued connectives $\wedge$, $\vee$, $\circledast_{\mathcal{L}}$, $\circledast_{\mathcal{S}}$, $\Rightarrow_{\mathcal{N}}$, $\Rightarrow_{\mathcal{L}}$ and $\Rightarrow_{\mathcal{S}}$, and the operations defined in 8, 11 and 12. It is provided by using the next definition and the next theorem.

Definition 13. *Let C be a covering of the universe U, and let $X \subseteq U$, we can define the function $F_X^C : C \mapsto \{0, \frac{1}{2}, 1\}$, where*

$$F_X^C(N) = \begin{cases} 1 & \text{if } N \subseteq X, \\ 0 & \text{if } N \cap X = \emptyset, \\ \frac{1}{2} & \text{otherwise.} \end{cases} \tag{1}$$

for each $N \in C$. We denote F_X^C with F_X, when C is clear from the context.

The following theorem states that each operation between orthopairs is obtained from the respective three-valued connective, by using function 1.

Theorem 4. *Let C be a covering of U, and let $X, Y \subseteq U$. Suppose that the operation $\circ$ belongs to $\{\wedge_{\mathcal{K}}, \vee_{\mathcal{K}}, *_{\mathcal{L}}, *_{\mathcal{S}}, \rightarrow_{\mathcal{N}}, \rightarrow_{\mathcal{L}}, \rightarrow_{\mathcal{S}}\}$, then*

$$(\mathcal{L}(X), \mathcal{E}(X)) \circ (\mathcal{L}(Y), \mathcal{E}(Y))$$

is the orthopair (A, B) such that

$$A = \bigcup \{N \in C \mid F_X(N) \circledcirc F_Y(N) = 1\}$$

and

$$B = \bigcup \{N \in C \mid F_X(N) \circledcirc F_Y(N) = 0\},$$

where $\circledcirc$ respectively belongs to $\{\wedge, \vee, \circledast_{\mathcal{L}}, \circledast_{\mathcal{S}}, \Rightarrow_{\mathcal{N}}, \Rightarrow_{\mathcal{L}}, \Rightarrow_{\mathcal{S}}\}$.

Proof. We provide the proof only for the operation $*_{\mathcal{S}}$, since the remaining cases can be similarly demonstrated.

Let $x \in U$ and suppose that $(\mathcal{L}(X), \mathcal{E}(X)) *_{\mathcal{S}} (\mathcal{L}(Y), \mathcal{E}(Y)) = (A, B)$. By Definition 11, $x \in A$ if and only if $x \in (\mathcal{L}(X) \setminus \mathcal{E}(Y)) \cup (\mathcal{L}(Y) \setminus \mathcal{E}(X))$, namely $x \in \mathcal{L}(X) \setminus \mathcal{E}(Y)$ or $x \in \mathcal{L}(Y) \setminus \mathcal{E}(X)$. This is equivalent to affirm that x belongs to a node N of C such that

- $N \subseteq X$ and $N \cap Y = \emptyset$, or
- $N \subseteq X$ and $N \cap Y = \emptyset$.

Then, $F_X(N) = 1$ and $F_Y(N) \neq 0$, or $F_Y(N) = 1$ and $F_X(N) \neq 0$. We conclude that $F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 1$, since $\circledast_{\mathcal{S}}$ is the Sobociński conjunction.

Similarly, $x \in B$ if and only if $x \in \mathcal{E}(X) \cup \mathcal{E}(Y)$, by 11; namely, x belongs to a node N of C such that $N \cap X = \emptyset$ or $N \cap Y = \emptyset$. Then, $F_X(N) = 0$ or $F_Y(N) = 0$. Hence, $F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 0$.

Remark 6. However, the previous operations can be also defined by considering orthopairs that correspond to rough sets (see Definition 3). In this case, it is necessary to introduce some closure properties in order to ensure that operations between rough sets always generate a rough set. But, it will be done in the next sections.

Moreover, in Section 4.5, we extend the operations defined in 8, 11 and 12 to sequences of orthopairs in order to obtain many-valued algebraic structures.

2.3 Ordered structures

Partial orders and lattices This section contains some important contents of partial order theory and lattice theory. Partial order and lattice theory play an important role in many disciplines of computer science and engineering [54] [14].

Definition 14 (Partially ordered set). *A* partially ordered set*, more briefly a* poset*, is a pair $(P, \leq)$, where P is a non empty set and $\leq$ is a binary relation on P satisfying the following properties.*

1. *$x \leq x$ (reflexivity),*
2. *if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry),*
3. *if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity),*

for each $x, y, z \in L$.

Moreover, if $(P, \leq)$ is a poset, then $(S, \leq)$ is also a poset, for each $S \subseteq P$.

An example of partially ordered set is the set 2^U of all subsets of U with the set inclusion $\subseteq$.

Let $(P, \leq)$ be a poset, and $x, y \in P$, we say that y is the *successor* of x in P, if $x < y$ and there is no $z \in P$ such that $x < z < y$. Furthermore, P has a *maximum* (or *greatest*) element if there exists $x \in P$ such that $y \leq x$ for all $y \in P$. An element $x \in P$ is *maximal* if there is no element $y \in P$ with $y > x$. Minimum and minimal elements are dually defined. P has a *minimum* (or *least*) element if there exists $x \in P$ such that $x \leq y$ for all $y \in P$. An element $x \in P$ is *minimal* if there is no element $y \in P$ with $y < x$.

We can draw the *Hasse diagram* of each finite poset $(P, \leq)$: the elements of P are represented by points in the plane, and a line is drawn from x up to y, when y is a successor of x. Smaller elements are drawn under their successors.

Definition 15 (Chain). *A partially ordered set $(P, \leq)$ is a* chain *if and only if $x \leq y$ or $y \leq x$, for each $x, y \in P$.*

Definition 16 (Downset and Upset). *Let $(P, \leq)$ be a partially ordered set, and let $S \subseteq P$. Then, S is a* downset *of P if and only if satisfies the following property:*

$$\text{for any } y \in P, \text{ if } y \leq x \text{ and } x \in S, \text{ then } y \in S.$$

Dually, S is an upset *of P if and only if satisfies the following property:*

$$\textit{for any } y \in P, \textit{ if } x \leq y \textit{ and } x \in S, \textit{ then } y \in S.$$

Moreover, we set

$\downarrow S = \{y \in P \mid y \leq x$ *for some* $x \in S\}$ *and*
$\uparrow S = \{y \in P \mid x \leq y$ *for some* $x \in S\}$.

Definition 17 (Forest). *A partially ordered set* $(P, \leq)$ *is a forest if and only if the downset of each element of* P *is a chain.*

Definition 18 (Tree). *A tree* $(P, \leq)$ *is a forest that has minimum.*

Example 3. Consider the following binary relation on the set $\mathbb{N}$ of positive integers defined as follows: let $x, y \in \mathbb{N}$,

$$x \preccurlyeq y \text{ if and only if } x \text{ divides } y. \tag{2}$$

Then, the Hasse diagrams of the partially ordered sets

$$(\{1, 2, 3\}, \preccurlyeq),\ (\{1, 2, 5, 10\}, \preccurlyeq) \text{ and } (\{2, 7, 14\}, \preccurlyeq)$$

are respectively represented as follows.

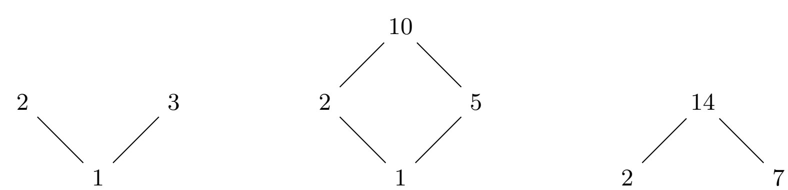

Fig. 2: Partially ordered sets

The poset $(\uparrow \{7\}, \preccurlyeq)$ is a chain. The poset $(\{1, 2, 3\}, \preccurlyeq)$ is a forest.

Minimal elements of a forest are called *roots*, while maximal elements are called *leaves*. A map $f : F \mapsto G$ between forests is *open* if, for $a \in G$ and $b \in F$, whenever $a \leq f(b)$ there exists $c \in F$ with $c \leq b$ such that $f(c) = a$. Equivalently, open maps carry upsets to upsets.

Let P be a poset, and let S be a subset of P. We say that an element $x \in P$ is an *upper bound* for S if $x \geq s$ for each $s \in S$. We can say that x is the *least upper bound* for S if x is an upper bound for S and $x \leq y$, for every upper bound y of S. Dually, x is a *lower bound* for S if $s \leq x$ for each $s \in S$; x is the *greatest lower bound* for S if x is a lower bound for S and $y \leq x$, for every lower bound y of S. If the least upper bound and the greatest lower bound of S exist, then they are unique.

Definition 19 (Lattice). *A* lattice *is a partially ordered set in which every pair of elements x and y has a least upper bound and a greatest lower bound, denoted with $x \wedge y$ and $x \vee y$, respectively.*

Lattices can also be defined as algebraic structures.

Definition 20 (Lattice). *[80] A* lattice *is an algebra $(L, \wedge, \vee)$ that satisfies the following proprieties.*

1. *$x \wedge x = x$ and $x \vee x = x$ (idempotent laws),*
2. *$x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (commutative laws),*
3. *$x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associative laws),*
4. *$x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ (absorption law),*

for each $x, y, z \in L$.

Remark 7. The latter two definitions are equivalent. Indeed, suppose that $(L, \leq)$ is a lattice, and $x \wedge y$ and $x \vee y$ denote the least upper bound and a greatest lower bound of x and y, respectively. Then, $(L, \wedge, \vee)$ satisfies the all proprieties of Definition 20.

Moreover, given a lattice $(L, \wedge, \vee)$, we can consider the following binary relation $\leq$ on L: let $x, y \in L$

$$x \leq y \text{ if and only if } x \wedge y = x \text{ (or } x \vee y = y).$$

We can prove that $(L, \leq)$ is a partially ordered set, in which every pair of elements has a greatest lower bound and a least upper bound.

An example of lattice is the structure $(2^U, \cap, \cup)$ of all subsets of a set U, with the usual set operations of intersection and union, or equivalently $(2^U, \subseteq)$, where $\subseteq$ is the set inclusion.

We are interested in *bounded distributive lattices* having the following definition.

Definition 21 (Bounded lattice). *A* bounded lattice *is a structure*

$$(L, \wedge, \vee, 0, 1)$$

such that $(L, \wedge, \vee)$ is a lattice, 0 is the identity element for $\vee$ ($x \vee 0 = 0$) and 1 is the identity element for $\wedge$ ($x \wedge 1 = x$). Moreover, 0 *and* 1 *are called* bottom *and* top *of L, respectively.*

Definition 22 (Distributive lattice). *[45] A lattice $(L, \wedge, \vee)$ is* distributive *if and only if the operations $\wedge$ and $\vee$ distribute over each other, namely*

1. *$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and*
2. *$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*

for each $x, y, z \in L$.

In 1937, the mathematician Garrett Birkhoff proved that there exists a one-to-one correspondence between distributive lattices and partial orders [15]. Namely, elements of a distributive lattice can be viewed as upsets, and the lattices operations correspond to intersection and union between sets.

Theorem 5 (Birkhoff's representation theorem). *Let $(P, \leq)$ be a partially ordered set, then the structure $(Up(P), \cap, \cup, \emptyset, P)$, where $Up(P)$ is the set of all upsets of P, and the operations $\cap$ and $\cup$ are respectively the intersection and the union between sets, is a bounded distributive lattice; furthermore, if $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, then there exists a partially ordered set $(P, \leq)$ such that $(Up(P), \cap, \cup, \emptyset, P)$ is isomorphic to $(L, \wedge, \vee, 0, 1)$.*

Definition 23 (Residuated lattice). *A* residuated lattice *is a structure*

$$(L, \wedge, \vee, *, \rightarrow, e, 0, 1)$$

such that

1. $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
2. $(L, *, e)$ *is a monoid,*
3. $x * y \leq z$ *if and only if* $x \leq z \rightarrow y$, *for each* $x, y, z \in L$ *($*$ and $\rightarrow$ satisfy the adjointness property).*

Kleene algebras Kleene algebras are a subclass of *De Morgan algebras*. The latter were introduced by Moisil [76] without the restriction including 0 and 1. Successively, they were studied by several authors, in particular, by Kalman [64] (under the name of *distributive i-lattices*), and by Bialynicki-Birula and Rasiowa, which called them *quasi-Boolean algebras* [12]. The notation that is still used was introduced by Monteiro [77].

Definition 24 (De Morgan algebra). *A* De Morgan algebra *is a structure* $(A, \wedge, \vee, \neg, 0, 1)$, *where*

1. $(A, \wedge, \vee, 0, 1)$ *is a bounded distributive lattice,*
2. $\neg(x \vee y) = \neg x \wedge \neg y$ *(the* Morgan's law*),*
3. $\neg\neg x = x$ *($\neg$ is an* involution*),*

for each $x, y \in A$.

Definition 25 (Kleene algebra). *[30] A* Kleene algebra $(A, \wedge, \vee, \neg, 0, 1)$ *is a De Morgan algebra such that the following property, called* Kleene property*, holds:*

$$x \wedge \neg x \leq y \vee \neg y \tag{3}$$

for each $x, y \in A$.

Kleene algebras are also called *normal i-lattices* by Kalman.

Example 4. The structure $(\{0, \frac{1}{2}, 1\}, \wedge, \vee, \neg, 0, 1)$ is a three-element Kleene algebra, where $\wedge$ and $\vee$ are respectively the Kleene conjunction and implication defined in Section 2.2, and $\neg x = 1 - x$ for each $x \in \{0, \frac{1}{2}, 1\}$.

Example 5. Let C be a partition of the finite universe U, and let O_C be the set of all orthopairs generated by C. Then, the structure

$$(O_C, \wedge_{\mathcal{K}}, \vee_{\mathcal{K}}, \neg, (\emptyset, U), (U, \emptyset))$$

is a Kleene algebra, where $\wedge_{\mathcal{K}}$ and $\vee_{\mathcal{K}}$ are given in Definition 8, and $\neg(A, B) = (B, A)$ for each $(A, B) \in O_C$.

We are interested in the family of *finite centered Kleene algebras with the interpolation property*, that are explored in [31].

From now on, we denote an algebraic structure having support A with $\mathbb{A}$.

Definition 26 (Centered Kleene algebra). *A Kleene algebra $\mathbb{A}$ is a* centered Kleene algebra *if there exists $c \in A$ such that $c = \neg c$. The element c is called* center *of A.*

By using the Kleene property (see Definition 25), it is easy to prove that if c is a center of A, then it is unique.

The following notion was introduced for the first time by Monteiro [78].

Definition 27. *Let $(A, \wedge, \vee, \neg, 0, 1)$ be a centered Kleene algebra. Let c be the center of A. We say that A has the* interpolation property *if and only if for every $x, y \geq c$ such that $x \wedge y \leq c$ there exists z such that $z \vee c = x$ and $\neg z \vee c = y$.*

In [27] the above definition is called (CK) property, but it is also noticed that it coincides with the interpolation property described in [31], so we will use this last name. Not every centered Kleene algebra has the interpolation property, see Example 5 in [27].

Definition 28. *As in [31], let $(A, \wedge, \vee, \neg, 0, 1)$ be a Kleene algebra, we set*

$$A^+ = \{x \in A \mid \neg x \leq x\} \quad \textit{and} \quad A^- = \{x \in A \mid x \leq \neg x\}.$$

We call A^+ and A^- positive *and* negative *cone, respectively.*

We can observe that the structure $(A^+, \wedge, \vee)$ is a sublattice of $(A, \wedge, \vee)$ containing 1, and dually, $(A^-, \wedge, \vee)$ is a sublattice of $(A, \wedge, \vee)$ containing 0.

Kalman construction The following construction is due to Kalman [64]. Let $(L, \wedge, \vee, 0, 1)$ be a bounded distributive lattice, we consider

$$\mathsf{K}(L) = \{(x, y) \in L \times L \mid x \wedge y = 0\} \tag{4}$$

and the operations $\sqcap$, $\sqcup$ and $\neg$ defined on $\mathsf{K}(L)$ as follows:

$$(x, y) \sqcap (u, v) = (x \wedge u, y \vee v) \tag{5}$$

$$(x, y) \sqcup (u, v) = (x \vee u, y \wedge v) \tag{6}$$

$$\neg(x, y) = (y, x) \tag{7}$$

for each $(x, y), (u, v) \in \mathsf{K}(L)$. Then,

$$\mathbb{K}(L) = (\mathsf{K}(L), \sqcap, \sqcup, \neg, (0,1), (1,0)) \tag{8}$$

is a centered Kleene algebra, with center $(0,0)$. Moreover,

$$\mathsf{K}(L)^+ = \{(x,0) \mid x \in L\} \ \text{ and } \ \mathsf{K}(L)^- = \{(0,x) \mid x \in L\}.$$

The following theorem, proved by Cignoli [31] states that centered Kleene algebras with the interpolation property are represented by bounded distributive lattices.

Theorem 6. *A Kleene algebra $\mathbb{A}$ is isomorphic to $\mathbb{K}(L)$ for some bounded distributive lattice $\mathbb{L}$ if and only if $\mathbb{A}$ is centered and satisfies the interpolation property. In this case $\mathbb{L}$ is isomorphic to the lattice $\mathbb{A}^+$.*

By Birkhoff representation theorem and by Theorem 6, the following result holds.

Theorem 7. *A Kleene algebra $\mathbb{A}$ is isomorphic to $\mathbb{K}(Up(P))$, for some partially ordered set $(P, \leq)$, if and only if $\mathbb{A}$ is centered and satisfies the interpolation property. In this case $(Up(P), \cap, \cup, \emptyset, P)$ is isomorphic to the lattice $\mathbb{A}^+$.*

Remark 8. Trivially, $\mathsf{K}(Up(P))$ is the set of all pairs of disjoint upsets of P, and the operations 5 and 6 are the following: let $(X^1, X^2), (Y^1, Y^2) \in \mathsf{K}(Up(P))$, then

$$(X^1, X^2) \sqcap (Y^1, Y^2) = (X^1 \cap Y^1, X^2 \cup Y^2), \tag{9}$$

$$(X^1, X^2) \sqcup (Y^1, Y^2) = (X^1 \cup Y^1, X^2 \cap Y^2). \tag{10}$$

In this thesis, we focus on some structures having Kleene algebras as reduct. Namely, they are Nelson algebras, Nelson lattices, Kleene lattices with implication and IUML-algebras. Moreover, we will require that they are centered and satisfy the interpolation property.

Nelson algebras Nelson algebras were introduced by Rasiowa [89], under the name of N-lattices, as the algebraic counterparts of the constructive logic with strong negation considered by Nelson and Markov [90]. The centered Nelson algebras with the interpolation property are represented by Heyting algebras, that are defined as follows.

Definition 29 (Pseudo-complement). *[31] Let $(L, \wedge, \vee, 0, 1)$ be a bounded distributive lattice, and let $x, y \in L$. Then, the* pseudo-complement of x with respect to y*, denoted with $x \to y$, is an element of L satisfying the following proprieties:*

1. *$x \wedge x \to y \leq y$ and*
2. *if $x \wedge z \leq y$, then $z \leq x \to y$, for each $z \in L$.*

Notice that, given a bounded distributive lattice $(L, \wedge, \vee, 0, 1)$, the pseudo-complement of x with respect to y does not always exist.

Definition 30 (Heyting algebra). *An* Heyting algebra *is a structure*

$$(H, \wedge, \vee, \rightarrow, 0, 1),$$

where the reduct $(H, \wedge, \vee, 0, 1)$ *is a bounded residuated lattice, and* $x \rightarrow y$ *is the pseudo-complement of* x *with respect to* y *given in Definition 29.*

The next theorem affirms that there exists a one-to-one correspondence between finite Heyting algebras and finite partially ordered sets.

Theorem 8. *[15] For each finite Heyting algebra* $\mathbb{H}$*, there exists a finite poset* $(P, \leq)$ *such that* $\mathbb{H}$ *is isomorphic to* $(Up(P), \cap, \cup, \rightarrow_P, \emptyset, P)$*, where*

$$X \rightarrow_P Y = P \setminus \downarrow (X \setminus Y), \tag{11}$$

for each $X, Y \in Up(P)$.

Definition 31 (Quasi-Nelson algebra). *A* quasi-Nelson algebra *is a structure*

$$(A, \wedge, \vee, \neg, \Rightarrow, 0, 1)$$

such that

1. $(A, \wedge, \vee, \neg, 0, 1)$ *is a Kleene algebra, and*
2. *for each* $x, y \in A$*, the pseudo-complement of* x *with respect to* $\neg x \vee y$*, denoted with* $x \Rightarrow y$*, exists.*

Definition 32 (Nelson algebra). *A Nelson algebra is a quasi Nelson algebra* $(A, \wedge, \vee, \neg, \Rightarrow, 0, 1)$*, that satisfies the following property: let* $x, y, z \in A$

$$(x \wedge y) \Rightarrow z = x \Rightarrow (y \Rightarrow z).$$

Example 6. The structure $(\{0, \frac{1}{2}, 1\}, \wedge, \vee, \neg, \Rightarrow_{\mathcal{N}}, 0, 1)$, where $\neg x = 1 - x$ for each $x \in \{0, \frac{1}{2}, 1\}$, and $\Rightarrow_{\mathcal{N}}$ is the Nelson implication on $\{0, \frac{1}{2}, 1\}$ defined in Section 2.2, is a three-element Nelson algebra.

Example 7. Let C be a partition of the finite universe U, and let O_C be the set of all orthopairs generated by C. Then, the structure

$$(O_C, \wedge_{\mathcal{K}}, \vee_{\mathcal{K}}, \neg, \rightarrow_{\mathcal{N}}, (\emptyset, U), (U, \emptyset))$$

is a finite Nelson algebra, where $\rightarrow_{\mathcal{N}}$ is given in Definition 12.

Manuel M. Fidel [47] and Dimiter Vakarelov [103] have shown independently that if $(H, \wedge, \vee, \rightarrow, 0, 1)$ is an Heyting algebra, then $(\mathbb{K}(H), \Rightarrow)$, that is the structure $(\mathsf{K}(H), \sqcap, \sqcup, \neg, \Rightarrow, (\emptyset, H), (H, \emptyset))$, is a Nelson algebra, where

$$(x, y) \Rightarrow (u, v) = (x \rightarrow u, x \wedge v) \tag{12}$$

for each $(x, y), (u, v) \in \mathsf{K}(H)$.

Moreover, Cignoli [31] proved the following result.

Theorem 9. *A finite Nelson algebra $\mathbb{A}$ is isomorphic to $(\mathbb{K}(H), \Rightarrow)$ for some finite Heyting algebra $\mathbb{H}$ if and only if $\mathbb{A}$ is centered and satisfies the interpolation property.*

By Theorem 8, Equation 12 and Theorem 9, the following result holds.

Theorem 10. *Let $\mathbb{A}$ be a Nelson algebra. Then, $\mathbb{A}$ is a finite centered Nelson algebra with the interpolation property if and only if there exists a finite poset $(P, \leq)$ such that $A \cong (\mathbb{K}(Up(P)), \rightarrow_1)$, where*

$$(X^1, X^2) \rightarrow_1 (Y^1, Y^2) = (P \setminus \downarrow (X^1 \setminus Y^1), X^1 \cap Y^2), \tag{13}$$

for each $(X^1, X^2), (Y^1, Y^2) \in \mathsf{K}(Up(P))$.

Nelson lattices Nelson lattices are algebraic models of constructive logic with strong negation [101]. They are particular involutive residuated lattices. Moreover, finite centered Nelson lattices are represented by Heyting algebras.

Definition 33 (Involutive residuated lattice). *An* involutive residuated lattice *is a bounded, integral and commutative residuated lattice*

$$(A, \wedge, \vee, *, \rightarrow, e, 0, 1)$$

such that the operation $\neg$, defined by $\neg x = x \rightarrow 0$ for each $x \in A$, is an involution.

The operations $*$ and $\rightarrow$ of an involutive residuated lattice with support A can be obtained one from each other as follows: let $x, y \in A$, then

$$x * y = \neg(x \rightarrow \neg y) \tag{14}$$

and

$$x \rightarrow y = \neg(x * \neg y). \tag{15}$$

Definition 34 (Nelson lattice). *A* Nelson lattice *is an involutive residuated lattice*

$$(A, \wedge, \vee, *, \rightarrow, e, 0, 1),$$

where the following inequality holds: let $x^2 = x * x$,

$$(x^2 \rightarrow y) \wedge ((\neg y^2) \rightarrow \neg x) \leq x \rightarrow y,$$

for each $x, y \in A$.

Example 8. The structure $(\{0, \frac{1}{2}, 1\}, \wedge, \vee, \circledast_{\mathcal{L}}, \Rightarrow_{\mathcal{L}}, \frac{1}{2}, 0, 1)$ is a three-element Nelson lattice, where $\circledast_{\mathcal{L}}$ and $\Rightarrow_{\mathcal{L}}$ are respectively the Łukasiewicz conjunction and implication on $\{0, \frac{1}{2}, 1\}$ defined in Section 2.2.

Example 9. Let C be a partition of the finite universe U, and let O_C be the set of all orthopairs generated by C. Then, the structure

$$(O_C, \wedge_{\mathcal{K}}, \vee_{\mathcal{K}}, *_{\mathcal{L}}, \rightarrow_{\mathcal{L}}, (\emptyset, \emptyset), (\emptyset, U), (U, \emptyset)),$$

where $*_{\mathcal{L}}$ and $\rightarrow_{\mathcal{L}}$ are defined in Section 2.2, is a finite Nelson lattice.

Remark 9. Centered Nelson algebras and Nelson lattices are equationally equivalent, namely they are obtained one from the other as follows [23].
If $(A, \wedge, \vee, \neg, \Rightarrow, 0, 1)$ is a centered Nelson algebra, then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a Nelson lattice, where

$$x * y = \neg(x \Rightarrow \neg y) \vee \neg(y \Rightarrow \neg x) \quad \text{and} \quad x \rightarrow y = (x \Rightarrow y) \wedge (\neg y \Rightarrow \neg x),$$

for each $x, y, z \in A$. Vice-versa, if $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a Nelson lattice, then $(A, \wedge, \vee, \neg, \Rightarrow, 0, 1)$ is a centered Nelson algebra, where

$$\neg x = x \rightarrow 0 \quad \text{and} \quad x \Rightarrow y = x^2 \rightarrow y,$$

for each $x, y \in A$.

We can notice that if $(H, \wedge, \vee, \rightarrow, 0, 1)$ is an Heyting algebra, then

$$(\mathbb{K}(H), *, \Rightarrow),$$

where $(\mathbb{K}(H), *, \Rightarrow)$ denotes $(\mathsf{K}(H), \sqcap, \sqcup, *, \Rightarrow, (\emptyset, \emptyset), (\emptyset, H), (H, \emptyset))$, is a Nelson lattice, such that

$$(x, y) * (u, v) = (x \wedge u, (x \rightarrow v) \wedge (u \rightarrow y)) \tag{16}$$

and

$$(x, y) \Rightarrow (u, v) = ((x \rightarrow u) \wedge (v \rightarrow y), x \wedge v), \tag{17}$$

for each $x, y, u, v \in H$.

Finite centered Nelson lattices with the interpolation property are represented by finite Heyting algebras [27].

Theorem 11. *A finite Nelson lattice $\mathbb{A}$ is isomorphic to $(\mathbb{K}(H), *, \Rightarrow)$ for some finite Heyting algebra $\mathbb{H}$ if and only if $\mathbb{A}$ is centered and satisfies the interpolation property.*

By Theorem 8, Equation 16, Equation 17 and Theorem 11, the following result holds.

Theorem 12. *Let $\mathbb{A}$ be a Nelson lattice. Then, $\mathbb{A}$ is a finite centered Nelson lattice with the interpolation property if and only if there exists a finite poset $(P, \leq)$ such that $A \cong (\mathbb{K}(Up(P)), \star_2 \rightarrow_2)$, where*

$$(X^1, X^2) \star_2 (Y^1, Y^2) = (X^1 \cap Y^1, P \setminus (\downarrow (X^1 \setminus Y^2) \cup \downarrow (Y^1 \setminus X^2))), \tag{18}$$

$$(X^1, X^2) \rightarrow_2 (Y^1, Y^2) = (P \setminus (\downarrow (X^1 \setminus Y^1) \cup \downarrow (Y^2 \setminus X^2)), X^1 \cap Y^2), \tag{19}$$

for each $(X^1, X^2), (Y^1, Y^2) \in \mathsf{K}(Up(P))$.

IUML-algebras IUML-algebras are the algebraic counterpart of the logic IUML, which is a substructural fuzzy logic that is an axiomatic extension of the multiplicative additive intuitionistic linear logic MAILL [74]. IUML-algebras can also be defined as *bounded odd Sugihara monoids*, where a Sugihara monoid is the equivalent algebraic semantics for the relevance logic RM^t of R-mingle as formulated with Ackermann constants. In [49] a dual categorical equivalence is shown between IUML-algebras and suitable topological spaces defined starting from Kleene spaces. In this dissertation, we focus only on finite IUML-algebras refers to [3] and [74].

Definition 35 (IUML-algebra). *An* idempotent uninorm mingle logic algebra (IUML-algebra) *[75] is an idempotent commutative bounded residuated lattice*

$$(A, \wedge, \vee, *, \rightarrow, e, \bot, \top),$$

satisfying the following properties:

1. $(x \rightarrow y) \vee (y \rightarrow x) \geq e$, *and*
2. $(x \rightarrow e) \rightarrow e = x$,

for every $x, y \in A$.

In any IUML-algebra, if we define the unary operation $\neg$ as $\neg x = x \rightarrow e$, then $\neg\neg x = x$ ($\neg$ is involutive) and $x \rightarrow y = \neg(x * \neg y)$.

Example 10. The structure $(\{0, \frac{1}{2}, 1\}, \wedge, \vee, \circledast_{\mathcal{S}}, \Rightarrow_{\mathcal{S}}, \frac{1}{2}, 0, 1)$ is a three-element IUML-algebra, , where $\circledast_{\mathcal{S}}$ and $\Rightarrow_{\mathcal{S}}$ are respectively the Sobociński conjunction and implication on $\{0, \frac{1}{2}, 1\}$ defined in Section 2.2.

Example 11. Let C be a partition of the finite universe U, and let O_C be the set of all orthopairs generated by C. Then, the structure

$$(O_C, \wedge_{\mathcal{K}}, \vee_{\mathcal{K}}, *_{\mathcal{S}}, \rightarrow_{\mathcal{S}}, (\emptyset, \emptyset), (\emptyset, U), (U, \emptyset)),$$

where $*_{\mathcal{S}}$ and $\rightarrow_{\mathcal{S}}$ are defined in Section 2.2, is a finite IUML-algebra.

Moreover, in [3] a dual categorical equivalence is described between finite forests F with order preserving open maps and finite IUML-algebras with homomorphisms.

Definition 36. *For any finite forest F, we consider $\mathsf{K}(Up(F))$, that is the set of pairs of disjoint upsets of F (it is the set defined by 4 starting from the lattice $(Up(F), \cap, \cup, \emptyset, F)$, and we define the following operations: if (X^1, X^2) and (Y^1, Y^2) belong to $\mathsf{K}(Up(F))$, we set:*

$$(X^1, X^2) \star_3 (Y^1, Y^2) = ((X^1 \cap Y^1) \cup (X \diamond Y), (X^2 \cup Y^2) \setminus (X \diamond Y)) \tag{20}$$

where, for each $U = (U^1, U^2), V = (V^1, V^2) \in \mathsf{K}(Up(F))$, letting
$U^0 = F \setminus (U^1 \cup U^2)$, we set

$$U \diamond V = \uparrow((U^0 \cap V^1) \cup (V^0 \cap U^1)).$$

$$(X^1, X^2) \rightarrow_3 (Y^1, Y^2) = \neg((X^1, X^2) \star_3 (Y^2, Y^1)). \tag{21}$$

Theorem 13. *[3] For every finite forest F, the structure*

$$(\mathbb{K}(Up(F)), \star_3, \to_3) = (\mathsf{K}(Up(F)), \sqcap, \sqcup, \star_3, \to_3, (\emptyset, \emptyset), (\emptyset, F), (F, \emptyset))$$

is an IUML-algebra. Vice-versa, for each finite IUML-algebra $\mathbb{A}$ there is a finite forest F_A such that $\mathbb{A}$ is isomorphic with $(\mathbb{K}(Up(F_A)), \star_3, \to_3)$.

Kleene lattices with implication Kleene lattices with implication are a class of Kleene algebras where an additional operation of implication can be defined in such a way to make them DLI-algebras, (i.e. *algebras with implication*). The latter generalize the Heyting algebras and are defined in [28].

Definition 37 (DLI-algebra). *A* DLI-algebra *is a structure*

$$(H, \vee, \wedge, \to, 0, 1),$$

where $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the following properties hold: let $x, y, z \in A$

1. $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$,
2. $(x \to z) \wedge (y \to z) = (x \vee y) \to z$,
3. $0 \to x = 1$,
4. $x \to 1 = 1$.

*Furthermore, a DLI^+-*algebra *is a DLI-algebra $(H, \vee, \wedge, \to, 0, 1)$ where the following inequality holds: $a \wedge (a \to b) \leq b$, for each $a, b \in H$.*

It is easy to prove that each Heyting algebra is also a DLI^+-algebra.

Definition 38 (DLI*-algebra). *A DLI^*-*algebra *is a structure*

$$(H, \wedge, \vee, \to, 0, 1),$$

where $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\to$ is defined as follows: let $x, y \in H$,

$$x \to y = \begin{cases} 1 & \text{if } x = 0, \\ y & \text{if } x \neq 0. \end{cases} \tag{22}$$

Proposition 1. *A DLI^*-algebra is a DLI^+-algebra.*

By Theorem 5, the following result holds.

Theorem 14. *The structure $(H, \wedge, \vee, \to, 0, 1)$ is a DLI^*-algebra if and only if $\mathbb{H} \cong (Up(P), \cap, \cup, \to_P^*, \emptyset, P)$, where*

$$X \to_P^* Y = \begin{cases} P & \text{if } X = \emptyset, \\ Y & \text{if } X \neq \emptyset, \end{cases} \tag{23}$$

for each $X, Y \in P$.

