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Extremal and typical results in Real Algebraic Geometry

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Ph.D. Thesis

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Abstract

In the first part of the dissertation we show that $2((d-1)^n-1)/(d-2)$ is the maximum possible number of critical points that a generic $(n-1)$-dimensional spherical harmonic of degree d can have. Our result in particular shows that there exist generic real symmetric tensors whose all eigenvectors are real. The results of this part are contained in Chapter 2.

In the second part of the thesis we are interested in expected outcomes in three different problems of probabilistic real algebraic and differential geometry.

First, in Chapter 3 we compute the volume of the projective variety $\Delta \subset \mathrm{PSym}(n,\mathbb{R})$ of real symmetric matrices with repeated eigenvalues. Our computation implies that the expected number of real symmetric matrices with repeated eigenvalues in a uniformly distributed projective 2-plane $L \subset \mathrm{PSym}(n,\mathbb{R})$ equals $\mathbb{E}\#(\Delta \cap L) = \binom{n}{2}$. The sharp upper bound on the number of matrices in the intersection $\Delta \cap L$ of Δ with a generic projective 2-plane L is $\binom{n+1}{3}$.

Second, in Chapter 4 we provide explicit formulas for the expected condition number for the polynomial eigenvalue problem defined by matrices drawn from various Gaussian matrix ensembles.

Finally, in Chapter 5 we are interested in the expected number of lines that are simultaneously tangent to the boundaries of several convex sets randomly positioned in the sphere. We express this number in terms of the integral mean curvatures of the boundaries of the convex sets.

To my wonderful wife Yana and our lovely daughters ...

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Introduction

Real algebraic geometry studies real solutions of polynomial systems with real coefficients. Most results that hold in complex algebraic geometry do not have direct analogs in the real setting mainly due to the fact that the field $\mathbb{R}$ of real numbers is not algebraically closed. For example, the classical fundamental theorem of algebra implies that a one-variable polynomial with sufficiently generic real coefficients has degree many complex roots. Some of these roots may be real and those which are not real always come in pairs of complex conjugate roots. Even though the number of real roots of a generic polynomial is not constant there exist *maximal* real polynomials all of whose roots are real. This is an example of an *extremal* result in real algebraic geometry.

In the space $\mathbb{R}^{d+1}$ of all real one-variable polynomials of degree d there is an algebraic hypersurface, called *the discriminant*, that consists of polynomials with repeated roots. Connected components of the complement to the discriminant hypersurface are open semialgebraic subsets of $\mathbb{R}^{d+1}$ and, in particular, any two polynomials belonging to the same connected component have the same number of real roots. Fixing a “reasonable” probability distribution in the space $\mathbb{R}^{d+1}$ one can see that the measure of each connected component of the complement to the discriminant hypersurface is positive. In other words, with a positive probability a random real polynomial of degree d has $d-2[d/2],\dots,d-2$ or d real roots. It is then natural to ask for the expected (typical, average) number of real roots. For example, Kac proved [48] that a real polynomial of degree d with independent standard Gaussian coefficients has

$$\frac{4}{\pi}\int_0^1 \frac{\sqrt{(1-x^{2(d+1)})^2-(d+1)^2x^4(1-x^2)^2}}{(1-x^2)(1-x^{2(d+1)})}\,dx \sim \frac{2}{\pi}\log(d+1), \quad d\to+\infty$$

real roots on average. This is an example of a *typical* result in (probabilistic) real algebraic geometry.

Below we give an overview of the main results of this dissertation.

Spherical harmonics with the maximum number of critical points

In Chapter 2 we construct generic spherical harmonics with the maximum possible number of critical points. Our result, in particular, gives a positive answer to the question addressed by Abo, Seigal and Sturmfels in [1, Sec. 6] (see Conjecture 6.5). This part of the dissertation is based on the work [51].

The study of geometric and topological properties of Laplace eigenfunctions on Riemannian manifolds has rich and interesting history, we refer the reader to the survey of the results [89]. Classical directions of investigation concern properties of the zero level hypersurfaces of Laplace eigenfunctions. Computation and estimation of the volume of these hypersurfaces [30, 31, 70, 59, 58] and study of their basic topological invariants [67, 56, 34, 64] remain active areas of research. Studies of critical points of Laplace eigenfunctions have appeared in [54, 47, 65].

In our work we are interested in the maximal number of critical points of eigenfunctions of the spherical Laplace operator. These functions are called *spherical harmonics* and they can be equivalently defined as the restriction to the sphere of harmonic homogeneous polynomials. In [11, Problem 1] Arnold asked to determine the largest number of local maxima that a Morse spherical harmonic $h \in \mathcal{H}_{d,3}$ can have on the sphere S^2. For even d the answer to this question is not known in general (to our knowledge). For odd d the answer $(d^2 - d + 2)/2$ was given by Kuznetsov and Kholshevnikov in [54], where they also proved that the maximum number $m_{d,3}$ of critical points of the restriction $f|_{S^2}$ to the sphere of a Morse (see Subsection 2.1.1 for the definition) real homogeneous polynomial $f \in \mathcal{P}_{d,3}$ of degree d equals:

$$m_{d,3} = 2(d^2 - d + 1)$$

and surprisingly enough this bound is attained by spherical harmonics. In the following theorem we generalize the result of Kuznetsov and Kholshevnikov to the case of any number of variables.

Theorem. *For any $d \geq 1$ and $n \geq 2$ the maximum number $m_{d,n}$ of critical points of the restriction $f|_{S^{n-1}}$ to the sphere of a Morse real homogeneous polynomial $f \in \mathcal{P}_{d,n}$ of degree d equals*

$$m_{d,n} = 2\frac{(d-1)^n - 1}{d-2} = 2((d-1)^{n-1} + \cdots + (d-1) + 1)$$

Moreover, for any $d \geq 1$ and $n \geq 2$ there exists a Morse spherical harmonic $h \in \mathcal{H}_{d,n}$ with $m_{d,n}$ critical points.

Critical points of restrictions to the sphere of real homogeneous polynomials reappeared in the context of *spectral theory* of high order tensors independently initiated by Lim [57] and Qi [69] in 2005. Several generalizations of the classical concept of an eigenvector of a matrix were introduced in [57, 69]. Critical points of the restrictions to the sphere of real homogeneous polynomials correspond to l^2-eigenvectors of Lim or Z-eigenvectors of Qi as we explain in Section 2.1.2.

Let $A = (a_{i_1\ldots i_d})_{i_j=1}^n$, $a_{i_1\ldots i_d} \in \mathbb{R}$ be a real n-dimensional tensor of order d (in the sequel, n^d-tensor). A non-zero vector $x \in \mathbb{C}^n \setminus \{0\}$ is called an *eigenvector* of A if there exists $\lambda \in \mathbb{C}$, the corresponding *eigenvalue*, such that

$$Ax^{d-1} = \lambda x, \quad Ax^{d-1} := \left(\sum_{i_2,\ldots,i_d=1}^{n} a_{1i_2\ldots i_d} x_{i_2} \cdots x_{i_d}, \ \ldots \ , \sum_{i_2,\ldots,i_d=1}^{n} a_{ni_2\ldots i_d} x_{i_2} \cdots x_{i_d} \right).$$

For $d = 2$ one recovers the classical definition of an eigenvector of an $n \times n$ matrix $A = (a_{i_1 i_2})_{i_j=1}^n$. The point $[x] \in \mathbb{C}\mathrm{P}^{n-1}$ defined by an eigenvector $x \in \mathbb{C}^n \setminus \{0\}$ is called an *eigenpoint* and the set of all eigenpoints is called an *eigenconfiguration.*

An n^d-tensor $A = (a_{i_1\ldots i_d})_{i_j=1}^n$, $a_{i_1,\ldots,i_d} \in \mathbb{R}$ is said to be *symmetric* if $a_{i_{\sigma_1}\ldots i_{\sigma_d}} = a_{i_1\ldots i_d}$ for any permutation $\sigma \in S_d$. Cartwright and Sturmfels [25] proved that the number of eigenpoints of a *generic* (for the definition see Subsection 2.1.2) symmetric n^d-tensor equals

$$\tilde{m}_{d,n} := \frac{(d-1)^n - 1}{d-2} = (d-1)^{n-1} + \cdots + (d-1) + 1$$

but, except for the case of real symmetric matrices ($d = 2$), not all eigenvectors of a general real symmetric tensor of order $d \geq 3$ are real. In fact, most of real symmetric tensors have eigenpoints in $\mathbb{C}\mathrm{P}^{n-1} \setminus \mathbb{R}\mathrm{P}^{n-1}$. Abo, Seigal and Sturmfels conjectured [1, Conjecture 6.5] that for any $d \geq 1$ and $n \geq 2$ there exists a generic real symmetric n^d-tensor having only real eigenvectors and proved it for $d \geq 1, n = 3$ and for $d = n = 4$. The cases $d \geq 1, n = 2$ and $d = 2, n \geq 2$ are elementary, the case of general d, n was unknown (see for example [82]). In the following theorem we cover the case of arbitrary d and n.

Theorem. *For any $d \geq 1$ and $n \geq 2$ there exists a generic real symmetric n^d-tensor all of whose $\tilde{m}_{d,n}$ eigenpoints are real.*

The above two theorems are proved in Section 2. In the end of that section we also discuss relation of our results to few other problems among which the estimation of the number of real zeros of a semidefinite polynomial (Theorem 9).

On the geometry of the set of symmetric matrices with repeated eigenvalues

In Chapter 3 we investigate some geometric properties of the set Δ (below called *discriminant*) of real symmetric matrices with repeated eigenvalues and of unit

Frobenius norm

$$\Delta = \{Q \in \mathrm{Sym}(n, \mathbb{R}) \text{ such that } \lambda_i(Q) = \lambda_j(Q) \text{ for some } i \neq j\} \cap S^{N-1},$$

where $N = \frac{n(n+1)}{2} = \dim(\mathrm{Sym}(n, \mathbb{R}))$ and S^{N-1} denotes the unit sphere with respect to the Frobenius norm $\|Q\|^2 = \mathrm{tr}(Q^2)$. The results presented in Chapter 3 are based on the work [21], in collaboration with Paul Breiding and Antonio Lerario.

The discriminant appears in several areas of mathematics, from mathematical physics to real algebraic geometry [7, 9, 8, 10, 83, 3, 4, 87].

The set Δ is an algebraic subset of S^{N-1} of codimension two. It is defined by the *discriminant polynomial*:

$$\mathrm{dis}(Q) := \prod_{i \neq j} (\lambda_i(Q) - \lambda_j(Q))^2$$

which is a non-negative homogeneous polynomial of degree $\deg(\mathrm{dis}) = n(n-1)$ in the entries of Q and, moreover, it is a sum of squares of real polynomials [46, 66]. The set Δ_{sm} of smooth points of Δ consists of matrices with *exactly* two repeated eigenvalues (in fact, Δ is stratified according to the multiplicity sequence of the eigenvalues [7]). Our first main result about the discriminant $\Delta \subset S^{N-1}$ is the explicit formula for its volume.

Theorem. *For any $n \geq 2$ we have*

$$\frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}. \tag{1}$$

Remark. *Results of this type (the computation of the volume of some relevant algebraic subsets of the space of matrices) have started appearing in the literature since the 90's [32, 33], with a particular emphasis on asymptotic studies and complexity theory, and have been crucial for the theoretical advance of numerical algebraic geometry, especially for what concerns the estimation of the so called* condition number *of numerical problems [29]. The very first result gives the volume of the set $\Sigma \subset \mathbb{R}^{n^2}$ of square matrices with zero determinant and Frobenius norm one; this was computed in [32, 33]:*

$$\frac{|\Sigma|}{|S^{n^2-1}|} = \sqrt{\pi} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \sim \sqrt{\frac{\pi}{2}} n^{1/2}. \tag{2}$$

For example (2) *is used in [32, Theorem 6.1] to compute the average number of zeroes of the determinant of a matrix of linear forms. Subsequently, this computation was extended to include the volume of the set of $n \times m$ matrices of given corank in [14] and the volume of the set of* symmetric *matrices with determinant zero in [55], with similar expressions.*

The proof of (3.1) requires the evaluation of the expectation of the square of the characteristic polynomial of a $\mathrm{GOE}(n)$ matrix (Theorem 13 from Section 3), which constitutes a result of independent interest.

The second main result of this part of the dissertation concerns *maximal cuts* of the discriminant. To state it let's denote by $\mathrm{P}\Delta \subset \mathrm{PSym}(n, \mathbb{R}) \simeq \mathbb{R}\mathrm{P}^{N-1}$ the projectivization of the discriminant. Since $\mathrm{P}\Delta$ has codimension two, the number $\#(L \cap \mathrm{P}\Delta)$ of symmetric matrices with repeated eigenvalues in a generic projective two-plane $L \simeq \mathbb{R}\mathrm{P}^2 \subset \mathbb{R}\mathrm{P}^{N-1}$ is finite. In the following theorem we provide a sharp upper bound on this number.