Definition 39 (Kleene lattice with implication). *A* Kleene lattice with implication *is a structure*

$$(A, \wedge, \vee, \neg, *, \rightarrow, 0, 1)$$

such that $(A, \wedge, \vee, \neg, 0, 1)$ *is a centered Kleene algebra and the following conditions hold: let* c *be the center of* A *and let* $x, y \in A$

1. $(A, \wedge, \vee, \rightarrow, 0, 1)$ *is a DLI-algebra,*
2. $(x \wedge (x \rightarrow y)) \vee c \leq y \vee c$*,*
3. $c \rightarrow c = 1$*,*
4. $(x \rightarrow y) \wedge c = (\neg x \vee y) \wedge c$*,*
5. $(x \rightarrow \neg y) \vee c = ((x \rightarrow (\neg x \vee c)))$*.*

By equation 14, we can define the operation $*$ *from* $\rightarrow$*. Vice-versa, by equation 15,* $\rightarrow$ *is obtained from* $*$*.*

It is easy to prove that each Nelson algebra is also a Kleene lattice with implication.

Let $(H, \wedge, \vee, \rightarrow, 0, 1)$ be a DLI^+-algebra, then $(\mathbb{K}(H), \star, \Rightarrow)$ is a Kleene lattice with implication, where $\Rightarrow$ is defined by 17 and $x \star y = \neg(x \Rightarrow \neg y)$. Moreover, the following theorem holds.

Theorem 15. *A Kleene lattice with implication* A *is isomorphic to the structure* $(\mathbb{K}(H), \star, \Rightarrow)$ *for some* DLI^+*-algebra* $\mathbb{H}$ *if and only if it has the interpolation property.*

Definition 40 (KLI*-algebra). *A* KLI*-algebra *is a structure*

$$(A, \wedge, \vee, \neg, *, \rightarrow, 0, 1),$$

where $(A, \wedge, \vee, \neg, 0, 1)$ *is a centered Kleene algebra and the operations* $*$ *and* $\rightarrow$ *are defined as follows: let* c *be the center of* A*, and let* $x, y \in A$

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq c \text{ and } y \geq c; \\ \neg x, & \text{if } x \leq c \text{ and } y \ngeq c; \\ y, & \text{if } x \nleq c \text{ and } y \geq c; \\ ((y \vee c) \wedge \neg x) \vee ((\neg x \vee c) \wedge y), & \text{if } x \nleq c \text{ and } y \ngeq c; \end{cases} \tag{24}$$

and $x * y = \neg(x \rightarrow y)$*.*

Proposition 2. *[27] A KLI*-algebra is a Kleene lattice with implication.*

The next result follows by Theorem 14 and Theorem 15.

Theorem 16. *The structure* $(A, \wedge, \vee, \neg, *, \rightarrow, 0, 1)$ *is a KLI*-algebra with the interpolation property if and only if* $\mathbb{A} \cong (\mathbb{K}(Up(P)), \star_4, \rightarrow_4)$*, where* $\star_4$ *and* $\rightarrow_4$ *are defined as follows.*

$$(X^1, X^2) \star_4 (Y^1, Y^2) = \begin{cases} (\emptyset, P), & \text{if } X^1 = \emptyset \text{ and } Y^1 = \emptyset; \\ (X^1, X^2), & \text{if } X^1 = \emptyset \text{ and } Y^1 \neq \emptyset; \\ (Y^1, Y^2), & \text{if } X^1 \neq \emptyset \text{ and } Y^1 = \emptyset; \\ (X^1 \cap Y^1, X^2 \cap Y^2), & \text{if } X^1 \neq \emptyset \text{ and } Y^1 \neq \emptyset; \end{cases} \tag{25}$$

and

$$(X^1, X^2) \rightarrow_4 (Y^1, Y^2) = \begin{cases} (P, \emptyset), & \text{if } X^1 = \emptyset \text{ and } Y^2 = \emptyset; \\ (X^2, X^1), & \text{if } X^1 = \emptyset \text{ and } Y^2 \neq \emptyset; \\ (Y^1, Y^2), & \text{if } X^1 \neq \emptyset \text{ and } Y^2 = \emptyset; \\ (Y^1 \cap X^2, X^1 \cap Y^2), & \text{if } X^1 \neq \emptyset \text{ and } Y^2 \neq \emptyset; \end{cases} \tag{26}$$

for each $(X^1, X^2), (Y^1, Y^2) \in \mathsf{K}(Up(P))$.

3 Sequences of refinements of orthopairs

Mathematical objects are not so directly given as physical objects. They are something between the ideal world and the empirical world.

Kurt Gödel

In this chapter, we introduce the definition of *refinement sequences of partial coverings* as special sequences of coverings representing situations where new information is gradually provided on ever smaller sets of objects. We provide examples of environments in which refinement sequences arise; in detail, we obtain refinement sequences starting from incomplete information tables and formal contexts. We identify some families of sequences considering how much the blocks of their coverings overlap. We identify refinement sequences as partially ordered sets. Moreover, we introduce the notion of *sequences of orthopairs*, in order to generalize the rough set theory. We represent each sequence of orthopairs as a pair of disjoint upsets of a partially ordered set, or equivalently, as a labelled poset. Finally, we provide a theorem that is fundamental to prove the results of Chapter 4. Preliminary versions of this chapter appeared in [1] [19] [20] [2].

3.1 Refinement sequences

In this section, we introduce the notion of *refinement sequence* of a universe.

Refinement sequences are special sequences of partial coverings of a given universe (a partial covering of U is a subset of 2^U, i.e. any set of subsets of U [35]). More precisely, the refinements sequences are defined as follows.

Definition 41. *A sequence $\mathcal{C} = (C_1, \dots, C_n)$ of partial coverings of U is a* refinement sequence *of U if each element of C_i is contained in an element of C_{i-1}, for $i = 2, \dots, n$.*

For simplicity, we omit to specify on which universe the refinement sequence is defined, when it is clear.

Example 12. Suppose that $U = \{a, b, c, d, e, f, g\}$ and that C_1 and C_2 are partial coverings of U respectively defined as follows:

- $C_1 = \{\{a, b, c, d\}, \{d, e, f, g\}\}$;
- $C_2 = \{\{a, b, c\}, \{c, d\}, \{d, e\}, \{f, g\}\}$.

Then, (C_1, C_2) is a refinement sequence of U.

Remark 10. We notice that a partial covering of U naturally defines a tolerance relation on a subset of U and the vice-versa also holds. Moreover, we call blocks both the elements of a partial covering and the tolerance classes. Therefore, a refinement sequence $(C_1, \dots, C_n)$ of partial coverings of U corresponds to a sequence $(R_1, \dots, R_n)$ of tolerance relations respectively defined on the subsets $U_1, \dots, U_n$ of U, where

– U_i is the union of the blocks of C_i, for each $i \in \{1, \ldots, n\}$;
– $U_i \subseteq U_j$, for each $j \leq i$;
– $R_i(u) \subseteq R_j(u)$, for each $j \leq i$ and $u \in U_i$.

In this thesis, we also consider refinement sequences of *partial partitions* of a universe, where a partition corresponds to an equivalence relation, and it is a covering such that its blocks are disjoint with each others.

As shown in the following example, the refinement sequences can be used for ontology construction.

Example 13. Suppose to start from a set of rocks (first covering) and then to specify our interest in *magmatic rocks* and *sedimentary rocks* that form a partial covering of the initial set of rocks (the latter also contains several elements that are metamorphic rocks, then the covering made of magmatic and sedimentary rocks is partial). Then, we intend to refine such classification by considering two groups of *magmatic rock* (*intrusive rocks* and *extrusive rocks*) and two groups of *sedimentary rocks* (*Chemical rocks* and *Clastic rocks*). The refinement sequence of partial coverings can be represented as follows.

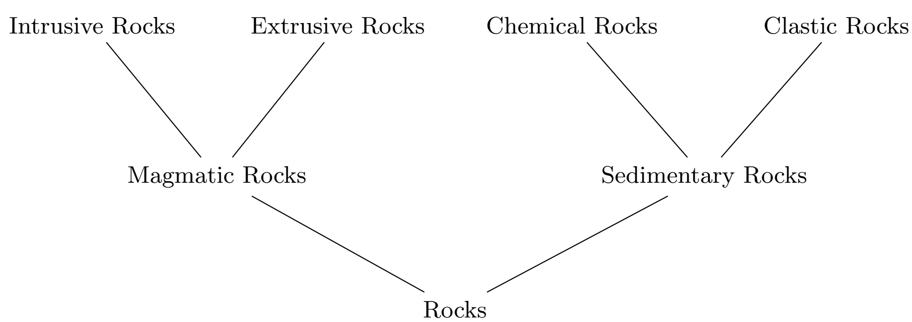

Fig. 3: Refinement sequence for rocks classification

The next example shows that a refinement sequence corresponds to an *incomplete information table*. The latter is a table where a set of objects is described by several attributes, but some data may be missing.

Example 14. Suppose that we have information about 22 users of Facebook, labelled with $u_1, \ldots, u_{22}$. In particular, we focus on information related to the place where each user declares to come from on its personal profile.

The available data are organized in the *information table* as in Table 9, (see [67]) where $U = \{u_1, \ldots, u_{22}\}$ is the universe and $\{Country, Region, City\}$ is the set of attributes.

	Country	Region	City		Country	Region	City
u_1	Italy	$\times$	$\times$	u_{12}	France	Brittany	Rennes
u_2	Italy	Lombardy	Varese	u_{13}	France	Brittany	Rennes
u_3	Italy	Lombardy	Varese	u_{14}	France	Brittany	$\times$
u_4	Italy	Lombardy	Milan	u_{15}	France	Brittany	$\times$
u_5	Italy	Lombardy	Milan	u_{16}	France	Grand Est	Strasbourg
u_6	Italy	Lombardy	Pavia	u_{17}	France	Grand Est	Strasbourg
u_7	Italy	Lombardy	Pavia	u_{18}	France	Grand Est	Mets
u_8	Italy	Campania	Naples	u_{19}	France	Grand Est	Mets
u_9	Italy	Campania	Naples	u_{20}	France	Grand Est	$\times$
u_{10}	Italy	Campania	$\times$	u_{21}	France	Grand Est	$\times$
u_{11}	Italy	Campania	$\times$	u_{22}	France	$\times$	$\times$

Table 9: Information table of the users

Observe that there are three equivalence relations between users determined respectively by considering users coming from the same country or the same region or the same city[1]. They are the so-called *indiscernibility relations* of Table 9 [67]. Moreover, their respective partial coverings (that are also partial partitions) are $C_1 = \{\{u_1, \ldots, u_{11}\}, \{u_{12}, \ldots, u_{22}\}\}$ (classes are sets of users coming from the same country); $C_2 = \{\{u_2, \ldots, u_7\}, \{u_8, \ldots, u_{11}\}, \{u_{12}, \ldots, u_{15}\}, \{u_{16}, \ldots, u_{21}\}\}$ (classes are set of users coming from the same region) and $C_3 = \{\{u_2, u_3\}, \{u_4, u_5\}, \{u_6, u_7\}, \{u_8, u_9\}, \{u_{12}, u_{13}\}, \{u_{16}, u_{17}\}, \{u_{18}, u_{19}\}\}$ (classes are set of users coming from the same city). It easy to see that $\mathcal{C} = (C_1, C_2, C_3)$ is a refinement sequence of U.

Refinement sequences and formal context There is a close connection between refinement sequences and *formal contexts*, which are mathematical structures used in *Formal Concept Analysis* and *Fuzzy Formal Concept Analysis* [50] [26]. A formal context is a triple (X, Y, I), where X is a set of objects, Y is a set of attributes, and I is a binary relation between X and Y. If I is a fuzzy relation, then (X, Y, I) is called *fuzzy formal context*, and $I(x, y)$ expresses the degree wherewith the object x has the attribute y. A formal context can be represented by a table with rows corresponding to objects, columns corresponding to attributes, and table entries containing each degree $I(x, y)$, with $x \in X$ and $y \in Y$. In particular, it is clear that if I is an ordinary relation, the table entries only contain the degrees 0 and 1. By using several techniques [10] [17], *formal*

[1] The equivalence relations *coming from the same region* and *coming from the same city* are defined on proper subsets of U, for there are missing data for some users.

concepts are extracted from every formal context. Formal concepts are particular clusters which represent natural human-like concepts such as "organism living in water", "car with all wheel drive system", etc.

Given a refinement sequence $\mathcal{C} = (C_1, \ldots, C_n)$, we can view a block b of C_i as the set of all elements of U that have a specific attribute y_b. Thus, $\mathcal{C}$ corresponds to a formal context $(U, Y_{\mathcal{C}}, I)$, where $Y_{\mathcal{C}} = \cup\{y_b \mid b \in C_i \text{ and } i \in \{1, \ldots, n\}\}$ and "$(u, y_b) \in I$ if and only if $u \in b$". For example, let $\mathcal{C} = (C_1 = \{b_1, b_2\}, C_2 = \{b_3, b_4, b_5\})$ be the refinement sequence of $\{a, b, c, d, e, f, g\}$ such that $b_1 = \{a, b, c\}, b_2 = \{d, e, f, g\}$, $b_3 = \{a, b, c\}, b_4 = \{d, e\}$ and $b_5 = \{f, g\}$. Then, the formal context associated to $\mathcal{C}$ is represented by Table 10.

I	y_{b_1}	y_{b_2}	y_{b_3}	y_{b_4}	y_{b_5}
a	1	0	1	0	0
b	1	0	1	0	0
c	1	0	1	0	0
d	0	1	0	1	0
e	0	1	0	1	0
f	0	1	0	0	1
g	0	1	0	0	1

Table 10: Formal context of $\mathcal{C}$

Vice-versa, starting from a formal context, we can build a refinement sequence as follows. For each $y \in Y$, we set $b_y = \{x \in X \mid (x, y) \in I\}$. Let $s = |Y|$, if $s = 1$, then the refinement sequence assigned to (X, Y, I) is trivially made of only one covering. Suppose that $s > 1$, then we set $C_s = \{b_y \mid b_{y'} \not\subset b_y, \text{ for each } y' \in Y\}$ and, let $i < s$, $C_i = \{b_y \mid \text{ there exists } b_{y'} \in C_{i+1} \text{ such that } b_{y'} \subset b_y \text{ and } b_{y'} \subset b_{y''} \subset b_y \text{ does not hold for each } y'' \in Y\}$. Therefore, $\mathcal{C} = (C_k, C_{k+1}, \ldots, C_s)$ is the refinement sequence assigned to (X, Y, I), where $k = max\{i \in \{1, \ldots, s - 1\} \mid C_i \neq C_{i+1}\}$. For example, we consider the formal context

$$K = (\{a_1, a_2, a_3, a_4, a_5\}, \{\text{feline, cat, tiger}\}, I),$$

where $\{a_1, a_2, a_3, a_4, a_5\}$ represents a set of 5 animals and I is defined by Table 11.

Then, the refinement sequence assigned to K is made of coverings C_1 and C_2 such that $C_1 = \{\{a_1, a_2, a_4, a_5\}\} = \{\text{animals that are felines}\}$ and $C_2 = \{\{a_1, a_2\}, \{a_4, a_5\}\} = \{\{\text{animals that are cats}\}, \{\text{animals that are tigers}\}\}$.

I	$feline$	cat	$tiger$
a_1	1	1	0
a_2	1	1	0
a_3	0	0	0
a_4	1	0	1
a_5	1	0	1

Table 11: Formal context K

3.2 Refinement sequences as Posets

In this section, we show that each refinement sequence is represented as a partially ordered set.

Definition 42. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a refinement sequence of U. We assign the partially ordered set $(P_{\mathcal{C}}, \leq_{\mathcal{C}})$ to $\mathcal{C}$, where:*

- *$P_{\mathcal{C}} = \bigcup_{i=1}^{n} C_i$ (the set of nodes is the set of all subsets of U belonging to the coverings $C_1, \ldots, C_n$), and*
- *$N \leq_{\mathcal{C}} M$ if and only if $M \subseteq N$, for $N, M \in P_{\mathcal{C}}$ (the partial ordered relation is the reverse inclusion between sets).*

Example 15. Let (C_1, C_2, C_3) be a refinement sequence of $\{a, b, c, d, e, f, g, h\}$, where

- $C_1 = \{\{a, b, c, d, e, f, g\}\}$,
- $C_2 = \{\{a, b, c, d\}, \{c, d, e, f\}\}$ and
- $C_3 = \{\{c, d\}, \{d, e, f\}\}$.

The poset assigned to (C_1, C_2, C_3) is shown in the following figure.

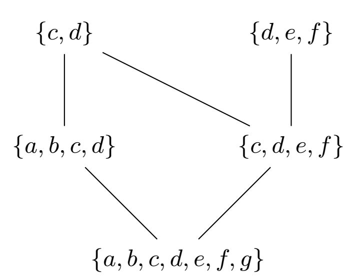

Fig. 4: Poset assigned to (C_1, C_2, C_3)

Proposition 3. *If $\mathcal{C}$ is a refinement sequence of partial partitions of U, then $(P_{\mathcal{C}}, \leq_{\mathcal{C}})$ is a forest.*

Proof. Let $N, M \in \downarrow X$, with $X \in P_{\mathcal{C}}$. Then, $N, M \leq_{\mathcal{C}} X$. By Definition 42, $X \subseteq N \cap M$. Suppose that $N \in C_i$ and $M \in C_j$, with $i \leq j$. By Definition 41, there exists $\tilde{N} \in C_j$ such that $\tilde{N} \subseteq N$. Since C_j is a partial partition of U, we have that $\tilde{N} = M$ or $\tilde{N} \cap M = \emptyset$. On the other hand, both M and $\tilde{N}$ contain X. Consequently, $\tilde{N} = M$ and so $N \leq_{\mathcal{C}} M$.

Example 16. If $\mathcal{C}$ is the refinement sequence of Example 14, then $(P_{\mathcal{C}}, \leq_{\mathcal{C}})$ is the following forest.

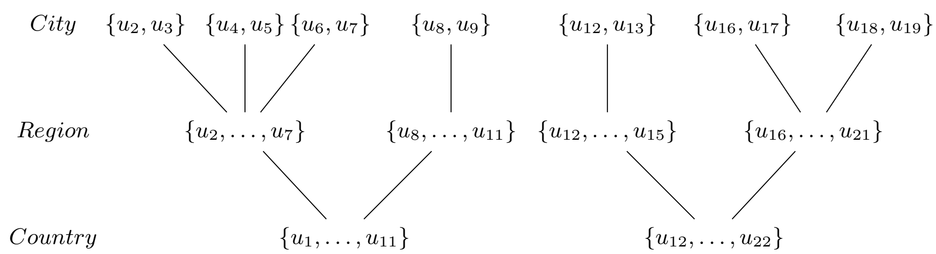

Fig. 5: Forest of the users

Remark 11. The maximal and minimal elements of $(P_{\mathcal{C}}, \leq_{\mathcal{C}})$ are all blocks of C_n and C_1, respectively.

Remark 12. The main difference between $\mathcal{C} = (C_1, \ldots, C_n)$ and the partially ordered set $P_{\mathcal{C}}$ is that the coverings $C_1, \ldots, C_n$ can also contain the same blocks, while each block appears only once in $P_{\mathcal{C}}$. For example, consider the refinement sequence $\mathcal{C} = (C_1, C_2)$ such that $C_1 = \{\{a, b\}, \{b, c, d, e\}\}$ and $C_2 = \{\{a, b\}, \{c, d\}\}$, then $P_{\mathcal{C}}$, that is represented by the following figure, has only one block $\{a, b\}$.

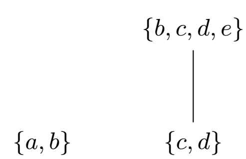

Fig. 6: Poset assigned to (C_1, C_2)

Remark 13. Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a refinement sequence of partial partitions of U and let $N \in C_i$, the successors of N are the nodes of C_{i+1} that are included in N if and only if $N \notin C_{i+1}$. More precisely, the successors of N are the blocks of C_j included in N, such that $j = min\{k > i \mid N \notin C_k\}$.

3.3 Some properties of refinement sequences

Now, we introduce several properties that a refinement sequence could have; so, we define what does it mean that a refinement sequence is *complete*, *safe* and *pairwise overlapping*. Given a refinement sequence $\mathcal{C}$, we denote by $\mathsf{K}(\mathcal{C})$ the set made of the pairs of disjoint upsets of $P_{\mathcal{C}}$. We notice that $\mathsf{K}(\mathcal{C})$ coincides with the set $\mathsf{K}(Up(P_{\mathcal{C}}))$ given by 4 (see Section 2.3) starting from the lattice $(Up(P_{\mathcal{C}}), \cap, \cup, \emptyset, P)$.

Definition 43. *A refinement sequence $\mathcal{C}$ of a universe U is* complete *if and only if*

$$\bigcup_{N \in A} N \ \cap \ \bigcup_{N \in B} N = \emptyset \tag{27}$$

for each pair (A, B) of $\mathsf{K}(\mathcal{C})$.

If the pair (A, B) belongs to $\mathsf{K}(\mathcal{C})$, and it satisfies the condition 27, then we say that "(A, B) is a pair of *totally disjoint* upsets of $P_{\mathcal{C}}$" and "A and B are *totally disjoint* from each other".

Example 17. Let $\mathcal{C} = (C_1, C_2, C_3)$ be a refinement sequence of the universe $\{a, b, c, d, e, f\}$, where

- $C_1 = \{\{a, b, c, d, e, f\}\}$,
- $C_2 = \{\{a, b, c, d\}, \{d, e, f\}\}$ and
- $C_3 = \{\{a, b\}\}$.

Also, we consider the sets $A^1 = \{\{a, b, c, d\}, \{a, b\}\}$ and $A^2 = \{\{d, e, f\}\}$, which are upsets of $P_{\mathcal{C}}$, and they are pairwise disjoint. We have that $\{d\}$ is the intersection between $\{a, b, c, d\} \cup \{a, b\}$ (the blocks of A^1) and $\{d, e, f\}$ (the only block of A^2). Indeed, the refinement sequence $\mathcal{C}$ is not complete.

Example 18. The refinement sequence of $\{a, b, c, d, e, f, g\}$ represented by the following forest is complete.

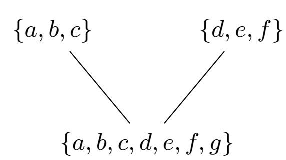

Fig. 7: Complete refinement sequence

Proposition 4. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a refinement sequence of U. If $C_1, \ldots, C_n$ are partial partitions of U, then $\mathcal{C}$ is complete.*

Proof. Let A^1 and A^2 be upsets of $P_{\mathcal{C}}$ such that $A^1 \cap A^2 = \emptyset$. Suppose that $b_1 \in A^1 \cap C_i$ and $b_2 \in A^2 \cap C_j$ with $i \leq j$. By Definition 41, there exists $\tilde{b}_2 \in C_i$ with $b_2 \subseteq \tilde{b}_2$. Since C_i is a partial partition, $b_1 \cap \tilde{b}_2 = \emptyset$ or $b_1 = \tilde{b}_2$. The equality $b_1 = \tilde{b}_2$ implies $b_2 \in A_1 \cap A_2$ which can not occur (A_2 is an upset). Consequently, $b_1 \cap \tilde{b}_2 = \emptyset$ and so $b_1 \cap b_2 = \emptyset$.

On the other hand, there exist complete refinement sequences made of coverings that are not partitions (see the following example).

Example 19. Let $\mathcal{C} = (C_1, C_2, C_3)$ be the refinement sequence of the universe $\{a, b, c, d, e, f, g\}$ such that

- $C_1 = \{\{a, b, c, d, e\}, \{f, g\}\}$,
- $C_2 = \{\{a, b, c\}, \{a, b, d\}, \{f, g\}\}$ and
- $C_3 = \{\{a, b\}, \{f, g\}\}$.

Then, $\mathcal{C}$ is complete.

Definition 44. *A refinement sequence $\mathcal{C}$ is* safe *if for each $N \in P_{\mathcal{C}}$ such that $N \subseteq N_1 \cup \ldots \cup N_r$ with $N_1, \ldots, N_r \in P_{\mathcal{C}}$, there exists $j \in \{1, \ldots r\}$ such that $N \subseteq N_j$.*

Therefore, given a safe refinement sequence $\mathcal{C}$, each node N of $P_{\mathcal{C}}$ is not included in the union of some other nodes of $P_{\mathcal{C}}$ that are all greater than N or disjoint with N.

The followings are two examples of refinement sequence: the first one is safe and the second one is not safe.

Example 20. Suppose that

$$C_1 = \{\{a, b, c, d, e\}, \{a, f, g, h\}\} \text{ and } C_2 = \{\{a, b, c\}, \{c, d\}, \{f, g\}\},$$

then the refinement sequence $\mathcal{C} = (C_1, C_2)$ is safe.

Example 21. The refinement sequence $(\tilde{C}_1, \tilde{C}_2)$ with

$$\tilde{C}_1 = \{\{a, b, c, d, e\}, \{c, d, e, f, g, h\}\} \text{ and } \tilde{C}_2 = \{\{a, b, c\}, \{c, d\}, \{e, f, g\}\},$$

is not safe, since $\{a, b, c, d, e\} \subseteq \{a, b, c\} \cup \{c, d\} \cup \{e, f, g\}$.

The next remark provides a condition that all nodes of $P_{\mathcal{C}}$ must satisfy so that the complete refinement sequence $\mathcal{C}$ is also safe.

Remark 14. By Definition 44, if $\mathcal{C}$ is safe and $N \in P_{\mathcal{C}}$, then there exists $x \in N$ such that $x \notin M$, for each $M \in P_{\mathcal{C}} \setminus \downarrow \{N\}$.

The following proposition yields a condition on nodes of $P_{\mathcal{C}}$, so that a complete refinement sequence $\mathcal{C}$ is also safe.

Proposition 5. *Let $\mathcal{C}$ be a complete refinement sequence of U. $\mathcal{C}$ is safe if and only if each node of $P_{\mathcal{C}}$ is not included in the union of its successors.*

Proof. ($\Rightarrow$). This implication is trivial, and it holds true even without the assumption that $\mathcal{C}$ is complete.

($\Leftarrow$). Suppose that $N \in P_{\mathcal{C}}$ and $N \subseteq N_1 \cup \ldots \cup N_r$, with $N_1, \ldots, N_r \in P_{\mathcal{C}}$ and $N_i \cap N \neq \emptyset$ for each $i \in \{1, \ldots, r\}$. Since $\mathcal{C}$ is complete, $N_i \subseteq N$ or $N \subseteq N_i$, for each $i \in \{1, \ldots, n\}$. By hypothesis, there exists $\tilde{N} \in \{N_1, \ldots, N_r\}$ such that $N_i \not\subseteq N$. Then, $N \subseteq N_i$.

By Proposition 4, we can say that a refinement sequence of partial partitions is safe if and only if each node of the respective forest is not equal the union of its successors.

Definition 45. *A refinement sequence $\mathcal{C} = C_1, \ldots, C_n$ is* pairwise overlapping *if there are not disjoint blocks in C_i, for each $i \in \{1, \ldots, n\}$.*

Example 22. The refinement sequence of Examples 15 is pairwise overlapping, since the element d belongs to each block of C_1, C_2 and C_3.

A pairwise overlapping refinement sequence differs more from the sequences of partial partitions than the other refinement sequences. Furthermore, refinement sequences of partial partitions are pairwise overlapping if and only if the forests assigned with them are chains.

We also notice that refinement sequences that are associated to forests are not complete, when are pairwise overlapping. As a consequence, a complete refinement sequence cannot also be pairwise overlapping.

3.4 Sequences of refinements of orthopairs

The main aim of this section is to define sequences of refinements of orthopairs.

Definition 46. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a refinement sequence of U and $X \subseteq U$. The* sequence of refinements of orthopairs *of X determined by $\mathcal{C}$ is the sequence*

$$\mathcal{O}_{\mathcal{C}}(X) = ((\mathcal{L}_1(X), \mathcal{E}_1(X)), \ldots, (\mathcal{L}_n(X), \mathcal{E}_n(X))),$$

where $(\mathcal{L}_i(X), \mathcal{E}_i(X))$ is the orthopair *of X determined by C_i.*

For short, $\mathcal{O}_{\mathcal{C}}(X)$ is also called *sequence of orthopairs* of X determined by $\mathcal{C}$.

Example 23. Let $U = \{a, b, c, d, e, f, g, h, i, j\}$ and $X = \{a, b, c, d, e\}$. If $\mathcal{C}$ is the refinement sequence of U made of $C_1 = \{\{a, b, c, d, e, f, g, h, i, j\}\}$, $C_2 = \{\{a, b, c, d, e\}, \{e, f, g, h, i\}\}$, $C_3 = \{\{a, b, c\}, \{c, d\}, \{e, f, g\}, \{g, h\}\}$, then

$$\mathcal{O}_{\mathcal{C}}(X) = ((\emptyset, \emptyset), (\{\{a, b, c, d, e\}\}, \emptyset), (\{\{a, b, c\}, \{c, d\}\}, \{\{g, h\}\}))\,.$$

Example 24. Suppose that we are interested to describe the set $X = \{u_1, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}\}$ with respect to the refinement sequence $\mathcal{C}$ of Example 14. We know that X contains all users that have the attributes Campania (hence Naples), Brittany (hence Rennes) and Strasbourg; while users that come from Lombardy (hence Varese, Milan and Pavia) and Mets do not belong to X. This means that the sequence of orthopairs of X is $(\mathcal{O}_{C_1}(X), \mathcal{O}_{C_2}(X), \mathcal{O}_{C_3}(X))$ where $\mathcal{O}_{C_1}(X) = (\emptyset, \emptyset), \mathcal{O}_{C_2}(X) = (\{u_8, \ldots, u_{15}\}, \{u_2, \ldots, u_7\})$ and $\mathcal{O}_{C_3}(X) = (\{u_8, u_9, u_{12}, u_{13}, u_{16}, u_{17}\}, \{u_2, \ldots, u_7, u_{18}, u_{19}\}))$.

We indicate the set of all sequences of orthopairs generated by $\mathcal{C}$ with $\mathsf{SO}(\mathcal{C})$; namely, we set

$$\mathsf{SO}(\mathcal{C}) = \{\mathcal{O}_{\mathcal{C}}(X) \mid X \subseteq U\}.$$

Given a refinement sequence $\mathcal{C} = (C_1, \ldots, C_n)$ of U, by Definition 46, the orthopair $(\mathcal{L}_i(X), \mathcal{E}_i(X))$ of $\mathcal{O}_{\mathcal{C}}(X)$ is generated by the covering C_i that is finer than C_{i-1}. Clearly, this does not imply that $(\mathcal{L}_i(X), \mathcal{E}_i(X))$ approximates better than $(\mathcal{L}_{i-i}(X), \mathcal{E}_{i-1}(X))$ the set X (we say that the orthopair $O(X) = (\mathcal{L}(X), \mathcal{E}(X))$ approximates better than the orthopair $\tilde{O}(X) = (\tilde{\mathcal{L}}(X), \tilde{\mathcal{E}}(X))$ the set X if and only if $\tilde{\mathcal{L}}(X) \subseteq \mathcal{L}(X)$ and $\tilde{\mathcal{E}}(X) \subseteq \mathcal{E}(X)$), since $X \cap U_i$ may be strictly included in $X \cap U_{i-1}$ (the sets $U_1, \ldots, U_n$ are defined in Remark 10).

Example 25. We consider the sequence of Example 24. We observe that $\mathcal{O}_{C_3}(X)$ is not a better approximation of X than $\mathcal{O}_{C_2}(X)$, despite C_3 is finer than C_2, since $u_{10}, u_{11}, u_{14}, u_{15}$ appear in $\mathcal{O}_{C_2}(X)$, but they do not appear in $\mathcal{O}_{C_3}(X)$. Trivially, this is the consequence of the fact that the sequence of partial coverings loses objects during the refinement process.

More precisely, the following proposition holds.

Proposition 6. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a refinement sequence of U and $X \subseteq U$. Suppose that $a \in \mathcal{L}_{i-1}(X)$ (or $a \in \mathcal{E}_{i-1}(X)$), with $i \in \{2, \ldots, n\}$. Then, $a \in \mathcal{L}_i(X)$ if and only if $a \in U_i$; (or $a \in \mathcal{E}_i(X)$ if and only if $a \in U_i$).*

Moreover, it is clear that two different subsets of the given universe can have the same sequences of orthopairs.

Example 26. Let $\mathcal{C} = (C_1, C_2)$ be the refinement sequence of Example 18. Suppose that $X = \{a, b, c, d\}$ and $Y = \{a, b, c, e\}$, then

$$\mathcal{O}_{\mathcal{C}}(X) = \mathcal{O}_{\mathcal{C}}(Y) = ((\emptyset, \emptyset), (\{a, b, c\}, \emptyset)).$$

At this is point, in order to show that each sequence of orthopairs is represented by a pair of disjoint upsets of the poset assigned to the given refinement sequence, we give the following definition.

Definition 47. *Let $\mathcal{C} = (C_1, \ldots, C_2)$ be a refinement sequence of U and $X \subseteq U$. We set*

$$(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) = (\{N \in P_{\mathcal{C}} \mid N \subseteq X\}, \{N \in P_{\mathcal{C}} \mid N \cap X = \emptyset\}).$$

Moreover, we set $\mathsf{K}_O(\mathcal{C}) = \{(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \mid X \subseteq U\}$.

From now, we write (X^1, X^2) instead of $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}})$, when $\mathcal{C}$ is clear from the context.

The following theorem shows that there is a one-to-one correspondence between the elements of $\mathsf{SO}(\mathcal{C})$ and $\mathsf{K}_O(\mathcal{C})$.

Theorem 17. *Given a refinement sequence $\mathcal{C} = (C_1, \ldots, C_n)$ of a universe U, the map*

$$\alpha : \mathcal{O}_{\mathcal{C}}(X) \in \mathsf{SO}(\mathcal{C}) \mapsto (X^1, X^2) \in \mathsf{K}_O(\mathcal{C})$$

is a bijection.

Proof. First of all, we prove that α is well defined and injective, namely $\mathcal{O}_{\mathcal{C}}(X) = \mathcal{O}_{\mathcal{C}}(Y)$ if and only if $(X^1, X^2) = (Y^1, Y^2)$.

($\Rightarrow$). We observe that $N \in X^1$ if and only if $N \in C_i$ and $N \subseteq X$ for some $i \in \{1, \ldots, n\}$, namely $N \in C_i$ and $N \subseteq \mathcal{L}_i(X)$. Consequently $N \in Y^1$, since $\mathcal{L}_i(X) = \mathcal{L}_i(Y)$. Dually, $N \in X^2$ if and only if $N \in Y^2$, since $\mathcal{E}_i(X) = \mathcal{E}_i(Y)$ for each $i \in \{1, \ldots, n\}$.

($\Leftarrow$). Let $i \in \{1, \ldots, n\}$. $x \in \mathcal{L}_i(X)$ if and only if there is $N \in P_{\mathcal{C}}$ such that $x \in N$ and $N \subseteq X$. By hypothesis, $N \subseteq Y$. Then, $x \in \mathcal{L}_i(Y)$. Dually, we can prove that $\mathcal{E}_i(X) = \mathcal{E}_i(Y)$ for each $i \in \{1, \ldots, n\}$, since $X^2 = Y^2$.

Surjectivity follows by the definition of $\mathsf{K}_O(\mathcal{C})$. Hence, α is a bijection.

Remark 15. Definition 42 and Theorem 17 allow us to view a sequence of orthopairs as a labelled poset. Indeed, we can graphically represent sequences of orthopairs. More precisely, given a refinement sequence $\mathcal{C}$, the sequence $\mathcal{O}_{\mathcal{C}}(X)$ corresponds to the poset $P_{\mathcal{C}}$ that has labels associated with its nodes through the function $l_X : P_{\mathcal{C}} \mapsto \{\bullet, \circ, ?\}$ such that

$$l_X(N) = \begin{cases} \bullet & \text{if } N \in X^1; \\ \circ & \text{if } N \in X^2; \\ ? & \text{if } N \in P_{\mathcal{C}} \setminus \{X^1 \cup X^2\}. \end{cases} \tag{28}$$

For example, consider the refinement sequence of Example 18. Assume that $X = \{d, e, f, g\}$, $Y = \{a, b, c, d, e, f\}$ and $Z = \{a\}$, then the sequences $\mathcal{O}_{\mathcal{C}}(X) = ((\emptyset, \emptyset), (\{d, e, f\}, \{a, b, c\}))$, $\mathcal{O}_{\mathcal{C}}(Y) = ((\emptyset, \emptyset), (\{a, b, c, d, e, f\}, \emptyset))$ and $\mathcal{O}_{\mathcal{C}}(Z) = ((\emptyset, \emptyset), (\emptyset, \{d, e, f\}))$ have the following labelled posets, respectively.

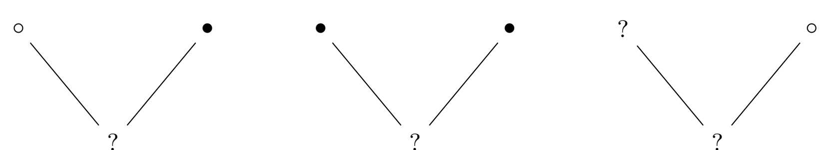

Fig. 8: Labelled posets

Trivially, by 28, if $l_X(N) = \bullet$ and $N \leq_{\mathcal{C}} M$, then $l_X(M) = \bullet$. Similarly, if $l_X(N) = \circ$ and $N \leq_{\mathcal{C}} M$, then $l_X(M) = \circ$. On the other hand, $l_X(M)$ can be anyone between $\bullet$, $\circ$ and ?, when $l_X(N) =?$ and $N \leq_{\mathcal{C}} M$.

Sequences of orthopairs and decision trees Sequences of orthopairs correspond to *decision trees*. These are graphical models widely used in machine learning for describing sequential decision problems. A decision tree generates a classification procedure that recursively partitions a universe into smaller subdivisions on the basis of a set of tests defined at each branch (or node) in the tree [48]. The tree is made of a *root node* (the universe), a set of *internal nodes* (splits), and a set of *terminal nodes* (leaves). A test is applied for the universe and for each internal node in order to split the set of objects into successively smaller groups. The terminal nodes are labelled with values corresponding to the final decisions. An example of decision tree can be viewed in Figure 9, where the labels A, B, C and D represent the final outcomes of the decision-making process.

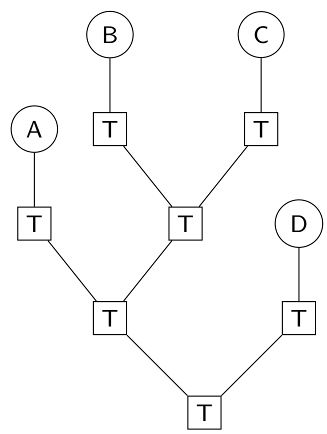

Fig. 9: Decision tree

Let $\mathcal{C}$ be a refinement sequence of *partial partitions* of U, and let $X \subseteq U$. The sequence of orthopairs $\mathcal{O}_{\mathcal{C}}(X)$ determines three pairwise disjoint subsets of U: $\cup\{N \in P_{\mathcal{C}} \mid l_X(N) = \bullet\}$, $\cup\{N \in P_{\mathcal{C}} \mid l_X(N) = \circ\}$ and $\cup\{N \in P_{\mathcal{C}} \mid l_X(N) =?\}$. This also corresponds to result produced by the decision tree $(\mathcal{T}_{\mathcal{C}}(X), \leq_{\mathcal{C}})$ such that

- $\mathcal{T}_{\mathcal{C}}(X) = (P_{\mathcal{C}} \cup \{U\}) \setminus H$, where

$$H = \{N \in P_{\mathcal{C}} \mid \text{ if } M \in P_{\mathcal{C}} \text{ and } M \leq_{\mathcal{C}} N \text{ then } l_X(M) \in \{\bullet, \circ\}\}, \text{ and}$$

- let N be a leaf of $\mathcal{T}_{\mathcal{C}}(X)$, then the label of N is $l_X(N)$.

Trivially, $\mathcal{T}_{\mathcal{C}}(X)$ can have three outcomes at most, which are $\bullet, \circ$ and ?. Hence, if $\mathcal{O}_{\mathcal{C}}(X)$ is the sequence of orthopairs having labelled poset as in Figure 10. Then, the tree decision $\mathcal{T}_{\mathcal{C}}(X)$ is shown in Figure 11.

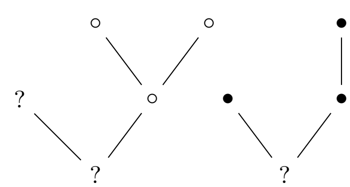

Fig. 10: Labelled poset of $\mathcal{O}_{\mathcal{C}}(X)$

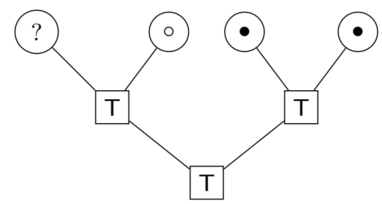

Fig. 11: Decision tree $\mathcal{T}_{\mathcal{C}}(X)$

Clearly, a decision tree with three outcomes determines a refinement sequence (by considering all nodes of the tree) and a sequence of orthopairs (by considerings all nodes and all labels of the tree).

From now, given a refinement sequence $\mathcal{C}$, we write $\mathsf{K}(\mathcal{C})$ to denote $\mathsf{K}(Up(P_{\mathcal{C}}))$, that is

$$\mathsf{K}(Up(P_{\mathcal{C}})) = \{(A, B) \in Up(P_{\mathcal{C}}) \times Up(P_{\mathcal{C}}) \mid A \cap B = \emptyset\},$$

where $Up(P_{\mathcal{C}})$ is the set of all upsets of $P_{\mathcal{C}}$ (see Section 2.3).

The next proposition shows that each element of $\mathsf{K}_O(\mathcal{C})$ also belongs to $\mathsf{K}(\mathcal{C})$.

Proposition 7. *Let $\mathcal{C}$ be a refinement sequence of U and $X \subseteq U$. Then, (X^1, X^2) is a pair of disjoint upsets of $P_{\mathcal{C}}$.*

Proof. By Definition 47, $X^1 \cap X^2 = \emptyset$. If $N \in X^1$ and $N \leq_{\mathcal{C}} M$, then $M \subseteq N \subseteq X$ (by Definition 47) hence $M \subseteq X$ and $M \in X^1$. Similarly, if $N \in X^2$ and $N \leq_{\mathcal{C}} M$ then $M \subseteq N$ and $N \cap X = \emptyset$, hence $M \cap X = \emptyset$ and $M \in X^2$.

By Proposition 7, $\mathsf{K}_O(\mathcal{C}) \subseteq \mathsf{K}(\mathcal{C})$. However, the opposite does not always hold.

Example 27. Consider the refinement sequence $\mathcal{C}$, where $P_{\mathcal{C}}$ is represented in Figure 12.

We have that $(\{\{a, b, c\}\}, \{\{c, d\}\}) \in \mathsf{K}_O(\mathcal{C})$, but $(\{\{a, b, c\}\}, \{\{c, d\}\}) \notin \mathsf{K}(\mathcal{C})$.

The next theorem (Theorem 18) provides the condition that a pair of disjoint upsets of $P_{\mathcal{C}}$ must have in order to belong to $\mathsf{K}_O(\mathcal{C})$, when $\mathcal{C}$ is safe. To prove Theorem 18, we need the following proposition.

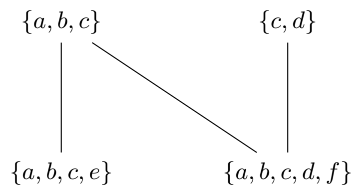

Fig. 12: Poset of $\mathcal{C}$

Proposition 8. *Let $\mathcal{C}$ be a safe refinement sequence of U and let A be an upset of $P_{\mathcal{C}}$. Suppose that $N \in P_{\mathcal{C}}$ and*

$$N \subseteq \bigcup_{M \in A} M.$$

Then, $N \in A$.

Proof. Since $\mathcal{C}$ is safe (see Definition 44), there exists $M \in A$ such that $N \subseteq A$. However, A is an upset of $P_{\mathcal{C}}$, then $N \in A$.

From now on, we only consider coverings that do not contain singletons, which are blocks with only one element. We stress that the imposition of this constraint concerns the very relations between coverings and orthopairs as approximation of sets, as shown in the following example.

Example 28. Let $U = \{a, b, c, d, e\}$ and consider the covering of U given by $C = \{\{a, b\}, \{c\}, \{d, e\}\}$. Then, $(X^1, X^2) = (\{a, b\}, \{d, e\})$ is an orthopair made of blocks of C, but (X^1, X^2) does not approximate any subset X of U, since either $c \in X$, and then $c \in X^1$ or $c \in X$, and then $c \in X^2$. More generally, each orthopair such that $\{c\}$ is not contained in one of the components of the pair does not approximate any subset of U.

In order to state the next theorem, we recall that two upsets A and B of a given poset are *totally disjoint* if and only if all blocks of A are disjoint from all blocks of B.

Theorem 18. *Let $\mathcal{C}$ be a safe refinement sequence of U and let $(A, B) \in \mathsf{K}(\mathcal{C})$. Then, $(A, B) \in \mathsf{K}_O(\mathcal{C})$ if and only if A and B are totally pairwise disjoint.*

Proof. ($\Rightarrow$). By Definition 47, if $(A, B) \in \mathsf{K}_O(\mathcal{C})$, then there exists $X \subseteq U$ such that $N \subseteq X$ for each $N \in A$ and $N \cap M = \emptyset$ for each $M \in B$. Trivially, each node of A is disjoint with each node of B, since there is not $x \in U$ such that $x \in X$ and $x \notin X$.

($\Leftarrow$). Suppose that each node of A is disjoint with each node of B. We set

$$D = \{N \in P_{\mathcal{C}} \setminus (A \cup B) \mid N \cap M = \emptyset \text{ for each } M \in A \text{ and if } M >_{\mathcal{C}} N \text{ then } M \in B\}.$$

Since $\mathcal{C}$ is safe, for each $N \in D$, we can pick an element $x_N \in N$ such that $x_N \notin M$, for each $M \in P_{\mathcal{C}} \setminus \{\downarrow N\}$ (see Remark 14). Then, we set

$$X = \bigcup_{N \in A} N \cup \{x_N \mid N \in D\}.$$

We prove that $(A, B) = (X^1, X^2)$. It is trivial that $A \subseteq X^1$ and $B \subseteq X^2$. Now, we suppose that $N \in X^1$, and we intend to prove that $N \in A$. Let $x \in N$. Then, $x = x_M$ with $M \in D$ or x belongs to some node of A. If $x = x_M$ with $M \in D$, then $N \in \downarrow M$ (see 14), and so $M \subseteq N$. Now, two cases can happen. If M is not a maximal element of $P_{\mathcal{C}}$, then M contains some elements of the nodes of B. However, by the hypothesis that A and B are totally pairwise disjoint, this is an absurd. In the other case, namely, if M is a maximal element of $P_{\mathcal{C}}$, then it contains at least another element that is not equal to x_M (we assumed that the blocks of refinement sequences are not singletons). By definition of D, such element is not in A and it is different from other elements x_N. It is clear that it is an absurd, since $N$is included in X, by hypothesis. We can conclude N is included in the union of blocks of A. Therefore, by Proposition 8, since $\mathcal{C}$ is safe, we have that $N \in A$. Now, we suppose that $N \in X^2$, and we intend to prove that $N \in B$. if $N \in X^2$, then $N \cap M = \emptyset$, for each $M \in A \cup D$. Consequently, $N \notin \ (\downarrow A) \cup (\downarrow D)$. Moreover, we can notice that $B = P_{\mathcal{C}} \setminus \{(\downarrow A) \cup (\downarrow D)\}$. Then, we can state that $N \in B$.

Theorem 18 permits us to prove the following result, which is relevant to regard sequences of orthopairs as Kleene algebras.

Theorem 19. *Let $\mathcal{C}$ be a complete and safe refinement sequence of U. Then, $\mathsf{K}_O(\mathcal{C}) = \mathsf{K}(\mathcal{C})$.*

Proof. We have that $\mathsf{K}_O(\mathcal{C}) \subseteq \mathsf{K}(\mathcal{C})$, by Proposition 7. Moreover, Let $(A, B) \in \mathsf{K}(\mathcal{C})$, then A and B are totally pairwise disjoint, since $\mathcal{C}$ is complete. By hypothesis that $\mathcal{C}$ is safe and by Theorem 18, $(A, B) \in \mathsf{K}_O(\mathcal{C})$.

As a consequence of the previous theorem, we can define several operations on sequences of orthopairs, using the operations already defined on sets of pairs of disjoint upsets of posets (see Section 2.3). However, we will explore this topic in the next chapter.

4 Sequences of orthopairs as Kleene algebras

Mathematics is the art of giving the same name to different things.

Henrie Poincaré

In this chapter, we equip sets of sequences of orthopairs with some operations in order to obtain finite many-valued algebraic structures (those are defined in Section 2.3). Furthermore, we prove theorems providing to represent such structures as sequences of orthopairs. We show that, when sequences of orthopairs are generated by one covering, our operations coincide with operations between orthopairs listed in Section 2.2. Also, we discover how to generate operations between sequences of orthopairs starting from those concerning individual orthopairs. Finally, we use a sequence of orthopairs to represent an examiner's opinion on a number of candidates applying for a job. Moreover, we show that opinions of two or more examiners can be combined using our operations in order to get a final decision on each candidate.

4.1 From a safe refinement sequence to a Kleene algebra

In the previous chapter, given a refinement sequence $\mathcal{C}$, we proved that each element of $\mathsf{K}_O(\mathcal{C})$ is a pair of disjoint upsets of $P_\mathcal{C}$ (see Proposition 7), and that $\mathsf{K}_O(\mathcal{C})$ coincides with $\mathsf{K}(\mathcal{C})$ if and only if $\mathcal{C}$ is safe and complete (see Example 27 and Theorem 19). As a consequence, we can equip $\mathsf{K}_O(\mathcal{C})$ with the operations $\sqcap$, $\sqcup$ and $\neg$ defined by 9, 10 and 7 (see Section 2.3), respectively, and so we can consider the following structure

$$\mathbb{K}_O(\mathcal{C}) = (\mathsf{K}_O(\mathcal{C}), \sqcap, \sqcup, \neg, (P_\mathcal{C}, \emptyset), (\emptyset, P_\mathcal{C})).$$

Unfortunately, $\mathbb{K}_O(\mathcal{C})$ is not always a lattice, since $\mathsf{K}_O(\mathcal{C})$ could not be closed under $\sqcap$ and $\sqcup$, when $\mathsf{K}_O(\mathcal{C}) \subset \mathsf{K}(\mathcal{C})$.

Example 29. Let $U = \{a, b, c, d\}$ and $\mathcal{C} = (C_1, C_2)$, where

- $C_1 = \{\{a, b, c, d\}\}$ and
- $C_2 = \{\{a, b\}, \{c, d\}\})$.

Then, it occurs that

- $(\emptyset, \{\{a, b\}\}) \sqcap (\emptyset, \{\{c, d\}\}) = (\emptyset, \{\{a, b\}, \{c, d\}\})$ and
- $(\{\{a, b\}\}, \emptyset) \sqcup (\{\{c, d\}\}, \emptyset) = (\{\{a, b\}, \{c, d\}\}, \emptyset)$.

However, $(\emptyset, \{\{a, b\}, \{c, d\}\}), (\{\{a, b\}, \{c, d\}\}, \emptyset) \notin \mathsf{K}_O(\mathcal{C})$.

On the other hand, the following theorem states that requiring that refinement sequences be safe is sufficient to obtain finite centered Kleene algebras.

Theorem 20. *Let $\mathcal{C}$ be a safe refinement sequence of U. Then,*

1. $\mathsf{K}_O(\mathcal{C}) \supseteq \mathsf{K}^+(\mathcal{C})$ *and*
2. $\mathbb{K}_O(\mathcal{C})$ *is a centered Kleene subalgebra of* $\mathbb{K}(\mathcal{C})$ *(see Definition 26), where*

$$\mathbb{K}(\mathcal{C}) = (\mathsf{K}(\mathcal{C}), \sqcap, \sqcup, \neg, (\emptyset, P_{\mathcal{C}}), (P_{\mathcal{C}}, \emptyset)),$$

and the center is $(\emptyset, \emptyset)$.

Proof. 1. Let $(A, B) \in \mathsf{K}^+(\mathcal{C})$, then $B = \emptyset$. Consequently, A and B are totally disjoint, namely satisfy Condition 27. Certainly, $(A, B) \in \mathsf{K}_O(\mathcal{C})$, by Theorem 18.
2. Since $\mathsf{K}^+(\mathcal{C}) \subseteq \mathsf{K}_O(\mathcal{C})$, we have that $(\emptyset, \emptyset) \in \mathsf{K}_O(\mathcal{C})$. Moreover, $\mathsf{K}_O(\mathcal{C})$ is closed under all operations of $\mathbb{K}(\mathcal{C})$, since both $(X^1 \cap Y^1, X^2 \cup Y^2)$ and $(X^1 \cup Y^1, X^2 \cap Y^2)$ are pairs of totally disjoint upsets of $P_{\mathcal{C}}$. Then, by Theorem 18, both belong to $\mathsf{K}_O(\mathcal{C})$.

Remark 16. Clearly, when $\mathcal{C}$ is a safe refinement sequence of U, then $\mathsf{K}^-(\mathcal{C})$ is also included in $\mathsf{K}_O(\mathcal{C})$.

When a safe refinement sequence $\mathcal{C}$ is also complete or pairwise overlapping, $\mathbb{K}_O(\mathcal{C})$ satisfies properties that are additional to those of Theorem 20. More precisely, the following theorem holds.

Theorem 21. *Let $\mathcal{C}$ be a safe refinement sequence of U,*

1. *if $\mathcal{C}$ is complete, then $\mathbb{K}_O(\mathcal{C})$ is a finite centered Kleene algebra with the interpolation property,*
2. *if $\mathcal{C}$ is pairwise overlapping, then $\mathsf{K}_O(\mathcal{C}) = \mathsf{K}^+(\mathcal{C}) \cup \mathsf{K}^-(\mathcal{C})$.*

Proof. 1. By Theorem 19, $\mathsf{K}_O(\mathcal{C}) = \mathsf{K}(\mathcal{C})$. Moreover, the structure $\mathbb{K}(\mathcal{C})$ is a centered Kleene algebra with the interpolation property (see Theorem 7).
2. By Definition 47, if $(A, B) \in \mathsf{K}_O(\mathcal{C})$, then A and B are totally disjoint. However, since $\mathcal{C}$ is pairwise overlapping, Vice-versa, by Theorem 20, if (A, B) is in $\mathsf{K}^+(\mathcal{C})$ or $\mathsf{K}^-(\mathcal{C})$, then belongs to $\mathsf{K}_O(\mathcal{C})$, also.

In the next example, we take three different refinement sequences such that their posets are isomorphic, and we show that the Hasse diagrams of their respective Kleene algebras are not isomorphic.

Example 30. We consider the refinement sequences $\mathcal{C} = (C_1, C_2)$ and $\mathcal{C}' = (C_1', C_2')$ of $\{a, b, c, d, e, f\}$, where

- $C_1 = \{\{a, b, c, d, e\}, \{c, d, f\}\}$,
- $C_2 = \{\{a, b\}, \{c, d\}\}$,
- $C_1' = \{\{a, b, d, e, f\}, \{c, d, e\}\}$ and
- $C_2' = \{\{b, d\}, \{d, e\}\}$.

As shown in Figure 13 and Figure 14, $P_{\mathcal{C}}$ and $P_{\mathcal{C}'}$ have the same Hasse diagram. Then, $\mathbb{K}(\mathcal{C}) \cong \mathbb{K}(\mathcal{C}')$.

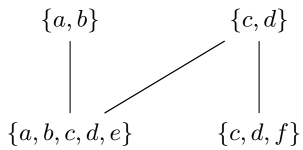

Fig. 13: Hasse diagram of $P_{\mathcal{C}}$

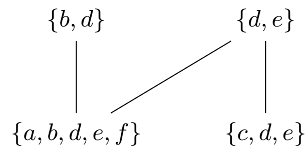

Fig. 14: Hasse diagram of $P'_{\mathcal{C}}$

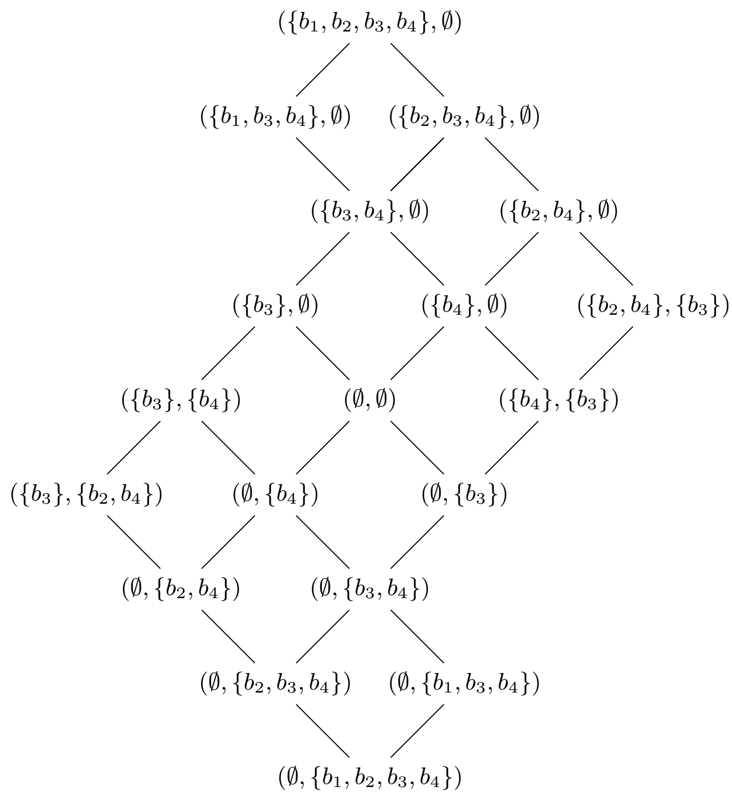

Fig. 15: Hasse diagram of $\mathbb{K}_O(\mathcal{C})$

We set $b_1 = \{a,b,c,d,e\}$, $b_2 = \{c,d,f\}$, $b_3 = \{a,b\}$, $b_4 = \{c,d\}$, $b'_1 = \{a,b,d,e,f\}$, $b'_2 = \{c,d,e\}$, $b'_3 = \{b,d\}$ and $b'_4 = \{d,e\}$. Then, the Hasse diagrams of $\mathbb{K}_O(\mathcal{C})$ and $\mathbb{K}_O(\mathcal{C}')$ are represented in Figure 15 and Figure 16, respectively.

Notice that $\mathsf{K}_O(\mathcal{C}) = \mathsf{K}(\mathcal{C})$, since $\mathcal{C}$ is safe and complete. Instead, since $\mathcal{C}'$ is safe but not complete, $\mathsf{K}_O(\mathcal{C}') \subset \mathsf{K}(\mathcal{C}')$ and $(\{b'_3\},\{b'_4\}), (\{b'_4\},\{b'_3\}), (\{b'_3\}, \{b'_2,b'_4\}), (\{b'_2,b'_4\},\{b'_3\}) \notin \mathsf{K}_O(\mathcal{C}')$. We stress that $\mathbb{K}_O(\mathcal{C}) \ncong \mathbb{K}_O(\mathcal{C}')$, despite $P_{\mathcal{C}} \cong P_{\mathcal{C}'}$.

Now, we consider the refinement sequence $\tilde{\mathcal{C}} = (\tilde{C}_1, \tilde{C}_2)$, where

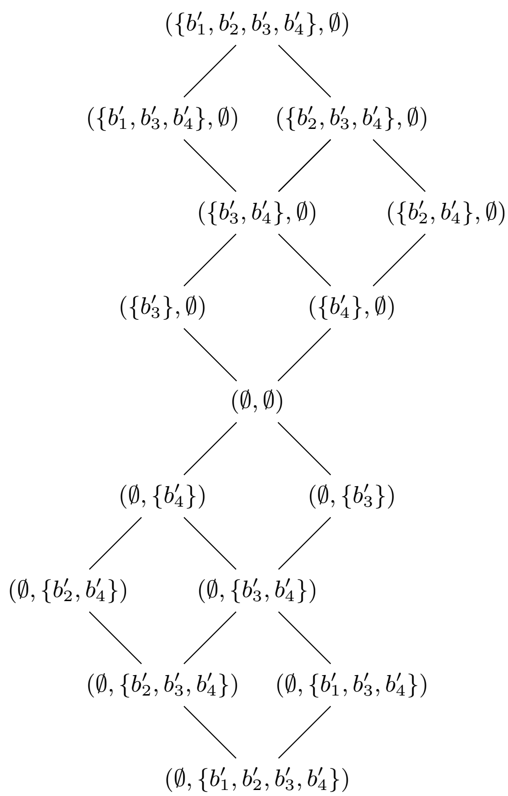

Fig. 16: Hasse diagram of $\mathbb{K}_O(\mathcal{C}')$

- $\tilde{C}_1 = \{\{a, b, c, d, e\}, \{c, d, f\}\}$ and
- $\tilde{C}_2 = \{\{a, b, c\}, \{c, d\}\}$.

Clearly, $\tilde{\mathcal{C}}$ is a safe and pairwise overlapping refinement sequence. If we set $\tilde{b}_1 = \{a, b, c, d, e\}$, $\tilde{b}_2 = \{c, d, f\}$, $\tilde{b}_3 = \{a, b, c\}$ and $\tilde{b}_4 = \{c, d\}$, then the Hasse diagrams of $P_{\tilde{\mathcal{C}}}$ and $\mathbb{K}_O(\tilde{\mathcal{C}})$ are respectively represented in Figure 17 and Figure 18.

We can observe that $\mathsf{K}_O(\tilde{\mathcal{C}}) = \mathsf{K}(\tilde{\mathcal{C}})^+ \cup \mathsf{K}(\tilde{\mathcal{C}})^-$. Moreover, $\mathbb{K}_O(\tilde{\mathcal{C}}) \ncong \mathbb{K}_O(\mathcal{C})$ and $\mathbb{K}_O(\tilde{\mathcal{C}}) \ncong \mathbb{K}_O(\mathcal{C}')$, despite $P_{\tilde{\mathcal{C}}} \cong P_{\mathcal{C}}$ and $P_{\tilde{\mathcal{C}}} \cong P_{\mathcal{C}'}$.

Remark 17. Let $\mathcal{C}$ be a refinement sequence, then $|\mathsf{K}_O(\mathcal{C})|$, that is the cardinality of $\mathsf{K}_O(\mathcal{C})$, depends from the number of blocks that pairwise overlap in every covering of $\mathcal{C}$. Consequently, if $\mathcal{C}$ is complete and safe, then $|\mathsf{K}_O(\mathcal{C})|$ is maximum, and it is equal to $|\mathsf{K}(\mathcal{C})|$. Furthermore, if $\mathcal{C}$ is pairwise overlapping and not safe, then $|\mathsf{K}_O(\mathcal{C})| \geq |\mathsf{K}(\mathcal{C})^+ \cup \mathsf{K}(\mathcal{C})^-|$.

We can extend the results shown in Theorem 21, by considering the operation $\rightarrow_1$ and the pairs of operations $(\star_2, \rightarrow_2)$, $(\star_3, \rightarrow_3)$ and $(\star_4, \rightarrow_4)$, defined in Section

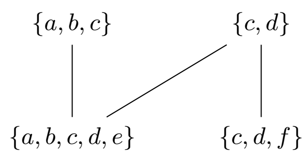

Fig. 17: Hasse diagram of $P_{\tilde{\mathcal{C}}}$

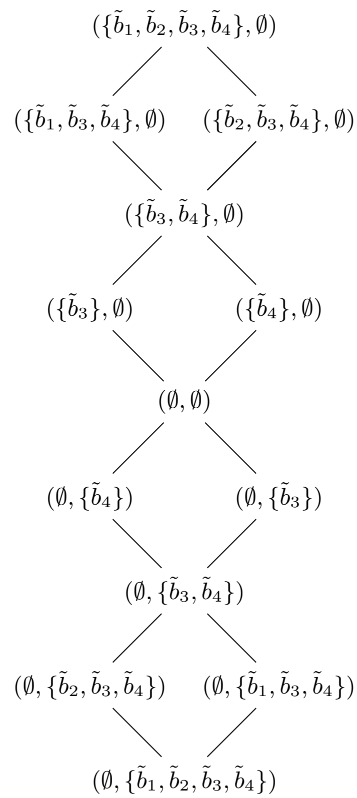

Fig. 18: Hasse diagram of $\mathbb{K}_O(\tilde{\mathcal{C}})$

2.3 (more exactly, see the equations 13, 18, 19, 20, 21, 25 and 26), on the set $\mathsf{K}_O(\mathcal{C})$. Then, let $i \in \{1, \ldots, 4\}$, we can use the notation $\mathbb{K}^i_O(\mathcal{C})$ to denote the structure $\mathbb{K}_O(\mathcal{C})$ with the additional operations $\star_i$ and $\rightarrow_i$.

Corollary 1. *If $\mathcal{C}$ is a safe and complete refinement sequence, then*

- $\mathbb{K}^1_O(\mathcal{C})$ *is a finite Nelson algebra,*
- $\mathbb{K}^2_O(\mathcal{C})$ *is a finite Nelson lattice and*
- $\mathbb{K}^4_O(\mathcal{C})$ *is a finite KLI* algebra.*

Regarding $\mathbb{K}^3_O(\mathcal{C})$, we need to add the extra condition that $\mathcal{C}$ must be composed by partial partitions.

Corollary 2. *If $\mathcal{C}$ is a safe refinement sequence of partial partitions, then $\mathbb{K}^3_O(\mathcal{C})$ is a finite IUML-algebra.*

If some coverings of $\mathcal{C}$ are not partitions, then the operations $\star_i$ and $\rightarrow_i$ cannot be defined on $\mathsf{K}_O(\mathcal{C})$. Clearly, this is a consequence that such operations are defined between pairs of disjoint upsets of a forest (see 20 and 21), and they can not be extended between pairs of disjoint upsets of a poset.

Example 31. Let $\mathcal{C}$ be the refinement sequence defined in Example 30. $\mathcal{C}$ is safe and complete, but

$$(\{b_3\},\{b_2,b_4\}) \star_3 (\{b_1,b_3,b_4\},\emptyset) = (\{b_1,b_3,b_4\},\{b_2\})$$

and

$$(\{b_3\},\{b_2,b_4\}) \to_3 (\emptyset,\{b_1,b_3,b_4\}) = (\{b_2\},\{b_1,b_3,b_4\})$$

that do not belong to $\mathsf{K}(\mathcal{C})$.

4.2 From a complete refinement sequence to a Kleene algebra

In this section, given a complete refinement sequence $\mathcal{C}$, we want to determine new operations on $\mathsf{K}_O(\mathcal{C})$, to obtain the same structure encountered in the previous section. In order to do this, starting from a complete refinement sequence $\mathcal{C}$, we build a new refinement sequence $\mathcal{C}'$ such that $\mathsf{K}_O(\mathcal{C}) = \mathsf{K}_O(\mathcal{C}') = \mathsf{K}(\mathcal{C}')$.