Theorem. *For a generic projective two-plane $L \simeq \mathbb{R}\mathrm{P}^2$ the following* sharp *upper bound holds:*

$$\#(L \cap \mathrm{P}\Delta) \leq \binom{n+1}{3}. \tag{3}$$

The formula (1) for the volume of Δ combined with Poincaré formula (Corollary 2 from Subsection 1.3.2) allows to compute the average number of symmetric matrices with repeated eigenvalues in a uniformly distributed projective two-plane $L \subset \mathbb{R}\mathrm{P}^{N-1}$:

$$\mathop{\mathbb{E}}_{L \in \mathbb{G}(2, N-1)} \#(L \cap \mathrm{P}\Delta) = \frac{|\mathrm{P}\Delta|}{|\mathbb{R}\mathrm{P}^{N-3}|} = \frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}. \tag{4}$$

Remark. *Consequence* (4) *is especially interesting because it "violates" a frequent phenomenon in random algebraic geometry, which goes under the name of* square root law*: for a large class of models of random systems, often related to the so called Edelman-Kostlan-Shub-Smale models [32, 74, 33, 50, 75, 73], the average number of solutions equals (or is comparable to) the square root of the maximum number; here this is not the case. We also observe that, surprisingly enough, the* average cut *of the discriminant is an integer number (there is no reason to even expect that it should be a rational number!).*

The computation (1) of the volume of Δ is obtained by a limiting procedure. Using the fact that the restriction of the $\mathrm{GOE}(n)$ measure to the unit sphere in $\mathrm{Sym}(n, \mathbb{R})$ gives the uniform measure, we will describe the volume of the ϵ-tube around Δ using the joint density of the eigenvalues of a $\mathrm{GOE}(n)$ matrix and then make a careful application of Weyl's tube formula to derive the asymptotic of this volume at zero (whose leading coefficient, up to a constant, equals $|\Delta|$). The main difficulties here are the explicit description of the tube, and the fact that the variety Δ is singular, which makes the application of Weyl's tube formula delicate. In this way we will prove the following result, which also includes information on the volume of the set Δ_1 of symmetric matrices whose smallest two eigenvalues are equal.

Theorem. *Let $\Delta_1 \subset \Delta \subset S^{N-1}$ denote the set of symmetric matrices with the smallest two eigenvalues repeated. Then we have the two following integral expressions:*

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\, n!} \binom{n}{2} \int_{\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(n-2)} \Big[\det(Q-u\mathbb{1})^2\Big] e^{-u^2}\, \mathrm{d}u, \tag{5}$$

$$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\, n!} \binom{n}{2} \int_{\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(n-2)} \Big[\det(Q-u\mathbb{1})^2 \mathbf{1}_{\{Q-u\mathbb{1}\succ 0\}}\Big] e^{-u^2}\, \mathrm{d}u. \tag{6}$$

(note the appearance of the characteristic function $\mathbf{1}_{\{Q-u\mathbb{1}\succ 0\}}$ in the second integral).

The exact evaluation of the integral in (5) (given in the theorem below) will take a considerable amount of work and is of independent interest. It is based on some key properties of Hermite polynomials. By contrast, we do not know whether there exists a closed form evaluation of (6).

Theorem. *For a fixed positive integer k we have*

$$\int_{\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(k)}[\det(Q-u\mathbb{1})^2]\, e^{-u^2}\, \mathrm{d}u = \sqrt{\pi}\, \frac{(k+2)!}{2^{k+1}}.$$

To proof the bound (3) we exploit an interesting duality between symmetric matrices with repeated eigenvalues in a 3-dimensional linear family and singularities of some algebraic surface. To be more specific, given three independent matrices $R_1, R_2, R_3 \in \mathrm{Sym}(n,\mathbb{R})$ denote by $L = \mathrm{P}(\mathrm{span}\{R_1, R_2, R_3\}) \subset \mathrm{PSym}(n,\mathbb{R})$ the projective two-plane that they generate and consider the projective *symmetroid surface*

$$\mathrm{P}\Sigma_{3,n} = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{R}\mathrm{P}^3 \mid \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\}.$$

The following result, which is a particular case of Proposition 9, describes the mentioned duality.

Proposition. *For generic matrices $R_1, R_2, R_3 \in \mathrm{Sym}(n,\mathbb{R})$ there is a one-to-one correspondence between singular points of the symmetroid surface $\mathrm{P}\Sigma_{3,n} \subset \mathbb{R}\mathrm{P}^2$ and symmetric matrices with repeated eigenvalues in the projective two-plane $L \simeq \mathbb{R}\mathrm{P}^2$.*

The above results about the discriminant are proved in Chapter 3.

The condition number for polynomial eigenvalues of random matrices

In Chapter 4 we explicitly compute the expected condition number for polynomial eigenvalues of random matrices drawn from various random matrix ensembles. Results of this part are presented in the joint work [15] of the author with Carlos Beltran.

First, following the ideas in [73, 17], we note that many numerical problems can be described within the following simple general framework. We consider a space of *inputs* and a space of *outputs* denoted by $\mathcal{I}$ and $\mathcal{O}$ respectively, and some equation of the form $ev(i, o) = 0$ stating when an output is a solution for a given input. Both $\mathcal{I}$ and $\mathcal{O}$, and the *solution variety*

$$\mathcal{V} = \{(i, o) \in \mathcal{I} \times \mathcal{O} : o \text{ is an output to } i\} = \{(i, o) \in \mathcal{I} \times \mathcal{O} : ev(i, o) = 0\}$$

are frequently real algebraic or just semialgebraic sets. The numerical problem to be solved can then be written as "given $i \in \mathcal{I}$, find $o \in \mathcal{O}$ such that $(i, o) \in \mathcal{V}$", or "find all $o \in \mathcal{O}$ such that $(i, o) \in \mathcal{V}$". One can have in mind the following examples:

1. Polynomial Root Finding: $\mathcal{I}$ is the set of univariate real polynomials of degree d, $\mathcal{O} = \mathbb{R}$ and $\mathcal{V} = \{(f, \zeta) : f(\zeta) = 0\}$.

2. Polynomial System Solving, which we can see as the homogeneous multivariate version of Polynomial Root Finding: $\mathcal{I}$ is the projective space of (dense or structured) systems of n real homogeneous polynomials of degrees $d_1, \ldots, d_n$ in variables $x_0, \ldots, x_n$, $\mathcal{O} = \mathbb{R}\mathrm{P}^n$ and $\mathcal{V} = \{(f, \zeta) : f(\zeta) = 0\}$.

3. EigenValue Problem: $\mathcal{I} = \mathbb{R}^{n\times n}$, $\mathcal{O} = \mathbb{R}$ and $\mathcal{V} = \{(A, \lambda) : \det(A - \lambda\,\mathrm{Id}) = 0\}$.

4. (Homogeneous) Polynomial EigenValue Problem (in the sequel called PEVP): $\mathcal{I}$ is the set of tuples of $d+1$ real $n\times n$ matrices $A = (A_0, \ldots, A_d)$, $\mathcal{O} = \mathbb{R}\mathrm{P}^1$ and $\mathcal{V} = \{(A, [\alpha : \beta]) : P(A, \alpha, \beta) = \det(\alpha^0\beta^d A_0 + \alpha^1\beta^{d-1}A_1 + \cdots + \alpha^d\beta^0 A_d) = 0\}$. One can force some of the matrices to be symmetric, a particularly important case in applications, or consider other structured problems, see [62, 28, 41, 85]. In cases $d = 1$ and $d = 2$ polynomial eigenvalues are often referred to as *generalized eigenvalues* and *quadratic eigenvalues* respectively.

The condition number of a numerical problem like the one above measures the sensibility of the solution o under an infinitesimal perturbation of the input i.

Definition. *Let $\mathcal{I}, \mathcal{O}$ and $\mathcal{V}$ be real algebraic sets such that the smooth loci of $\mathcal{I}, \mathcal{O}$ are endowed with Riemannian structures and let $(i, o) \in \mathcal{V}$ be a smooth point of $\mathcal{V}$ such that $i \in \mathcal{I}, o \in \mathcal{O}$ are smooth points of $\mathcal{I}$ and $\mathcal{O}$ respectively. Moreover, assume that $D_{(i,o)}p_1 : T_{(i,o)}\mathcal{V} \to T_i\mathcal{I}$ is invertible. Then the* condition number $\mu(i, o)$ *of $(i, o) \in \mathcal{V}$ is defined as*

$$\mu(i, o) = \left\| D_{(i,o)}p_2 \circ D_{(i,o)}p_1^{-1} \right\|_{\mathrm{op}},$$

where $p_1 : \mathcal{V} \to \mathcal{I}$, $p_2 : \mathcal{V} \to \mathcal{O}$ are the projections and $\|\cdot\|_{\mathrm{op}}$ is the operator norm. For points $(i, o) \in \mathcal{V}$ not satisfying the above assumptions the condition number is set to ∞.

Remark. *This definition is intrinsic in $\mathcal{I}$, i.e., changing $\mathcal{I}$ to some subvariety $\mathcal{I}' \subset \mathcal{I}$ leads (in general) to different, smaller, value of the condition number, since perturbations of the input are only allowed in the direction of the tangent space to the input set. Note also that the condition number depends on choices of Riemannian structures on $\mathcal{I}$ and $\mathcal{O}$.*

In the PEVP the input space $\mathcal{I}$ is endowed with the following Riemannian structure: $\langle \dot{A}, \dot{B} \rangle_A = ((\dot{A}_0, \dot{B}_0) + \cdots + (\dot{A}_d, \dot{B}_d))/((A_0, A_0) + \cdots + (A_d, A_d))$, where $(\cdot, \cdot)$ is the Frobenius inner product, $A = (A_0, \ldots, A_d)$ and $\dot{A} = (\dot{A}_0, \ldots, \dot{A}_d), \dot{B} = (\dot{B}_0, \ldots, \dot{B}_d) \in T_A\mathcal{I}$. The output space $\mathcal{O} = \mathbb{R}\mathrm{P}^1$ possesses the standard metric and the solution variety $\mathcal{V} = \{(A, [\alpha : \beta]) : P(A, \alpha, \beta) = 0\}$ is endowed with the induced product Riemannian structure. An explicit formula for the condition number for the Homogeneous PEVP was derived in [28, Th. 4.2] (we write here the relative condition number version):

$$\mu(A, (\alpha, \beta)) = \left(\sum_{k=0}^{d} \alpha^{2k} \beta^{2d-2k} \right)^{1/2} \frac{\|r\| \|\ell\|}{|\ell^t v|} \|A\|,$$

where $A = (A_0, \ldots, A_d)$, $(\alpha, \beta) \in \mathbb{R}^2$ is a polynomial eigenvalue of A, r and ℓ are the corresponding right and left eigenvectors and

$$v = \beta \frac{\partial}{\partial \alpha} P(A, \alpha, \beta) r - \alpha \frac{\partial}{\partial \beta} P(A, \alpha, \beta) r.$$

A given tuple A can have up to nd real isolated polynomial eigenvalues. We define the condition number of A simply as the sum of the condition numbers over all these PEVs:

$$\mu(A) = \sum_{[\alpha:\beta] \in \mathbb{R}\mathrm{P}^1 \text{is a PEV of } A} \mu(A, (\alpha, \beta)).$$

(If $A = (A_0, \ldots, A_d)$ has infinitely many polynomial eigenvalues, then we set $\mu(A) = \infty$). In Chapter 4 we prove a general result (Theorem 16) which is designed to provide exact formulas for the expected value of the condition number in the PEVP and other problems. A simple particular case of our general theorem is as follows.

Theorem. *If $A_0, \ldots, A_d \in M(n, \mathbb{R})$ are independent $n \times n$ real matrices whose entries are independent identically distributed standard Gaussian variables, then*

$$\begin{aligned} \underset{A_0,\ldots,A_d \sim \text{i.i.d. Gaussian matrices}}{\mathbb{E}} \mu(A) &= \pi \frac{\Gamma\left(\frac{(d+1)n^2}{2}\right)}{\Gamma\left(\frac{(d+1)n^2-1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{\pi}{2} \sqrt{(d+1)n^3} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \ n \to +\infty \end{aligned} \tag{7}$$

Remark. *Recently in [6] Armentano and Beltran investigated the expectation of the squared condition number for polynomial eigenvalues of complex Gaussian matrices. Our result* (7) *establishes the "asymptotic square root law" for the considered problem, i.e., when $n \to +\infty$ (and up to the factor $\pi/2$) our answer in* (7) *equals the square root of the answer in [6].*

Another particular instance of our general Theorem 16 is the computation of the expected condition number for matrices drawn from the Gaussian Orthogonal Ensemble.

Theorem. *If $A_0, \ldots, A_d \in \mathrm{Sym}(n, \mathbb{R})$ are independent* $\mathrm{GOE}(n)$*-matrices and n is even, then*

$$\begin{aligned} \underset{A_0,\ldots,A_d \sim \text{ i.i.d. GOE(n)-matrices}}{\mathbb{E}} \mu(A) &= \sqrt{2}n \frac{\Gamma\left(\frac{(d+1)n(n+1)}{4}\right)}{\Gamma\left(\frac{(d+1)n(n+1)-2}{4}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \\ &= \sqrt{(d+1)n^3}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right), \ n \to +\infty \end{aligned}$$

If n is odd the explicit formula is more complicated and is given in (4.13)*. However, the above asymptotic formula is valid for both even and odd n.*

More generally, Theorem 17 from Chapter 4 gives the exact expression of the expected condition number for Gaussian matrices constrained to stay in a given linear subspace $V \subset M(n, \mathbb{R})$ of the space of all real square matrices. The key point of the proof of this general theorem is the expression of the expected condition number in terms of the volume of the set of singular matrices in V.

On the number of flats tangent to convex hypersurfaces in random position

The last but not the least chapter of the dissertation is about enumerative geometry of tangents to hypersurfaces randomly positioned in real projective space. Results of Chapter 5 are presented in the joint work [52] of the author with Antonio Lerario.