Definition 48. *Let $\mathcal{C} = (C_1,\dots,C_n)$ be a refinement sequence of U. Then, we build the sequence $\mathcal{C}' = (C'_1,\dots,C'_n)$ in the following way.*

- $C'_n = C_n$,
- *for every $i \in \{1,\dots,n-1\}$ and $N \in C_i$, if there are not $N_1,\dots,N_l \in C'_{i+1}$ such that $N = N_1 \cup \ldots \cup N_l$ then $N \in C'_i$, otherwise $N \notin C'_i$ but $N_j \in C'_i$ for each $j = 1,\dots,l$.*

Example 32. Let $\mathcal{C}$ be the refinement sequence of Example 14. Then, $\mathcal{C}' = (C'_1, C'_2, C'_3)$, where

$C'_1 = \{\{u_1,\dots,u_{11}\},\{u_{12},\dots,u_{22}\}\}$;
$C'_2 = \{\{u_2,u_3\},\{u_4,u_5\},\{u_6,u_7\},\{u_8,\dots,u_{11}\},\{u_{12},\dots,u_{15}\},\{u_{16},\dots,u_{21}\}\}$;
$C'_3 = \{\{u_2,u_3\},\{u_4,u_5\},\{u_6,u_7\},\{u_7,u_8\},\{u_{12},u_{13}\},\{u_{16},u_{17}\},\{u_{18},u_{19}\}$.

Observe that $\mathcal{C}'$ is still a refinement sequence of U, so we can associate it with a poset $P_{\mathcal{C}'}$.

Example 33. Let $\mathcal{C}$ be the refinement sequence of Example 14. The poset $P_{\mathcal{C}'}$ assigned to the new refinement sequence $\mathcal{C}'$ represented in Figure 19.

We notice that the node $\{u_2,\dots,u_7\}$ of $P_{\mathcal{C}}$ (see Example 16) does not belong to $P_{\mathcal{C}'}$, and it is equal to the union of its successors $\{u_2,u_3\}$, $\{u_4,u_5\}$ and $\{u_6,u_7\}$.

Remark 18. In general, $P_{\mathcal{C}'}$ is obtained by removing from $P_{\mathcal{C}}$ all the nodes equal to the union of their successors (cfr. the operation of elimination in [24]). That is, we delete *reducible* elements, according to the terminology given in [112], in the covering generated by all sets in the forest $P_{\mathcal{C}}$.

By the previous remark follows this proposition.

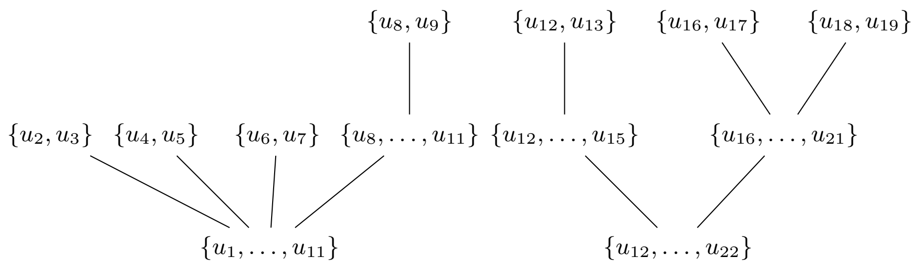

Fig. 19: Forest of the users

Proposition 9. *Let $\mathcal{C}$ be a refinement sequence of U and let $N \in P_{\mathcal{C}}$. Then, $N \in P_{\mathcal{C}'}$ if and only if $N \neq N_1 \cup \ldots \cup N_r$, where $N_1, \ldots, N_r$ are the successors of N in $P_{\mathcal{C}}$.*

Clearly, $\mathsf{K}_O(\mathcal{C}') \subseteq \mathsf{K}_O(\mathcal{C})$. Moreover, it is clear that the following proposition holds.

Proposition 10. *Let $\mathcal{C}$ be a complete refinement sequence. Then, $\mathcal{C}'$ is also complete.*

The following proposition shows that there exists an order isomorphism between $\mathsf{K}_O(\mathcal{C})$ and $\mathsf{K}_O(\mathcal{C}')$, when $\mathcal{C}$ is complete.

Theorem 22. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a complete refinement sequence of U. If $\mathcal{C}'$ is the refinement sequence of U built in Definition 48, then the function*

$$\beta : \mathsf{K}_O(\mathcal{C}) \mapsto \mathsf{K}_O(\mathcal{C}'),$$

where $\beta((X^1_{\mathcal{C}}, X^2_{\mathcal{C}})) = (X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'})$ for each $X \subseteq U$, is an order isomorphism.

Proof.

– The function β is injective. Let $X, Y \subseteq U$, we suppose that

$$\beta((X^1_{\mathcal{C}}, X^2_{\mathcal{C}})) = \beta((Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}})).$$

Then,

$$(X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) = (Y^1_{\mathcal{C}'}, Y^2_{\mathcal{C}'}). \tag{29}$$

Firstly, we intend to prove that $X^1_{\mathcal{C}} = Y^1_{\mathcal{C}}$. By Definition 48, each node N of $P_{\mathcal{C}}$ is equal to $N_1 \cup \ldots \cup N_r$, where $N_1 \cup \ldots \cup N_r \in P_{\mathcal{C}'}$. Let $N \in X^1_{\mathcal{C}}$, then $N = N_1 \cup N_r \subseteq X$ and so $N_i \subseteq X$ for each $i \in \{1, \ldots, r\}$. Therefore, $N_1, \ldots, N_r \in X^1_{\mathcal{C}'} = Y^1_{\mathcal{C}'}$. Consequently, N is included in Y and so belongs to $Y^1_{\mathcal{C}}$. The proof that $X^2_{\mathcal{C}} = Y^2_{\mathcal{C}}$ is analogous.

– The function β is surjective. Let $X \subseteq U$ and $(X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) \in \mathsf{K}_O(\mathcal{C}')$. We consider the set

$$H = \{N \in P_{\mathcal{C}} \ : \ N = N_1 \cup \ldots \cup N_r, \text{ where } N_i \in X^1_{\mathcal{C}'} \text{ for each } i \in \{1, \ldots, r\}\}$$

and

$K = \{N \in P_{\mathcal{C}} \,:\, N = N_1 \cup \ldots \cup N_r,$ where $N_i \in X^2_{\mathcal{C}'}$ for each $i \in \{1, \ldots, r\}\}$.

Since $\mathcal{C}$ is complete, we have that $(X^1_{\mathcal{C}'} \cup H, X^2_{\mathcal{C}'} \cup K)$ belongs to $\mathsf{K}_O(\mathcal{C})$. Moreover, it is clear that $\beta((X^1_{\mathcal{C}'} \cup H, X^2_{\mathcal{C}'} \cup K)) = (X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'})$.

– It is trivial that $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \leq (Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}})$ if and only if $(X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) \leq (Y^1_{\mathcal{C}'}, Y^2_{\mathcal{C}'})$ (we remember that, let (X^1, X^2) and (Y^1, Y^2) be two pairs of disjoint upsets, then $(X^1, X^2) \leq (Y^1, Y^2)$ if and only if $X^1 \subseteq Y^1$ and $Y^2 \subseteq Y^1$).

By Proposition 5 and Proposition 9, the next result follows.

Proposition 11. *Let $\mathcal{C}$ be a complete refinement sequence, then $\mathcal{C}'$ is safe.*

Consequently, by Theorem 19, $\mathsf{K}_O(\mathcal{C}')$ coincides with $\mathsf{K}(\mathcal{C}')$. Therefore, we can consider $\mathsf{K}_O(\mathcal{C}')$ equipped with the operations defined in the previous section. By using this result and Theorem 22, we can introduce the following new operations on $\mathsf{K}_O(\mathcal{C})$.

Definition 49. *Let $\mathcal{C}$ be a complete refinement sequence of U and let β be the function defined in Theorem 22. Then, we set*

– $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \cap_{\mathsf{K}_O} (Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}}) := \beta^{-1}((X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) \sqcap (Y^1_{\mathcal{C}'}, Y^2_{\mathcal{C}'}))$,
– $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \cup_{\mathsf{K}_O} (Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}}) := \beta^{-1}((X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) \sqcup (Y^1_{\mathcal{C}'}, Y^2_{\mathcal{C}'}))$,
– $\neg_{\mathsf{K}_O}(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) := \beta^{-1}(\neg(X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}))$,
– $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \star^i_{\mathsf{K}_O} (Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}}) := \beta^{-1}((X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) \star_i (Y^1_{\mathcal{C}'}, Y^2_{\mathcal{C}'}))$, *for each* $i \in \{2, 3, 4\}$,
– $(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \to^i_{\mathsf{K}_O} (Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}}) := \beta^{-1}((X^1_{\mathcal{C}'}, X^2_{\mathcal{C}'}) \to_i (Y^1_{\mathcal{C}'}, Y^2_{\mathcal{C}'}))$, *for each* $i \in \{1, 2, 3, 4\}$.

As a consequence of the previous definition and the results of Section 4.1, we obtain the following theorem.

Theorem 23. *Let $\mathcal{C}$ be a complete refinement sequence of U, then*

$$\mathbb{K}'_O(\mathcal{C}) = (\mathsf{K}_O(\mathcal{C}), \cap_{\mathsf{K}_O}, \sqcup_{\mathsf{K}_O}, \neg_{\mathsf{K}_O}, (\emptyset, P_{\mathcal{C}'}), (P_{\mathcal{C}'}, \emptyset))$$

is a centered Kleene algebra with the interpolation property and if $\mathcal{C}$ is pairwise overlapping, then $\mathsf{K}_O(\mathcal{C}) \cong \mathsf{K}(\mathcal{C}')^+ \cup \mathsf{K}(\mathcal{C}')^-$. Moreover,

– $(\mathbb{K}'_O(\mathcal{C}), \to^1_{\mathsf{K}_O})$ *is a finite Nelson algebra;*
– $(\mathbb{K}'_O(\mathcal{C}), \star^2_{\mathsf{K}_O}, \to^2_{\mathsf{K}_O})$ *is a finite Nelson lattice;*
– $(\mathbb{K}'_O(\mathcal{C}), \star^4_{\mathsf{K}_O}, \to^4_{\mathsf{K}_O})$ *is a finite KLI*-algebra.*

If $\mathcal{C}$ is a refinement sequence of partial partitions, then

– $(\mathbb{K}'_O(\mathcal{C}), \star^3_{\mathsf{K}_O}, \to^3_{\mathsf{K}_O})$ *is a finite IUML-algebra.*

Remark 19. Trivially, if $\mathcal{C}$ is also safe, then $\mathcal{C} = \mathcal{C}'$ and so $\mathbb{K}_O(\mathcal{C}) = \mathbb{K}'_O(\mathcal{C})$.

Example 34. Let $\mathcal{C}$ be the refinement sequence defined in Example 29. Trivially, $\mathcal{C}' = \{\{a, b\}, \{c, d\}\}$. The Hasse diagram of $\mathbb{K}(\mathcal{C})$, $\mathbb{K}_O(\mathcal{C})$ and $\mathbb{K}_O(\mathcal{C}')$ (which is the same as that of $\mathbb{K}(\mathcal{C}')$) are respectively represented in Figure 20, Figure 21 and Figure 22.

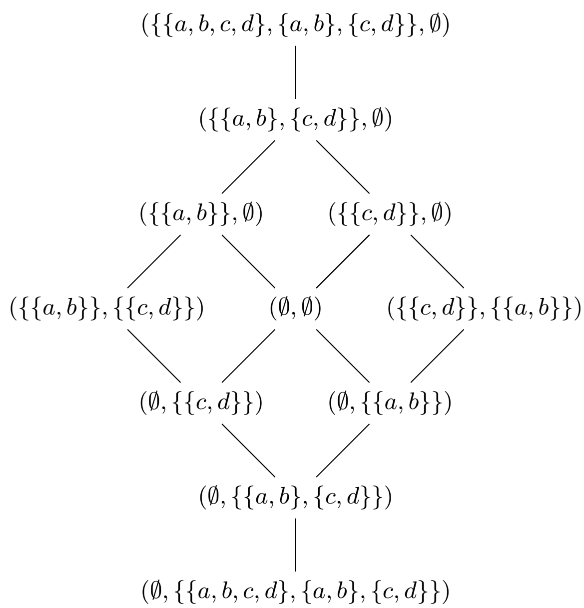

Fig. 20: Hasse diagram of $\mathbb{K}(\mathcal{C})$

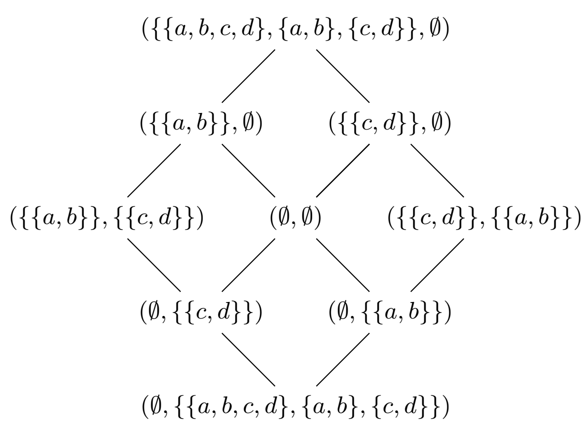

Fig. 21: Hasse diagram of $\mathbb{K}_O(\mathcal{C})$

Now, we consider $(\{\{a,b\}\},\emptyset)$ and $(\{\{c,d\}\},\emptyset)$ in $\mathsf{K}_O(\mathcal{C})$. Then $(\{\{a,b\}\},\emptyset) \sqcup (\{\{c,d\}\},\emptyset)$ is equal $(\{\{a,b\},\{c,d\}\},\emptyset)$, which does not belong to $\mathsf{K}_O(\mathcal{C})$. However, $(\{\{a,b\}\},\emptyset) \cup_{\mathsf{K}_O} (\{\{c,d\}\},\emptyset) = \beta^{-}1((\{\{a,b\},\{c,d\}\},\emptyset)) = (\{\{a,b,c,d\},\{a,b\},\{c,d\}\},\emptyset) \in \mathsf{K}_O(\mathcal{C})$.

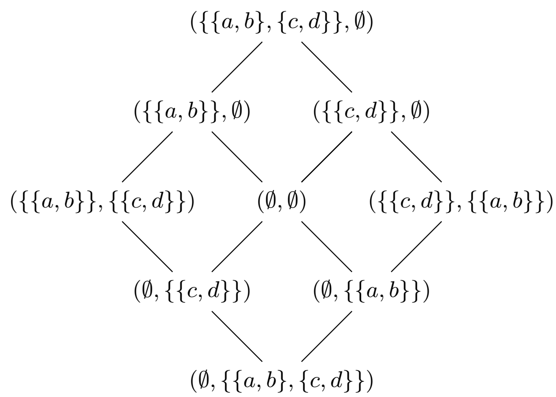

Fig. 22: Hasse diagram of $\mathbb{K}_O(\mathcal{C}')$

4.3 From a Kleene algebra to a refinement sequence

In this section, we associate a finite Kleene algebra with a refinement sequence and the respective sequences of orthopairs.

Let $(P, \leq)$ be a finite partially ordered set, and let n be the maximum number of elements of a chain in P. For each $i \in \{1,\ldots,n\}$ we define the i-th level of P as

$$P^i = \{\, N \in P \mid i = max\{|h| \mid h \text{ is a chain of } \downarrow N\}\,\}. \tag{30}$$

We denote by $\mathcal{M}(P)$ the set of maximal elements of P and we set $U_P = \{x_1,\ldots,x_m\}$, where $m = |P| + |\mathcal{M}(P)|$. We call *maximal sequence* of P the sequence $\mathcal{C} = (C_1,\ldots,C_n)$ built as follows. Suppose $\mathcal{M}(P)$ consists of nodes $N_1,\ldots,N_u$, where $u = |\mathcal{M}(P)| \leq \lfloor m/2 \rfloor$ since $u < 2u \leq |\mathcal{M}(P)| + |P| = m$. We set

$$b_{N_i} = \{x_{2i-1}, x_{2i}\} \tag{31}$$

for every $i = 1,\ldots,u$ and

$$C_n = \{b_{N_i} \mid N_i \in \mathcal{M}(P)\}. \tag{32}$$

Since $|P \backslash \mathcal{M}(P)| = m - 2u$, we denote by $N_{u+1},\ldots,N_{m-u}$ the nodes of $P \backslash \mathcal{M}(P)$ and we set $\alpha_P(N_i) = x_{i+u}$ for any $i \in \{u+1,\ldots,m-u\}$.

For each $N \notin \mathcal{M}(P)$, let

$$b_N = \bigcup_{M > N} b_M \cup \{\alpha_P(N)\} \tag{33}$$

and, for each $j \in \{1,\ldots,n-1\}$,

$$C_j = \{b_N \mid N \in P^j\} \cup \{b_M \mid M \in \mathcal{M}(P) \text{ and } \downarrow M \cap P^j = \emptyset\}. \tag{34}$$

It is trivial to see that for each $N, M \in P$

$$b_N \cap b_M = \cup\{\, b_L \mid L \in \uparrow N \cap \uparrow M\,\}. \tag{35}$$

Example 35. Let P be the partially ordered set with the following Hasse diagram.

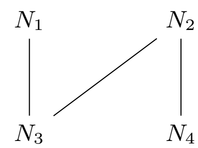

Fig. 23: Hasse diagram of P

$U_P = \{x_1, \ldots, x_6\}$, since $6 = 4 + 2$, where $|P| = 4$ and $|\mathcal{M}(P)| = 2$. We have $\alpha_P(N_3) = x_5$ and $\alpha_P(N_4) = x_6$. Then, we have $b_{N_1} = \{x_1, x_2\}$, $b_{N_2} = \{x_3, x_4\}$, $b_{N_3} = \{x_1, x_2\} \cup \{x_3, x_4\} \cup \{\alpha_P(N_3)\} = \{x_1, x_2, x_3, x_4, x_5\}$ and $b_{N_4} = \{x_3, x_4\} \cup \{\alpha_P(N_4)\} = \{x_3, x_4, x_6\}$. Moreover, $n = 2$, then the maximal sequence is made of two partial coverings of $\{x_1, \ldots, x_6\}$ that are $C_1 = \{\{x_1, x_2, x_3, x_4, x_5\}, \{x_3, x_4, x_6\}\}$ and $C_2 = \{\{x_1, x_2\}, \{x_3, x_4\}\}$.

Proposition 12. *Let P be a finite partially ordered set. Then, the maximal sequence $\mathcal{C}$ of P is a complete and safe refinement sequence of U_P and $\mathsf{SO}(\mathcal{C}) \cong \mathsf{K}(Up(P))$.*

Proof. Firstly, we prove that $\mathcal{C}$ is a refinement sequence of U_P. Then, suppose that $b \in C^i$ with $i > 1$, we have $b = b_N$ where $N \in P$. Since $b_N \in C^i$, two cases are possible: if $N \in P^i$, then there exists at least a node M of P^{i-1} such that $M < N$ (see 30), hence $b_M \in C^{i-1}$ (see 34) and $b_N \subset b_M$ (see 33); if $N \notin P^i$, then $N \in \mathcal{M}(P)$ and $\downarrow N \cap P^i = \emptyset$. In this latter case, we have two subcases to consider: $\downarrow N \cap P^{i-1} = \emptyset$ which implies $b_N \in C^{i-1}$ and $\downarrow N \cap P^{i-1} \neq \emptyset$ which implies that there exists $M \in P^{i-1}$ with $M \leq N$, hence $b_N \subseteq b_M$ where $b_M \in C^{i-1}$.

$\mathcal{C}$ is complete, since if $b_N \cap b_M \neq \emptyset$ with $b_N, b_M \in P_{\mathcal{C}}$, then $b_N \cap b_M \supseteq b_L$ with $L \in \uparrow N \cap \uparrow M$ (see 35), hence b_N and b_M can not belong to two upsets that are disjoint. To prove that $\mathcal{C}$ is safe, we consider the blocks $b_N, b_{N_1}, \ldots, b_{N_k}$ of coverings of $\mathcal{C}$ with $b_N \subseteq b_{N_1} \cup \ldots \cup b_{N_k}$. Then, we pick a subset $\{b_{N'_1}, \ldots, b_{N'_h}\}$ of $\{b_{N_1}, \ldots, b_{N_k}\}$ such that $b_N \subseteq b_{N'_1} \cup \ldots \cup b_{N'_h}$ and $b_N \cap b_{N'_i} \neq \emptyset$ for each $i \in \{1, \ldots, h\}$. Trivially, $b_N \cap b \neq \emptyset$ if and only if $b_N \subseteq b$, when $N \in \mathcal{M}(P)$. Otherwise, if $N \notin \mathcal{M}(P)$, by 33 we have that $\alpha_P(N) \in b_N$, hence $\alpha_P(N)$ belongs to b'_{N_i} for some $i \in \{1, \ldots, h\}$, then $b_N \subseteq b_{N'_i}$ since $N'_i \leq N$ (see 33).

By Proposition 7, $\mathsf{K}_O(\mathcal{C}) \subseteq \mathsf{K}(\mathcal{C})$. Vice-versa, let $(A, B) \in \mathsf{K}(\mathcal{C})$, then $A^* \cap B^* = \emptyset$, since otherwise, by 35, there exist $N, M, L \in P$ such that $b_L \subseteq b_N \cap b_M$, then $b_L \in A \cap B$ that is an absurd. By Theorem 19, $(A, B) \in \mathsf{K}_O(\mathcal{C})$. Therefore, $\mathsf{K}(\mathcal{C}) \subseteq \mathsf{K}_O(\mathcal{C})$.

Furthermore, observe that if $\mathcal{C} = (C_1, \ldots, C_n)$ is the maximal sequence of the poset P, then C_n is a partial partition of the respective universe U_P.

We remark that the maximal sequence $\mathcal{C} = (C_1, \ldots, C_n)$ of a given partially ordered set P is not the only complete and safe refinement sequence having the assigned poset isomorphic to P. We can generate such sequences in addressing numerous ways. For example, we can build a sequence $\mathcal{C}^*$ by adopting the previous procedure, but by assigning a set A_i made of at least three elements to the maximal node N_i of P, for each $i \in \{1, \ldots, m\}$. Trivially, if the sets $A_1, \ldots, A_m$ are pairwise disjoints, then $\mathcal{C}^*$ is a complete and safe refinement sequence satisfying $P_{\mathcal{C}^*} \cong P_{\mathcal{C}}$. Clearly, we can also generate a safe and complete refinement with its poset isomorphic to P by starting from the maximal sequence $\mathcal{C}$. For example, we can add a finite set disjoint with U_P to each block of an upsets of $\mathcal{C}$. On the other hand, we observe that the universe covered by any safe and complete refinement sequence with its poset isomorphic to P has cardinality grater that $|U_P|$.

By Theorem 9 and Proposition 12, the following theorem holds.

Theorem 24. *Let P be a partially ordered set and $\mathcal{C}$ its maximal sequence. Then, $\mathbb{K}_O(\mathcal{C})$ is a centered Kleene algebra that satisfies the interpolation property.*

4.4 Representation theorems

Considering that $\mathsf{K}_O(\mathcal{C})$ coincides with the set of sequences of orthopairs generated by $\mathcal{C}$ (see Theorem 17), we can define on $\mathsf{SO}(\mathcal{C})$ the following operations.

Definition 50. *Let $\mathcal{C}$ be a refinement sequence of U and let α be the function defined in 17. Then, let $X, Y \subseteq U$, we set*

- $\mathcal{O}(X) \curlywedge \mathcal{O}(Y) := \alpha^{-1}((X^1, X^2) \cap_{\mathsf{K}_O} (Y^1, Y^2))$;
- $\mathcal{O}(X) \curlyvee \mathcal{O}(Y) := \alpha^{-1}((X^1, X^2) \cup_{\mathsf{K}_O} (Y^1, Y^2))$;
- $\sim\ \mathcal{O}(X) := \alpha^{-1}(\neg_{\mathsf{K}_O} (X^1, X^2))$;
- $\mathcal{O}(X) \odot_i \mathcal{O}(Y) := \alpha^{-1}((X^1, X^2) \star^i_{\mathsf{K}_O} (Y^1, Y^2))$, *for* $i \in \{2, 3, 4\}$;
- $\mathcal{O}(X) \hookrightarrow_i \mathcal{O}(Y) := \alpha^{-1}((X^1, X^2) \to^i_{\mathsf{K}_O} (Y^1, Y^2))$, *for* $i \in \{1, 2, 3, 4\}$.

Moreover, given a refinement sequence $\mathcal{C} = (C_1, \ldots, C_n)$, we set

$$\bot_{\mathcal{C}} = (\bot_1, \ldots, \bot_n) \text{ and } \top_{\mathcal{C}} = \sim \bot_{\mathcal{C}},$$

where $\bot_i = (\emptyset, \{x \in b \mid b \in C_i\})$, for each i from 1 to n. Then, it is clear that $\bot_{\mathcal{C}}$ and $\top_{\mathcal{C}}$ are respectively the minimum and the maximum of $\mathsf{SO}(\mathcal{C})$. Moreover, we set $e_{\mathcal{C}} = ((\emptyset, \emptyset), \ldots, (\emptyset, \emptyset))$, that is $\alpha^{-1}((\emptyset, \emptyset))$.

Theorem 25. *Let $\mathbb{S}$ be a Kleene algebra. $\mathbb{S}$ is a finite centered Kleene algebra with interpolation property if and only if*

$$\mathbb{S} \cong (\mathsf{SO}(\mathcal{C}), \curlywedge, \curlyvee, \sim, \bot_{\mathcal{C}}, \top_{\mathcal{C}}),$$

where $\mathcal{C}$ is a complete refinement sequence of a finite universe U.

Proof. ($\Rightarrow$). If $\mathbb{S}$ is a centered Kleene algebra with interpolation property, then there exists a bounded distributive lattice $L_{\mathbb{S}}$ such that $\mathbb{S} \cong K(L_{\mathbb{S}})$, by Theorem 9. By Birkhoff representation theorem, there exists a poset $P_{L_{\mathbb{S}}}$ such that $L_{\mathbb{S}} \cong U(P_{L_{\mathbb{S}}})$. Consequently, $\mathbb{S} \cong \mathsf{K}(U(P_{L_{\mathbb{S}}}))$. By Proposition 12, $\mathcal{C}$ is the maximal sequence of $P_{L_{\mathbb{S}}}$, that is a complete and safe refinement sequence of $U_{P_{L_{\mathbb{S}}}}$.

($\Leftarrow$). By the theorems 17 and 23, if $\mathcal{C}$ is complete, then $(\mathsf{SO}(\mathcal{C}), \curlywedge, \curlyvee, \sim, \bot_{\mathcal{C}}, \top_{\mathcal{C}})$ is a centered Kleene algebra with the interpolation property.

Similarly, by using the theorems of Section 2.3, we can present some classes of finite many-valued structures such that their reduct is a centered Kleene algebra with the interpolation property as sequences of orthopairs. More precisely, the following theorems hold.

Theorem 26. *Let $\mathbb{S}$ be a Nelson algebra. $\mathbb{S}$ is a finite centered Nelson algebra with interpolation property if and only if*

$$\mathbb{S} \cong (\mathsf{SO}(\mathcal{C}), \curlywedge, \curlyvee, \sim, \odot_1, \hookrightarrow_1, \bot_{\mathcal{C}}, \top_{\mathcal{C}}),$$

where $\mathcal{C}$ is a complete refinement sequence of a finite universe U.

Theorem 27. *Let $\mathbb{S}$ be a Nelson lattice. $\mathbb{S}$ is a finite centered Nelson lattice with interpolation property if and only if*

$$\mathbb{S} \cong (\mathsf{SO}(\mathcal{C}), \curlywedge, \curlyvee, \sim, \odot_2, \hookrightarrow_2, e_{\mathcal{C}}, \bot_{\mathcal{C}}, \top_{\mathcal{C}}),$$

where $\mathcal{C}$ is a complete refinement sequence of a finite universe U.

Theorem 28. *Let $\mathbb{S}$ be a IUML-algebra. $\mathbb{S}$ is a finite IUML-algebra if and only if*

$$\mathbb{S} \cong (\mathsf{SO}(\mathcal{C}), \curlywedge, \curlyvee, \sim, \odot_3, \hookrightarrow_3, e_{\mathcal{C}}, \bot_{\mathcal{C}}, \top_{\mathcal{C}}),$$

where $\mathcal{C}$ is a refinement sequence of partial partitions of a finite universe U.

Theorem 29. *Let $\mathbb{S}$ be a KLI^*-algebra. $\mathbb{S}$ is finite and satisfies the interpolation property if and only if*

$$\mathbb{S} \cong (\mathsf{SO}(\mathcal{C}), \curlywedge, \curlyvee, \sim, \odot_4, \hookrightarrow_4, \bot_{\mathcal{C}}, \top_{\mathcal{C}}),$$

where $\mathcal{C}$ is a complete refinement sequence of a finite universe U.

4.5 Operations between sequences of orthopairs

In this section, we focus on operations between sequences of orthopairs. In particular, we show how they can be obtained starting from the operations between orthopairs of an individual covering. The latter are listed in Section 2.2.

Theorem 30. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a safe and complete refinement sequence of U and let $X, Y \subseteq U$, then*

1. $\mathcal{O}_{\mathcal{C}}(X) \curlywedge \mathcal{O}_{\mathcal{C}}(Y) = ((A_1, B_1), \ldots, (A_n, B_n))$,

2. $\mathcal{O}_{\mathcal{C}}(X) \curlyvee \mathcal{O}_{\mathcal{C}}(Y) = ((D_1, E_1), \dots, (D_n, E_n))$,
3. $\sim \mathcal{O}_{\mathcal{C}}(X) = ((F_1, G_1), \dots, (F_n, G_n))$,

where

1. $(A_i, B_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) \wedge_{\mathcal{K}} (\mathcal{L}_i(Y), \mathcal{E}_i(Y))$
2. $(D_i, E_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) \vee_{\mathcal{K}} (\mathcal{L}_i(Y), \mathcal{E}_i(Y))$
3. $(F_i, G_i) = \neg(\mathcal{L}_i(X), \mathcal{E}_i(X))$,

for each $i \in \{1, \dots, n\}$*. The operations* $\wedge_{\mathcal{K}}$ *and* $\vee_{\mathcal{K}}$ *are given in Definition 8, and* $\neg(A, B) = (B, A)$.

Proof. We only provide the proof of point 1, since we can demonstrate the remaining cases in a similar way. Then, we suppose that Z is the subset of U such that $\mathcal{O}_{\mathcal{C}}(X) \curlywedge \mathcal{O}_{\mathcal{C}}(Y) = \mathcal{O}_{\mathcal{C}}(Z)$. Since $\mathcal{C}$ is safe, $\mathcal{O}_{\mathcal{C}}(X) \curlywedge \mathcal{O}_{\mathcal{C}}(Y) = \alpha^{-1}((X^1, X^2) \sqcap (Y^1, Y^2)) = \alpha^{-1}((X^1 \cap Y^1, X^2 \cup Y^2))$. Then, $Z^1 = X^1 \cap Y^1$ and $Z^2 = X^2 \cup Y^2$. On the other hand, we recall that

$$(\mathcal{L}_i(X), \mathcal{E}_i(X)) \wedge_{\mathcal{K}} (\mathcal{L}_i(Y), \mathcal{E}_i(Y)) = (\mathcal{L}_i(X) \cap \mathcal{L}_i(Y), \mathcal{E}_i(X) \cup \mathcal{E}_i(Y)).$$

So, fixed $i \in \{1, \dots, n\}$, $x \in \mathcal{L}_i(Z)$ if and only if there exists $N \in P_{\mathcal{C}}$ such that $N \subseteq Z$. Therefore, there exists $N \in P_{\mathcal{C}}$ such that $N \in X^1 \cap Y^1$, and so $N \subseteq X \cap Y$. This is equivalent to say that $x \in \mathcal{L}_i(X) \cap \mathcal{E}_i(Y)$. Similarly, we can prove that $x \in \mathcal{E}_i(Z)$ if and only if $\mathcal{E}_i(X) \cup \mathcal{E}_i(Y)$.

Example 36. Let $\mathcal{C} = (C_1, C_2)$ be the refinement sequence of $\{a, b, c, d, e\}$, such that $C_1 = \{\{a, b, c, d, e\}\}$ and $C_2 = \{\{a, b\}, \{c, d\}\}$. Since $\mathcal{C}$ is safe and complete, the previous theorem holds. Then,

$$\mathcal{O}_{\mathcal{C}}(\{a, b\}) \curlywedge \mathcal{O}_{\mathcal{C}}(\{a, b, c\}) = ((\emptyset, \emptyset), (\{a, b\}, \{c, d\})),$$

where

$(\mathcal{L}_1(\{a, b\}), \mathcal{E}_1(\{a, b\})) \wedge_{\mathcal{K}} (\mathcal{L}_1(\{a, b, c\}), \mathcal{E}_1(\{a, b, c\})) = (\emptyset, \emptyset) \wedge_{\mathcal{K}} (\emptyset, \emptyset) = (\emptyset, \emptyset)$.
$(\mathcal{L}_2(\{a, b\}), \mathcal{E}_2(\{a, b\})) \wedge_{\mathcal{K}} (\mathcal{L}_2(\{a, b, c\}), \mathcal{E}_2(\{a, b, c\})) = (\{a, b\}, \{c, d\}) \wedge_{\mathcal{K}} (\{a, b\}, \emptyset) = (\{a, b\}, \{c, d\}))$.

Moreover,

$$\mathcal{O}_{\mathcal{C}}(\{a, b\}) \curlyvee \mathcal{O}_{\mathcal{C}}(\{a, b, c\}) = ((\emptyset, \emptyset), (\{a, b\}, \emptyset)),$$

where

$(\mathcal{L}_1(\{a, b\}), \mathcal{E}_1(\{a, b\})) \vee_{\mathcal{K}} (\mathcal{L}_1(\{a, b, c\}), \mathcal{E}_1(\{a, b, c\})) = (\emptyset, \emptyset) \vee_{\mathcal{K}} (\emptyset, \emptyset) = (\emptyset, \emptyset)$.
$(\mathcal{L}_2(\{a, b\}), \mathcal{E}_2(\{a, b\})) \vee_{\mathcal{K}} (\mathcal{L}_2(\{a, b, c\}), \mathcal{E}_2(\{a, b, c\})) = (\{a, b\}, \{c, d\}) \vee_{\mathcal{K}} (\{a, b\}, \emptyset) = (\{a, b\}, \emptyset))$.

Moreover,

$$\sim \mathcal{O}_{\mathcal{C}}(\{a, b\}) = ((\emptyset, \emptyset), (\{c, d\}, \{a, b\})),$$

where

$(\mathcal{L}_1(\{a,b\}), \mathcal{E}_1(\{a,b\})) = \neg(\emptyset,\emptyset) = (\emptyset,\emptyset)$;
$(\mathcal{L}_2(\{a,b\}), \mathcal{E}_2(\{a,b\})) = \neg(\{a,b\},\{c,d\}) = (\{c,d\},\{a,b\})$.