Given $d_{k,n} = (k+1)(n-k)$ projective hypersurfaces $X_1, \ldots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ a classical problem in enumerative geometry is to determine how many k-dimensional projective subspaces of $\mathbb{R}\mathrm{P}^n$ (called *k-flats*) are simultaneously tangent to $X_1, \ldots, X_{d_{k,n}}$.

Geometrically we can formulate this problem as follows. Let $\mathbb{G}(k, n)$ denote the Grassmannian of k-dimensional projective subspaces of $\mathbb{R}\mathrm{P}^n$ (note that $d_{k,n} = \dim \mathbb{G}(k, n)$). If $X \subset \mathbb{R}\mathrm{P}^n$ is a smooth hypersurface, we denote by $\Omega_k(X) \subset \mathbb{G}(k, n)$ the variety of k-tangents to X, i.e. the set of k-flats that are tangent to X

at some point. The number of k-flats simultaneously tangent to hypersurfaces $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ equals

$$\#\, \Omega_k(X_1) \cap \dots \cap \Omega_k(X_{d_{k,n}}).$$

Of course this number depends on the mutual position of the hypersurfaces $X_1, \dots, X_{d_{k,n}}$ in the projective space $\mathbb{R}\mathrm{P}^n$.

In [77] F.Sottile and T.Theobald proved that there are at most $3 \cdot 2^{n-1}$ real lines tangent to $2n-2$ general spheres in $\mathbb{R}^n$ and they found a configuration of spheres with $3 \cdot 2^{n-1}$ common tangent lines. They also studied [78] the problem of k-flats tangent to $d_{k,n}$ many general quadrics in $\mathbb{R}\mathrm{P}^n$ and proved that the "complex bound" $2^{d_{k,n}} \cdot \deg(\mathbb{G}_{\mathbb{C}}(k,n))$ can be attained by real quadrics. See also [19, 60, 61, 79] for other interesting results on real enumerative geometry of tangents.

An exciting point of view comes by adopting a random approach: one asks for the *expected value* for the number of tangents to hypersurfaces in *random position*. We say that the hypersurfaces $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ are in *random position* if each one of them is randomly translated by elements $g_1, \dots, g_{d_{k,n}}$ sampled independently from the orthogonal group $O(n+1)$ endowed with the uniform distribution. The average number $\tau_k(X_1, \dots, X_{d_{k,n}})$ of k-flats tangent to $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ in random position is then given by

$$\tau_k(X_1, \dots, X_{d_{k,n}}) := \mathbb{E}_{\,g_1, \dots, g_{d_{k,n}} \in O(n+1)} \#\, \Omega_k(g_1 X_1) \cap \dots \cap \Omega_k(g_{d_{k,n}} X_{d_{k,n}}).$$

The computation and study of properties of this number is precisely the goal of Chapter 5.

A special feature of our study is that we concentrate on the case when the hypersurfaces (not necessarily algebraic) are boundaries of convex sets. Part of the results we present, however, hold in higher generality as we discuss in Section 5.5.

Definition (Convex hypersurface)**.** *A subset C of $\mathbb{R}\mathrm{P}^n$ is called (strictly) convex if C does not intersect some hyperplane L and it is (strictly) convex in the affine chart $\mathbb{R}\mathrm{P}^n \setminus L \simeq \mathbb{R}^n$. A smooth hypersurface $X \subset \mathbb{R}\mathrm{P}^n$ is said to be* convex *if it bounds a strictly convex open set of $\mathbb{R}\mathrm{P}^n$.*

Recently, Bürgisser and Lerario [24] have studied the similar problem of determining the average number of k-flats that simultaneously intersect $d_{k,n}$ many $(n-k-1)$-flats in random position in $\mathbb{R}\mathrm{P}^n$. They have called this number the *expected degree* of the real Grassmannian $\mathbb{G}(k,n)$, here denoted by $\delta_{k,n}$, and have claimed that this is the key quantity governing questions in random enumerative geometry of flats. (The name comes from the fact that the number of solutions of the analogous problem over the complex numbers coincides with the degree of $\mathbb{G}_{\mathbb{C}}(k,n)$ in the Plücker embedding. Note however that the notion of expected degree is intrinsic and does not require any embedding.)

For reasons that will become more clear later, it is convenient to introduce the *special Schubert variety* $\mathrm{Sch}(k,n) \subset \mathbb{G}(k,n)$ consisting of k-flats in $\mathbb{R}\mathrm{P}^n$ intersecting a fixed $(n-k-1)$-flat. The volume of the special Schubert variety is computed in [24, Theorem 4.2] and equals

$$|\mathrm{Sch}(k,n)| = |\mathbb{G}(k,n)| \cdot \frac{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)},$$

where $|\mathbb{G}(k,n)|$ denotes the volume of the Grassmannian (see Subection 1.3.1). The following theorem relates our main problem to the expected degree (this is Theorem 20 from Section 5.3).

Theorem. *The average number of k-flats in $\mathbb{R}\mathrm{P}^n$ simultaneously tangent to convex hypersurfaces $X_1,\ldots,X_{d_{k,n}}$ in random position equals*

$$\tau_k(X_1,\ldots,X_{d_{k,n}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|\mathrm{Sch}(k,n)|}, \tag{8}$$

where $|\Omega_k(X)|$ denotes the volume of the manifold of k-tangents to X.

The number $\delta_{k,n}$ equals (up to a multiple) the volume of a convex body for which the authors of [24] coined the name *Segre zonoid.* Except for $\delta_{0,n} = \delta_{n-1,n} = 1$, the exact value of this quantity is not known, but it is possible to compute its asymptotic as $n \to \infty$ for fixed k. For example, in the case of the Grassmannian of lines in $\mathbb{R}\mathrm{P}^n$ one has [24, Theorem 6.8]

$$\delta_{1,n} = \frac{8}{3\pi^{5/2}} \cdot \frac{1}{\sqrt{n}} \cdot \left(\frac{\pi^2}{4}\right)^n \cdot \left(1 + \mathcal{O}(n^{-1})\right).$$

The number $\delta_{1,3}$ (the average number of lines meeting four random lines in $\mathbb{R}\mathrm{P}^3$) can be written as an integral [24, Proposition 6.7], whose numerical approximation is $\delta_{1,3} = 1.7262....$ It is an open problem whether this quantity has a closed formula (possibly in terms of special functions).

The above theorem reduces our study to the investigation of the geometry of the manifold of tangents, for which we prove the following result (Proposition 11 and Corollary 6 from Chapter 5).

Proposition (The volume of the manifold of k-tangents)**.** *For a convex hypersurface $X \subset \mathbb{R}\mathrm{P}^n$ we have*

$$\frac{|\Omega_k(X)|}{|\mathrm{Sch}(k,n)|} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_X \sigma_k(x) dV_X. \tag{9}$$

where $\sigma_k : X \to \mathbb{R}$ is the k-th elementary symmetric polynomial of the principal curvatures of the embedding $X \hookrightarrow \mathbb{R}\mathrm{P}^n$.

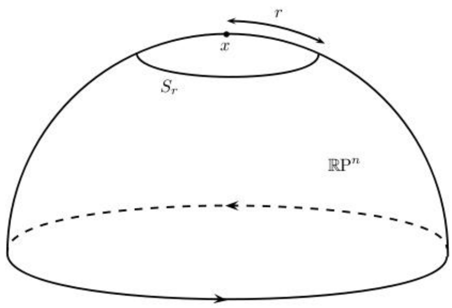

Figure 1: The equation $x_1^2+\cdots+x_n^2=(\tan r)^2 x_0^2$ defines in $\mathbb{R}\mathrm{P}^n$ a metric sphere of radius r, i.e. the set of all points at distance r from a fixed point.

Remark. *After this result was obtained P. Bürgisser has pointed out to us that it can be also derived using a limiting argument from [5], where the tube neighborhood around $\Omega_k(X)$ is described.*

In the case of spheres in projective space we are able to compute explicitly the volume of the manifold of tangents and hence also the expected number of flats that are simultaneously tangent to all the spheres.

Example (Spheres in projective space). *Let $S_{r_i}=\{x_1^2+\cdots+x_n^2=(\tan r_i)^2\, x_0^2\}\subset \mathbb{R}\mathrm{P}^n$ be a metric sphere in $\mathbb{R}\mathrm{P}^n$ of radius $r_i\in(0,\pi/2)$, $i=1,\dots,d_{k,n}$ (see Figure 1). Since all principal curvatures of S_{r_i} are constants equal to $\cot r_i$ and since $|S_{r_i}|=\frac{2\sqrt{\pi}^n}{\Gamma(\frac{n}{2})}(\sin r_i)^{n-1}$ by* (9) *we have*

$$\frac{|\Omega_k(S_r)|}{|Sch(k,n)|}=\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}\cdot(\cos r_i)^k(\sin r_i)^{n-k-1},$$

and combining it with (8) *we obtain*

$$\tau_k(S_{r_1},\dots,S_{r_{d_{k,n}}})=\delta_{k,n}\cdot\prod_{i=1}^{d_{k,n}}\left(\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}\cdot(\cos r_i)^k(\sin r_i)^{n-k-1}\right)$$

Observe that a hypersurface $S_{y,r}$ which is a sphere in some affine chart $U\simeq\mathbb{R}^n$, i.e. $S_{y,r}=\{x\in\mathbb{R}^n:\sum_{i=1}^n(x_i-y_i)^2=r^2\}$, is a convex hypersurface in $\mathbb{R}\mathrm{P}^n$, but it is not a sphere with respect to the projective metric unless it's centered at the origin ($y=0$); and, viceversa, a metric sphere in $\mathbb{R}\mathrm{P}^n$ needs not be a sphere in an affine chart. In fact, (5.3) *tells that Sottile and Theobald's upper bound $3\cdot 2^{n-1}$ for the number of lines tangent to $d_{1,n}$ affine spheres in $\mathbb{R}^n$* does not *apply to the case of spheres in $\mathbb{R}\mathrm{P}^n$: since $\frac{2\pi}{e}>2$, when n is large* (5.3) *is larger than $3\cdot 2^{n-1}$; as a* *consequence there must be a configuration of $d_{1,n}$ projective spheres in $\mathbb{R}\mathrm{P}^n$ with (exponentially) more common tangent lines.*

Remark (The semialgebraic case)**.** *The theorem above remains true in the case of semialgebraic hypersurfaces $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ satisfying some mild non-degeneracy conditions (see Section 5.5 for more details). Specifically it still holds true that*

$$\tau_k(X_1, \dots, X_{d_{k,n}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|Sch(k,n)|},$$

but the volume of the manifold of k-tangents has a more complicated description:

$$\frac{|\Omega_k(X)|}{|Sch(k,n)|} = \frac{\binom{n-1}{k}\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_X \mathbb{E}_{\,\Lambda \in Gr_k(T_xX)} |B_x(\Lambda)| dV_X,$$

where $|B_x(\Lambda)|$ denotes the absolute value of the determinant of the matrix of the second fundamental form of $X \hookrightarrow \mathbb{R}\mathrm{P}^n$ restricted to $\Lambda \in Gr_k(T_xX)$ and written in an orthonormal basis of Λ (see Subsection 5.2.1), and the expectation is taken with respect to the uniform distribution on $Gr_k(T_xX) \simeq Gr(k, n-1)$.

The quantities $|\Omega_k(X)|$ offer an alternative interesting interpretation of the classical notion of *intrinsic volumes*. If C is a convex set in $\mathbb{R}\mathrm{P}^n$, the spherical Steiner's formula [38, (9)] allows to write the volume of the ϵ-neighborhood $\mathcal{U}_{\mathbb{R}\mathrm{P}^n}(C,\epsilon)$ of C in $\mathbb{R}\mathrm{P}^n$ as

$$|\mathcal{U}_{\mathbb{R}\mathrm{P}^n}(C,\epsilon)| = |C| + \sum_{k=0}^{n-1} f_k(\epsilon)|S^k||S^{n-k-1}|V_k(C),$$

where

$$f_k(\epsilon) = \int_0^\epsilon (\cos t)^k (\sin t)^{n-1-k} dt.$$

The quantities $V_0(C), \dots, V_{n-1}(C)$ are called intrinsic volumes of C. What is remarkable is that when C is smooth and strictly convex, $|\Omega_k(\partial C)|$ coincides, up to a constant depending on k and n only, with the $(n-k-1)$-th intrinsic volume of C (again this property can be derived by a limiting argument from the results in [5]).

Proposition (The manifold of k-tangents and intrinsic volumes)**.** *Let $C \subset \mathbb{R}\mathrm{P}^n$ be a smooth strictly convex set. Then*

$$|V_{n-k-1}(C)| = \frac{1}{4} \cdot \frac{|\Omega_k(\partial C)|}{|\mathrm{Sch}(k,n)|}, \quad k = 0, \dots, n-1.$$

This interpretation offers possible new directions of investigation and allows to prove the following universal upper bound (see Corollary 9 from Chapter 5)

$$\tau_k(X_1, \ldots, X_{d_{k,n}}) \leq \delta_{k,n} \cdot 4^{d_{k,n}},$$

where the right-hand side depends only on k and n. However, already for $n = 3$, as observed by T. Theobald [84] there is no upper bound on the number of lines that can be simultaneously tangent to four convex hypersurfaces in $\mathbb{R}\mathrm{P}^3$ in general position (see Section 5.4 for a proof of this fact).

Part of the results presented in the dissertation are obtained in collaboration of the author with other people: Chapter 3 is based on the joint work [21] with Paul Breiding and Antonio Lerario, the results of Chapter 4 are presented in the joint work [15] with Carlos Beltrán and Chapter 5 is a result of the joint work [52] with Antonio Lerario. In all mentioned works the contribution of the authors was comparable or equal.

Chapter 1

Notations and preliminary results

In this chapter we set our notations, give main definitions, state some classical results about spherical harmonics and semialgebraic sets and define the probabilistic framework for Chapters 3, 4 and 5.

1.1 Spherical harmonics

Let us denote by $\mathcal{P}_{d,n}$ the set of real homogeneous polynomials of degree d in n variables $x_1, \dots, x_n$. It is a real vector space of dimension

$$\dim(\mathcal{P}_{d,n}) = \binom{n+d-1}{d}$$

Denote by

$$\mathcal{H}_{d,n} = \left\{ h = f|_{S^{n-1}} : f \in \mathcal{P}_{d,n},\ \Delta f = \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_n^2} = 0 \right\}$$

the space of restrictions to the sphere S^{n-1} of harmonic polynomials of degree d. Functions in $\mathcal{H}_{d,n}$ are called *spherical harmonics* and the dimension of $\mathcal{H}_{d,n}$ equals

$$\dim \mathcal{H}_{d,n} = \binom{n+d-1}{d} - \binom{n+d-3}{d-2} \quad \text{if} \quad d \geq 2, \quad \dim \mathcal{H}_{0,n} = 1,\ \dim \mathcal{H}_{1,n} = n$$

Definition 1. The spherical Laplace operator $\Delta_{S^{n-1}}$ *is defined as follows. If* $f \in C^2(S^{n-1})$ *is a* C^2*-differentiable function on* S^{n-1} *and* $\tilde{f}$ *its degree* 0 *homogeneous extension to* $\mathbb{R}^n \setminus \{0\}$ *, i.e.,* $\tilde{f}(x) = f(x/\|x\|), x \in \mathbb{R}^n \setminus \{0\}$*, then*

$$\Delta_{S^{n-1}} f := (\Delta \tilde{f})|_{S^{n-1}}$$

Using spherical coordinates it is straightforward to check that any spherical harmonic $h \in \mathcal{H}_{d,n}$ is an eigenfunction of the spherical Laplace operator $\Delta_{S^{n-1}}$ corresponding to the eigenvalue $-d(d+n-2)$. Remarkably, $\Delta_{S^{n-1}}$ has no other eigenfunctions [76, Thm. 22.1].

Let us now consider the standard action of the orthogonal group $SO(n)$ on the space $C(\mathbb{R}^n)$ of continuous functions on $\mathbb{R}^n$:

$$g \in SO(n), f \in C(\mathbb{R}^n) \mapsto f \circ g^{-1}$$

Since this action commutes with the Laplace operator:

$$\Delta f \circ g^{-1} = \Delta(f \circ g^{-1}), \quad f \in C(\mathbb{R}^n),\ g \in SO(n)$$

the space of harmonic homogeneous polynomials of degree d is invariant under it and hence the spaces $\mathcal{H}_{d,n}, d \geq 0$ of spherical harmonics are finite-dimensional representations of $SO(n)$. It turns out that these representations are actually irreducible [35, Thm. 9.3.4].

There is a special family of orthogonal polynomials that is intimately related to spherical harmonics.

Definition 2 (Gegenbauer polynomials)**.** *Let $n \geq 2$. Gegenbauer polynomials of parameter $\frac{n-2}{2}$ are defined via the recurrence relation [2, 22.4.2, 22.7.3]:*

$$\begin{aligned}
&G_{0,n}(x) = 1,\\
&G_{1,n}(x) = (n-2)x,\\
&G_{d,n}(x) = \frac{1}{d}\left[2x(d+\frac{n}{2}-2)G_{d-1,n}(x) - (d+n-4)G_{d-2,n}(x)\right]
\end{aligned}$$

The polynomials $\{G_{d,n}\}_{d\geq 0}$ form an orthogonal family on the interval $[-1,1]$ with respect to the measure $(1-z^2)^{\frac{n-3}{2}}\,dz$ [2, 22.2.3]:

$$\int_{-1}^{1} G_{d_1,n}(z)\, G_{d_2,n}(z)\, (1-z^2)^{\frac{n-3}{2}} dz = 0, \quad d_1 \neq d_2$$

For any point $y \in S^{n-1}$ and any $d \geq 0$ there exists a spherical harmonic $Z_d^y \in \mathcal{H}_{d,n}$, called *zonal*, which is invariant under the rotations preserving y:

$$Z_d^y(g^{-1}x) = Z_d^y(x), \quad g \in \mathrm{SO}(n),\ gy = y$$

The function $Z_d^y(x)$ is determined uniquely up to a constant and it is proportional to $G_{d,n}(\langle x, y\rangle)$ [81, Thm. 2.14],[1] where $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ is the standard scalar product in $\mathbb{R}^n$.

[1] According to the usual definition [81, page 143] a zonal harmonic Z_d^y is determined uniquely by some normalization condition. Since a normalization is unimportant for our purposes we abuse the terminology and call zonal any spherical harmonic with the mentioned invariance property.

The inclusion map

$$
\begin{aligned}
i : \mathcal{P}_{d,n} &\hookrightarrow \mathcal{P}_{d,n+1} \\
f &\mapsto i(f)(x_1, \dots, x_n, x_{n+1}) = f(x_1, \dots, x_n)
\end{aligned}
$$

induces the linear inclusion

$$
\hat{\ } : \mathcal{H}_{d,n} \hookrightarrow \mathcal{H}_{d,n+1}
$$

$$
h = f|_{S^{n-1}} \mapsto \hat{h} = i(f)|_{S^n}
$$

We call *tesseral* any spherical harmonic of the form $\hat{h} \in \mathcal{H}_{d,n+1}$ for some $h \in \mathcal{H}_{d,n}$.

1.2 Semialgebraic geometry

The material of this section is covered by [18, 27].

Definition 3. *A subset $S \subset \mathbb{R}^n$ is* semialgebraic *if it can be written as*

$$
S = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n : f_{i,j} *_{i,j} 0\},
$$

where $f_{i,j} \in \mathbb{R}[x_1, \dots, x_n]$ are some real polynomials and $*_{i,j}$ is either $<$ or $=$ for $i = 1, \dots, s$, $j = 1, \dots, r_i$.

Remark 1. *If $S \subset \mathbb{R}^n$ is algebraic, i.e., $S = \{x \in \mathbb{R}^n : f(x) = 0 \ \forall f \in B\}$ for some set of real polynomials $B \subset \mathbb{R}[x_1, \dots, x_n]$, then it is also semialgebraic since by Hilbert's basis theorem S can be defined by a finite collection of polynomials.*

Note that, by definition, the family of semialgebraic subsets of $\mathbb{R}^n$ is closed under taking finite intersections, finite unions and complements and the product $S_1 \times \dots \times S_r \subset \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_r}$ of semialgebraic subsets $S_i \subset \mathbb{R}^{n_i}$, $i = 1, \dots, r$ is as well semialgebraic. On of the most important statements about semialgebraic sets is Tarski-Seidenberg theorem.

Theorem 1 (Tarski-Seidenberg)**.** *If $S \subset \mathbb{R}^n$ is semialgebraic and $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is the projection on the first k coordinates, then $\pi(S) \subset \mathbb{R}^k$ is semialgebraic.*

Morphisms between semialgebraic sets are defined as follows.

Definition 4. *Let $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^k$ be semialgebraic sets. A mapping $f : S \to T$ is said to be* semialgebraic *if its graph*

$$
\Gamma_f = \{(x, y) \in S \times T : f(x) = y\}
$$

is a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}^k$.

Tarski-Seidenberg theorem implies that the direct image and the inverse image of a semialgebraic set under a semialgebraic mapping are semialgebraic.

It is easy to describe the semialgebraic subsets of $\mathbb{R}$: they are the unions of finitely many points and open intervals. The following proposition asserts that as in the case $n = 1$ semialgebraic subsets of $\mathbb{R}^n$ are nothing but finite unions of open "cubes" of different dimensions.

Proposition 1. *Every semialgebraic subset $S \subset \mathbb{R}^n$ can be represented as the disjoint union $S = \sqcup_{i=1}^r C_i$ of a finite number of semialgebraic subsets C_i (called* cells*), where each cell C_i is a smooth submanifold of $\mathbb{R}^n$ semialgebraically diffeomorphic to an open cube $(0,1)^{d_i}, d_i \in \mathbb{N}$ (with $(0,1)^0$ being a point). The number d_i is the* dimension *of the cell C_i and the* dimension *of S is defined as* $\dim(S) = \max\{d_i : i = 1, \ldots, r\}$*. This definition does not depend on a particular decomposition of S.*

Given a semialgebraic set $S \subset \mathbb{R}^n$ of dimension $k \leq n$ and given a decomposition of S into cells (as in Proposition 1) we denote by S_{top} the union of all k-dimensional cells and by S_{low} the union of the cells of dimension less than k. The sets $S_{\text{top}}, S_{\text{low}} \subset \mathbb{R}^n$ are semialgebraic and S_{top} is a smooth k-dimensional submanifold of $\mathbb{R}^n$.

One of the central results about semialgebraic mappings is Hardt's theorem.

Theorem 2 (Hardt's semialgebraic triviality)**.** *Let $S \subset \mathbb{R}^n$ be a semialgebraic set and let $f : S \to \mathbb{R}^k$ be a continuous semialgebraic mapping. Then there exists a finite partition of $\mathbb{R}^k$ into semialgebraic sets $T_1, \ldots, T_r \subset \mathbb{R}^k$ such that f is semialgebraically trivial over each T_i, i.e., there are a semialgebraic set F_i and a semialgebraic homeomorphism $h_i : f^{-1}(T_i) \to T_i \times F_i$ such that the composition of h_i with the projection $T_i \times F_i \to T_i$ equals $f|_{f^{-1}(T_i)}$.*

The following corollary of Hardt's theorem is frequently used to estimate dimensions of semialgebraic sets.

Corollary 1. *Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}^k$ be as in Theorem 2. Then the set $\{x \in \mathbb{R}^k : \dim(f^{-1}(x)) = d\}$ is semialgebraic and has dimension not greater than* $\dim(S) - d$*. In particular,* $\dim(f(S)) \leq \dim(S)$*.*

The classical Sard's theorem asserts that the set of critical values of a smooth mapping between two smooth manifolds is of measure zero. Its semialgebraic version allows to say a bit more.

Theorem 3 (Semialgebraic Sard's theorem)**.** *Let $f : S \to T$ be a smooth semialgebraic mapping between smooth semialgebraic sets. Then the set of its critical values is a semialgebraic subset of T of dimension strictly less than* $\dim(T)$*.*

1.3 Probabilistic framework

In this section we set up the probabilistic framework we will work in.

1.3.1 Conventions on metrics and volumes

The real projective space $\mathbb{R}\mathrm{P}^{n-1}$ is endowed with the Riemannian metric with respect to which the double covering $S^{n-1} \xrightarrow{2:1} \mathbb{R}\mathrm{P}^{n-1}$ is a local isometry, where the unit sphere $S^{n-1} \subset \mathbb{R}^n$ inherits its metric from the Euclidean space $\mathbb{R}^n$. We refer to this Riemannian metric on $\mathbb{R}\mathrm{P}^{n-1}$ as "projective metric" or "standard metric".

The Grassmannian manifold $Gr(k,n)$ of k-planes in $\mathbb{R}^n$ is endowed with an $O(n)$-invariant Riemannian metric through the Plücker embedding

$$i : Gr(k,n) \hookrightarrow \mathrm{P}\left(\bigwedge^k \mathbb{R}^n\right)$$

where $\mathrm{P}(\bigwedge^k \mathbb{R}^n)$, the projectivization of the vector space $\bigwedge^k \mathbb{R}^n \simeq \mathbb{R}^{\binom{n}{k}}$, is endowed with the standard metric. Using the Plücker embedding we locally identify $Gr(k,n)$ with the set of unit simple k-vectors $v_1 \wedge \cdots \wedge v_k$, where $v_1, \ldots, v_k$ are orthonormal in $\mathbb{R}^n$ (see [53] for more details).

A canonical left-invariant metric on the orthogonal group $O(n)$ is defined as

$$\langle A, B \rangle := \frac{1}{2}\mathrm{tr}(A^t B), \ A, B \in T_{\mathbf{1}}O(n)$$

Denoting by $|M|$ the total volume of a Riemannian manifold M (whenever it is finite) one can prove the following formulas:

$$|Gr(k,n)| = \frac{|O(n)|}{|O(k)||O(n-k)|}, \quad \frac{|O(n+1)|}{|O(n)|} = |S^n|, \quad |O(1)| = 2, \quad |S^n| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}.$$

We will also need a notion of the volume of a semialgebraic set.

Definition 5 (Volume of a semialgebraic set)**.** *Let $S \subset \mathbb{R}^n$ be a semialgebraic subset of dimension $k \leq n$ and S_{top} be the union of all k-dimensional cells of some cell decomposition of S (see Proposition 1). The k-dimensional manifold S_{top} inherits a Riemannian structure from $\mathbb{R}^n$ and by $|S| := |S_{\mathrm{top}}|$ we denote the volume of S_{top}.*

One can check that this definition is independent of a particular cell decomposition of the semialgebraic subset $S \subset \mathbb{R}^n$.