The following theorems allow us to express the operations $\hookrightarrow_1$, $\star_2$, $\hookrightarrow_2$, $\star_3$ and $\hookrightarrow_3$ through the operations between orthopairs of an individual covering (see Definition 11 and Definition 12). We present the proof only for the operation $\odot_3$ of Theorem 33, because it is possible to give the proof for the other operations with similar procedures. We recall that, given a refinement sequence $\mathcal{C} = (C_1, \ldots, C_n)$, in Remark 10, we denote the union of all blocks of C_i with U_i, for each $i \in \{1, \ldots, n\}$.

Theorem 31. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a safe and complete refinement sequence of U. Then,*

$$\mathcal{O}_{\mathcal{C}}(X) \hookrightarrow_1 \mathcal{O}_{\mathcal{C}}(Y)$$

is the sequence $((A_1,B_1), \ldots, (A_n,B_n))$ defined as follows. Firstly, we set

$$(A'_i, B'_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) \to_{\mathcal{N}} (\mathcal{L}_i(Y), \mathcal{E}_i(Y)),$$

for each i from 1 to n. Then, we set $(A_n, B_n) = (A'_n, B'_n)$ and

$$A_i = A'_i \setminus \cup\{N \in C_i \mid N' \subseteq N \text{ with } N' \in C_{i+1} \text{ and } N' \subseteq U_{i+1} \setminus A_{i+1}\},$$

and $B_i = B'_i$ for each $i < n$.

Theorem 32. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a safe and complete refinement sequence of U. Then,*

$$\mathcal{O}_{\mathcal{C}}(X) \odot_2 \mathcal{O}_{\mathcal{C}}(Y)$$

is the sequence $((A_1,B_1), \ldots, (A_n,B_n))$ defined as follows. Firstly, we set

$$(A'_i, B'_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) *_{\mathcal{L}} (\mathcal{L}_i(Y), \mathcal{E}_i(Y)),$$

for each i from 1 to n. Then, we set $(A_n, B_n) = (A'_n, B'_n)$, $A_i = A'_i$, and

$$B_i = B'_i \setminus \cup\{N \in C_i \mid N' \subseteq N \text{ with } N' \in C_{i+1} \text{ and } N' \subseteq U_{i+1} \setminus B_{i+1}\}$$

for each $i < n$. Moreover,

$$\mathcal{O}_{\mathcal{C}}(X) \hookrightarrow_2 \mathcal{O}_{\mathcal{C}}(Y)$$

is the sequence defined as follows. Firstly, we set

$$(A'_i, B'_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) \to_{\mathcal{L}} (\mathcal{L}_i(Y), \mathcal{E}_i(Y)),$$

for each i from 1 to n. Then, we set $(A_n, B_n) = (A'_n, B'_n)$,

$$A_i = A'_i \setminus \cup\{N \in C_i \mid N' \subseteq N \text{ with } N' \in C_{i+1} \text{ and } N' \subseteq U_{i+1} \setminus A_{i+1}\},$$

and $B_i = B'_i$, for each $i < n$.

Theorem 33. *Let $\mathcal{C} = (C_1, \dots, C_n)$ be a safe refinement sequence of partial partitions of U, then*

$$\mathcal{O}_{\mathcal{C}}(X) \odot_3 \mathcal{O}_{\mathcal{C}}(Y)$$

is the sequence of orthopairs $((A_1, B_1), \dots, (A_n, B_n))$ defined as follows. Firstly we set

$$(A'_i, B'_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) *_{\mathcal{S}} (\mathcal{L}_i(Y), \mathcal{E}_i(Y))$$

for each i from 2 to n. Then, we set $(A_1, B_1) = (A'_1, B'_1)$,

$$A_i = A'_i \cup \{N \in C_i \mid N \subseteq A_{i-1}\},\ \textit{and}\ B_i = B'_i \setminus A_i,$$

for each $i > 0$.

Moreover,

$$\mathcal{O}_{\mathcal{C}}(X) \hookrightarrow_3 \mathcal{O}_{\mathcal{C}}(Y)$$

is the sequence of orthopairs $((A_1, B_1), \dots, (A_n, B_n))$ defined as follows. Firstly, we set

$$(A'_i, B'_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) \rightarrow_{\mathcal{S}} (\mathcal{L}_i(Y), \mathcal{E}_i(Y))$$

for each $i > 2$. Then, we set

$$(A_1, B_1) = (A'_1, B'_1),\ B_i = B'_i \cup \{N \in P_i \mid N \subseteq B_{i-1}\},\ \textit{and}\ A_i = A'_i \setminus B_i,$$

for each $i > 0$.

In order to prove Theorem 33, we need to move from sequences of orthopairs to pairs of disjoint upsets. Let $\mathcal{C}$ be a refinement sequence of U such that $\mathcal{C} = \mathcal{C}'$. Then, the operation $\star^3_{\mathsf{K}_O}$ coincides with $\star_3$ on $\mathsf{K}(\mathcal{C})$. Indeed, $\mathcal{C} = \mathcal{C}'$ implies that β is the identity function (β is defined in Theorem 22). Consequently, for any $X, Y \subseteq U$, we have $(X^1, X^2) \star^3_{\mathsf{K}_O} (Y^1, Y^2) = \beta^{-1}((X^1, X^2) \star_3 (Y^1, Y^2)) = (X^1, X^2) \star_3 (Y^1, Y^2)$.

On the other hand, if $\mathcal{C} \neq \mathcal{C}'$ the IUML-algebras $\mathsf{K}_O(\mathcal{C})$ and $\mathsf{K}_O(\mathcal{C}')$ are not isomorphic. In any case, we can find a relationship between operations in $\mathsf{K}_O(\mathcal{C}')$ and Sobociński conjunction, as follows.

Proposition 13. *Let $\mathcal{C}$ be a refinement sequence of partial partitions of U, let $X, Y \subseteq U$, and let $F^{\mathcal{C}}_X$ be the function defined by Equation 1. Then,*

$$(X^1_{\mathcal{C}}, X^2_{\mathcal{C}}) \star^3_{\mathsf{K}_O} (Y^1_{\mathcal{C}}, Y^2_{\mathcal{C}}) = \beta^{-1}((Z^1_{\mathcal{C}'}, Z^2_{\mathcal{C}'})),$$

where

$$Z^1_{\mathcal{C}'} = \uparrow \{N \in P_{\mathcal{C}'} \mid F^{\mathcal{C}'}_X(N) \circledast_{\mathcal{S}} F^{\mathcal{C}'}_Y(N) = 1\}$$

and

$$Z^2_{\mathcal{C}'} = \{N \in P_{\mathcal{C}'} \mid F^{\mathcal{C}'}_X(N) \circledast_{\mathcal{S}} F^{\mathcal{C}'}_Y(N) = 0\} \setminus Z^1_{\mathcal{C}'}.$$

Proof. By Definition 49, we must prove that $Z^1_{\mathcal{C}'} = (X^1_{\mathcal{C}'} \cap Y^1_{\mathcal{C}'}) \cup (X \diamond Y)$ and $Z^2_{\mathcal{C}'} = (X^2_{\mathcal{C}'} \cup Y^2_{\mathcal{C}'}) \backslash (X \diamond Y)$, where $X \diamond Y$ is related to $\mathcal{C}'$.

A node N belongs to $(X^1_{\mathcal{C}'} \cap Y^1_{\mathcal{C}'}) \cup (X \diamond Y)$ if and only if $F_X(N) = 1$ and $F_Y(N) = 1$, or there exists $M \in P_{\mathcal{C}'}$ such that $N \subseteq M$ and $F_X(M) = 1$ and $F_Y(M) = 1\backslash 2$, or $F_X(M) = 1\backslash 2$ and $F_Y(M) = 1$. This is equivalent to affirm that $F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 1$ or there exists $M \in P_{\mathcal{C}'}$ such that $N \subseteq M$ and $F_X(M) \circledast_{\mathcal{S}} F_Y(M) = 1$, since $\circledast_{\mathcal{S}}$ is the Sobociński conjunction.

Similarly, N belongs to $(X^2_{\mathcal{C}'} \cup Y^2_{\mathcal{C}'}) \backslash (X \diamond Y)$ if and only if $F_X(N) = 0$ or $F_Y(N) = 0$ and there does not exist $M \in P_{\mathcal{C}'}$ such that $N \subseteq M$ and $F_X(M) \circledast_{\mathcal{S}} F_Y(M) = 1$. Then, $N \in \{N \in P_{\mathcal{C}'} \mid F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 0\} \setminus Z^1$.

Proof (Theorem 33). By definition of α (see Theorem 17), we have $(X^1, X^2) = \alpha(\mathcal{O}_{\mathcal{C}}(X))$, $(Y^1, Y^2) = \alpha(\mathcal{O}_{\mathcal{C}}(Y))$. Let Z be the subset of U such that

$$(Z^1, Z^2) = \alpha(\mathcal{O}_{\mathcal{C}}(X)) \odot_3 \alpha(\mathcal{O}_{\mathcal{C}}(Y)).$$

By induction on i we prove that $(\mathcal{L}_i(Z), \mathcal{E}_i(Z)) = (A_i, B_i)$.

Let $i = 1$. By definition and recalling that $Z^1 = \{N \in P_{\mathcal{C}} \mid N \subseteq Z\}$, we have

$$\mathcal{L}_1(Z) = \bigcup \{N \in C_1 \mid N \subseteq Z\} = \bigcup \{N \in C_1 \cap Z^1\}.$$

By Proposition 13, $Z^1 = \uparrow \{N \in P_{\mathcal{C}} \mid F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 1\}$, hence $Z^1 \cap C_1 = \{N \in C_1 \mid F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 1\}$. We have, by Proposition 4:

$$\mathcal{L}_1(Z) = \bigcup \{N \in C_1 \mid F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 1\} = A_1.$$

Now, we fix $i > 1$ and suppose by induction hypothesis that $A_{i-1} = \mathcal{L}_{i-1}(Z)$. Then by Proposition 4 and 13,

$$\mathcal{L}_i(Z) = \bigcup_{N \in Z^1 \cap C_i} N =$$

$$= \bigcup \{N \in C_i \mid F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 1\} \cup \bigcup \{N \in C_i \mid N \subseteq M \text{ with } M \in Z^1 \cap C_{i-1}\}.$$

We notice that $A'_i = \cup\{N \in C_i \mid F_X(N) \circledast_{\mathcal{S}} F_Y(N) = 1\}$ and $A_{i-1} = \mathcal{L}_{i-1}(Z) = \cup\{M \mid M \in Z^1 \cap C_{i-1}\}$. Consequently,

$$\mathcal{L}_i(Z) = A'_i \cup \{N \in C_i \mid N \subseteq A_{i-1}\}.$$

Similarly, by Propositions 4 and 13, we can prove that $B_i = B'_i \setminus A_i$, for each $i \in \{1, \ldots, n\}$.

In other words, the operation $\odot_3$ maps each pair of sequences of orthopairs to the sequence of orthopairs given by applying the Sobociński conjunction between orthopairs relative to the same partition and then closing with respect to the inclusion in the first component.

Hence, we can say that if we apply $\odot_3$ to sequences of orthopairs, the indeterminate value is always overcome by the determined ones, and in addition, as soon as a determined value is reached with respect to a given level of partial partitions, it is automatically given to all the blocks in the next refinements.

Example 37. Let $\mathcal{C}'$ be the refinement sequence of U of Example 16. We consider $X, Y \subseteq U$ such that $\mathcal{O}_{\mathcal{C}'}(X)$ is equal to $\mathcal{O}_{\mathcal{C}}(X)$ defined in Example 24 and $\mathcal{O}_{\mathcal{C}'}(Y) = (\mathcal{O}_{C'_1}(Y), \mathcal{O}_{C'_2}(Y), \mathcal{O}_{C'_3}(Y))$, where

$\mathcal{O}_{C'_1}(Y) = (\emptyset, \emptyset)$,
$\mathcal{O}_{C'_2}(Y) = (\{u_3, u_4\}, \{u_5, u_6, u_{15}, \ldots, u_{20}\})$ and
$\mathcal{O}_{C'_3}(Y) = (\{u_3, u_4, u_7, u_8\}, \{u_5, u_6, u_{11}, u_{12}, u_{15}, \ldots, u_{18}\})$.

Hence,

$\mathcal{O}_{C'_1}(X) *_{\mathcal{S}} \mathcal{O}_{C'_1}(Y) = (\emptyset, \emptyset)$,
$\mathcal{O}_{C'_2}(X) *_{\mathcal{S}} \mathcal{O}_{C'_2}(Y) = (\{u_7, \ldots, u_{14}\}, \{u_1, \ldots, u_6, u_{15}, \ldots, u_{20}\})$,
$\mathcal{O}_{C'_3}(X) *_{\mathcal{S}} \mathcal{O}_{C'_3}(Y) = (\{u_7, u_8\}, \{u_1, \ldots, u_6, u_{11}, u_{12}, u_{15}, \ldots, u_{18}\})$.

Then, in order to close with respect to the inclusion in the first component, we add the elements of block $\{u_{11}, u_{12}\}$ to the first component of $\mathcal{O}_{C'_3}(X) *_{\mathcal{S}} \mathcal{O}_{C'_3}(Y)$ and we subtract them from the second component of $\mathcal{O}_{C'_3}(X) *_{\mathcal{S}} \mathcal{O}_{C'_3}(Y)$.

Finally, we obtain that $\mathcal{O}_{\mathcal{C}'}(X) \odot_3 \mathcal{O}_{\mathcal{C}'}(Y)$ is the sequence of $\mathsf{SO}(\mathcal{C}')$ made of the following pairs.

$(\emptyset, \emptyset)$,
$(\{u_7, \ldots, u_{14}\}, \{u_1, \ldots, u_6, u_{15}, \ldots, u_{20}\})$ and
$(\{u_7, u_8, u_{11}, u_{12}\}, \{u_1, \ldots, u_6, u_{15}, \ldots, u_{18}\})$.

We observe that $\mathcal{O}_{\mathcal{C}'}(X) \odot_3 \mathcal{O}_{\mathcal{C}'}(Y)$ provides precise information about the blocks $\{u_{15}, \ldots, u_{20}\}$, $\{u_1, u_2\}$, $\{u_7, \ldots, u_{10}\}$ and $\{u_{11}, \ldots, u_{14}\}$, while we do not know what happens to the elements u_{19} and u_{20} in $\mathcal{O}_{\mathcal{C}'}(X)$ and to the elements u_1, u_2, u_9, u_{10},
u_{13} and u_{14} in $\mathcal{O}_{\mathcal{C}'}(Y)$. Hence, the uncertainty represented by the sequence $\mathcal{O}_{\mathcal{C}'}(X) \odot_3 \mathcal{O}_{\mathcal{C}'}(Y)$ is smaller than uncertainty that is in $\mathcal{O}_{\mathcal{C}'}(X)$ and $\mathcal{O}_{\mathcal{C}'}(Y)$.

Remark 20. The operations $\odot_4$ and $\hookrightarrow_4$ are not obtained by the generalization of some three-valued connectives. On the other hand, they allow us to define a new pair of operations between orthopairs that is following.

Let C be a covering of U, and let $X, Y \subseteq U$. Then,

$$(\mathcal{L}(X), \mathcal{E}(X)) \odot_4 (\mathcal{L}(Y), \mathcal{E}(Y)) = \begin{cases} (\emptyset, U), & \text{if } \mathcal{L}(X)=\emptyset \text{ and } \mathcal{L}(Y)=\emptyset; \\ (\mathcal{L}(X), \mathcal{E}(X)), & \text{if } \mathcal{L}(X)=\emptyset \text{ and } \mathcal{L}(Y)\neq\emptyset; \\ (\mathcal{L}(Y), \mathcal{E}(Y)), & \text{if } \mathcal{L}(X)\neq\emptyset \text{ and } \mathcal{L}(Y)=\emptyset; \\ (\mathcal{L}(X)\cap\mathcal{L}(Y), \mathcal{E}(X)\cap\mathcal{E}(Y)), & \text{if } \mathcal{L}(X)\neq\emptyset \text{ and } \mathcal{L}(Y)\neq\emptyset. \end{cases} \tag{36}$$

and

$$(\mathcal{L}(X), \mathcal{E}(X)) \hookrightarrow_4 (\mathcal{L}(Y), \mathcal{E}(Y)) = \begin{cases} (U, \emptyset), & \text{if } \mathcal{L}(X)=\emptyset \text{ and } \mathcal{E}(Y)=\emptyset; \\ (\mathcal{E}(X), \mathcal{L}(X)), & \text{if } \mathcal{L}(X)=\emptyset \text{ and } \mathcal{E}(Y)\neq\emptyset; \\ (\mathcal{L}(Y), \mathcal{E}(Y)), & \text{if } \mathcal{L}(X)\neq\emptyset \text{ and } \mathcal{E}(Y)=\emptyset; \\ (\mathcal{E}(X)\cap\mathcal{L}(Y), \mathcal{L}(X)\cap\mathcal{E}(Y)), & \text{if } \mathcal{L}(X)\neq\emptyset \text{ and } \mathcal{E}(Y)\neq\emptyset. \end{cases} \tag{37}$$

4.6 Application scenario

In this section, we explain how an examiner's opinion on a number of candidates applying for a job can be represented by a sequence of orthopairs. Also, we show how opinions of two or more examiners can be combined by employing the operations $\curlywedge$, $\curlyvee$, $\odot_2$, $\odot_3$ and $\odot_4$ in order to get a final decision on each candidate. Moreover, such results are found in [21].

Imagine that a food company needs to recruit staff through a commission composed of several examiners, and managed by a committee chair. We indicate the set of twenty-four candidates with $\{c_1, \ldots, c_{24}\}$. The first selection will be to investigate the curriculum vitae of each candidate, after that all shortlisted applicants will be called for the first job interview. We suppose that the chair identifies some groups of applicants of $\{c_1, \ldots, c_{24}\}$ that have some specific characteristics which in his/her opinion are useful to work for the given company. Step by step, as it will be explained, the chair continues to refine each of these groups by identifying other suitable characteristics to work for the company. We underline that the chair selects sets made of applicants that have a specific characteristic in order to allow to each examiner to express his / her opinion on groups of candidates and not on every individual candidate. In this way, the first selection process is simplified.

In detail, the refinement process is made as follows. Initially, the chair identifies two characteristics: "to have a master degree in chemistry" and "to have a master degree in biology". Consequently, the covering $C_1 = \{b_1, b_2\}$ of $\{c_1, \ldots, c_{24}\}$ is determined, where $b_1 = \{c_1, \ldots, c_{12}\}$ is made of candidates with a master degree in chemistry and $b_2 = \{c_{13}, \ldots, c_{23}\}$ is made of candidates with a master degree in biology. Successively, the chair decides that the best candidates of b_1 are those specialized in "industrial chemistry", namely those of the set $b_3 = \{c_1, \ldots, c_5\}$ or in "pharmaceutical technology", namely the candidates of the set $b_4 = \{c_6, \ldots, c_{11}\}$. Moreover, the chair thinks that the best candidates of b_2 are those of $b_5 = \{c_{13}, \ldots, c_{17}\}$ that are specialized in "Biology of immunology" and those of $b_6 = \{c_{18}, \ldots, c_{22}\}$ that are specialized in "Food biology". In this way, the partial covering $C_2 = \{b_3, b_4, b_5, b_6\}$ of $\{c_1, \ldots, c_{24}\}$ is determined. Eventually, the chair considers $b_7 = \{c_1, c_2\}, b_8 = \{c_3, c_4\}, b_9 = \{c_6, c_7\}, b_{10} = \{c_8, c_9\}, b_{11} = \{c_{13}, c_{14}\}, b_{12} = \{c_{15}, c_{16}\}$ and $b_{13} = \{c_{18}, c_{19}\}$ and $b_{14} = \{c_{20}, c_{21}\}$, where b_7, b_9, b_{11} and b_{13} are respectively the subsets of b_3, b_4, b_5 and b_6 of candidates that have a certificate of Spanish language, instead b_8, b_{10}, b_{12} and b_{14} are respectively the subsets of b_3, b_4, b_5, b_6 of candidates that have a certificate of French language. Trivially, $C_3 = \{b_7, \ldots, b_{14}\}$ is also a partial covering of $\{c_1, \ldots, c_{24}\}$, and $\mathcal{C} = (C_1, C_2, C_3)$ is a refinement sequence of $\{c_1, \ldots, c_{24}\}$. More precisely, C_1, C_2 and C_3 are partial partitions of $\{c_1, \ldots, c_{24}\}$. The data used for the chair's classification are contained in the incomplete information table as Table 12, where $\{c_1, \ldots, c_{24}\}$ is the universe and {Master degree, Specialization, Language certification} is the set of attributes. The poset assigned to $\mathcal{C}$ is a forest, and it is shown in Figure 24.

It is easy to notice that $\mathcal{C}$ is safe and complete.

Clearly, $P_{\mathcal{C}}$ is isomorphic to the forest of Figure 25.

Each node of Figure 25 is the set of all values contained in Table 12 that characterizes the block of candidates of the respective node in $P_{\mathcal{C}}$ (we set

	Master degree	*Specialization*	*Language certification*
c_1	Chemistry	Industrial Chemistry	Spanish
c_2	Chemistry	Industrial Chemistry	Spanish
c_3	Chemistry	Industrial Chemistry	French
c_4	Chemistry	Industrial Chemistry	French
c_5	Chemistry	Industrial Chemistry	$\times$
c_6	Chemistry	Pharmaceutical Technology	Spanish
c_7	Chemistry	Pharmaceutical Technology	Spanish
c_8	Chemistry	Pharmaceutical Technology	French
c_9	Chemistry	Pharmaceutical Technology	French
c_{10}	Chemistry	Pharmaceutical Technology	$\times$
c_{11}	Chemistry	Pharmaceutical Technology	$\times$
c_{12}	Chemistry	$\times$	$\times$
c_{13}	Biology	Immunology	Spanish
c_{14}	Biology	Immunology	Spanish
c_{15}	Biology	Immunology	Spanish
c_{16}	Biology	Immunology	French
c_{17}	Biology	Immunology	$\times$
c_{18}	Biology	Food Biology	Spanish
c_{19}	Biology	Food Biology	Spanish
c_{20}	Biology	Food Biology	French
c_{21}	Biology	Food Biology	French
c_{22}	Biology	Food Biology	$\times$
c_{23}	Biology	$\times$	$\times$
c_{24}	$\times$	$\times$	$\times$

Table 12: Information table of the candidates

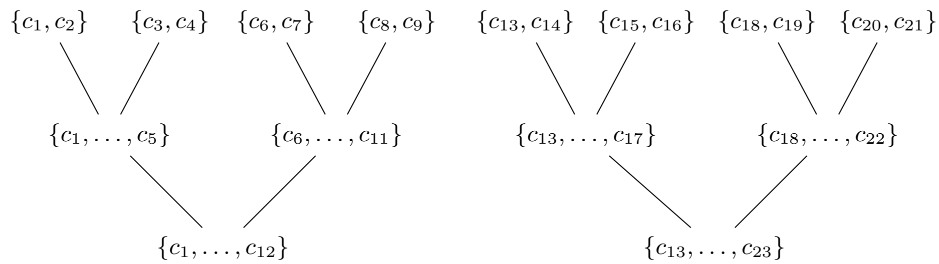

Fig. 24: Forest of the candidates

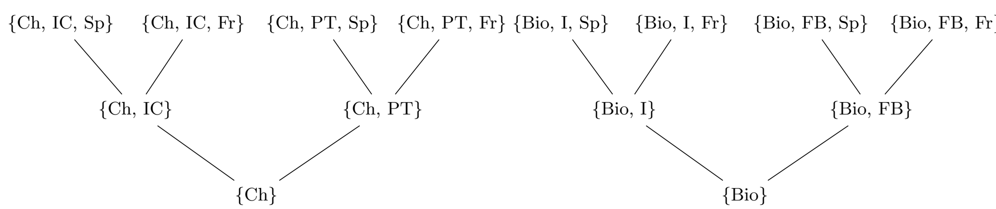

Fig. 25: Forest of the values of the candidates

Ch=Chemistry, IC=Industrial Chemistry, PT=Pharmaceutical Technology, Bio= Biology, I=Immunology, FB=Pharmaceutical Technology, Sp=Spanish, Fr=French). As an example, $\{Ch, IC, Fr\}$ is the set of the values that characterize the block $\{c_3, c_4\}$.

Once the classification process is completed, the chair invites every examiner to express his / her opinion about every block of $P_{\mathcal{C}}$, starting from the blocks that are minimal elements of $P_{\mathcal{C}}$ to those that are maximal elements of $P_{\mathcal{C}}$. Namely, examiners must first reveal their point of view on the nodes of level 0 of $P_{\mathcal{C}}$, then on those of level 1 of $P_{\mathcal{C}}$, and finally on those of level 2 of $P_{\mathcal{C}}$. For example, they can evaluate the blocks of $P_{\mathcal{C}}$ by following this order: $\{c_1, \dots, c_{12}\}$, $\{c_{13}, \dots, c_{23}\}$, $\{c_1, \dots, c_5\}$, $\{c_6, \dots, c_{11}\}$, $\{c_{13}, \dots, c_{17}\}$, $\{c_{18}, \dots, c_{22}\}$, $\{c_1, c_2\}$, $\{c_3, c_4\}$, $\{c_6, c_7\}$, $\{c_8, c_9\}$, $\{c_{13}, c_{14}\}$, $\{c_{15}, c_{16}\}$, $\{c_{18}, c_{19}\}$, $\{c_{20}, c_{21}\}$. Moreover, given a block b of $P_{\mathcal{C}}$ and an examiner E, we assume that three possibilities can occur: E could be in favour of the recruitment of all candidates in b, or E could not want to hire them, or E could be doubtful about them. Trivially, if E is in favour of the applicants of b, then E is also in favour of the candidates of all blocks included in b. For example, if E wants to recruit all candidates having a master degree in Chemistry, namely those of $\{c_1, \dots, c_{12}\}$, then E is also in favour of hiring the candidates of $\{c_1, \dots, c_5\}$ and $\{c_6, \dots, c_{11}\}$, regardless of their specialization, and consequently also all candidates of $\{c_1, c_2\}$, $\{c_3, c_4\}$, $\{c_6, c_7\}$, and $\{c_8, c_9\}$, regardless of their language certification. Similarly, if E is not in favour of the applicants of b, then E is against hiring all candidates of every block included in b. Therefore, the opinion of E about all blocks of candidates in $P_{\mathcal{C}}$ is represented by the sequence of orthopairs $\mathcal{O}_{\mathcal{C}}(\mathsf{E})$ belonging to $\mathsf{SO}(\mathcal{C})$, that is

$$\mathcal{O}_{\mathcal{C}}(\mathsf{E}) = ((\mathcal{L}_1(\mathsf{E}), \mathcal{E}_1(\mathsf{E})), (\mathcal{L}_2(\mathsf{E}), \mathcal{E}_2(\mathsf{E})), (\mathcal{L}_3(\mathsf{E}), \mathcal{E}_3(\mathsf{E}))),$$

such that

$$\mathcal{L}_j(\mathsf{E}) = \cup\{b \in C_j \mid \mathsf{E} \text{ is in favour of hiring the candidates of } b\} \text{ and}$$
$$\mathcal{E}_j(\mathsf{E}) = \cup\{b \in C_j \mid \mathsf{E} \text{ is not in favour of hiring the candidates of } b\},$$

for $j = 1, 2, 3$.

Once examiners give their opinions, the chair can combine these through some operations defined between sequences of orthopairs. Hence, if $\mathsf{E}_1, \dots, \mathsf{E}_m$ are our examiners, then the chair can consider the sequence

$$\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \star \dots \star \mathcal{O}_{\mathcal{C}}(\mathsf{E}_m),$$

where $\star \in \{\curlywedge, \curlyvee, \odot_2, \odot_3, \odot_4\}$ (these operations are defined in Section 4.5).

So, if a candidate belongs at least to one of the first components of the pairs in $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \star \dots \star \mathcal{O}_{\mathcal{C}}(\mathsf{E}_m)$, then he / she will pass the first selection; if he / she belongs to at least one of the second components of the pairs in $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \star \dots \star \mathcal{O}_{\mathcal{C}}(\mathsf{E}_m)$, then he / she will be excluded; otherwise, the chair will decide about him / her.

In order to provide the reader with a more intuitive representation of the examiners opinion and their combinations through our operations, we can describe sequences of orthopairs as labelled graphs defined in Remark 15. Thus, the labelled poset assigned to the sequence $\mathcal{O}_{\mathcal{C}}(X)$ of $\mathsf{SO}(\mathcal{C})$ is determined by the function

$$l_X : P_{\mathcal{C}} \mapsto \{\bullet, \circ, ?\}$$

such that

$$l_X(b) = \begin{cases} \bullet & \text{if } b \subseteq \mathcal{L}_i(X) \text{ for some } i \in \{1,2,3\}, \\ \circ & \text{if } b \subseteq \mathcal{E}_i(X) \text{ for some } i \in \{1,2,3\}, \\ ? & \text{otherwise,} \end{cases}$$

where $(\mathcal{L}_i(X), \mathcal{E}_i(X))$ denotes the i-th orthopair of $\mathcal{O}_{\mathcal{C}}(X)$.

Now, we assume that the examiners of the commission are two: E_1 and E_2. Moreover, the opinions of E_1 and E_2 are respectively expressed by the following labelled posets.

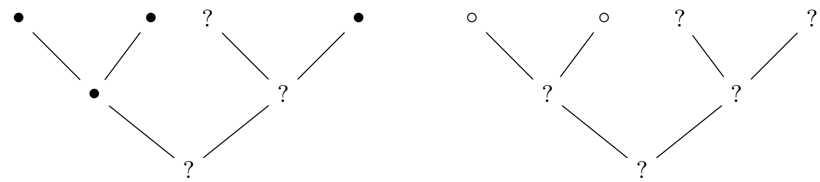

Fig. 26: Labelled forest of $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1)$

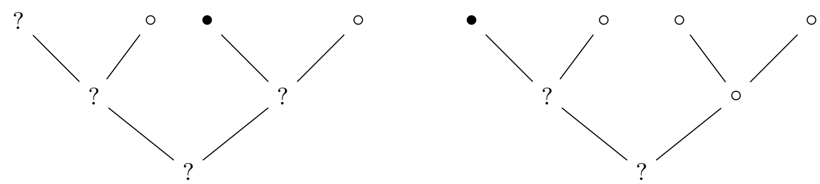

Fig. 27: Labelled forest of $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$

The labelled posets assigned to $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \curlywedge \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$, $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \curlyvee \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$, $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \odot_2 \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$, $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \odot_3 \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$ and $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \odot_4 \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$ are respectively the following.

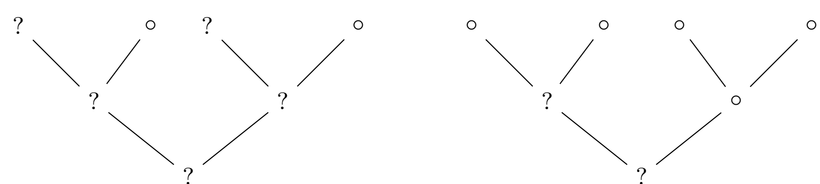

Fig. 28: Labelled forest of $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \curlywedge \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$

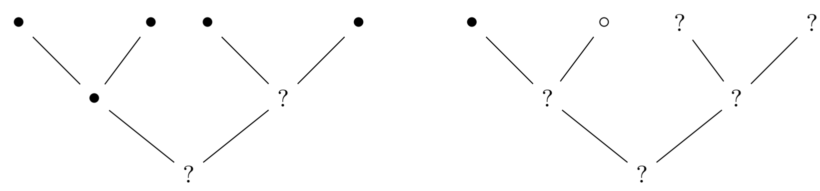

Fig. 29: Labelled forest of $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \curlyvee \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$

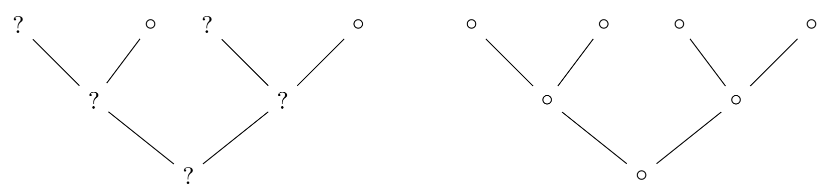

Fig. 30: Labelled forest of $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \odot_2 \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$

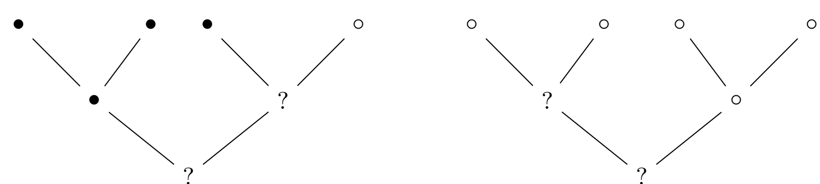

Fig. 31: Labelled forest of $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \odot_3 \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$

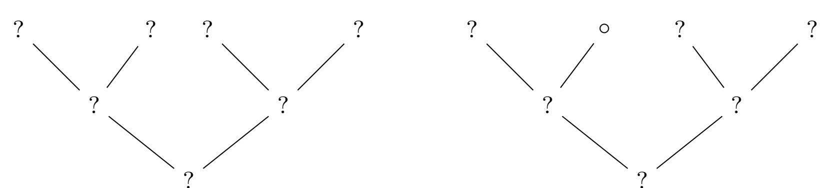

Fig. 32: Labelled forest of $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1) \odot_4 \mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$

We can observe that each of the previous operation determines the choice or the exclusion of some candidates of $\{c_1, \ldots, c_{24}\}$ with respect to the first selection. For example, $\odot_2$ involves the exclusion of candidates $c_3, c_4, c_8, c_9, c_{13}, \ldots,$ c_{23}, and it does not allow any candidate to be admitted.