Given a Riemannian manifold M and a measurable function $f : M \to \mathbb{R}$ on it we denote by $\int_M f(y)\, dV_M$ the integration of f with respect to the Riemannian volume density of M. We recall that there is a unique $O(n)$-invariant probability distribution on $O(n), Gr(k,n)$ and S^n called *uniform* (see [24, 53] for more details). For a measurable subset $A \subset M \in \{O(n), Gr(k,n), S^n\}$ it is defined as

$$\mathbb{P}(A) := \frac{1}{|M|} \int_M \mathbf{1}_A\, dV_M.$$

Remark 2. *For a measurable $A \subset Gr(k,n)$ the set $\hat{A} = \{g \in O(n) : g^{-1}\mathbb{R}^k \in A\}$ is measurable in $O(n)$ and*

$$\mathbb{P}(A) = \frac{1}{|Gr(k,n)|} \int_{Gr(k,n)} \mathbf{1}_A\, dV_{Gr(k,n)} = \frac{1}{|O(n)|} \int_{O(n)} \mathbf{1}_{\hat{A}}\, dV_{O(n)} = \mathbb{P}(\hat{A})$$

We will implicitly use this identification when needed.

1.3.2 Integral geometry formula

The classical Poincaré's integral geometry formula allows to compute the average number of intersection points of two planar curves, where one is fixed and the other is moved around the plane $\mathbb{R}^2$ by a randomly chosen isometry of the plane.

Theorem 4 (Poincaré formula)**.** *Let $c_0, c_1 \subset \mathbb{R}^2$ be two smooth planar curves and let the isometry group $I(\mathbb{R}^2) = SO(2) \rtimes \mathbb{R}^2$ of the plane be endowed with the product Riemannian structure. Then*

$$\int_{g \in I(\mathbb{R}^2)} \#(c_0 \cap gc_1)\, dV_{I(\mathbb{R}^2)} = 4|c_0||c_1|$$

There are generalizations due to Brothers [22] and Howard [45] of Poincaré formula to the case of two submanifolds in an arbitrary homogeneous space endowed with an invariant metric. In Chapters 3 and 4 we will only need a version of this formula (that we state below) for submanifolds of spheres and real projective spaces and in Chapter 5 a generalized Poincaré formula for hypersurfaces in Grassmannian from [24] will be used (for this see Section 5.1).

Theorem 5 (Poincaré formula for submanifolds of a sphere)**.** *Let $M, N \subset S^n$ be smooth submanifolds of the sphere of dimensions* $\dim(M) = m, \dim(N) = k$ *and let $m + k \geq n$. Then for almost all $g \in SO(n+1)$ the intersection $M \cap gN$ is transverse and*

$$\frac{1}{|SO(n+1)|} \int_{g \in SO(n+1)} |M \cap gN|\, dV_{SO(n+1)} = |S^{m+k-n}| \frac{|M|}{|S^m|} \frac{|N|}{|S^k|}$$

The above formula also holds in case when $M, N \subset S^n$ are algebraic subsets (even singular).

The following corollary of Theorem 5 will be particularly useful for us.

Corollary 2. *Let $M \subset \mathbb{R}\mathrm{P}^n$ be a smooth submanifold of dimension $\dim(M) = m$, $N \simeq \mathbb{R}\mathrm{P}^k \subset \mathbb{R}\mathrm{P}^n$ be a k-dimensional projective subspace and let $m + k \geq n$. Then for almost all $g \in SO(n+1)$ the intersection $M \cap gN$ is transverse and*

$$\mathbb{E}_{\,L\in\mathbb{G}(k,n)}|M\cap L| = \frac{1}{|SO(n+1)|}\int\limits_{g\in SO(n+1)} |M\cap gN|\, dV_{SO(n+1)} = |\mathbb{R}\mathrm{P}^{m+k-n}|\frac{|M|}{|\mathbb{R}\mathrm{P}^m|},$$

where $\mathbb{G}(k,n) \simeq Gr(k+1, n+1)$ denotes the Grassmannian of projective k-subspaces in $\mathbb{R}\mathrm{P}^n$. Again, the above formula also holds in case when $M \subset \mathbb{R}\mathrm{P}^n$ is an algebraic subset (even singular).

1.3.3 Smooth coarea formula

Let $f : M \to N$ be a smooth map between two smooth manifolds. A point $x \in M$ is called *a regular point of f* if the differential $D_x f : T_x M \to T_{f(x)} N$ is surjective, otherwise $x \in M$ is called *a critical point*. The point $y \in N$ is called *a regular value of f* if its preimage $f^{-1}(y)$ consists only of regular points, otherwise $y \in N$ is called *a critical value*. By convention, any point $y \in N \setminus f(M)$ outside the image of f is a regular value. Note that if $\dim(M) < \dim(N)$, then all points $x \in M$ are critical for f.

Sard's theorem asserts that the set of critical values of f has zero measure.

Let now $f : M \to N$ be a smooth map between two Riemannian manifolds of dimensions $\dim(M) = m, \dim(N) = k$.

Definition 6. The normal Jacobian $NJ_x f$ *of f at $x \in M$ is defined as follows. If $x \in M$ is a critical point of f then $NJ_x f = 0$, otherwise*

$$NJ_x f = \|D_x f(e_1) \wedge \cdots \wedge D_x f(e_k)\|,$$

where $e_1, \dots e_k \in (\ker(D_x f))^\perp$ is any orthonormal basis of the orthogonal complement to the kernel of the differential of f at x.

The smooth coarea formula is a far going generalization of the classical formula for the change of variables in the integral.

Theorem 6 (Smooth coarea formula)**.** *Let $f : M \to N$ be a smooth map between Riemannian manifolds of dimensions* $\dim(M) = m, \dim(N) = k$. *Then*

$$\int\limits_{y\in N}\int\limits_{x\in f^{-1}(y)} h(x)\, dV_{f^{-1}(y)}\, dV_N = \int\limits_{x\in M} h(x)\, NJ_x f\, dV_M$$

for any Borel measurable function h defined almost everywhere on M and such that the integral on the right is finite.

Remark 3. *Note that when almost all points $x \in M$ are regular one can rewrite the smooth coarea formula as follows:*

$$\int\limits_{y\in N}\int\limits_{x\in f^{-1}(y)} h(x)(NJ_x f)^{-1} dV_{f^{-1}(y)}\, dV_N = \int\limits_{x\in M} h(x)\, dV_M$$

1.3.4 Random matrices

The space $M(n,\mathbb{R})$ of $n \times n$ real matrices is endowed with the Frobenius inner product and the associated norm:

$$(A,B) = \mathrm{tr}(A^tB), \quad \|A\|^2 = (A,A), \quad A,B \in M(n,\mathbb{R}).$$

Any k-dimensional vector subspace $V \subset M(n,\mathbb{R})$ is endowed with *the standard normal probability distribution* $\mathcal{N}_V$:

$$\mathbb{P}_{\mathcal{N}_V}(U) = \frac{1}{\sqrt{2\pi}^k}\int_U e^{-\frac{\|v\|^2}{2}}\, dv,$$

where dv is the Riemannian volume density on $(V,(\cdot,\cdot))$ and $U \subset V$ is a measurable subset.

In case when $V = \mathrm{Sym}(n,\mathbb{R})$ is the space of real symmetric $n \times n$ matrices the probability space $(\mathrm{Sym}(n,\mathbb{R}), \mathcal{N}_{\mathrm{Sym}(n,\mathbb{R})})$ is called *the Gaussian Orthogonal Ensemble* and random matrices taking values in it are called $\mathrm{GOE}(n)$-matrices. Denoting by $dQ = \prod_{1\le i\le j\le n} dQ_{ij}$ the Lebesgue measure of the entries of a symmetric matrix $Q \in \mathrm{Sym}(n,\mathbb{R})$ one can write

$$\mathbb{P}_{\mathrm{GOE}(n)}\{U\} = \frac{1}{\sqrt{2}^n\sqrt{\pi}^N}\int_U e^{-\frac{\|Q\|^2}{2}}\, dQ,$$

where $N = \dim(\mathrm{Sym}(n,\mathbb{R})) = \binom{n+1}{2}$ and $U \subset \mathrm{Sym}(n,\mathbb{R}) \simeq \mathbb{R}^N$ is a measurable subset. Due to the invariance under conjugations $Q \mapsto C^tQC$, $C \in O(n)$ by orthogonal matrices, the GOE(n) measure on $\mathrm{Sym}(n,\mathbb{R})$ induces a probability measure on the space of eigenvalues $\mathbb{R}^n$ [63, Sect. 3.1]:

$$\mathbb{P}\{U\} := \frac{1}{Z_n}\int_U e^{-\frac{\|\lambda\|^2}{2}}|\Delta(\lambda)|\,d\lambda$$

where $d\lambda = \prod_{i=1}^n d\lambda_i$ is the Lebesgue measure on $\mathbb{R}^n$, $U \subset \mathbb{R}^n$ is a measurable subset, $\|\lambda\|^2 = \lambda_1^2 + \cdots + \lambda_n^2$ is the Euclidean norm of $\lambda \in \mathbb{R}^n$, $\Delta(\lambda) := \prod_{1\le i<j\le n}(\lambda_j - \lambda_i)$ is the Vandermonde determinant and Z_n is the normalization constant. The value of Z_n equals [63, (17.5.9)]

$$Z_n := \int_{\mathbb{R}^n} e^{-\frac{\|\lambda\|^2}{2}}|\Delta(\lambda)|\,\mathrm{d}\lambda = \sqrt{2\pi}^n \prod_{i=1}^n \frac{\Gamma(1+\frac{i}{2})}{\Gamma(\frac{3}{2})} \tag{1.1}$$

The unit sphere $S^{N-1} := \{Q \in \mathrm{Sym}(n,\mathbb{R}) : \|Q\| = 1\}$ in the space of symmetric matrices is endowed with the uniform measure. Then the normalized volume $|E|/|S^{N-1}|$ of any measurable set $E \subset S^{N-1}$ equals the GOE(n) measure of the cone $C(E) := \{Q \in \mathrm{Sym}(n,\mathbb{R}) : Q/\|Q\| \in E\}$ over E:

$$\frac{|E|}{|S^{N-1}|} = \mathbb{P}\{C(E)\} \tag{1.2}$$

This property of the GOE measure will be useful for us.

Chapter 2

Spherical harmonics with the maximum number of critical points

Spherical harmonics are the restrictions to the standard unit sphere of harmonic homogeneous polynomials. They can be equivalently defined as eigenfunctions of the spherical Laplace operator. Spherical harmonics play an important role in mathematics (the theory of special functions, spectral geometry) and physical sciences (quantum mechanics, cosmology, geodesy).

The study of geometric and topological properties of Laplace eigenfunctions on Riemannian manifolds has rich and interesting history, we refer the reader to the survey of the results [89]. Being eigenfunctions of the spherical Laplace operator spherical harmonics have many remarkable properties that are not shared by general homogeneous polynomials. For example, the algebraic hypersurface they define cannot have arbitrarily small volume. In fact, from the result [30, Thm. 1.2] of Donnelly and Fefferman it follows that if $h \in \mathcal{H}_{d,n+1}$ is a spherical harmonic corresponding to a harmonic homogeneous polynomial of degree d in $n+1$ variables then the $(n-1)$-dimensional Hausdorff volume of *the nodal hypersurface* $N_h = \{x \in S^n : h(x) = 0\}$ of the spherical harmonic h satisfies

$$c_n\sqrt{\lambda_{d,n}} \leq H^{n-1}(N_h),$$

where $c_n > 0$ is a positive constant that depends only on n and $-\lambda_{d,n} = -d(d+n-1)$ is the eigenvalue of the spherical Laplace operator Δ_{S^n} that corresponds to h.

Another interesting result concerns topological properties of nodal sets of spherical harmonics. Connected components of the complement in the sphere to the nodal hypersurface of a spherical harmonic are called *nodal domains*. In [67] Pleijel proved the following asymptotic bound on the number $\mu(h)$ of nodal domains in S^2 of a spherical harmonic $h \in \mathcal{H}_{d,3}$:

$$\lim_{d\to+\infty} \frac{\mu(h)}{d(d-1)} < 0.69 \tag{2.1}$$

By the results of Harnack [42] a real plane algebraic curve of degree d, i.e., the zero set in $\mathbb{R}\mathrm{P}^2$ of a real homogeneous polynomial of degree d, cannot have more than $(d-1)(d-2)/2+1$ connected components in $\mathbb{R}\mathrm{P}^2$ when it is non-singular and this bound is attained by some non-singular curves which are called *M-curves*. The result (2.1) in particular implies that the non-singular real plane algebraic curves defined by harmonic homogeneous polynomials cannot be M-curves (at least when the degree d is large enough). In [11, Problem 1] Arnold asked to determine for a given d the maximum number of nodal domains that a spherical harmonic $h \in \mathcal{H}_{d,3}$ can have. In [56] trying to answer Arnold's question Leydold conjectured the following formula:

$$\max_{h\in\mathcal{H}_{d,3}} \mu(h) = \begin{cases} \frac{1}{2}(d+1)^2, & \text{if } d \text{ is odd} \\ \frac{1}{2}d(d+2), & \text{if } d \text{ is even} \end{cases}$$

and he proved it for $d \leq 6$. For $d > 6$ the problem is still open to our knowledge.