We can make the following remarks, in order to compare the results generated with $\curlywedge$, $\curlyvee$, $\odot_2$ and $\odot_3$. By theorems proved in Section 4.5, by Theorem 1, and by Theorem 2, we can affirm that $\curlywedge, \curlyvee, \odot_2$ and $\odot_3$ are respectively obtained starting from the three-valued operations $\wedge$, $\vee$, $\circledast_{\mathcal{L}}$ and $\circledast_{\mathcal{S}}$. Therefore, we obtain more excluded candidates with $\odot_2$ than with $\curlywedge$, $\curlyvee$ and $\odot_3$; indeed, $\odot_2$ is determined starting from the Łukasiewicz conjunction $\circledast_{\mathcal{L}}$, where $\frac{1}{2} \circledast_{\mathcal{L}} \frac{1}{2} = 0$, instead of $\frac{1}{2} \vee \frac{1}{2} = \frac{1}{2} \circledast_{\mathcal{S}} \frac{1}{2} = \frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2}$. More candidates pass the first selection with $\odot_3$ than with $\curlywedge$ and $\odot_2$, since $\odot_3$ is obtained from the Sobociński conjunction $\circledast_{\mathcal{S}}$, where $\frac{1}{2} \circledast_{\mathcal{S}} 1 = 1 \circledast_{\mathcal{S}} \frac{1}{2} = 1$, instead of $\frac{1}{2} \circledast_{\mathcal{L}} 1 = 1 \circledast_{\mathcal{L}} \frac{1}{2} = \frac{1}{2} \wedge 1 = 1 \wedge \frac{1}{2} = \frac{1}{2}$. On the other hand, the operation $\curlywedge$ refers more candidates to the chair's decision than $\odot_2$ and $\odot_3$, since it is defined starting from the Kleene conjunction $\wedge$, where $\frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2} \wedge 1 = 1 \wedge \frac{1}{2} = \frac{1}{2}$.

In this context, the operation $\odot_4$ can be interpreted as follows. Given $j \in \{1, 2\}$, we say that the opinion of E_j is *overall positive*, when E_j is in favour of recruiting of at least one block of candidates of $P_{\mathcal{C}}$, otherwise E_j's opinion is *overall negative*. If the opinions of E_1 and E_2 are both overall negative, then all candidates of $\{c_1, \ldots, c_{24}\}$ are excluded. If only the E_1' s opinion (or the E_2' s opinion) is overall positive, then the candidates that are negative for E_2 (or E_1) are excluded (by negative candidates for E_2 (or E_1), we mean those belonging to each block b such that $l_{\mathsf{E}_2}(b) = \circ$ (or $l_{\mathsf{E}_1}(b) = \circ$)), and the chairman decides for the remaining applicants. If the opinions of E_1 and E_2 are both overall positive, then the candidates of each block b in $P_{\mathcal{C}}$ such that $l_{\mathsf{E}_1}(b) = l_{\mathsf{E}_2}(b) = \bullet$ pass the first selection, the candidate of each block b in $P_{\mathcal{C}}$ such that $l_{\mathsf{E}_1}(b) = l_{\mathsf{E}_2}(b) = \circ$ are excluded, and the chairman decides for the remaining applicants.

We can notice that each operation belonging to $\{\curlywedge, \curlyvee, \odot_2, \odot_3, \odot_4\}$ represents a way to repartition the universe $\{c_1, \ldots, c_{24}\}$ in three sets of candidates: the selected candidates (those belonging to some blocks with label $\bullet$), the excluded candidates (those belonging to some blocks with label $\circ$), and the remaining candidates on which the evaluation is uncertain (those belonging to blocks that all with label ?). More generally, each sequence of orthopairs of $\mathsf{SO}(\mathcal{C})$ determines a tri-partition (i.e. partition made of three elements) of $\{c_1, \ldots, c_{24}\}$. For example, $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1)$ and $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_2)$ generate respectively the following partitions of $\{c_1, \ldots, c_{24}\}$.

$$\begin{aligned}
&P_{\mathsf{E}_1} = \{\{c_1, \ldots, c_5, c_8, c_9\}, \{c_{13}, \ldots, c_{16}\}, \{c_6, c_7, c_{10}, c_{11}, c_{12}, c_{17}, \ldots, c_{24}\}\},\\
&P_{\mathsf{E}_2} = \{\{c_6, c_7, c_{13}, c_{14}\}, \{c_3, c_4, c_8, c_9, c_{15}, c_{16}, c_{18}, \ldots, c_{22}\},\\
&\quad \{c_1, c_2, c_5, c_{10}, c_{11}, c_{12}, c_{17}, c_{23}, c_{24}\}\}.
\end{aligned}$$

Tri-partitions are at the basis of three-way decision (3WD) theory proposed by Yao [107]. A three-way decision procedure mainly consists in two steps: *dividing* the universe in three regions and then *acting*, i.e. taking a different strategy on objects belonging to different regions. In 3WD theory, the standard tools to trisect the universe are the classical rough sets and orthopairs, namely those generated by a partition [108]. Then, the lower approximation, the impossibility domain and the boundary region are called *acceptance region*, *rejection region* and *uncertain region*, respectively. On the other hand, a sequence of orthopairs divides the universe in a more precise way also starting from an incomplete information table, in which the data are missing. For example, if we focus on the labelled forest assigned to $\mathcal{O}_{\mathcal{C}}(\mathsf{E}_1)$, then we can observe that level 2 gives arise the tri-partition $\{\{c_1, c_2, c_3, c_4, c_8, c_9\}, \{c_{13}, c_{14}, c_{15}, c_{16}\}, \{c_6, c_7, c_{18}, c_{19}, c_{20}, c_{21}\}\}$, but level 1 allows us to put in the acceptance region also the element c_5.

Furthermore, operations between sequences of orthopairs represent several ways to aggregate different tri-partitions of the same universe. For example, if we consider $\curlyvee$, then the tri-partition made of $\{c_1, \ldots, c_9, c_{13}, c_{14}\}$, $\{c_{15}, c_{16}\}$ and $\{c_{10}, c_{11}, c_{12}, c_{17}, \ldots, c_{24}\}$ is generated starting from P_{E_1} and P_{E_2}.

Once the three regions have been obtained, one might need to expand or reduce one of them. For example, it could occur that the accepted candidates with $\curlyvee$ may be too many. Then, we can assign a weight to every object of the universe, by considering the labels of each block to which it belongs. Let $P_{\mathcal{C}}^{j}$ be the j-th level of $P_{\mathcal{C}}$ defined in 30 such that $j \in \{1, \ldots, n\}$, where n is the maximum number of elements of a chain in $P_{\mathcal{C}}$. For each $c \in \{c_1, \ldots, c_{24}\}$, we set

$$p_j(c) = \begin{cases} 1 & \text{if } c \in b \text{ where } b \in P_{\mathcal{C}}^{k} \text{ with } k \leq j \text{ and it is labelled with } \bullet; \\ 0 & \text{if } c \in b \text{ where } b \in P_{\mathcal{C}}^{k} \text{ with } k \leq j \text{ and it is labelled with } \circ; \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Moreover, we assign to c, the following final weight.

$$w(c) = \frac{\sum_{j=1}^{n} p_j(c)}{n}.$$

If we focus on the sequences of orthopairs obtained starting from operation $\odot_3$, we have

- $w(c_1) = w(c_2) = w(c_3) = w(c_4) = w(c_5) = \dfrac{\frac{1}{2} + 1 + 1}{3} = \dfrac{5}{6}$;
- $w(c_6) = w(c_7) = \dfrac{\frac{1}{2} + \frac{1}{2} + 1}{3} = \dfrac{2}{3}$;
- $w(c_8) = w(c_9) = w(c_{13}) = w(c_{14}) = w(c_{15}) = w(c_{16}) = \dfrac{\frac{1}{2} + \frac{1}{2} + 0}{3} = \dfrac{1}{3}$;
- $w(c_{18}) = w(c_{19}) = w(c_{20}) = w(c_{21}) = w(c_{22}) = \dfrac{\frac{1}{2} + 0 + 0}{3} = \dfrac{1}{6}$;
- $w(c_{10}) = w(c_{11}) = w(c_{12}) = w(c_{17}) = w(c_{23}) = w(c_{24}) = \dfrac{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}}{3} = \dfrac{1}{2}$.

Trivially, $w(c)$ belongs to the real interval $[0, 1]$, and it expresses how much the candidate c must pass the first selection from 0 to 1.

The weights $w(c_1), \ldots, w(c_{24})$ can be used in several ways. For example, the chair could decide that the candidates with weight greater than $\frac{2}{3}$, and so

c_1, c_2, c_3, c_4, c_5 pass the first selection, and that the remaining candidates are excluded. Moreover, he could choose two thresholds α and β in $[0, 1]$ such that $\alpha \leq \beta$. Successively, he can redefine the following tri-partition of $\{c_1, \ldots, c_{24}\}$

- $\{c \in \{c_1, \ldots, c_{24}\} : w(c) \leq \alpha\}\}$ (*rejection region*),
- $\{c \in \{c_1, \ldots, c_{24}\} : \alpha < w(c) < \beta\}$ (*uncertain region*),
- $\{c \in \{c_1, \ldots, c_{24}\} : w(c) \geq \beta\}$ (*acceptance region*).

We observe that our procedure can be also applied for sequences of orthopairs generated by a sequence of equivalence relations that is not a refinement sequence. However, the advantage of considering sequences of refinements of orthopairs is that once we know that a block N is included in the acceptance region (or in the rejection region), we also know that all blocks included in N are included in the acceptance region (or in the rejection region). Similarly, if we know that $p_j(c) = 1$ (or $p_j(c) = 0$), we also know that $p_{j+1}(c) = 1$ (or $p_{j+1}(c) = 0$).

5 Modal logic and sequences of orthopairs

"Then you should say what you mean," the March Hare went on. "I do," Alice hastily replied; "at least–at least I mean what I say–that's the same thing, you know." "Not the same thing a bit!" said the Hatter. "You might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!" "You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like'!" "You might just as well say," added the Dormouse, who seemed to be talking in his sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"

Lewis Carroll *(Alice's Adventures in Wonderland)*

In this chapter, firstly, we recall some basic notions of modal logic and the existing connections between modal logic and rough sets (see Section 5.1). In Section 5.2, we develop the original modal logic SO_n, defining its language, introducing its Kripke models, and providing its axiomatization. Moreover, we investigate the properties of our logic system, such as the consistency, the soundness and the completeness with respect to Kripke semantics. In Section 5.3 we explore the relationships between modal logic SO_n and sequences of orthopairs. Also, we consider the operations between orthopairs and between sequences of orthopairs from the logical point of view. In the last section of this chapter, we employ modal logic SO_n to represent the knowledge of an agent that increases over time, as new information is provided.

5.1 Modal logic $S5$ and rough sets

Modal logic is the logic of *necessity* and *possibility* [38]. It is characterized by the symbols $\Box$ and $\Diamond$, called *modal operators*, such that the formula $\Box\varphi$ means "it is necessary that φ" or, in other words, "φ is the case in every possible circumstance", and the formula $\Diamond\varphi$ means "it is possible that φ" or, in other words, "φ is the case in at least one possible circumstance". However, *necessity* and *possibility* are not the only modalities, since the term *modal logic* is used more broadly to cover a family of logics with similar rules and a variety of different symbols [51]. In this thesis, we are interested in propositional modal logic $S5$, that was proposed by Clarence Irving Lewis and Cooper Harold Langford in their book *Symbolic Logic* [69].

Now, we briefly describe the syntax and the semantics of modal logic $S5$ [29]. The $S5$-language contains all symbols of propositional logic, plus the modalities $\Box$ and $\Diamond$. In terms of semantics, the formulas of $S5$-language are interpreted with the *Kripke models*. A Kripke model of $S5$ is a triple consisting of a universe U (its element are named *possible worlds*), an equivalence relation R on U, and an evaluation function v, that assigns to a propositional variable p the set of all worlds of U in which p is true. We can extend v on the formulas of propositional logic as usual and on the modal formulas as following. Let p be a propositional variable, and let $u \in U$,

$\Box p$ *is true in* u if and only if "p is true in every world v of U such that uRv", and

$\Diamond p$ *is true in* u if and only if "p is true at least in a world v of U such that uRv".

The axiom schemas are obtained by adding the following schemas to those of propositional logic.

Definition 51 (Axioms of $S5$).

K. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *(distribution axiom);*
T. $\Box\varphi \to \varphi$ *(necessitation axiom);*
5. $\Diamond\varphi \to \Box\Diamond\varphi$.

We notice that Axiom 5 it is equivalent to the set of axioms made of

B. $\varphi \to \Box\Diamond\varphi$ *and*
4. $\Box\varphi \to \Box\Box\varphi$.

The inference rules are the *modus ponens* and the *necessitation rule* $(\varphi/\Box\varphi)$. We stress that $S5$ belongs to the family of *normal modal logics*, that are characterized by adding the necessitation rule, and a list of axiom schemas Ax including **K** to the principles of propositional logic. The weakest normal modal logic is named K in honour of Saul Kripke, where $Ax=\{\mathbf{K}\}$. Thus, $S5$, as every normal modal logic, is an extension of K. A further example of normal modal logic is $S4$, that is obtained by adding to system K the axiom schemas **T** a and **4**.

The system $S5$ is sound and complete with respect to the class of all Kripke models of $S5$.

Moreover, propositional modal logic is also interpreted as an extension of classical propositional logic with two added operators expressing modality [56]. Since Pawlak rough set algebra is an extension of Boolean algebra (see Remark 3), the relationship between propositional modal logic and rough sets appears intuitive. In particular, modal logic $S5$ is connected with rough set theory, since the necessity and possibility can be interpreted as the lower and the upper approximation [82] [81]. Hence, let (U, R, v) be a Kripke model of $S5$, we have that

$$||\Box\varphi||_{\mathsf{v}} = \mathcal{L}_R(||\varphi||_{\mathsf{v}}) \text{ and } ||\Diamond\varphi||_{\mathsf{v}} = \mathcal{U}_R(||\varphi||_{\mathsf{v}}),$$

where $||\varphi||_{\mathsf{v}}$, $||\Box\varphi||_{\mathsf{v}}$ and $||\Diamond\varphi||_{\mathsf{v}}$ are made of possible worlds in which φ, $\Box\varphi$ and $\Diamond\varphi$ are true, respectively.

It is important to recall that $S5$ can be considered as an epistemic logic in the sense that it is suitable for representing and reasoning about the knowledge of an individual agent [46] [68] [42]. Indeed, the formula $\Box\varphi$ can be read as "the agent knows φ". Moreover, the axioms of $S5$ express the properties of the knowledge. For instance, Schema **4** expresses the fact that if an agent knows φ, then she knows that she knows φ (*the positive introspection axiom*).

5.2 Modal logic SO_n

In this section, the novel modal logic SO_n is developed.

From now, by refinement sequence, we mean a refinement sequence of *partial partitions* of the given universe, and we fix an integer $n > 0$.

Language of SO_n

We indicate the language of SO_n with L. Then, the alphabet of L consists of

- a set Var of propositional variables;
- the logical connectives $\wedge$ and $\neg$;
- the sequences of modal operators $(\Box_1, \dots, \Box_n)$ and $(\bigcirc_1, \dots, \bigcirc_n)$.

The propositional variables are typically denoted with $p, q, r, \dots$ and refer to the statements that are considered basic, for example "the book is red". The symbols $\wedge$ and $\neg$ are respectively the *conjunction* and *negation* of classical propositional logic. Fixed $i \in \{1, \dots, n\}$, we call *i-box* and *i-circle* the modal operators $\Box_i$ and $\bigcirc_i$, respectively.

We denote the well formed formulas of L with Greek letters. As usual, the set Form of all well formed formulas of L is the smallest set that contains Var and satisfies the following conditions. Let $\varphi, \psi \in \mathsf{Form}$,

- if $\varphi \in \mathsf{Form}$, then $\neg\varphi$, $\Box_i\varphi$, $\bigcirc_i\varphi \in \mathsf{Form}$, for each $i \in \{1, \dots, n\}$;
- if $\varphi, \psi \in \mathsf{Form}$, then $\varphi \wedge \psi \in \mathsf{Form}$.

We simply call the elements of Form *formulas* or *sentences*. Moreover, the alphabet of L also contains the brackets "(" and ")" to establish the order wherewith the connectives work in the complex formulas. In this way, the language is clear and has no ambiguity.

The abbreviations introduced in the next definition, except the last one, are the standard abbreviations defined for the classical propositional logic [65].

Definition 52 (Abbreviations in L). *Let $\varphi, \psi \in \mathsf{Form}$ and $p \in \mathsf{Var}$,*

1. $\bot := p \wedge \neg p$ *(false);*
2. $\top := \neg\bot$ *(true);*
3. $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$ *(disjunction);*
4. $\varphi \to \psi := \neg\varphi \vee \psi$ *(implication);*
5. $\varphi \equiv \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$ *(equivalence);*
6. $\triangle_i\varphi := \Box_i\neg\varphi$, *(i-triangle) with* $i \in \{1, \dots, n\}$.

We employ the convention that $\leftrightarrow$ dominates $\to$, and $\to$ dominates the remaining symbols. For example, the formula $\Box_i p \to q$ is understood as $(\Box_i p) \to q$.

By *schema*, we mean a set of formulas all having the same form. For example, the schema $\varphi \wedge \psi$ is the set $\{\varphi \wedge \psi \mid \varphi, \psi \in \mathsf{Form}\}$.

Semantics of SO_n

We define the Kripke models of SO_n, which we also call *orthopaired Kripke models* or *SO_n-models.*

Definition 53. *A Kripke model of SO_n is a triple*

$$\mathcal{M} = (U, (R_1, \dots, R_n), \mathsf{v}),$$

where

1. *U is a non-empty set of objects,*
2. *$(R_1, \ldots, R_n)$ is a sequence of equivalence relations on U (i.e. for i from 1 to n, $R_i \subseteq (U \times U)$ and R_i is reflexive, symmetric and transitive) such that, let $u \in U$,*
   - *$R_1(u) \neq \{u\}$, and*
   - *$R_{i+1}(u) \subseteq R_i(u)$, for each $i < n$;*
3. *v is an evaluation function that assigns a subset of U to each element of Var (i.e. $\mathsf{v} : \mathsf{Var} \mapsto 2^U$, where 2^U is the power set of U).*

We say that U is the *domain* or the *universe* of $\mathcal{M}$, the elements of U are the *states* or the *possible worlds* of $\mathcal{M}$, and $R_1, \ldots, R_n$ are the *accessibility relations* of $\mathcal{M}$. The pair $(U, (R_1, \ldots, R_n))$ is called Kripke frame of SO_n. Moreover, let $p \in \mathsf{Var}$, if $u \in \mathsf{v}(p)$, then we can say that *p is true at u in $\mathcal{M}$*.

Remark 21. The domain of an orthopaired Kripke model has at least two elements.

Example 38. Let $\mathsf{Var} = \{p, q, r\}$, we suppose that

- $U = \{a, b, c, d\}$,
- $R_1 = \{(a, b), (b, a), (c, d), (d, c)\} \cup \{(u, u) \mid u \in U\}$,
- $R_2 = \{(a, b), (b, a)\} \cup \{(u, u) \mid u \in U\}$,
- v is a function from Var to 2^U such that $\mathsf{v}(p) = \{a, b, c\}$, $\mathsf{v}(q) = \{c, d\}$ and $\mathsf{v}(r) = \{a, c\}$.

Then, $\mathcal{M} = (U, (R_1, R_2), \mathsf{v})$ is a Kripke model of SO_n.

Orthopaired Kripke models are also models of modal logic $S5^n$ developed in [46]. However, a Kripke model of $S5^n$ is not always a Kripke model of SO_n; in fact, the accessibility relations of each $S5^n$-model have only the property to be equivalence relations.

Definition 54 (Kripke models of SO_n as graphs). *A Kripke model $\mathcal{M} = (U, (R_1, \ldots, R_n), \mathsf{v})$ of SO_n is represented by the graph $\mathcal{G}_{\mathcal{M}}$, where*

- *the set of the vertices is U,*
- *two vertices are connected with the labeled edge i if and only if*

$$i = \mathit{max}\{j \in \{1, \ldots, n\} \mid (a, b) \in R_j\}.$$

- *the label of $u \in U$ is the list of the propositional variables that are true at u in $\mathcal{M}$.*

Example 39. Suppose that $\mathsf{Var} = \{p\}$ and $\mathcal{M} = (U, (R_1, R_2), \mathsf{v})$ is a Kripke model of SO_n, where

- $U = \{a, b, c, d, e\}$;
- $R_1 = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d)\} \cup \{(u, u) \mid u \in U\}$,
- $R_2 = \{(a, b), (b, a)\} \cup \{(u, u) \mid u \in U\}$,

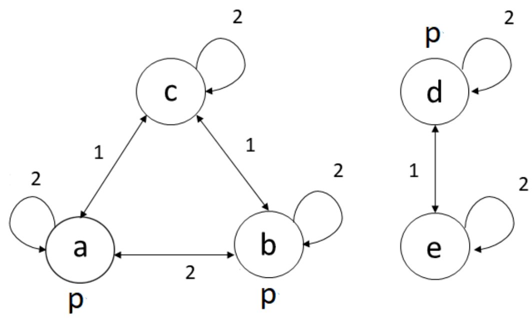

Fig. 33: Graph $\mathcal{G}_{\mathcal{M}}$

– $\mathsf{v}(p) = \{a, b, d\}$.

The graph $\mathcal{G}_{\mathcal{M}}$ is as in Figure 33.

The notion of *truth* of a formula in a Kripke model of SO_n is given by the next definition.

Definition 55. *Let $\mathcal{M} = (U, (R_1, \dots, R_n), \mathsf{v})$ be a Kripke model of SO_n. The notion of $(\mathcal{M}, u) \models \varphi$ is inductively defined as follows.*

1. $(\mathcal{M}, u) \models p$, *with* $p \in \mathsf{Var}$ *iff* "$u \in \mathsf{v}(p) = ||p||_{\mathsf{v}}$";
2. $(\mathcal{M}, u) \models (\varphi \wedge \psi)$ *iff* "$(\mathcal{M}, u) \models \varphi$ *and* $(\mathcal{M}, u) \models \psi$";
3. $(\mathcal{M}, u) \models \neg\varphi$ *iff* "$(\mathcal{M}, u) \not\models \varphi$";
4. $(\mathcal{M}, u) \models \Box_i\varphi$ *iff* "$R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$ *and* $R_i(u) \neq \{u\}$";
5. $(\mathcal{M}, u) \models \bigcirc_i\varphi$ *iff* "$u \models \varphi$ *and* $R_i(u) \neq \{u\}$";

where $||\varphi||_{\mathsf{v}}$ is the truth set *of φ, that is*

$$||\varphi||_{\mathsf{v}} = \{u \in U \mid (\mathcal{M}, u) \models \varphi\}.$$

$(\mathcal{M}, u) \models \varphi$ can be read as "φ is true at u in $\mathcal{M}$" or "φ holds at u in $\mathcal{M}$" or "$(\mathcal{M}, u)$ satisfies φ". Moreover, we say that "φ is false at u in $\mathcal{M}$" if and only if $(\mathcal{M}, u) \not\models \varphi$. We can write $u \models \varphi$, instead of $(\mathcal{M}, u) \models \varphi$, when $\mathcal{M}$ is clear from the context.

Remark 22. The points 1, 2 and 3 of Definition 55 are given for standard Kripke semantics too. Also, once fixed $i \in \{1, \dots, n\}$, $u \models \Box_i\varphi$ differs from $u \models \Box\varphi$, where $\Box$ is the necessity operator of $S5$ logic interpreted by R_i, since the additional condition $R_i(u) \neq \{u\}$ is required.

The next proposition follows by Definition 52 and Definition 55.

Proposition 14. *Let $\mathcal{M} = (U, (R_1, \dots, R_n), \mathsf{v})$ be a Kripke model of SO_n. Then,*

1. $(\mathcal{M}, u) \models (\varphi \vee \psi)$ *iff* "*either* $(\mathcal{M}, u) \models \varphi$ *or* $(\mathcal{M}, u) \models \psi$";

2. $(\mathcal{M}, u) \models \triangle_i \varphi$ *iff* "$R_i(u) \cap ||\varphi||_{\mathsf{v}} = \emptyset$ *and* $R_i(u) \neq \{u\}$";
3. $(\mathcal{M}, u) \models \varphi \to \psi$ *iff* "$(\mathcal{M}, u) \models \varphi$ *implies that* $(\mathcal{M}, u) \models \psi$";
4. $(\mathcal{M}, u) \models \varphi \equiv \psi$ *iff* "$(\mathcal{M}, u) \models \varphi$ *if and only if* $(\mathcal{M}, u) \models \psi$";

for each $u \in U$, $\varphi, \psi \in \mathsf{Form}$ *and* $i \in \{1, \ldots, n\}$.

Remark 23. It is clear that

- $(\mathcal{M}, u) \models \Box_1 \varphi$ iff $R_1(u) \subseteq ||\varphi||_{\mathsf{v}}$;
- $(\mathcal{M}, u) \models \triangle_1 \varphi$ iff $R_1(u) \cap ||\varphi||_{\mathsf{v}} = \emptyset$;
- $(\mathcal{M}, u) \models \varphi$ iff $(\mathcal{M}, u) \models \bigcirc_1 \varphi$;
- If $(\mathcal{M}, u) \models \bigcirc_i \varphi$, then $(\mathcal{M}, u) \models \varphi$;
- If $(\mathcal{M}, u) \models \Box_i \varphi$, then $(\mathcal{M}, u) \models \bigcirc_i \varphi$;

for each i from 1 to n.

The following theorem expresses the connection between the logical connectives of L and the set-theoretic operations.

Theorem 34. *Let* $\mathcal{M} = (U, (R_1, \ldots, R_n), \mathsf{v})$ *be a Kripke model of* SO_n. *Then,*

1. $||\bot||_{\mathsf{v}} = \emptyset$;
2. $||\top||_{\mathsf{v}} = U$;
3. $||\neg\varphi||_{\mathsf{v}} = U \setminus ||\varphi||_{\mathsf{v}}$;
4. $||\varphi \wedge \psi||_{\mathsf{v}} = ||\varphi||_{\mathsf{v}} \cap ||\psi||_{\mathsf{v}}$;
5. $||\varphi \vee \psi||_{\mathsf{v}} = ||\varphi||_{\mathsf{v}} \cup ||\psi||_{\mathsf{v}}$;
6. $||\varphi \to \psi||_{\mathsf{v}} = (U \setminus ||\varphi||_{\mathsf{v}}) \cup ||\psi||_{\mathsf{v}}$;
7. $||\varphi \equiv \psi||_{\mathsf{v}} = ((U \setminus ||\varphi||_{\mathsf{v}}) \cup ||\psi||_{\mathsf{v}}) \cap ((U \setminus ||\psi||_{\mathsf{v}}) \cup ||\varphi||_{\mathsf{v}})$;
8. $||\Box_i \varphi||_{\mathsf{v}} = \{u \in U \mid R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$ *and* $R_i(u) \neq \{u\}\}$;
9. $||\triangle_i \varphi||_{\mathsf{v}} = \{u \in U \mid R_i(u) \cap ||\varphi||_{\mathsf{v}} = \emptyset$ *and* $R_i(u) \neq \{u\}\}$; *for* i *from 1 to* n.

Let Cl_n be the class of the Kripke models of SO_n, we define the notion of validity in the models that belong to Cl_n.

Definition 56. *Let* $\mathcal{M} \in \mathsf{Cl}_n$. *Then, for each* $\varphi \in \mathsf{Form}$, *we write*

- $\models^{\mathcal{M}} \varphi$ *iff* "$(\mathcal{M}, u) \models \varphi$, *for every world* u *in* $\mathcal{M}$", *and we say that* φ *is valid in* $\mathcal{M}$;
- $\models^{\mathsf{Cl}_n} \varphi$ *iff* "$\models^{\mathcal{M}} \varphi$, *for every model* $\mathcal{M}$ *in* Cl_n", *and we say that* φ *is valid in* Cl_n.

From the previous notions of validity, two logical consequence relations can be formally defined.

Definition 57. *For each* $\mathcal{M} \in \mathsf{Cl}_n$, $\varphi \in \mathsf{Form}$ *and* $\Gamma \subseteq \mathsf{Form}$, *we write*

- $\Gamma \models^{\mathcal{M}} \varphi$ *iff* "*if* $\models^{\mathcal{M}} \Gamma$, *then* $\models^{\mathcal{M}} \varphi$", *and*
- $\Gamma \models^{\mathsf{Cl}_n} \varphi$ *iff* "*if* $\models^{\mathsf{Cl}_n} \Gamma$, *then* $\models^{\mathsf{Cl}_n} \varphi$".

Proposition 15. *Let* $i \in \{1, \ldots, n\}$, *the instances of the following schemes are* SO_n*-tautologies.*

Ab$_{\triangle_1}$**.** $\triangle_1\bot$.
Dist$_{\Box_i}$**.** $\Box_i(\varphi \wedge \psi) \equiv \Box_i\varphi \wedge \Box_i\psi$.
Dist$_{\triangle_i}$**.** $\triangle_i(\varphi \vee \psi) \equiv \triangle_i\varphi \wedge \triangle_i\psi$.
P$_1$**.** $\neg \bigcirc_i \varphi \to (\neg\Box_i\varphi \vee \neg\triangle_i\varphi)$.
P$_2$**.** $(\neg \bigcirc_i \varphi \wedge \varphi) \to (\neg\Box_i\varphi \wedge \neg\triangle_i\varphi)$.

Proof. Let $\mathcal{M} = (U, (R_1, \dots, R_n), \mathsf{v}) \in \mathsf{Cl}_n$, and let $u \in U$.

Ab$_{\triangle_1}$**.** By Definition 53, $R_1(u) \neq \{u\}$; moreover, by Theorem 34, $||\bot||_\mathsf{v} = \emptyset$. Then, $(\mathcal{M}, u) \models \triangle_1\bot$.

Dist$_{\Box_i}$**.** By Theorem 34, $||\varphi \wedge \psi||_\mathsf{v} = ||\varphi||_\mathsf{v} \cap ||\psi||_\mathsf{v}$. Trivially, $R_i(u) \subseteq ||\varphi \wedge \psi||_\mathsf{v}$ if and only if $R_i(u) \subseteq ||\varphi||_\mathsf{v}$ and $R_i(u) \subseteq ||\psi||_\mathsf{v}$. Then, $(\mathcal{M}, u) \models \Box_i(\varphi \wedge \psi)$ if and only if $(\mathcal{M}, u) \models \Box_i\varphi \wedge \Box_i\psi$.

Dist$_{\triangle_i}$**.** $(\mathcal{M}, u) \models \triangle_i(\varphi \vee \psi)$ if and only if $R_i(u) \subseteq ||\varphi \vee \psi||_\mathsf{v}$ and $R_i(u) \neq \{u\}$. By Proposition 14, $R_i(u) \cap ||\varphi \vee \psi||_\mathsf{v} = R_i(u) \cap (||\varphi||_\mathsf{v} \cup ||\psi||_\mathsf{v})$. Since $R_i(u) \cap (||\varphi||_\mathsf{v} \cup ||\psi||_\mathsf{v}) = (R_i(u) \cap ||\varphi||_\mathsf{v}) \cup (R_i(u) \cap ||\psi||_\mathsf{v})$, we have that $R_i(u) \cap ||\varphi \vee \psi||_\mathsf{v} = \emptyset$ if and only if $R_i(u) \cap ||\varphi||_\mathsf{v} = \emptyset$ and $R_i(u) \cap ||\psi||_\mathsf{v} = \emptyset$. Then, $(\mathcal{M}, u) \models \triangle_i\varphi$ and $(\mathcal{M}, u) \models \triangle_i\psi$.

P$_1$**.** Suppose that $(\mathcal{M}, u) \models \neg \bigcirc_i \varphi$. Then, $(\mathcal{M}, u) \not\models \varphi$ or $R_i(u) = \{u\}$. If $(\mathcal{M}, u) \not\models \varphi$, then $\neg\Box_i\varphi$ is true at u in $\mathcal{M}$. If $R_i(u) = \{u\}$, then both $\neg\Box_i\varphi$ and $\neg\triangle_i\varphi$ are true at u in $\mathcal{M}$.

P$_2$**.** If $(\mathcal{M}, u) \models \neg \bigcirc_i \varphi \wedge \varphi$, then $R_i(u) = \{u\}$. Consequently, both $\neg\Box_i\varphi$ and $\neg\triangle_i\varphi$ are true at u in $\mathcal{M}$.

Axiomatic system of SO_n

The orthopaired modal logic SO_n is the smallest set of sentences that contains the instances of the axiom schemes of propositional logic and the instances of the axiom schemes of Definition 58, and that is closed under the inference rules of Definition 59.

Definition 58 (Axioms of SO_n).

Z$_{\Box_1}$**.** $\Box_1\top$.
Def$_1$**.** $\Box_i\varphi \equiv \triangle_i\neg\varphi$.
Def$_2$**.** $\bigcirc_i\varphi \equiv \bigcirc_i\top \wedge \varphi$.
K$_{\Box_i}$**.** $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$.
T$_{\Box_i}$**.** $\Box_i\varphi \to \varphi$.
B$_{\Box_i}$**.** $\bigcirc_i\varphi \to \Box_i\neg\triangle_i\varphi$.
4$_{\Box_i}$**.** $\Box_i\varphi \to \Box_i\Box_i\varphi$.
Eq. $\bigcirc_i\top \equiv \Box_i\top$.
R1$_{\bigcirc_i}$**.** $\bigcirc_i\varphi \to (\Box_j\varphi \to \Box_i\varphi)$, *with* $j \leq i$.
R2$_{\bigcirc_i}$**.** $\Box_i\varphi \to \bigcirc_i\varphi$.
Nst$_{\bigcirc_i}$**.** $\bigcirc_i\varphi \to \bigcirc_j\varphi$, *with* $0 < j \leq i$.

Definition 59 (Inference rules of SO_n).

MP. $\dfrac{\varphi, \varphi \to \psi}{\psi}$ *(Modus Ponens).*

$\Box_i$**Mn.** $\dfrac{\varphi \to \psi}{\Box_i\varphi \to \Box_i\psi}$, *for each* $i \in I$.

We notice that Schema $\mathbf{Z}_{\Box_1}$ ensures that all equivalence classes of the first accessibility relation of the SO_n-models are not singletons. Furthermore, fixed $i \in \{1, \ldots, n\}$, Schema $\mathbf{Def}_1$ allows us to obtain $\Box_i$ through the modal operator $\triangle_i$; vice-versa, we also have that $\triangle_i\varphi \equiv \Box_i\neg\varphi$. Trivially, $\mathbf{Def}_2$ is introduced to individuate the possible worlds of which the i-th equivalence class is a singleton. Schemas $\mathbf{K}_{\Box_i}$, $\mathbf{T}_{\Box_i}$ and $\mathbf{4}_{\Box_i}$ are respectively the schemas $\mathbf{K}$, $\mathbf{T}$, and $\mathbf{4}$ that characterized $S4$ (see Definition 51), where $\Box = \Box_i$ and $\Diamond = \neg\triangle_i$. Thus, $\mathbf{K}_{\Box_i}$ states that the operator $\Box_i$ distributes over the implication $\to$; $\mathbf{T}_{\Box_i}$ and $\mathbf{4}_{\Box_i}$ express respectively that the accessibility relations of all SO_n-models are reflexive and transitive relations. On the other hand, taking $\Box_i = \Box$, $\mathbf{B}_{\Box_i}$ is not equal to $\mathbf{B}$; they are different because the hypothesis of $\mathbf{B}_{\Box_i}$ ($\bigcirc_i\varphi$) is stronger than the hypothesis of $\mathbf{B}$ (φ); so, we can say that each relation of each Kripke model of SO_n is a *strongly symmetric* relation. Furthermore, $\mathbf{B}_{\Box_1}$ is equal to $\mathbf{B}$, since $\mathbf{Z}_{\Box_1}$ requires that the condition $R_1(u) \neq \{u\}$ is satisfied, for each possible world u, and for each accessibility relation R_1 of the SO_n-models. Moreover, by Schema $\mathbf{B}_{\Box_i}$, we can observe that the accessibility relations of the SO_n-models satisfy the *euclidean* property. Also, we have to stress that the modal operator $\triangle_i$ corresponds to the negation of the *possibility operator* $\Diamond$ of every modal logic. In addition, the schemas $\mathbf{Eq}$, $\mathbf{R1}_{\bigcirc_i}$, $\mathbf{R2}_{\bigcirc_i}$ and $\mathbf{Nst}_{\bigcirc_i}$ provide some connections between the operators $\bigcirc_i$ and $\Box_i$. More precisely, $\mathbf{Eq}$ affirms that both $(\mathcal{M}, u) \models \bigcirc_i\top$ and $(\mathcal{M}, u) \models \Box_i\top$ mean that $R_i(u)$ is not a singleton. $\mathbf{R1}_{\bigcirc_i}$ guarantees that each relation is finer than the previous one, namely $R_{i+1}(u) \subseteq R_i(u)$ for each $i > 1$. By $\mathbf{R2}_{\bigcirc_i}$, we have that $\bigcirc_i$ follows from $\Box_i$. On the other side, $\mathbf{Nst}_{\bigcirc_i}$ states that if $R_i(u)$ is not a singleton, then all equivalence classes of the previous relations to R_i containing u are not singletons. Finally, we can notice that $\mathbf{T}_{\Box_i}$ is obtained from $\mathbf{Def}_2$ and $\mathbf{R2}_{\bigcirc_i}$.

Remark 24. Suppose that Schema $\mathbf{Z}_{\Box_1}$ is substituted by the schemas $\neg \bigcirc_1 \top$, $\ldots$, $\neg \bigcirc_n \top$. Then, each equivalence class of each accessibility relation of the SO_n-models is a singleton. In this case, it is clear that all axiom schemas of Definition 58 are trivially satisfied by each SO_n-model. Moreover, if $n = 1$, then the axiom schemas $\mathbf{Eq}$, $\mathbf{R1}_{\bigcirc_1}$, $\mathbf{R2}_{\bigcirc_1}$ and $\mathbf{Nst}_{\bigcirc_1}$ are trivially satisfied by each SO_1-model. Thus, the axiom schemas of our logic is obtain by adding $\mathbf{Z}_{\Box_1}$ to those of modal logic $S5$ and by setting $\Box_1 = \Box$ and $\triangle_1 = \neg\Diamond$. Clearly, in this case, the Kripke models of SO_1 are all Kripke models of $S5$ such that the equivalence classes of their accessibility relations are not singletons.

Soundness and Completeness of SO_n

Next, we prove the soundness of SO_n system with respect to the class of models Cl_n already defined.

Theorem 35. *The axiom schemes of SO_n are valid in the class Cl_n, and the rules preserve the validity in this class.*

Proof. Let $\mathcal{M} = (U, (R_1, \dots, R_n), \mathsf{v})$ be a model of Cl_n. Fixed $u \in U$, we prove that each instance of the axiom schemas of SO_n is true at u in $\mathcal{M}$.

$\mathbf{Z}_{\Box_1}$. By Definition 53, $R_1(u) \neq \{u\}$, and by Theorem 34, $||\top||_{\mathsf{v}} = U$. Then, $(\mathcal{M}, u) \models \Box_1 \top$.

$\mathbf{Def}_1$. $(\mathcal{M}, u) \models \Box_i \varphi$ if and only if $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$ and $R_i(u) \neq \{u\}$, by Definition 55. Moreover, $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$ if and only if $R_i(u) \cap (U \setminus ||\varphi||_{\mathsf{v}}) = \emptyset$. However, by Theorem 34, $U \setminus ||\varphi||_{\mathsf{v}} = ||\neg\varphi||_{\mathsf{v}}$, So, it is clear that $(\mathcal{M}, u) \models \triangle_i \neg\varphi$.

$\mathbf{Def}_2$. It is trivial.

$\mathbf{K}_{\Box_i}$. Suppose that $(\mathcal{M}, u) \models \Box_i(\varphi \to \psi)$ and $(\mathcal{M}, u) \models \Box_i \varphi$. Then, $R_i(u) \neq \{u\}$, $R_i(u) \subseteq ||\varphi \to \psi||_{\mathsf{v}}$ and $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$. By Theorem 34, $||\varphi \to \psi||_{\mathsf{v}} = (U \setminus ||\varphi||_{\mathsf{v}}) \cup ||\psi||_{\mathsf{v}}$. Therefore, it is obvious that $R_i(u) \subseteq ||\psi||_{\mathsf{v}}$ and so $(\mathcal{M}, u) \models \Box_i \psi$.

$\mathbf{T}_{\Box_i}$. Suppose that $(\mathcal{M}, u) \models \Box_i \varphi$. Then, $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$. By Definition 53, R_i is reflexive and so $u \in R_i(u)$. Consequently, $(\mathcal{M}, u) \models \varphi$.

$\mathbf{B}_{\Box_i}$. Suppose that $(\mathcal{M}, u) \models \bigcirc_i \varphi$. Then, $(\mathcal{M}, u) \models \varphi$ and $R_i(u) \neq \{u\}$. Since $u \in ||\varphi||_{\mathsf{v}}$, we have that

$$R_i(u) \cap ||\varphi||_{\mathsf{v}} \neq \emptyset. \tag{38}$$

On the other hand,

$$||\triangle_i \varphi||_{\mathsf{v}} = \{v \in U \mid R_i(v) \neq \{v\} \text{ and } R_i(v) \cap ||\varphi||_{\mathsf{v}} = \emptyset\}. \tag{39}$$

By 38 and 39, $R_i(u) \cap ||\triangle_i \varphi||_{\mathsf{v}} = \emptyset$. Therefore, $R_i(u) \subseteq U \setminus ||\triangle_i \varphi||_{\mathsf{v}}$ and so $R_i(u) \subseteq ||\neg\triangle_i \varphi||_{\mathsf{v}}$. Consequently, $(\mathcal{M}, u) \models \neg\triangle_i \varphi$.

$\mathbf{4}_{\Box_i}$. If $(\mathcal{M}, u) \models \Box_i \varphi$, then $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$ and $R_i(u) \neq \{u\}$. On the other hand, $||\Box_i \varphi||_{\mathsf{v}} = \cup_{u \in U}\{R_i(u) \mid R_i(u) \neq \{u\}\}$. Then, $R_i(u) \subseteq ||\Box_i \varphi||_{\mathsf{v}}$. Therefore, $(\mathcal{M}, u) \models \Box_i \Box_i \varphi$.

Eq. By Theorem 34, we have that $||\top||_{\mathsf{v}} = U$. Then, both $\Box_i \top$ and $\bigcirc_i \top$ are true at u in $\mathcal{M}$ if and only if $R_i(u) \neq \{u\}$.

$\mathbf{R1}_{\bigcirc_i}$. Suppose that $(\mathcal{M}, u) \models \bigcirc_i \varphi$ and $(\mathcal{M}, u) \models \Box_j \varphi$. Then $R_j(u) \subseteq ||\varphi||_{\mathsf{v}}$. Since $j \leq i$, $R_i(u) \subseteq R_j(u)$. Therefore, $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$. Since $(\mathcal{M}, u) \models \bigcirc_i \varphi$, we also have that $R_i(u) \neq \{u\}$. Then, $(\mathcal{M}, u) \models \Box_i \varphi$.

$\mathbf{R2}_{\bigcirc_i}$. Trivially, $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$ implies that $u \in ||\varphi||_{\mathsf{v}}$, since R_i is a reflexive relation.

$\mathbf{Nest}_{\bigcirc_i}$. Let $j \leq i$, if $R_i(u) \neq \{u\}$ then $R_j(u) \neq \{u\}$, since $R_i(u) \subseteq R_j(u)$; indeed $(\mathcal{M}, u) \models \bigcirc_i \varphi \to \bigcirc_j \varphi$.

We prove that if the hypothesis of the inference rules are true at u in $\mathcal{M}$, then the thesis is also true at u in $\mathcal{M}$.

MP. It is trivial.

$\Box_i$**Mn.** By Theorem 34, if $(\mathcal{M}, u) \models \varphi \to \psi$, then $||\varphi||_{\mathsf{v}} \subseteq ||\psi||_{\mathsf{v}}$. If $(\mathcal{M}, u) \models \Box_i\varphi$, then $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$ and $R_i(u) \neq \{u\}$. Then, it is clear that $(\mathcal{M}, u) \models \psi$.

Corollary 3. *The SO_n system is sound with respect to the class of models Cl_n (i.e. if $\vdash_{SO_n} \varphi$ then $\models_{\mathsf{Cl}_n} \varphi$, for each $\varphi \in \mathsf{Form}$).*

We usually write "$\vdash_{SO_n} \varphi$" to mean that φ is a theorem of SO_n, this is $\vdash_{SO_n} \varphi$.

In terms of theoremhood, we can characterize notions of deducibility and consistency.

Definition 60. *A formula φ of Form is* deductible *or* derivable *from a set of sentences Γ in the system SO_n, written $\Gamma \vdash_{SO_n} \varphi$, if we have*

$$\vdash_{SO_n} (\varphi_1 \wedge ... \wedge \varphi_n) \to \varphi,$$

where $\varphi_1, \ldots, \varphi_n$ are formulas in Γ.

Definition 61. *A subset Γ of Form is* consistent *in SO_n, written $Con_{SO_n}\Gamma$, if and only if the* falsum *is not deducible from Γ in SO_n, namely $\Gamma \nvdash_{SO_n} \bot$.*

Thus, Γ is inconsistent in SO_n just when $\Gamma \vdash_{SO_n} \bot$.

Next, we define the idea of a canonical model for axiomatic system SO_n, and we prove some fundamental theorems about completeness. Before of introducing the concept of canonical model, we need to define the concept of maximality. Intuitively, a set of formulas is maximal if it is consistent, and it contains as many formulas as it can without becoming inconsistent. We write $Max_{SO_n}\Gamma$ to indicate that Γ is SO_n-maximal, and we formally give the definition as follows.

Definition 62. *Let $\Gamma \subseteq \mathsf{Form}$, $Max_{SO_n}\Gamma$ if and only if*

1. *$Cons_{SO_n}\Gamma$, and*
2. *for each $\varphi \in \mathsf{Form}$, if $Cons_{SO_n}(\ \Gamma \bigcup \{\varphi\}\ )$ then $\varphi \in \Gamma$.*

Now, we have to recall Theorem 36, the Lindenbaum's lemma and its two corollaries (found in [29]) for the maximal consistent sets of logical systems. By *logical system*, we mean be any set which contains certain initial axioms and which is closed under certain rules of inference. Moreover, we write $Max_\Sigma\Gamma$ to denote that Γ is Σ-maximal.

Theorem 36. *Let Σ be a logical system, and let $Max_\Sigma\Gamma$, then*

1. *$\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$;*
2. *$\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$;*
3. *$\varphi \to \psi \in \Gamma$ iff if $\varphi \in \Gamma$, then $\psi \in \Gamma$.*

Theorem 37 (Lindenbaum's lemma). *[93] Let Σ be a logical system. If $Con_\Sigma\Gamma$, then there is a $Max_\Sigma\Delta$ such that $\Gamma \subseteq \Delta$*

Corollary 4. *Let Σ be a logical system. Then,*

$$\vdash_\Sigma \varphi \text{ if and only if } \varphi \in \Delta,$$

for every $Max_\Sigma \Delta$.

Corollary 5. *Let* Σ *be a logical system. Then,* $\Gamma \vdash_\Sigma \varphi$ *if and only if* φ *is an element of every* $Max_\Sigma \Delta$ *such that* $\Gamma \subseteq \Delta$.

In terms of maximality we can define what we shall call the proof set of a formula. Relative to system SO_n, the proof set of a formula φ (denoted by $|\varphi|_{SO_n}$) is the set of SO_n-maximal sets containing φ.

Definition 63. *Let* $\varphi \in \mathsf{Form}$, *we set*

$$|\varphi|_{SO_n} = \{Max_{SO_n}\Gamma \mid \varphi \in \Gamma\}.$$

We can state that a formula is deducible from a set of formulas if and only if it belongs to every maximal extension of the set.

Theorem 38. *Let* $\Gamma \subseteq \mathsf{Form}$, *and let* $\varphi \in \mathsf{Form}$. *Then,*

$$\Gamma \vdash_{SO_n} \varphi \text{ if and only if } \varphi \in \Delta \text{ for every } \Delta \in |\Gamma|_{SO_n}$$

Proof. It follows from the Lindenbaum's Lemma.

Definition 64. *The* canonical model *of* SO_n *is the structure*

$$\mathcal{M}^* = (U^*, (R_1^*, \ldots, R_n^*), \mathsf{v}^*)$$

that satisfies the following conditions.

1. $U^* = \{\Gamma \subseteq \mathsf{Form} : Max_{SO_n}\Gamma\}$;
2. *For every* $w', w \in U^*, w' \in R_i^*(w)$ *iff* $\{\varphi | \Box_i\varphi \in w\} \subseteq w'$ *(namely,* wR_i^*w' *if and only if every formula* φ *belongs to* w'*, whenever* $\Box_i\varphi$ *belongs to* w*), and* $\bigcirc_i\top \in w$;
3. $\mathsf{v}^*(p) = |p|_{SO_n}$, *for each* $p \in \mathsf{Var}$.

The canonical model has this property: if $w \in U^*$, then the formulas that are true at w in $\mathcal{M}^*$ are all and only the formulas belonging to w. More precisely, the following theorem holds.

Theorem 39. *Let* $\mathcal{M}^*$ *be the canonical model of* SO_n. *Then, for every possible world* w *of* $\mathcal{M}^*$ *and for every formula* φ *of* Form,

$$(\mathcal{M}^*, w) \models \varphi \quad \text{if and only if} \quad \varphi \in w. \tag{40}$$

Proof. In order to prove the statement 40, we use the induction on the length of the formulas. By the definition of v^* and by Definition 63, the propositional variables satisfy 40 (case base). Suppose that the statement 40 holds for the formulas φ and ψ (induction hypothesis), we intend to prove that $\neg\varphi$, $\varphi \wedge \psi$, $\Box_i\varphi$ and $\bigcirc_i\varphi$ satisfy 40for each $i \in \{1, \ldots, n\}$ (induction step).

($\neg\varphi$). By Definition 55, $(\mathcal{M}^*, w) \models \neg\varphi$ if and only if $(\mathcal{M}^*, w) \not\models \varphi$. By induction hypothesis, we have that $\varphi \notin w$, namely $\neg\varphi \notin w$, since Theorem 36 holds.

($\varphi \wedge \psi$). By Definition 55, $(\mathcal{M}^*, w) \models \varphi \wedge \psi$ if and only if $(\mathcal{M}^*, w) \models \varphi$ and $(\mathcal{M}^*, w) \models \psi$. By induction hypothesis, we have that $\varphi \in w$ and $\psi \in w$, namely $\varphi \wedge \psi \in w$, since Theorem 36 holds.

($\Box_i\varphi$). Suppose that $(\mathcal{M}^*, w) \models \Box_i\varphi$. Then, by Definition 55, $R_i^*(u) \subseteq ||\varphi||_{\mathsf{v}^*}$. Therefore, if $w' \in U^*$ and $\{\psi \mid \Box_i\psi \in w\} \subseteq w'$, then $(\mathcal{M}^*, w') \models \varphi$. By induction hypothesis, $\varphi \in w'$. Then, $w' \vdash_{SO_n} \varphi$, by Theorem 36. By Corollary 5, $\{\psi \mid \Box_i\psi \in w\} \vdash_{SO_n} \varphi$. So, by Definition 60, $\vdash_{SO_n} \psi_1 \wedge \ldots \wedge \psi_n \to \varphi$. By rule $\Box_i$**Mn**, $\vdash \Box_i\psi_1 \wedge \ldots \wedge \Box_n\psi \to \Box_i\varphi \in w$. Moreover, by modus ponens, $\Box_i\varphi \in w$.

Let $\Box_i\varphi \in w$, we intend to prove that $R_i^*(w) \subseteq ||\varphi||_{\mathsf{v}^*}$ and $R_i^*(w) \neq \{w\}$. Firstly, suppose that $w' \in R_i^*(w)$, then $\{\psi \mid \Box_i\psi \in w\} \subseteq w'$. Thus, $\varphi \in w$, since $\Box_i\varphi \in w$. Then, $w \in ||\varphi||_{\mathsf{v}^*}$.

By schema $\mathbf{R2}_{\bigcirc_i}$, $\Box_i\varphi \to \bigcirc_i\varphi \in w$ and by hypothesis $\bigcirc_i\varphi \in w$. Then, by modus ponens, $\bigcirc_i\varphi \in w$, and so $R_i^*(w) \neq \{w\}$.

($\bigcirc_i\varphi$). $(\mathcal{M}^*, w) \models \bigcirc_i\varphi$ if and only if $(\mathcal{M}^*, w) \models \varphi$ and $(\mathcal{M}^*, w) \models \bigcirc_i\top$. Then, by induction hypothesis, $\varphi \in w$ and by definition of canonical model $\bigcirc_i\top \in w$. They are equivalent to say that $\varphi \wedge \bigcirc_i\top \in w$, namely $\bigcirc_i\varphi \in w$.

Theorem 40. *The canonical model* $\mathcal{M}^* = (U^*, (R_1^*, \ldots, R_n^*), \mathsf{v}^*)$ *is a Kripke model of* SO_n.

Proof. (R_i^* *is reflexive*). Let $w \in U^*$ such that $\Box_i\varphi \in w$. By the schema $\mathbf{T}_i$ of Definition 58 ($\Box_i\varphi \to \varphi$) and by Theorem 36, we have that $\varphi \in w$. Then, wR_i^*w.

(R_i^* *is symmetric*). Suppose that wR_i^*w', with $w \neq w'$. Therefore, $R_i^*(w) \neq \{w\}$ (consequently, $\bigcirc_i\top \in w$), and $\{\varphi \in \mathsf{Form} \mid \Box_i\varphi \in w\} \subseteq w'$. Let $\varphi \in \mathsf{Form}$ such that $\Box_i\varphi \in w'$. We have to prove that $\varphi \in w$. If $\varphi \notin w$, then $\neg\varphi \in w$. By Schema $\mathbf{Def}_2$, $\bigcirc_i\neg\varphi \in w$. By Schema $\mathbf{B}_{\Box_i}$ and by Theorem 36, $\Box_i\neg\triangle_i\neg\varphi \in w$. By hypothesis, $\neg\triangle_i\neg\varphi \in w'$, namely $\triangle_i\neg\varphi \notin w'$. By Schema $\mathbf{Def}_1$, $\Box_i\varphi \notin w'$. The latter is an absurd, since we have assumed that $\Box_i\varphi \in w'$.

(R_i^* *is transitive*). Suppose that wR_i^*w' and $w'R_i^*w''$. Consequently, $\{\varphi \in \mathsf{Form} \mid \Box_i\varphi \in w\} \subseteq w'$ and $\{\varphi \in \mathsf{Form} \mid \Box_i\varphi \in w'\} \subseteq w''$. Let $\varphi \in \mathsf{Form}$ such that $\Box_i\varphi \in w$, we have to prove that $\varphi \in w''$. By schema $\mathbf{4}_{\Box_i}$ of Definition 58 and Theorem 36, if $\Box_i\varphi \in w$, then $\Box_i\Box_i\varphi \in w$. By hypothesis, $\Box_i\varphi \in w'$ and so $\varphi \in w''$.

($R_1^*(w) \neq \{w\}$, *for each* $w \in U^*$). We consider $w \in U^*$. By Definition 64, $\bigcirc_i\top \in w$. Then, $\bigcirc_1\top$
$\in w$ and so $R_1^*(w) \neq \{w\}$.

($R_{i+1}^*(w) \subseteq R_i^*(w)$, *for each* $i \in \{1, \ldots, n-1\}$). Let $w' \in R_{i+1}^*(w)$ and $\varphi \in \mathsf{Form}$ such that $\Box_i\varphi \in w$. We have to prove that $\varphi \in w'$. By Schema $\mathbf{T}_{\Box_i}$, the hypothesis that $\Box_i\varphi \in w$ implies that $\varphi \in w$. By Definition 64, $\bigcirc_{i+1}\top \in w$. Consequently, $\bigcirc_i\top \wedge \varphi \in w$ and so $\bigcirc_{i+1}\varphi \in w$.

Since $R_{i+1}(w) \neq \{w\}$, then $\bigcirc_{i+1}\top \in w$. By schema $\mathbf{R1}_{\bigcirc_i}$ of Definition 58 and Theorem 36, $\Box_{i+1}\varphi \in w$. Then, $\varphi \in w'$.

5.3 Orthopaired Kripke model and sequences of orthopairs

In this section, we intend to investigate on the connections between sequences of orhopairs and modal logic SO_n. The relationships between rough sets and modal logic have been explored by several authors (see [70] for a list); the most studied one concerns Pawlak set theory and modal logic $S5$ [8] [94]. As we have already said in Section 5.1, the intuition behind this link is that the lower and the upper approximations can be regarded as two unary operations on subsets of the given universe. Thus, let U be a universe, and let R be an equivalence relation on U, the *Pawlak rough set algebra* $(2^U, \cap, \cup, \neg, \mathcal{L}_R, \mathcal{U}_R, \emptyset, U)$ is an extension of the Boolean algebra $(2^U, \cap, \cup, \neg, \emptyset, U)$ (see Remark 3), and then it may be interpreted in terms of the notions of topological space and topological Boolean algebra [8].

Firstly, we prove that there is a one-to-one correspondence between refinement sequences and Kripke frames of SO_n.

Without loss of generality, let be $\mathcal{C} = (C_1, \ldots, C_n)$ a refinement sequence of U, we suppose that its first partition C_1 covers U.

Let n be a positive integer. We denote the set of all refinement sequences made of n partial partitions with RS_n, and the set of all Kripke frames of SO_n made of n equivalence relations with F_n.

Definition 65. *We consider the map $f : \mathsf{RS}_n \mapsto \mathsf{F}_n$, where, let $\mathcal{C} \in \mathsf{RS}_n$, $f(\mathcal{C}) = (U, (R_1, \ldots, R_n)) \in \mathsf{F}_n$ such that*

1. $U = \cup\{b \mid b \in C_1\}$,
2. *uR_iv if and only if $u = v$ or $\{u, v\} \subseteq b$, with $b \in C_i$; for each $u, v \in U$ and $i \in \{1, \ldots, n\}$.*

Clearly, let $(U, (R_1, \ldots, R_n)) \in \mathsf{F}_n$, then $f^{-1}((U, (R_1, \ldots, R_n)))$ is the refinement sequence $(C_1, \ldots, C_n)$ of U such that

$$C_i = \{R_i(u) \mid u \in U \text{ and } R_i(u) \neq \{u\}\}.$$

Proposition 16. *The function f is a bijection.*

Proof. It is trivial.

Let $\mathcal{C} \in \mathsf{RS}_n$, we denote $f(\mathcal{C})$ with $\mathcal{F}_{\mathcal{C}}$. vice versa, let $\mathcal{F} \in \mathsf{F}_n$, we denote $f^-1(\mathcal{C})$ with $\mathcal{C}_{\mathcal{F}}$.

Example 40. Let $\mathcal{C} = (C_1 = \{\{a, b, c\}, \{d, e\}\}, C_2 = \{\{a, b\}\})$ be a refinement sequence of $\{a, b, c, d, e\}$. Then, $f(\mathcal{C}) = (\{a, b, c, d, e\}, (R_1, R_2))$, where

1. $R_1 = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d)\} \cup \{(u, u) \mid u \in \{a, b, c, d, e\}\}$ and
2. $R_2 = \{(a, b), (b, a)\} \cup \{(u, u) \mid u \in \{a, b, c, d, e\}\}$.

Vice versa, $f^{-1}((\{a, b, c, d, e\}, (R_1, R_2)) = \mathcal{C}$.

Therefore, function f allows us to identify Kripke frames of SO_n logic having U as universe with refinement sequences of partial partitions of U. Furthermore, we can observe that Kripke frame $(U, (R_1, \ldots, R_n))$ corresponds to the sequences of Pawlak spaces $((U, R_1), \ldots, (U, R_n))$.

The following theorem establishes a connection between sequences of orthopairs and the modal operators $(\Box_1, \ldots, \Box_n)$ and $(\triangle_1, \ldots, \triangle_n)$ of SO_n logic.

Theorem 41. *Let $\mathcal{F} = (U, (R_1, \ldots, R_n)) \in \mathsf{F}_n$ and $(\mathcal{F}, \mathsf{v}) \in \mathcal{C}_n$. Then, $(||\Box_i\varphi||_{\mathsf{v}}, ||\triangle_i\varphi||_{\mathsf{v}})$ is the orthopair of $||\varphi||_{\mathsf{v}}$ generated by the i-th partition of $\mathcal{C}_{\mathcal{F}}$. Therefore,*

$$(\ (||\Box_1\varphi||_{\mathsf{v}}, ||\triangle_1\varphi||_{\mathsf{v}}), \ldots, (||\Box_n\varphi||_{\mathsf{v}}, ||\triangle_n\varphi||_{\mathsf{v}})\ )$$

is the sequence of orthopairs of $||\varphi||_{\mathsf{v}}$ generated by $\mathcal{C}_{\mathcal{F}}$.

Proof. The proof follows by Definition 55 (point 4), Proposition 14 (point 2) and Definition 65.

Example 41. Let $\mathcal{F}$ be the Kripke frame of Example 40. We suppose that $\mathsf{Var} = \{p, q\}$ and we consider the Kripke model $(\mathcal{F}, \mathsf{v})$ such that $\mathsf{v}(p) = \{a, b, c\}$, and $\mathsf{v}(q) = \{a, b, d\}$. Then, $||p \wedge q||_{\mathsf{v}} = \{a, b\}$. Moreover,

$$(\ (||\Box_1\ p\wedge q||_{\mathsf{v}}, ||\triangle_1\ p\wedge q||_{\mathsf{v}}), (||\Box_2\ p\wedge q||_{\mathsf{v}}, ||\triangle_2\ p\wedge q||_{\mathsf{v}})\ ) = ((\emptyset, \{d, e\}), (\{a, b\}, \emptyset)),$$

that is the sequence $\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}})$.

Trivially, let v and v' be two evaluation functions such that $\mathsf{v} \neq \mathsf{v}'$, then the sequence $\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}})$ is not usually equal to $\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}'})$.

Example 42. We consider the Kripke model $(\mathcal{F}, \mathsf{v})$ of Example 41 and the Kripke model $(\mathcal{F}, \mathsf{v}')$ such that $\mathsf{v}'(p) = \{a, d, e\}$ and $\mathsf{v}' = \{d, e\}$.
Then, $||p \wedge q||_{\mathsf{v}'} = \{d, e\}$ and so $\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}'}) = ((\{d, e\}, \{a, b, c\}), (\emptyset, \{a, b\}))$, that is not equal to the sequence $\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}})$.

Given a Kripke model $(\mathcal{F}, \mathsf{v})$ of SO_n and two formulas φ and ψ, there exists a formula obtained from φ and ψ that is valid in $(\mathcal{F}_{\mathcal{C}}, \mathsf{v})$ if and only if the sequences of orthopairs of $||\varphi||_{\mathsf{v}}$ and $||\psi||_{\mathsf{v}}$ generated by $\mathcal{C}_{\mathcal{F}}$ are equal to each other. More precisely, the following theorem holds.

Theorem 42. *Let $\varphi, \psi \in \mathsf{Form}$ and $(\mathcal{F}, \mathsf{v}) \in \mathcal{C}_n$, then*

$$\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) = \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) \quad \textit{iff} \quad \models^{(\mathcal{F},\mathsf{v})} \bigwedge_{i=1}^{n} (\Box_i\varphi \equiv \Box_i\psi) \wedge (\triangle_i\varphi \equiv \triangle_i\psi).$$

Proof. Notice that, by Proposition 14, $\models^{(\mathcal{F},\mathsf{v})} (\Box_i\varphi \equiv \Box_i\psi)$ if and only if $||\Box_i\varphi||_{\mathsf{v}} = ||\Box_i\psi||_{\mathsf{v}}$, for each $i \in \{1, \ldots, n\}$. Then, the thesis clearly follows.

The following remark shows that the modal operators $\bigcirc_1, \ldots, \bigcirc_n$ allow us to understand what are the elements that are lost during the refinement process.

Remark 25. Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a refinement sequence of U, through the modal operator $\bigcirc_i$, it is easy to check whether an element of U belongs to a block of the C_i; thus, let $u \in U$ and $i \in \{1, \ldots, n\}$, we have that

$$u \in \bigcup_{b \in C_i} b \text{ if and only if } ((\mathcal{F}_{\mathcal{C}}, \mathsf{v}), u) \models \bigcirc_i \top,$$

for each evaluation function v.

Furthermore, we can express the property of safety of refinement sequences of partial partitions by using the modal operators $(\Box_1, \ldots, \Box_n)$ and $(\bigcirc_1, \ldots, \bigcirc_n)$ (the meaning of safe refinement sequence is given in Definition 44).

Theorem 43. *Let $\mathcal{C}$ be a refinement sequence of U. Then, $\mathcal{C}$ is safe if and only if the following condition holds:*

"if $(\mathcal{M}, u) \models \Box_i \varphi$ and $i \leq j$, then $R_i(u) = R_j(u)$ or there exists $u' \in R_i(u)$ such that $(\mathcal{M}, u') \models \neg \bigcirc_j \varphi$" (or "if $(\mathcal{M}, u) \models \triangle_i \varphi$, then $R_i(u) = R_j(u)$ or there exists $u' \in R_i(u)$ such that $(\mathcal{M}, u') \models \neg \bigcirc_j \neg\varphi$"), for each $\varphi \in \mathsf{Form}$, $\mathcal{M} = (\mathcal{F}_{\mathcal{C}}, \mathsf{v}) \in \mathcal{C}_n$, $u \in U$ and $i \in \{1, \ldots, n-1\}$.

Proof. $(\Rightarrow)$. We suppose that $(\mathcal{M}, u) \models \Box_i \varphi$ and $R_i(u) \neq R_j(u)$, with $j > i$. We notice that $R_i(u) \in C_i$, since $R_i(u) \neq \{u\}$. On the other hand, $R_i(u) \notin C_j$, since $R_i(u) \neq R_j(u)$. So, we call $N_1, \ldots, N_m$ the blocks of C_j that are included in $R_i(u)$. By Remark 13, the successors $N'_1, \ldots, N'_l$ of $R_i(u)$ belong to C_k, where $i < k \leq j$. Since $\mathcal{C}$ is safe, there exists $u' \in R_i(u)$ such that $u' \notin N'_1 \cup \ldots \cup N'_l$ (see Definition 44). Then, $u' \notin \cup\{b \mid b \in C_k\}$ and so $u' \notin \cup\{b \mid b \in C_j\}$. Then, $R_j(u') = \{u'\}$ and this means that $(\mathcal{M}, u') \models \neg \bigcirc_j \varphi$.

$(\Leftarrow)$. Let $N \in P_{\mathcal{C}}$. Suppose that $N_1, \ldots, N_m$ are the successors of N in $P_{\mathcal{C}}$. We intend to prove that $N_1 \cup \ldots \cup N_m \subset N$. We consider the evaluation function v such that $\mathsf{v}(p) = N$, where $p \in \mathsf{Var}$. If $N \in C_i$, then there exists $u \in U$ such that $N = R_i(u)$. Trivially, we have that $((\mathcal{F}_{\mathcal{C}}, \mathsf{v}), u) \models \Box_i p$. We notice that $N_1, \ldots, N_m$ belong to C_j, with $j > i$. By hypothesis, there exists $u' \in R_i(u) (= N)$ such that $((\mathcal{F}_{\mathcal{C}}, \mathsf{v}), u) \models \neg \bigcirc_i p$. Then $R_j(u') \neq \{u'\}$ and so u' does not belong to some nodes of C_j. Therefore, $u' \in N$, but $u' \notin N_1 \cup \ldots \cup N_m$ and so by Definition 44, $\mathcal{C}$ is safe.

As a consequence of the previous theorem, we can express the results of Corollary 2 for refinement sequences of partial partitions by using the modal operators $(\Box_1, \ldots, \Box_n)$ and $(\bigcirc_1, \ldots, \bigcirc_n)$ as follows.