Also in [11, Problem 1] Arnold asked to determine the largest number of local maxima that a Morse spherical harmonic $h \in \mathcal{H}_{d,3}$ can have on the sphere S^2. For even d the answer to this question is not known in general (to our knowledge). For odd d the answer $(d^2-d+2)/2$ was given by Kuznetsov and Kholshevnikov in [54], where they also proved that the maximum number $m_{d,3}$ of critical points of the restriction $f|_{S^2}$ to the sphere of a Morse (see Subsection 2.1.1) real homogeneous polynomial $f \in \mathcal{P}_{d,3}$ of degree d equals:

$$m_{d,3} = 2(d^2-d+1)$$

and surprisingly enough this bound is attained by spherical harmonics. In the following theorem we generalize the result of Kuznetsov and Kholshevnikov to the case of any number of variables.

Theorem 7. *For any $d \geq 1$ and $n \geq 2$ the maximum number $m_{d,n}$ of critical points of the restriction $f|_{S^{n-1}}$ to the sphere of a Morse real homogeneous polynomial $f \in \mathcal{P}_{d,n}$ of degree d equals*

$$m_{d,n} = 2\frac{(d-1)^n-1}{d-2} = 2((d-1)^{n-1}+\cdots+(d-1)+1)$$

Moreover, for any $d \geq 1$ and $n \geq 2$ there exist a Morse spherical harmonic $h \in \mathcal{H}_{d,n}$ with $m_{d,n}$ critical points.

Critical points of restrictions to the sphere of real homogeneous polynomials reappeared in the context of *spectral theory* of high order tensors independently initiated by Lim [57] and Qi [69] in 2005. Several generalizations of the classical concept of an eigenvector of a matrix were introduced in [57, 69]. Critical points of the restrictions to the sphere of real homogeneous polynomials correspond to l^2-eigenvectors of Lim or Z-eigenvectors of Qi as we explain in Section 2.1.2.

Let $A = (a_{i_1\dots i_d})_{i_j=1}^n$, $a_{i_1\dots i_d} \in \mathbb{R}$ be a real n-dimensional tensor of order d (in the sequel, n^d-tensor). A non-zero vector $x \in \mathbb{C}^n \setminus \{0\}$ is called an *eigenvector* of A if there exists $\lambda \in \mathbb{C}$, the corresponding *eigenvalue*, such that

$$Ax^{d-1} = \lambda x, \quad Ax^{d-1} := \left(\sum_{i_2,\dots,i_d=1}^{n} a_{1 i_2\dots i_d} x_{i_2} \cdots x_{i_d}, \ \dots\ , \sum_{i_2,\dots,i_d=1}^{n} a_{n i_2\dots i_d} x_{i_2} \cdots x_{i_d} \right).$$

For $d = 2$ one recovers the classical definition of an eigenvector of an $n \times n$ matrix $A = (a_{i_1 i_2})_{i_j=1}^n$. The point $[x] \in \mathbb{C}\mathrm{P}^{n-1}$ defined by an eigenvector $x \in \mathbb{C}^n \setminus \{0\}$ is called an *eigenpoint* and the set of all eigenpoints is called an *eigenconfiguration.*

An n^d-tensor $A = (a_{i_1\dots i_d})_{i_j=1}^n$, $a_{i_1,\dots,i_d} \in \mathbb{R}$ is said to be *symmetric* if $a_{i_{\sigma_1}\dots i_{\sigma_d}} = a_{i_1\dots i_d}$ for any permutation $\sigma \in S_d$. Cartwright and Sturmfels [25] proved that the number of eigenpoints of a *generic* (for the definition see Subsection 2.1.2) symmetric n^d-tensor equals

$$\tilde{m}_{d,n} := \frac{(d-1)^n - 1}{d-2} = (d-1)^{n-1} + \dots + (d-1) + 1$$

but, except for the case of real symmetric matrices ($d = 2$), not all eigenvectors of a general real symmetric tensor of order $d \geq 3$ are real. In fact, “most”[1] of real symmetric tensors have eigenpoints in $\mathbb{C}\mathrm{P}^{n-1} \setminus \mathbb{R}\mathrm{P}^{n-1}$. Abo, Seigal and Sturmfels conjectured [1, Conjecture 6.5] that for any $d \geq 1$ and $n \geq 2$ there exists a generic real symmetric n^d-tensor having only real eigenvectors and proved it for $d \geq 1, n = 3$ and for $d = n = 4$. The cases $d \geq 1, n = 2$ and $d = 2, n \geq 2$ are elementary, the case of general d, n was unknown (see for example [82]). In the following theorem which is a trivial corollary of Theorem 7 we cover the case of arbitrary $d \geq 1$ and $n \geq 2$.

Theorem 8. *For any $d \geq 1$ and $n \geq 2$ there exists a generic real symmetric n^d-tensor all of whose $\tilde{m}_{d,n}$ eigenpoints are real.*

Before passing to the proof of Theorem 7 we need few auxiliary results: first in Subsection 2.1.1 we recall some basic facts about Morse functions; then, in Subsection 2.1.2, we give the definition of a generic tensor, recall the classical

[1]As it often happens in real algebraic geometry problems the objects of “maximal complexity” are rare and “numerically invisible”.

correspondence between symmetric tensors and homogeneous polynomials and show that under this correspondence unit real eigenvectors of a real symmetric tensor are exactly critical points of the restriction to the sphere of the corresponding real homogeneous polynomial; finally, we describe critical points of zonal and tesseral spherical harmonics in Subsection 2.1.3. In the end of this chapter we also discuss several related problems.

2.1 Preliminaries

2.1.1 Morse functions

Let $f : M \to \mathbb{R}$ be a smooth function on a smooth n-dimensional manifold M.

The *differential* $\mathrm{d}_x f$ of f at a point $x \in M$ is a linear form on $T_x M$: it sends $v = \gamma'(0) \in T_x M$ to $\mathrm{d}_x f(v) := (f \circ \gamma)'(0)$, where $\gamma = \gamma(t) \subset M$ is a smooth curve passing through $x = \gamma(0)$ with the tangent vector v at $t = 0$. The point $x_* \in M$ is said to be *critical* for f if $\mathrm{d}_{x_*} f = 0$.

The *second differential* $\mathrm{d}^2_{x_*} f$ of f at a critical point $x_* \in M$ is a quadratic form on $T_{x_*} M$: it sends $v = \gamma'(0) \in T_{x_*} M$ to $\mathrm{d}^2_{x_*} f(v) := (f \circ \gamma)''(0)$. In local coordinates $x = (x_1, \dots, x_n)$ near x_* one has

$$\mathrm{d}^2_{x_*} f(v) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_*) v_i v_j$$

The matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_*)\right)$ is called the *Hessian matrix*.

A critical point $x_* \in M$ of $f : M \to \mathbb{R}$ is said to be *(non-)degenerate* if the second differential $\mathrm{d}^2_{x_*} f$ is a (non-)degenerate (as quadratic form). Equivalently, a critical point $x^* \in M$ of f is (non-)degenerate if the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_*)\right)$ written in any local coordinates is (non-)singular.

Classical Morse lemma asserts that in some local coordinates $x = (x_1, \dots, x_n)$ near a non-degenerate critical point x_* the function f takes the form

$$f(x) = f(x_*) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

where the number k, called the *index* of f at x_*, equals the dimension of a maximal subspace of $T_{x_*} M$ on which $\mathrm{d}^2_{x_*} f$ is negative definite.

A smooth function $f : M \to \mathbb{R}$ with only non-degenerate critical points is called *Morse*. Non-degenerate critical points are isolated, hence on a compact manifold a Morse function can have only finitely many critical points.

In the following we also say that a real homogeneous polynomial $f \in \mathcal{P}_{d,n}$ is *Morse* if its restriction to the sphere S^{n-1} is a Morse function.

2.1.2 Symmetric tensors and homogeneous polynomials

A generic symmetric $n\times n$ matrix has n real simple eigenvalues and n corresponding eigenpoints. Moreover, in the space of all symmetric $n \times n$ matrices those which have repeated eigenvalues form a real algebraic subvariety, that we call the *discriminant*, and a generic matrix belongs to its complement. The codimension of the discriminant is two and this justifies the fact that the number of real eigenpoints is the same for all generic matrices.

Let $A=(a_{i_1\dots i_d})_{i_j=1}^n$, $a_{i_1\dots i_d}\in\mathbb{R}$ be an n-dimensional symmetric tensor of order d. Recall that a complex number $\lambda \in \mathbb{C}$ is an *eigenvalue* associated to an eigenvector $x \in \mathbb{C}^n$ if $Ax^{d-1} = \lambda x$. In this case the pair $(x, \lambda) \in \mathbb{C}^n \setminus \{0\} \times \mathbb{C}$ is called an *eigenpair* of A. Two eigenpairs (x, λ) and (x', λ') of A are said to be equivalent if they define the same eigenpoint $[x] = [x'] \in \mathbb{C}\mathrm{P}^{n-1}$. Theorem 1.2 in [25] asserts that the number of eigenpoints (equivalence classes of eigenpairs) of a sufficiently generic symmetric n^d-tensor equals $\tilde{m}_{d,n} = ((d-1)^n - 1)/(d-2)$. *Non-generic* tensors are cut out by an algebraic hypersurface, called the *eigendiscriminant* [1], and the number of eigenpoints of a non-generic tensor is not equal to the expected $\tilde{m}_{d,n}$. On each connected component of the complement of the eigendiscriminant the number of real eigenpoints (equivalence classes of real eigenpairs) is constant.

There is a well-known one-to-one correspondence between the set $\mathcal{P}_{d,n}$ of real homogeneous polynomials of degree d in n variables and the set of real symmetric n^d-tensors:

$$f_A = \sum_{i_1,\dots,i_d=1}^{n} a_{i_1\dots i_d} x_{i_1}\dots x_{i_d} \quad\longleftrightarrow\quad A = (a_{i_1\dots i_d})_{i_j=1}^n$$

The critical points of the restriction $f_A|_{S^{n-1}}$ of a homogeneous polynomial f_A to the unit sphere are precisely unit real eigenvectors of the corresponding symmetric tensor A. Indeed, by the method of Lagrange multipliers [16, Section 1.4], if $x \in S^{n-1}$ then

$$\mathrm{d}_x f_A|_{S^{n-1}} = 0 \quad\Leftrightarrow\quad \mathrm{d}_x f_A = \lambda\,\mathrm{d}_x\left(\frac{\|x\|^2-1}{2}\right) \quad\Leftrightarrow\quad Ax^{d-1} = (\lambda/d)x$$

Note that the Lagrange multiplier λ corresponds to the eigenvalue λ/d associated to the unit eigenvector x. In the terminology of Lim [57] and Qi [69] unit real eigenvectors are l^2-eigenvectors and Z-eigenvectors respectively. Theorem 1.2 in [25] thus gives an upper bound on the number of critical points of the restriction to the sphere of a Morse homogeneous polynomial.

Lemma 1. *If a real homogeneous polynomial $f \in \mathcal{P}_{d,n}$ defines a Morse function $f|_{S^{n-1}}$ on the sphere then the number of critical points of $f|_{S^{n-1}}$ is bounded by $m_{d,n} = 2\tilde{m}_{d,n} = 2((d-1)^n - 1)/(d-2)$.*

Proof. If $f_A|_{S^{n-1}}$ is a Morse function and the tensor A is generic then A has $\tilde{m}_{d,n}$ eigenpoints in $\mathbb{C}\mathrm{P}^{n-1}$ which implies that the number of unit real eigenvectors of A (that is equal to the number of critical points of $f_A|_{S^{n-1}}$) is bounded by $m_{d,n} = 2\tilde{m}_{d,n}$.

Suppose now that $f_A|_{S^{n-1}}$ is a Morse function but the tensor A is not generic. Since non-generic tensors form a hypersurface in the space of symmetric tensors any open neighbourhood of A contains a generic tensor $\tilde{A}$. Moreover, if $\tilde{A}$ is sufficiently close to A by [12, Cor. 5.24] the function $f_{\tilde{A}}|_{S^{n-1}}$ is Morse and it has the same number of critical points as $f_A|_{S^{n-1}}$. □

Remark 4. *It is not difficult to see that a real symmetric matrix is generic, i.e., all of its eigenvalues are simple, if and only if the corresponding quadratic homogeneous polynomial is Morse. However, in the case of higher order $d \geq 3$ there can exist non-generic tensors that correspond to Morse homogeneous polynomial.*

2.1.3 Critical points of zonal and tesseral spherical harmonics

Let $y \in S^{n-1}$ and $Z_d^y(x) = G_{d,n}(\langle x, y\rangle) \in \mathcal{H}_{d,n}$ be a zonal spherical harmonic, where $G_{d,n}$ is the degree d Gegenbauer polynomial of parameter $\frac{n-2}{2}$ (for details see Section 1.1). Since polynomials $\{G_{d,n}\}_{d\geq 0}$ are orthogonal on the interval $[-1, 1]$ with respect to the measure $(1-z^2)^{\frac{n-3}{2}}dz$ [2, 22.2.3] by [36, Prop. I.1.1] $G_{d,n}$ has d simple real roots in $(-1, 1)$ and hence its derivative $G'_{d,n}$ has $d-1$ roots in $(-1, 1)$ which we denote by $\alpha_{d,1}, \dots, \alpha_{d,d-1}$. The following lemma characterizes the critical points of a zonal spherical harmonic.

Lemma 2. *The set of critical points of Z_d^y consists of $y, -y$ and $d-1$ affine hyperplane sections of the sphere $\{x \in S^{n-1} : \langle x, y\rangle = \alpha_{d,i}\},\ i = 1, \dots, d-1$. The critical points y and $-y$ are non-degenerate.*

Proof. A point $x \in S^{n-1}$ is critical for $G_{d,n}(\langle x, y\rangle)$ if and only if $G'_{d,n}(\langle x, y\rangle)y$ is proportional to x. This is possible either if $\langle x, y\rangle$ is a root of $G'_{d,n}$ or $x = \pm y$. To prove the non-degeneracy of $x = \pm y$ we assume without loss of generality that $y = (0, \dots, 0, 1) \in S^{n-1}$ and then in local coordinates

$$(x_1, \dots, x_{n-1}) \mapsto \left(x_1, \dots, x_{n-1}, \pm\sqrt{1 - x_1^2 - \dots - x_{n-1}^2}\right) \in S^{n-1}$$

around $x = \pm y$ our function $G_{d,n}(\langle x, y\rangle)$ takes the form $G_{d,n}\left(\pm\sqrt{1 - x_1^2 - \dots - x_{n-1}^2}\right)$. One can easily verify that its Hessian matrix at $(x_1, \dots, x_{n-1}) = (0, \dots, 0)$ is non-singular. □

Let $h = f|_{S^{n-1}} \in \mathcal{H}_{d,n}$, where $f \in \mathcal{P}_{d,n}$ is a harmonic polynomial in n variables. Recall that the tesseral spherical harmonic $\hat{h} \in \mathcal{H}_{d,n+1}$ is the restriction to S^n of the polynomial $f \in \mathcal{P}_{d,n}$ viewed as an element of $\mathcal{P}_{d,n+1}$ (see Section 1.1). Critical points of tesseral spherical harmonics are described as follows.

Lemma 3. *Assume that $d, n \geq 2$ and $h \in \mathcal{H}_{d,n}$.*

(i) If the zero locus $\{h = 0\} \subset S^{n-1}$ is regular then the set of critical points of $\hat{h} \in \mathcal{H}_{d,n+1}$ consists of $\pm(0, \ldots, 0, 1) \in S^n$ and the points $(x_1, \ldots, x_n, 0)$, where $(x_1, \ldots, x_n) \in S^{n-1}$ is critical for h. Moreover, for $d \geq 3$ the points $\pm(0, \ldots, 0, 1) \in S^n$ are always degenerate.

(ii) If $\{h = 0\}$ is singular then, additionally, for each singular point $(x_1, \ldots, x_n) \in \{h = 0\}$ the great circle $\{(tx_1, \ldots, tx_n, \pm\sqrt{1 - t^2}) : 0 \leq t \leq 1\} \subset S^n$ consists of critical points of $\hat{h}$.

Proof. If $h = f|_{S^{n-1}}$ for some harmonic polynomial $f \in \mathcal{P}_{d,n}$ the critical points of $\hat{h} = i(f)|_{S^n} \in \mathcal{H}_{d,n+1}$ are characterized by

$$\frac{\partial f}{\partial x_1} = \lambda x_1, \quad \ldots \quad \frac{\partial f}{\partial x_n} = \lambda x_n, \quad \frac{\partial f}{\partial x_{n+1}} = 0 = \lambda x_{n+1} \tag{2.2}$$

Obviously $(x_1, \ldots, x_n, 0) \in S^n$ is a critical point of $\hat{h}$ if $(x_1, \ldots, x_n) \in S^{n-1}$ is critical for h. Now if $\lambda = 0$ and $\{h = 0\} \subset S^{n-1}$ is regular then $x_1 = \cdots = x_n = 0$ and $x_{n+1} = \pm 1$. If, instead, $\{h = 0\}$ is singular and $(x_1, \ldots, x_n) \in \{h = 0\}$ is a solution of $\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0$ then due to the homogeneity of f any point $(tx_1, \ldots, tx_n, \pm\sqrt{1 - t^2})$, $0 \leq t \leq 1$ is a solution of the system (2.2) with $\lambda = 0$. $\square$

2.2 Proof of main results

In this section we prove Theorem 7 and Theorem 8.

2.2.1 Proof of Theorem 7

Denote by $Z_{d,n}$ a zonal spherical harmonic $Z_d^y(x) = G_{d,n}(\langle x, y \rangle) = G_{d,n}(x_n) \in \mathcal{H}_{d,n}$ corresponding to the point $y = (0, \ldots, 0, 1) \in S^{n-1}$ and let $M_{d,n} \in \mathcal{H}_{d,n}$ be any Morse spherical harmonic with the maximum possible number of critical points. Note that by Lemma 1 this number is bounded by $m_{d,n} = 2((d-1)^n - 1)/(d-2)$. In dimension $n = 2$ any $h \in \mathcal{H}_{d,2}$ is just a trigonometric polynomial

$$h = a\cos(d\theta) + b\sin(d\theta), \quad a, b \in \mathbb{R}, \quad \theta \in [0, 2\pi)$$

and hence it is a Morse function on S^1 with $m_{d,2} = 2d$ critical points. For $n, d \geq 3$ the number of critical points of a general spherical harmonic $h \in \mathcal{H}_{d,n}$ is not anymore a constant and depends significantly on the choice of h. In the proposition below we exhibit for any $n, d \geq 2$ a Morse spherical harmonic $M_{d,n} \in \mathcal{H}_{d,n}$ having $m_{d,n}$ critical points. In fact, we construct $M_{d,n}$ by induction on n starting from a trigonometric polynomial $M_{d,2} \in \mathcal{H}_{d,2}$.

Proposition 2. *For any $d, n \geq 2$ and a sufficiently small $\varepsilon > 0$ the spherical harmonic $M_{d,n+1} := Z_{d,n+1} + \varepsilon \hat{M}_{d,n} \in \mathcal{H}_{d,n+1}$ is a Morse function on S^n with $m_{d,n+1}$ critical points.*

Proof. As observed above one can take $M_{d,2} = a\cos(d\theta) + b\sin(d\theta)$. Suppose that for some $n \geq 2$, we have already constructed a Morse spherical harmonic $M_{d,n} \in \mathcal{H}_{d,n}$ with $m_{d,n}$ critical points on S^{n-1}. By Lemmas 2 and 3 we have that the points $\pm(0, \ldots, 0, 1) \in S^n$ are critical for both $Z_{d,n+1}$ and $\hat{M}_{d,n}$ and hence also for the perturbation $Z_{d,n+1} + \varepsilon\hat{M}_{d,n}$. Since the points $\pm(0, \ldots, 0, 1)$ are non-degenerate for $Z_{d,n+1}$ they remain non-degenerate for the perturbation for small enough $\varepsilon > 0$.

We prove that each of the $d-1$ critical circles $\{x \in S^n : \langle x, y\rangle = \alpha_{d,i}\}$, $i = 1, \ldots, d-1$ of $Z_{d,n+1}$ breaks into $m_{d,n}$ non-degenerate critical points when $Z_{d,n+1}$ is slightly perturbed by $\hat{M}_{d,n}$. The idea is shown on Figure 2.2.1, where the red/purple color represents positive/negative values of functions. In spherical coordinates

$$\begin{aligned}
x_1 &= \sin\theta_n \cdot \tilde{x}_1 = \sin\theta_n \sin\theta_{n-1} \cdots \sin\theta_2 \sin\theta_1 \\
x_2 &= \sin\theta_n \cdot \tilde{x}_2 = \sin\theta_n \sin\theta_{n-1} \cdots \sin\theta_2 \cos\theta_1 \\
x_3 &= \sin\theta_n \cdot \tilde{x}_3 = \sin\theta_n \sin\theta_{n-1} \cdots \cos\theta_2 \\
&\vdots \\
x_n &= \sin\theta_n \cdot \tilde{x}_n = \sin\theta_n \cos\theta_{n-1} \\
x_{n+1} &= \cos\theta_n
\end{aligned}$$

on S^n, where $(\tilde{x}_1, \ldots, \tilde{x}_n) \in S^{n-1} = \{x \in S^n : x_{n+1} = 0\}$, we have

$$\begin{aligned}
Z_{d,n+1}(x_1, \ldots, x_{n+1}) &= G_{d,n+1}(x_{n+1}) = G_{d,n+1}(\cos\theta_n) \\
\hat{M}_{d,n}(x_1, \ldots, x_{n+1}) &= \sin^d\theta_n \, M_{d,n}(\tilde{x}_1, \ldots, \tilde{x}_n)
\end{aligned}$$

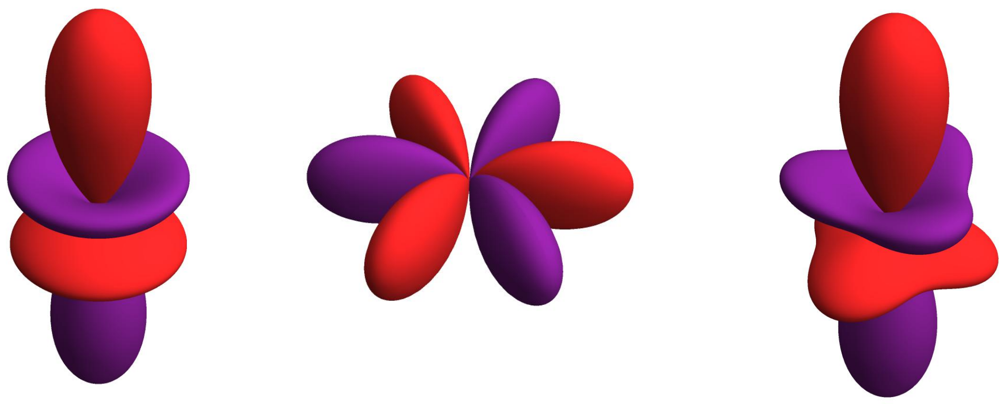

Figure 2.1: Plots of the absolute values on the sphere S^2 of a zonal harmonic $Z_{3,3}$ (left), a function $\hat{M}_{3,2}$ with 6 critical points on the circle (middle) and a perturbation of $Z_{3,3}$ by $\hat{M}_{3,2}$ with $14 = 2 + 2 \cdot 6$ non-degenerate critical points (right).

and hence the critical points of $Z_{d,n+1} + \varepsilon \hat{M}_{d,n}$ are described by the equations

$$\begin{aligned} \varepsilon \sin^d \theta_n \frac{\partial}{\partial \theta_1} M_{d,n}(\tilde{x}_1, \ldots, \tilde{x}_n) &= 0 \\ &\vdots \\ \varepsilon \sin^d \theta_n \frac{\partial}{\partial \theta_{n-1}} M_{d,n}(\tilde{x}_1, \ldots, \tilde{x}_n) &= 0 \\ \frac{\partial}{\partial \theta_n} \left[G_{d,n+1}(\cos \theta_n) + \varepsilon \sin^d \theta_n \, M_{d,n}(\tilde{x}_1, \ldots, \tilde{x}_n) \right] &= 0 \end{aligned} \tag{2.3}$$

Since the $d-1$ zeroes of $G'_{d,n+1}$ are non-degenerate, then for a fixed $(\tilde{x}_1, \ldots, \tilde{x}_n) \in S^{n-1}$ the equation (2.3) has $d-1$ non-degenerate solutions provided that ε is small enough. It follows that each critical point $(\tilde{x}_1, \ldots, \tilde{x}_n) \in S^{n-1}$ of $M_{d,n}$ gives rise to $d-1$ critical points of $Z_{d,n+1} + \varepsilon \hat{M}_{d,n}$. In spherical coordinates the Hessian matrix of $Z_{d,n+1} + \varepsilon \hat{M}_{d,n}$ computed at a critical point $\theta = (\theta_1, \ldots, \theta_{n-1}, \theta_n)$ has the block-diagonal form:

$$\begin{pmatrix} \varepsilon \sin^d \theta_n \frac{\partial^2 M_{d,n}}{\partial \theta_1^2}(\theta) & \dots & \varepsilon \sin^d \theta_n \frac{\partial^2 M_{d,n}}{\partial \theta_1 \partial \theta_{n-1}}(\theta) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \varepsilon \sin^d \theta_n \frac{\partial^2 M_{d,n}}{\partial \theta_{n-1} \partial \theta_1}(\theta) & \dots & \varepsilon \sin^d \theta_n \frac{\partial^2 M_{d,n}}{\partial \theta_{n-1}^2}(\theta) & 0 \\ 0 & \dots & 0 & \begin{matrix} \frac{\partial^2}{\partial \theta_n^2} \left[G_{d,n+1}(\cos\theta_n) + \right. \\ \left. \varepsilon \sin^d \theta_n \, M_{d,n}(\tilde{x}_1, \dots, \tilde{x}_n) \right] \end{matrix} \end{pmatrix}$$

It is non-singular since the function $M_{d,n}$ is, by assumption, Morse and for a small ε the solutions of (2.3) are non-degenerate. Thus, the function $Z_{d,n+1} + \varepsilon \hat{M}_{d,n}$ has $2 + (d-1) \cdot 2((d-1)^n - 1)/(d-2) = 2((d-1)^{n+1} - 1)/(d-2) = m_{d,n+1}$ non-degenerate critical points. □

Theorem 7 follows now from Lemma 1 and Proposition 2.

2.2.2 Proof of Theorem 8

In light of the correspondence between symmetric tensors and homogeneous polynomials described in Subsection 2.1.2 Theorem 8 follows from Theorem 7.

It is worth to note that the inductive construction from Proposition 2 can be generalized as follows.