Theorem 44. *Let $\mathcal{C} = (C_1, \ldots, C_n)$ be a refinement sequence of U. Then, $\mathbb{K}^3_{\mathcal{C}}$ is a finite IUML-algebra if and only if the following condition holds:*

"if $(\mathcal{M}, u) \models \Box_i \varphi$ and $i \leq j$, then $R_i(u) = R_j(u)$ or there exists $u' \in R_i(u)$ such that $(\mathcal{M}, u') \models \neg \bigcirc_j \varphi$" (or "if $(\mathcal{M}, u) \models \triangle_i \varphi$, then $R_i(u) = R_j(u)$ or there exists $u' \in R_i(u)$ such that $(\mathcal{M}, u') \models \neg \bigcirc_j \neg\varphi$"), for each $\varphi \in \mathsf{Form}$, $\mathcal{M} = (\mathcal{F}_{\mathcal{C}}, \mathsf{v}) \in \mathcal{C}_n$, $u \in U$ and $i \in \{1, \ldots, n-1\}$.

However, by using modal logic, we can also express the results obtained for the structures $\mathbb{K}^1_{\mathcal{C}}$, $\mathbb{K}^2_{\mathcal{C}}$ and $\mathbb{K}^4_{\mathcal{C}}$ in Section 4, but only when $\mathcal{C}$ is a refinement sequence of partial partitions (we recall that such algebraic structures, except $\mathbb{K}^3_{\mathcal{C}}$, are generated by refinement sequences of partial coverings of the given universe).

At the end of this section, we intend to include the operations $\curlywedge$, $\curlyvee$, $\hookrightarrow_1$, $\odot_2$, $\hookrightarrow_2$, $\odot_3$ and $\hookrightarrow_3$ defined on sequences of orthopairs of partial partitions (see 50) in our modal logic. [2]

Theorem 45. *Let $\varphi, \psi \in \mathsf{Form}$ and $(\mathcal{F}, \mathsf{v}) \in \mathsf{Cl}_n$. If $\mathcal{C}_{\mathcal{F}}$ is safe, then*

$$\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \curlywedge \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) = ((A_1, B_1), \ldots, (A_n, B_n)),$$

where $(A_i, B_i) = (||\Box_i\varphi \wedge \Box_i\psi||_{\mathsf{v}}{}^3, ||\triangle_i\varphi \vee \triangle_i\psi||_{\mathsf{v}})$, for each $i \in \{1, \ldots, n\}$, and

$$\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \curlyvee \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) = ((C_1, D_1), \ldots, (C_n, D_n)),$$

where $(C_i, D_i) = (||\Box_i\varphi \vee \Box_i\psi||_{\mathsf{v}}, ||\triangle_i\varphi \wedge \triangle_i\psi||_{\mathsf{v}}{}^4)$, for each $i \in \{1, \ldots, n\}$.

Proof. By Theorem 30, $\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \curlywedge \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) = ((A_1, B_1), \ldots, (A_n, B_n))$, such that $(A_i, B_i) = (\mathcal{L}_i(||\varphi||_{\mathsf{v}}), \mathcal{E}_i(||\varphi||_{\mathsf{v}})) \wedge_{\mathcal{K}} (\mathcal{L}_i(||\psi||_{\mathsf{v}}), \mathcal{E}_i(||\psi||_{\mathsf{v}})) = (\mathcal{L}_i(||\varphi||_{\mathsf{v}}) \cap \mathcal{L}_i(||\psi||_{\mathsf{v}}), \mathcal{E}_i(||\varphi||_{\mathsf{v}}) \cup \mathcal{E}_i(||\psi||_{\mathsf{v}}))$. Suppose that $u \in U$, we have that $u \in \mathcal{L}_i(||\varphi||_{\mathsf{v}}) \cap \mathcal{L}_i(||\psi||_{\mathsf{v}})$ if and only if $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$, $R_i(u) \subseteq ||\psi||_{\mathsf{v}}$ and $R_i(u) \neq \{u\}$, namely $u \models \Box_i\varphi \wedge \Box_i\psi$. Moreover, $u \in \mathcal{E}_i(||\varphi||_{\mathsf{v}}) \cup \mathcal{E}_i(||\psi||_{\mathsf{v}})$ if and only if $R_i(u) \neq \{u\}$ and either $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}$ or $R_i(u) \subseteq ||\psi||_{\mathsf{v}}$, namely $u \models \triangle_i\varphi \vee \triangle_i\psi$. The proof for the operation $\curlyvee$ is analogous.

Definition 66. *Let $\varphi, \psi \in \mathsf{Form}$, we recursively define the sequences of formulas $(\alpha_1(\varphi, \psi), \ldots, \alpha_n(\varphi, \psi))$, $(\beta_1(\varphi, \psi), \ldots, \beta_n(\varphi, \psi))$, $(\gamma_1(\varphi, \psi), \ldots, \gamma_n(\varphi, \psi))$, $(\delta_1(\varphi, \psi), \ldots, \delta_n(\varphi, \psi))$, $(\epsilon_1(\varphi, \psi), \ldots, \epsilon_n(\varphi, \psi))$, $(\zeta_1(\varphi, \psi), \ldots, \zeta_n(\varphi, \psi))$, $(\eta_1(\varphi, \psi), \ldots, \eta_n(\varphi, \psi))$, $(\theta_1(\varphi, \psi), \ldots, \theta_n(\varphi, \psi))$, $(\iota_1(\varphi, \psi), \ldots, \iota_n(\varphi, \psi))$ and $(\kappa_1(\varphi, \psi), \ldots, \kappa_n(\varphi, \psi))$ as follows.*

- $\alpha_n(\varphi, \psi) := \neg\Box_n\varphi \vee \Box_n\psi$;
- $\alpha_i(\varphi, \psi) := (\neg\Box_i\varphi \vee \Box_i\psi) \wedge \neg\alpha_{i+1}(\varphi, \psi)$, *with* $i \in \{1, \ldots, n-1\}$;
- $\beta_i(\varphi, \psi) := \Box_i\varphi \wedge \Box_i\psi$, *with* $i \in \{1, \ldots, n\}$;
- $\gamma_i(\varphi, \psi) := \Box_i\varphi \wedge \Box_i\psi$, *with* $i \in \{1, \ldots, n\}$;
- $\delta_n(\varphi, \psi) := \lambda_n(\varphi, \psi)$;
- $\delta_i(\varphi, \psi) := \lambda_i(\varphi, \psi) \wedge \neg\delta_{i+1}(\varphi, \psi)$, *with* $i \in \{1, \ldots, n-1\}$, *where*

$$\lambda_i(\varphi, \psi) := \neg(\Box_i\varphi \vee \Box_i\psi) \vee \Box_i\varphi \vee \Box_i\psi.$$

- $\epsilon_n(\varphi, \psi) := \mu_n(\varphi, \psi)$;
- $\epsilon_i(\varphi, \psi) := \mu_i(\varphi, \psi) \wedge \neg\epsilon_{i+1}(\varphi, \psi)$, *with* $i \in \{1, \ldots, n-1\}$, *where*

$$\mu_i(\varphi, \psi) := (\neg\Box_i\varphi \vee \Box_i\psi) \wedge (\triangle_i\varphi \vee \neg\triangle_i\psi).$$

- $\zeta_i(\varphi, \psi) := \Box_i\varphi \wedge \triangle_i\psi$, *with* $i \in \{1, \ldots, n\}$;

[2] We exclude the operations $\odot_4$ and $\hookrightarrow_4$, since they can not be obtained starting from operations between the orthopairs.

[3] By 15, $\Box_i\varphi \wedge \Box_i\psi = \Box_i(\varphi \wedge \psi)$.

[4] By 15, $\triangle_i\varphi \wedge \triangle_i\psi = \triangle_i(\varphi \vee \psi)$.

– $\eta_1(\varphi,\psi) := \nu_1(\varphi,\psi)$;
– $\eta_i(\varphi,\psi) := \nu_i(\varphi,\psi) \vee \Box_i \eta_{i-1}(\varphi,\psi)$, *with* $i>1$ *and*

$$\nu_i(\varphi,\psi) = (\Box_i\varphi \wedge \neg\triangle_i\psi) \;\vee\; (\Box_i\psi \wedge \neg\triangle_i\varphi).^5$$

– $\theta_i(\varphi,\psi) := (\triangle_i\varphi \vee \triangle_i\psi) \wedge \neg\, \eta_i(\varphi,\psi)$, *with* $i \in \{1,\ldots,n\}$;
– $\iota_i(\varphi,\psi) := ((\neg\Box_i\varphi \vee \Box_i\psi) \wedge (\triangle_i\varphi \vee \neg\triangle_i\psi)) \wedge \kappa_i(\varphi,\psi)$, *for each* $i \in \{1,\ldots,n\}$;
– $\kappa_1(\varphi,\psi) := \Box_1\varphi \wedge \triangle_1\psi$;
– $\kappa_i(\varphi,\psi) := (\Box_i\varphi \wedge \triangle_i\psi) \vee \kappa_{i-1}(\varphi,\psi)$, *for each* $i \in \{2,\ldots,n\}$.

Theorem 46. *Let* $\varphi,\psi \in \mathsf{Form}$ *and* $(\mathcal{F},\mathsf{v}) \in \mathcal{C}_n$. *If* $\mathcal{C}_{\mathcal{F}}$ *is safe, then*

$$\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \hookrightarrow_1 \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) = ((E_1,F_1),\ldots,(E_n,F_n)),$$

where $(E_i,F_i) = (||\alpha_i(\varphi,\psi)||_{\mathsf{v}}, ||\beta_i(\varphi,\psi)||_{\mathsf{v}})$, *for each* $i \in \{1,\ldots,n\}$.

$$\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \odot_2 \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) = ((G_1,H_1),\ldots,(G_n,H_n)),$$

where $(G_i,H_i) = (||\gamma_i(\varphi,\psi)||_{\mathsf{v}}, ||\delta_i(\varphi,\psi)||_{\mathsf{v}})$, *for each* $i \in \{1,\ldots,n\}$.

$$\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \hookrightarrow_2 \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) = ((I_1,J_1),\ldots,(I_n,J_n)),$$

where $(I_i,J_i) = (||\epsilon_i(\varphi,\psi)||_{\mathsf{v}}, ||\zeta_i(\varphi,\psi)||_{\mathsf{v}})$, *for each* $i \in \{1,\ldots,n\}$.

$$\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \odot_3 \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) = ((K_1,L_1),\ldots,(K_n,L_n)),$$

where $(K_i,L_i) = (||\eta_i(\varphi,\psi)||_{\mathsf{v}}, ||\theta_i(\varphi,\psi)||_{\mathsf{v}})$, *for each* $i \in \{1,\ldots,n\}$.

$$\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \hookrightarrow_3 \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}}) = ((M_1,N_1),\ldots,(M_n,N_n)),$$

where $(M_i,N_i) = (||\iota_i(\varphi,\psi)||_{\mathsf{v}}, ||\kappa_i(\varphi,\psi)||_{\mathsf{v}})$, *for each* $i \in \{1,\ldots,n\}$.

Proof. We only provide the proof for the operation $\odot_3$, since those of the remaining cases are analogous.

Let $u \in U$,

$$((\mathcal{F},\mathsf{v}),u) \models \nu_i \;\text{ iff }\; ((\mathcal{F},\mathsf{v}),u) \models \Box_1\varphi \wedge \neg\triangle_1\psi \text{ or } ((\mathcal{F},\mathsf{v}),u) \models \Box_1\psi \wedge \neg\triangle_1\varphi,$$

that is

– $R_i(u) \subseteq ||\varphi||_{\mathsf{v}}, R_i(u) \neq \{u\}$ and $R_i(u) \cap ||\psi||_{\mathsf{v}} \neq \emptyset$, or
– $R_i(u) \subseteq ||\psi||_{\mathsf{v}}, R_i(u) \neq \{u\}$ and $R_i(u) \cap ||\varphi||_{\mathsf{v}} \neq \emptyset$.

Consequently, we obtain that

$$((\mathcal{F},\mathsf{v}),u) \models \nu_i \;\text{ if and only if }\; u \in (\mathcal{L}_i(\varphi) \setminus \mathcal{E}_i(\psi)) \cup (\mathcal{L}_i(\psi) \setminus \mathcal{E}_i(\varphi)).$$

[5] Observe that this expression is equivalent to $(\Box_i\varphi \setminus \triangle_i\psi \;\wedge\; \Box_i\psi \setminus \triangle_i\varphi)$

Trivially, we can observe that

$$((\mathcal{F},\mathsf{v}),u) \models \Box_i \eta_{i-1}(\varphi,\psi) \ \text{ iff } \ R_i(u) \subseteq ||\eta_{i-1}(\varphi,\psi)||_{\mathsf{v}} \text{ and } R_i(u) \neq \{u\},$$

and

$$((\mathcal{F},\mathsf{v}),u) \models \theta_i(\varphi,\psi) \ \text{ iff } \ u \in \mathcal{E}_i(||\varphi||_{\mathsf{v}}) \cup \mathcal{E}_i(||\psi||_{\mathsf{v}}).$$

By Theorem 33 and by $(X,Y) *_{\mathcal{S}} (Z,W) = ((X \setminus W) \cup (Z \setminus Y), Y \cup W)$ (see Definition 11), we obtain that the i-th component of the sequence $\mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\varphi||_{\mathsf{v}}) \odot_3 \mathcal{O}_{\mathcal{C}_{\mathcal{F}}}(||\psi||_{\mathsf{v}})$ is $(||\eta_i(\varphi,\psi)||_{\mathsf{v}}, ||\theta_i(\varphi,\psi)||_{\mathsf{v}})$.

5.4 Epistemic logic SO_n

In this section, we employ modal logic SO_n and describe the knowledge of an agent during a sequence $(t_1,\dots,t_n)$ of consecutive instants of time. Also, we intend to establish whether the given agent is *interested in knowing* the truth or falsity of the sentences at every instant of $(t_1,\dots,t_n)$. In detail, we represent situations in which, given an agent $\mathcal{A}$ and a sequence $(t_1,\dots,t_n)$,

- $\mathcal{A}$ knows more information at time t_{i+1} than at time t_i, and
- $\mathcal{A}$ is less interested in knowing at time t_{i+1} than at time t_i.

Example 43. We suppose that a restaurant owner manages seven restaurants in seven Italian cities: Viterbo, Rieti, Rome, Latina, Frosinone, Potenza and Matera. He needs to know the weather report for tomorrow in order to decide whether to set up the gardens of his restaurants. At time t_1, he knows by speaking with a friend, that it is cloudy throughout Lazio, consequently it is cloudy in Viterbo, Rieti, Rome, Latina and Frosinone, but he does not know the weather in Potenza and Matera. At time $t_2 > t_1$, he finds the weather report on Internet, and he knows that it is cloudy with a chance of rain in Viterbo and Rieti, it is cloudy without rain in Latina and Frosinone, and it is sunny in Matera and Potenza. Since he decides that the restaurant will be close in Rome, he does not look for any information about the weather there. This situation is synthesized in Table 13, where C, C + R, C - R and S denote respectively cloudy, cloudy with rain, cloudy without rain and sunny. Moreover, the symbol $\times$ means that the restaurant owner excludes Rome from all cities he is interested in knowing the weather, and ? means that he has not information about the respective cities.

	Viterbo	*Rieti*	*Rome*	*Latina*	*Frosinone*	*Potenza*	*Matera*
t_1	C	C	C	C	C	?	?
t_2	C + R	C + R	$\times$	C - R	C - R	S	S

Table 13: Information about the weather

Table 13 corresponds to a refinement sequence made of the partial partitions C_1 and C_2, where

$C_1 = \{\{\text{Viterbo, Rieti, Rome, Latina, Frosinone}\}, \{\text{Potenza, Matera}\}\}$ and
$C_2 = \{\{\text{Viterbo, Rieti}\}, \{\text{Latina, Frosinone}\}, \{\text{Potenza, Matera}\}\}$.

Then, each block of C_1 is the set of the cities that, at time t_1, have the same weather with respect to the knowledge of the restaurant owner, and C_2 is made of the cities that, at time t_2, have the same weather with respect to the knowledge of the restaurant owner. We underline that the owner has more information about the weather in cities of Table 13 at time t_2 than at time t_1 (for example, at time t_1, he knows that it is cloudy in Viterbo, and at time t_2, he knows that it is cloudy with rain there); however, he is interested in knowing the weather in less cities at time t_2 than at time t_1 (precisely, at time t_2, he excludes Rome).

The finite sequences $(\Box_1, \ldots, \Box_n)$ and $(\bigcirc_1, \ldots, \bigcirc_n)$ of SO_n correspond to a sequence $(t_1, \ldots, t_n)$ made of consecutive instants of time, or of consecutive time intervals. In addition, let $i \in \{1, \ldots, n\}$, the interpretation of the modality $\Box_i$ with respect to an orthopaired Kripke model allows us to represent the knowledge of an agent at time t_i. Furthermore, the semantic interpretation of the modality $\bigcirc_i$ establishes whether the agent is interested in knowing the truth or falsity of a sentence at each initial possible world at time t_i. Thus, each Kripke frame $\mathcal{M} = (U, (R_1, \ldots, R_n))$ of SO_n is associated with a pair $(\mathcal{A}, (t_1, \ldots, t_n))$ such that $\mathcal{A}$ is an agent, and $(t_1, \ldots, t_n)$ is a sequence of successive instants of time. More precisely, let $u \in U, i \in \{1, \ldots, n\}$ and $\varphi \in \mathsf{Form}$, if $u \models \Box_i\varphi$, we can say that

" at time t_i, the agent $\mathcal{A}$ knows that φ is true at u".

Moreover, if $u \models \bigcirc_i\varphi$, then we can say that

"φ is true at u, but at time t_i, $\mathcal{A}$ is not interested in knowing it".

When $R_i(u) \neq \{u\}$ (i.e. $u \models \bigcirc_i\top$), at time t_i, the agent $\mathcal{A}$ is not able to distinguish the elements of $R_i(u)$ from one another; on the contrary, that is $R_i(u) = \{u\}$ (i.e. $u \models \neg \bigcirc_i \top$), at time t_i, the agent $\mathcal{A}$ ignores whether a formula is true or false at u. The epistemic interpretation that we give to modal logic SO_n is better explained through the following example.

Example 44. We consider a game where a player selects a card x in $\mathcal{D}$ that is a deck of French playing cards which are left face down, and he/she tries to guess the identity of x. He/she repeats these actions (i.e. select and try to guess a card) for up to three times, exactly at times t_1, t_2 and t_3, with $t_1 < t_2 < t_3$. If he/she guesses the identity of the choice card at least once, then he/she wins; otherwise, he/she loses. Trivially, let $i \in \{1, 2\}$, if he/she guesses the selected card at time t_i, then the game finishes without considering the time t_{i+1}. Furthermore, during the game, a referee, that knows the identity of all cards of $\mathcal{D}$, provides the player with information on several properties of the cards in $\mathcal{D}$ at each time of the sequence (t_1, t_2, t_3), as it will be shown.

We suppose that Alice and Bob are respectively the player and the referee of this game. Then, it occurs that

1. at time t_1, Bob divides the deck $\mathcal{D}$ into two stacks: red cards and black cards;
2. at time $t_2 > t_1$, he also brings together all cards that have the same suit in each group of cards that have the same colours;
3. at time $t_3 > t_2$, he divides each group of cards obtained at time t_2 into two stacks: the cards whose number is less than 7 and the cards whose number is greater or equal to 7.

The classification made by Bob to cards of $\mathcal{D}$ at times t_1, t_2 and t_3 is represented in the following figure, where $c(x)$ and $s(x)$ respectively denote the colour and the suit of card x.

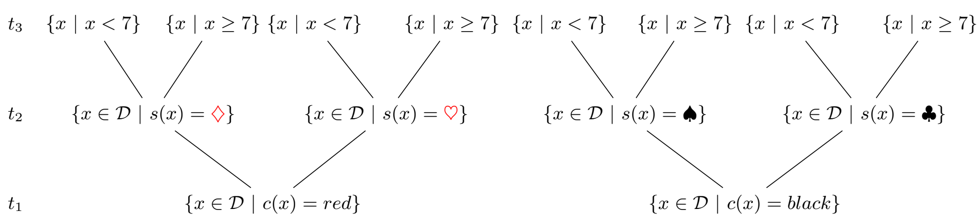

Fig. 34: Forest of Bob's classification at times t_1, t_2 and t_3

We set $B_1 = \{x \in \mathcal{D} \mid c(x) = red\}$, $B_2 = \{x \in \mathcal{D} \mid c(x) = black\}$, $B_3 = \{x \in \mathcal{D} \mid s(x) = \diamondsuit\}$, $B_4 = \{x \in \mathcal{D} \mid s(x) = \heartsuit\}$, $B_5 = \{x \in \mathcal{D} \mid s(x) = \spadesuit\}$, $B_6 = \{x \in \mathcal{D} \mid s(x) = \clubsuit\}$, $B_7 = \{x \in \mathcal{D} \mid s(x) = \diamondsuit$ and $x < 7\}$, $B_8 = \{x \in \mathcal{D} \mid s(x) = \diamondsuit$ and $x \geq 7\}$, $B_9 = \{x \in \mathcal{D} \mid s(x) = \heartsuit$ and $x < 7\}$, $B_{10} = \{x \in \mathcal{D} \mid s(x) = \heartsuit$ and $x \geq 7\}$, $B_{11} = \{x \in \mathcal{D} \mid s(x) = \spadesuit$ and $x < 7\}$, $B_{12} = \{x \in \mathcal{D} \mid s(x) = \spadesuit$ and $x \geq 7\}$, $B_{13} = \{x \in \mathcal{D} \mid s(x) = \clubsuit$ and $x < 7\}$, $B_{14} = \{x \in \mathcal{D} \mid s(x) = \clubsuit$ and $x \geq 7\}$.

We also assume that, let $i \in \{1, 2, 3\}$, at time t_i, Bob informs Alice about the properties that characterize each cards group corresponding to t_i. For example, at time t_2, he says to Alice that the cards of B_4 are all cards of $\mathcal{D}$ whose suit is $\heartsuit$ (then they are also red). Consequently, when Alice chooses a card x in B_i, despite she does not know the identity of x, she knows that x has the proprieties characterizing B_i. Thus, if she chooses a card x at time t_2 in B_4, then she knows that the suit of x is $\heartsuit$, and so that the colour of x is red.

In this framework, Alice represents the agent of the knowledge, and $\mathcal{D}$ is the universe of possible worlds of the Kripke frame assigned to Alice. We notice that each block of the forest in the previous figure is a set of cards which are indistinguishable for Alice at the respective time. For example, at time t_2, she still does not have enough information to distinguish $2\heartsuit$ from $8\heartsuit$. Moreover, it is easy to notice that the information that Bob gives to Alice defines three equivalence relations on $\mathcal{D}$, one for each time in (t_1, t_2, t_3), as follows: let $x, y \in \mathcal{D}$

- $xR_1y \;\Leftrightarrow\; c(x) = c(y)$,

- $xR_2y \;\Leftrightarrow\; s(x) = s(y)$,
- $xR_3y \;\Leftrightarrow\; xR_2y$ and $\{max(x, y) < 7$ or $min(x, y) \geq 7\}$.

Now, we imagine that at time t_2, in order to further help Alice, Bob removes from $\mathcal{D}$ a group D^2 of cards. Again, at time t_3, he removes from $\mathcal{D} \setminus \mathcal{D}^2$ the group $\mathcal{D}^3$ of cards. We suppose that he also informs Alice what cards belong to $\mathcal{D}^2$ (at time t_2) and $\mathcal{D}^3$ (at time t_3). These actions allow us to define three new equivalent relations, R'_1, R'_2 and R'_3, as follows. Let $x, y \in \mathcal{D}$

- $xR'_1y \;\Leftrightarrow\; xR_1y$
- $xR'_2y \;\Leftrightarrow\; \begin{cases} xR_2y, & \text{if } x, y \notin \mathcal{D}_2 \\ x = y, & \text{otherwise} \end{cases}$
- $xR'_3y \;\Leftrightarrow\; \begin{cases} xR_3y, & \text{if } x, y \notin \mathcal{D}_2 \cup \mathcal{D}_3 \\ x = y, & \text{otherwise} \end{cases}$

We suppose that Bob chooses $\mathcal{D}^2$ and $\mathcal{D}^3$ so that each group B_i without the cards of $\mathcal{D}^2 \cup \mathcal{D}^3$ is not made of one card.

Then, we can observe that, let $i \in \{1, 2, 3\}$, a cards is removed from $\mathcal{D}$ at time t_i if and only if its equivalent class with respect to R'_i is a singleton.

From now on, we indicate the card with number or face i, and suit j with ij, and we write $[ij]_k$ to denote the equivalence class of ij with respect to R'_k. Therefore, let φ be the proposition "the card is black", trivially, we have that

$$i\diamondsuit, i\heartsuit \models \Box_1\neg\varphi \text{ and } \; i\spadesuit, i\clubsuit \models \Box_1\varphi,$$

for each $i \in \{1, \ldots, 10\} \cup \{J, Q, K\}$. We respectively read the previous expressions as follows.

– "At time t_1, Alice knows that $i\diamondsuit$ is not black";
– "at time t_1, Alice knows that $i\heartsuit$ is not black";
– "at time t_1, Alice knows that $i\spadesuit$ is black";
– "at time t_1, Alice knows that $i\clubsuit$ is black".

On the other hand, if φ' is the proposition "the card is a two" and $j \in \{\diamondsuit, \heartsuit, \spadesuit, \clubsuit\}$, we have that

$$2j \models \neg\Box_1\varphi',$$

since $[2j]_1$ is equal to $\{ij \in \mathcal{D} \mid c(ij) = red\}$ or $\{ij \in \mathcal{D} \mid c(ij) = black\}$, and both are not contained in $||\varphi'|| = \{2j \mid j \in \{\diamondsuit, \heartsuit, \spadesuit, \clubsuit\}\}$. Then, $2j \models \neg\Box_1\varphi'$ means that

"at time t_1, Alice does not know that the number of $2j$ is a two".

We recall that all cards od $\mathcal{D}$ are left face down, and so Alice does not know the identity of $2j$. The previous sentences correspond to the fact that, at time t_1, Alice only knows the colour of all cards of $\mathcal{D}$, but she does not have more information about them; for example, she knows that $2\heartsuit$ is red, but no that it is a two. We suppose that $\mathcal{D}^2$ is made of all cards of $\mathcal{D}$ with face J, Q, K.

Consequently, let ψ be the proposition "the suit of the card is a spade", the sentence

$$K\spadesuit \models \neg\Box_2\psi$$

that we read as follows,

"at time t_2, Alice does not know that the suit of card is a spade",

is true, since $[K\spadesuit]_2$ is a singleton.

Moreover, the sentence

$$K\spadesuit \models \neg \bigcirc_2 \psi$$

that we read as follows,

"the suit of card is a spade, but at time t_2, Alice is not interested in knowing it",

is also true.

The latter two propositions correspond to the fact that at time t_2 Alice has information on suit of cards of $\mathcal{D}$, but she ignores $K\spadesuit$, since it is removed from the deck.

Furthermore,

$$5\heartsuit \models \bigcirc_2\neg\varphi$$

holds, and we read it as "the card is not black and at time t_2 Alice is interested to know it".

At this point, we assume that at time t_3 Bob removes $1\diamondsuit$, $2\diamondsuit$, $6\diamondsuit$, $8\diamondsuit$, $10\diamondsuit$, $2\heartsuit$, $4\heartsuit$, $5\heartsuit$, $6\heartsuit$, $7\heartsuit$, $1\spadesuit$, $2\spadesuit$, $3\spadesuit$, $7\spadesuit$, $10\spadesuit$, $3\clubsuit$, $5\clubsuit$, $6\clubsuit$, $7\spadesuit$ and $8\spadesuit$ from $\mathcal{D}\setminus D^2$. Then, let ψ' be the proposition "the number of the card is greater than or equal to 7", these sentences hold:

$$7\diamondsuit \models \Box_3\psi' \quad \text{and} \quad 9\spadesuit \models \bigcirc_3\psi'.$$

On the other hand, we have that

$$9\spadesuit \models \neg\Box_2\psi' \quad \text{and} \quad 7\heartsuit \models \neg \bigcirc_3 \psi'.$$

They say that

– "at time t_3, Alice knows that the number of $7\diamondsuit$ is greater than or equal to 7",
– the number of $9\spadesuit$ is greater than or equal to 7, and at time t_3, Alice is interested in knowing it",
– "at time t_2, Alice does not know that the number of $9\spadesuit$ is greater than or equal to 7",
– "$7\heartsuit$ is greater than or equal to 7, but at time t_3, Alice is not interested in knowing it".

The pair $(\mathcal{D}, (R_1', R_2', R_3'))$ is a Kripke frame of SO_3 logic, and it is assigned to Alice and to the sequence (t_1, t_2, t_3). Furthermore, $(\mathcal{D}, (R_1', R_2', R_3'))$ corresponds to the refinement sequence whose forest is represented in the following figure.

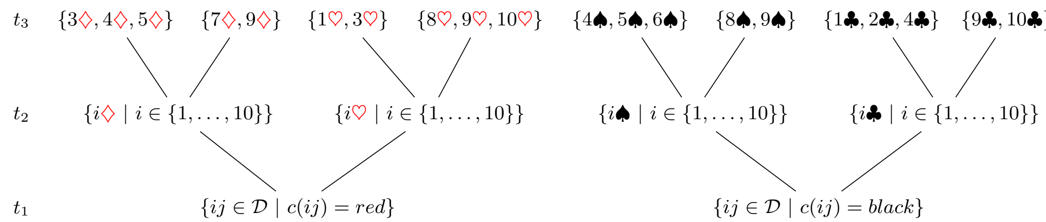

Fig. 35: Forest corresponding to $(\mathcal{D}, (R'_1, R'_2, R'_3))$

The next proposition states that at time t_i, Alice has the information acquired at time t_i, plus all information acquired at previous times.

Proposition 17. *Let φ be a formula, for each $i \geq j$, $\vdash \Box_i \Box_j \varphi \leftrightarrow \Box_j \varphi$.*

Finally, we can notice that by using theorems of SO_n, we can investigate on the properties of the knowledge of Alice during the sequence (t_1, t_2, t_3). For example, by Schema $\Box_i \varphi \to \bigcirc_i \varphi$, we can deduce that "at time t_i, if Alice knows φ, then she is also interested in knowing it".

6 Conclusions and future directions

I hope that we continue with exploration
Margaret H. Hamilton

In this thesis, we developed and studied a generalization of the rough set theory. In detail, we introduced the *sequences of orthopairs* generated by *refinement sequences*, that are special sequences of coverings representing situations where new information is gradually provided on smaller and smaller sets of objects. Refinement sequences can be viewed as formal contexts, so in the future, we propose to explore the connections between sequences of orthopairs and the *fuzzy concept lattices* [106]. Moreover, we want to consider *fuzzy sequences of orthopairs*, by generalizing the notion of *fuzzy rough sets* [43]. In particular, we would like to define novel sequences of orthopairs starting from the *Atanassov intuitionistic fuzzy sets* [5]. Another way to introduce novel sequences of orthopairs is to consider pairs of disjoint upsets such that intersection between their components has cardinality equal to an integer $k \geq 0$. In this case, the identity $\mathsf{K}_O(\mathcal{C}) = \mathsf{K}(\mathcal{C})$ could also hold for a refinement sequence $\mathcal{C}$ that is not complete and safe.

Also, we would like to deepen the relationships between sequences of orthopairs and decision trees by considering the so-called *three-way decision trees* [73], [25].

In Chapter 4, we investigated several operations between sequences of orthopairs, that allowed us to provide concrete representations of the following classes of many-valued structures: *finite centered Kleene algebras with interpolation property*, *finite centered Nelson algebras with the interpolation property*, *finite centered Nelson lattices with the interpolation property*, *finite IUML-algebras* and *finite KLI^*-algebras with the interpolation property.* Consequently, we found a way to interpret the operations in these algebraic structures in terms of approximations of sets. As a future direction, we intend to discover other algebraic structures that can be interpreted as sequences of orthopairs. Also, given the refinement sequences $\mathcal{C}_1$ and $\mathcal{C}_2$ of the universes U_1 and U_2, respectively, it would be interesting to consider the product of the Kleene algebras $\mathbb{K}_O(\mathcal{C}_1)$ and $\mathbb{K}_O(\mathcal{C}_2)$, and to discover the universe and the class of refinement sequences corresponding it. Moreover, we can notice that rough sets can also be interpreted by a temporal semantics, as done for NM-algebras in [13]. Therefore, another topic of future works is to provide a pure logical temporal semantics in these structures and their related logics.

Furthermore, we will focus on the novel operations between orthopairs $\odot_4$ and $\hookrightarrow_4$, defined by equations 36 and 37, in order to connect them with a three-valued propositional logic having a non-deterministic semantics [37].

In the previous chapter, we presented the original modal logic SO_n, with semantics based on sequences of orthopairs. The Kripke models of SO_n are characterized by a sequence $(R_1, \ldots, R_n)$ of equivalence relations corresponding to a refinement sequence of partitions. In the future, we intend to consider a new modal logic, that extends SO_n, since the sequences of the accessibility relations of its Kripke models are related to refinement sequences of coverings.

Sequences of orthopairs corresponds to decision trees with three outcomes, so we could investigate their relationship. Also, we could employ operations between sequences of orthopairs to combine several decision trees.

Eventually, we interpreted SO_n logic as an epistemic logic; namely, we used SO_n to represent the knowledge of an agent that increases over time, as new information is provided. Then, we also wish to compare SO_n with some other existing epistemic logics, especially the logics where time and multiple epistemic operators are involved [46] [44], and to investigate the potential extensions of SO_n. As a future application, we also intend to study SO_n to predict the interest of users of a social network for a given piece of advertisement in a given time window. Indeed, in this case, each block of a partition can represent topics that received the same amount of interest by a user [18] [41]. By refining the information about the user, it is possible to obtain a refinement sequence of partitions. Hence, the logic permits to express complex sentences about the user's interests and to tailor advertisements in a very effective way.

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