Proposition 3. *Let $f = f(x_1, \dots, x_n) \in \mathcal{P}_{d,n}$ be a Morse homogeneous polynomial with $m_{d,n}$ critical points on S^{n-1} and $p = p(x_{n+1})$ be an even or odd (depending on the parity of d) univariate polynomial of degree d whose derivative $p' = p'(x_{n+1})$ has $d-1$ simple roots in $(-1,1)$. Then for a small enough $\varepsilon > 0$ the function $p(x_{n+1}) + \varepsilon f(x_1, \dots, x_n) \in \mathcal{P}_{d,n+1}|_{S^n}$ has $m_{d,n+1}$ critical points on S^n.*

The proof of this proposition is identical to the proof of Proposition 2.

2.3 Relation to other problems

Eigenvectors of tensors or, equivalently, critical points of homogeneous polynomials on the sphere arise in many areas of research in pure mathematics and the applied sciences. We discuss three problems to which our work is closely related.

2.3.1 Real zeroes of semi-definite polynomials

A polynomial $f \in \mathcal{P}_{d,n}$ is said to be *positive (negative) semi-definite* if $f(x) \geq 0$ (respectively $f(x) \leq 0$) for any $x \in \mathbb{R}^n$. Note that polynomials of only even degree

d can be semi-definite. In [26] Choi, Lam and Reznick asked to determine the largest possible finite number $B_{d,n}$ of real zeros $x \in \mathbb{R}\mathrm{P}^{n-1}$ that a semi-definite polynomial $f \in \mathcal{P}_{d,n}$ can have. In the case $d = 2$ of quadratic polynomials one can immediately see that $B_{2,n} = 1$. It is also easy to prove that $B_{2d,2} = d$. But already in the case of $n = 3$ variables the exact value of $B_{d,3}$ is not known for most values of d. The following partial results about $B_{d,3}$ were obtained in [26]:

$$B_{4,3} = 4, \quad B_{6,3} = 10, \quad \frac{d^2}{4} \leq B_{d,3} \leq \frac{(d-1)(d-2)}{2}, \ d \geq 6 \tag{2.4}$$

It was also proved in [26] that the limit $B_{d,3}/d^2$ as $d \to +\infty$ exists and

$$\frac{5}{18} \leq \lim_{d \to +\infty} \frac{B_{d,3}}{d^2} \leq \frac{1}{2}$$

In the following Theorem we establish an upper bound on $B_{d,n}$.

Theorem 9. *Let $d \geq 2$ be even and $n \geq 2$. Then*

$$B_{d,n} \leq \frac{1}{2}(\tilde{m}_{d,n} - n) + 1 \tag{2.5}$$

where $\tilde{m}_{d,n} = ((d-1)^n - 1)/(d-2)$ is the number of complex eigenpoints of a generic symmetric tensor.

Proof. For a Morse homogeneous polynomial $f \in \mathcal{P}_{d,n}$ let $\mu_k(f)$ denote the number of critical points of $f|_{S^{n-1}}$ of index $k = 0, \dots, n-1$. In particular, $\mu_0(f)$ ($\mu_{n-1}(f)$) equals the number of local minima (respectively maxima) of $f|_{S^{n-1}}$ and the numbers $\mu_k(f)$ can be used to compute the Euler characteristics of the sphere S^{n-1}:

$$\sum_{k=0}^{n-1} (-1)^k \mu_k(f) = \chi(S^{n-1}) = 1 + (-1)^{n-1}$$

Since d is even by [39, Thm. A] the even Morse function $f|_{S^{n-1}}$ must have at least two critical points of index k for each $k = 0, \dots, n-1$. This together with the bound

$$\sum_{k=0}^{n-1} \mu_k(f) \leq 2\tilde{m}_{d,n}$$

from Lemma 1 implies that

$$\mu_0(f) + 2\left(\frac{n-2}{2}\right) \leq \sum_{k=0}^{(n-2)/2} \mu_{2k}(f) \leq \tilde{m}_{d,n}$$

if n is even and

$$\mu_0(f) + 2\binom{n-1}{2} \leq \sum_{k=0}^{(n-1)/2} \mu_{2k}(f) \leq \tilde{m}_{d,n} + 1$$

if n is odd. Consequently, we obtain the following upper bound on the number of local minima:

$$\mu_0(f) \leq \tilde{m}_{d,n} - n + 2 \tag{2.6}$$

independently of the parity of n.

Let now $f \in \mathcal{P}_{d,n}$ be any polynomial with finitely many local minima on S^{n-1} and let $\tilde{f} \in \mathcal{P}_{d,n}$ be a small perturbation of f such that $\tilde{f}|_{S^{n-1}}$ is a Morse function. Every local minimum of $f|_{S^{n-1}}$ gives rise to at least one (non-degenerate) minimum of $\tilde{f}|_{S^{n-1}}$. Hence the bound (2.6) on the number $\mu_0(f)$ of local minima of $f|_{S^{n-1}}$ applies even if $f|_{S^{n-1}}$ is not a Morse function. In particular, if $f \in \mathcal{P}_{d,n}$ is a non-negative polynomial with finitely many real zeros in $\mathbb{R}\mathrm{P}^{n-1}$ we have

$$\#\{x \in \mathbb{R}\mathrm{P}^{n-1} : f(x) = 0\} \leq \frac{\tilde{m}_{d,n} - n}{2} + 1$$

which completes the proof. □

Remark 5. *The bound* (2.5) *is not sharp. For example, for* $n = 3$ *it is worse than the bounds* (2.4) *from [26]. For* $n > 3$ *though we are not aware of any better bound on* $B_{d,n}$ *than* (2.5).

2.3.2 Low rank approximations

Eigenvectors and eigenvalues of a symmetric tensor can be used to find its best rank one approximation. A real symmetric n^d-tensor $A = (a_{i_1\ldots i_d})_{i_j=1}^n$ is said to be of *rank one* if $a_{i_1\ldots i_d} = \lambda x_{i_1}\cdots x_{i_d}$ for some vector $x \in S^{n-1}$ and constant $\lambda \in \mathbb{R}$. Consider the set

$$X_{d,n} := \{\lambda(x_{i_1}\cdots x_{i_d})_{i_j=1}^n : \lambda \in \mathbb{R}, x \in S^{n-1}\}$$

of real symmetric n^d-tensors of rank one and for a given real symmetric n^d-tensor $A = (a_{i_1\ldots i_d})_{i_j=1}^n$ define the function:

$$\begin{aligned} \mathrm{dist}_A : X_{d,n} &\to \mathbb{R} \\ \lambda(x_{i_1}\cdots x_{i_d})_{i_j=1}^n &\mapsto \sum_{i_1,\ldots,i_d=1}^n (a_{i_1\ldots i_d} - \lambda x_{i_1}\cdots x_{i_d})^2 \end{aligned}$$

(this function measures the Euclidean distance of a rank one tensor from A).

A rank one tensor $\lambda(x_{i_1}\cdots x_{i_d})_{i_1,=1}^n \in X_{d,n}$ is a critical point of dist_A if and only if $x \in S^{n-1}$ is a unit eigenvector of A and $\lambda \in \mathbb{R}$ is the corresponding eigenvalue. In this context a *best rank one approximation* to A, a tensor $\lambda(x_{i_1}\cdots x_{i_d})_{i_j=1}^n \in X_{d,n}$ which is a global minimizer of dist_A, corresponds to the greatest (in absolute value) eigenvalue $|\lambda|$ [68, Thm. 2 (d)].

Theorem 8 is then equivalent to the existence for any $d \geq 1$ and $n \geq 2$ of a real symmetric n^d-tensor A such that the function $\mathrm{dist}_A : X_{d,n} \to \mathbb{R}$ has the maximum possible generic number of critical points that is equal to $\tilde{m}_{d,n}$.

Remark 6. *The problem of finding a best rank one approximation to a real symmetric n^d-tensor $A = (a_{i_1\dots i_d})_{i_j=1}^n$ is equivalent to the problem of maximizing the absolute value $|f_A(x)|$ of the homogeneous polynomial $f_A(x) = \sum_{i_j=1}^n a_{i_1\dots i_d} x_{i_1}\cdots x_{i_d}$ constrained on the sphere S^{n-1}.*

2.3.3 Complex dynamics

Let $f : \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ be a non constant holomorphic map. Then in homogeneous coordinates one can write $f = [f_1 : \cdots : f_n]$, where

$$f_i(x) = \sum_{i_2,\dots,i_d=1}^{n} a_{ii_2\dots i_n} x_{i_2}\cdots x_{i_d},\ i = 1,\dots,n$$

are complex homogeneous polynomials of certain degree $d-1$ having no common zeroes in $\mathbb{C}\mathrm{P}^{n-1}$. Moreover, the polynomials $f_1,\dots,f_n$ are determined uniquely up to a common constant multiple. It is straightforward to see that the fixed points $\{x \in \mathbb{C}\mathrm{P}^{n-1} : f(x) = x\}$ of $f = [f_1 : \cdots : f_n]$ are precisely the eigenpoints $x \in \mathbb{C}\mathrm{P}^{n-1}$ of the tensor $A = (a_{i_1\dots i_d})_{i_j=1}^n$. The number of fixed points for a generic map f equals $\tilde{m}_{d,n}$ [37, 25].

When the polynomials $f_1,\dots,f_n$ are real, $f = [f_1 : \cdots : f_n]$ preserves $\mathbb{R}\mathrm{P}^{n-1} \subset \mathbb{C}\mathrm{P}^{n-1}$ and the real fixed points of this map are precisely the real eigenpoints of A. Theorem 8 implies that for some generic real map f all of its (a priori complex) fixed points are real.

Chapter 3

On the geometry of the set of symmetric matrices with repeated eigenvalues

In this chapter we investigate some geometric properties of the set Δ (below called *discriminant*) of real symmetric matrices with repeated eigenvalues and of unit Frobenius norm

$$\Delta = \{Q \in \mathrm{Sym}(n, \mathbb{R}) \text{ such that } \lambda_i(Q) = \lambda_j(Q) \text{ for some } i \neq j\} \cap S^{N-1},$$

where $N = \frac{n(n+1)}{2} = \dim(\mathrm{Sym}(n, \mathbb{R}))$ and S^{N-1} denotes the unit sphere with respect to the Frobenius norm $\|Q\|^2 = \mathrm{tr}(Q^2)$.

The discriminant appears in several areas of mathematics, from mathematical physics to real algebraic geometry [7, 9, 8, 10, 83, 3, 4, 87].

The set Δ is an algebraic subset of S^{N-1} of codimension two. It is defined by the *discriminant polynomial*:

$$\mathrm{dis}(Q) := \prod_{i\neq j}(\lambda_i(Q) - \lambda_j(Q))^2$$

which is a non-negative homogeneous polynomial of degree $\deg(\mathrm{dis}) = n(n-1)$ in the entries of Q and, moreover, it is a sum of squares of real polynomials [46, 66]. The set Δ_{sm} of smooth points of Δ consists of matrices with *exactly* two repeated eigenvalues (in fact, Δ is stratified according to the multiplicity sequence of the eigenvalues [7]). In the following theorem we compute the volume of $\Delta \subset S^{N-1}$.

Theorem 10 (The volume of the discriminant)**.**

$$\frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}. \tag{3.1}$$

Remark 7. *Results of this type (the computation of the volume of some relevant algebraic subsets of the space of matrices) have started appearing in the literature since the 90's [32, 33], with a particular emphasis on asymptotic studies and complexity theory, and have been crucial for the theoretical advance of numerical algebraic geometry, especially for what concerns the estimation of the so called* condition number *of linear problems [29]. The very first result gives the volume of the set $\Sigma \subset \mathbb{R}^{n^2}$ of square matrices with zero determinant and Frobenius norm one; this was computed in [32, 33]:*

$$\frac{|\Sigma|}{|S^{n^2-1}|} = \sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \sim \sqrt{\frac{\pi}{2}}n^{1/2}. \tag{3.2}$$

For example (3.2) *is used in [32, Theorem 6.1] to compute the average number of zeroes of the determinant of a matrix of linear forms. Subsequently, this computation was extended to include the volume of the set of $n \times m$ matrices of given corank in [14] and the volume of the set of* symmetric *matrices with determinant zero in [55], with similar expressions.*

The proof of (3.1) requires the evaluation of the expectation of the square of the characteristic polynomial of a $\mathrm{GOE}(n)$ matrix (Theorem 13 below), which constitutes a result of independent interest.

Next we discuss the problem of *the maximal cut* of the discriminant. Let's denote by $\mathrm{P}\Delta \subset \mathrm{PSym}(n,\mathbb{R}) \simeq \mathbb{R}\mathrm{P}^{N-1}$ the projectivization of the discriminant. Since $\mathrm{P}\Delta$ has codimension two, the number $\#(L \cap \mathrm{P}\Delta)$ of symmetric matrices with repeated eigenvalues in a generic projective two-plane $L \simeq \mathbb{R}\mathrm{P}^2 \subset \mathbb{R}\mathrm{P}^{N-1}$ is finite. In the following theorem we provide a sharp upper bound on this number.

Theorem 11 (The maximal cut of the discriminant)**.** *For a generic projective two-plane $L \simeq \mathbb{R}\mathrm{P}^2$ the following* sharp *upper bound holds:*

$$\#(L \cap \mathrm{P}\Delta) \leq \binom{n+1}{3}. \tag{3.3}$$

Remark 8. *This result has already appeared in [72, Cor. 15]. The proof we present below is a bit different from the one given in [72].*

Theorem 10 combined with Poincaré formula (Corollary 2) allows to compute the average number of symmetric matrices with repeated eigenvalues in a uniformly distributed projective two-plane $L \subset \mathbb{R}\mathrm{P}^{N-1}$:

$$\mathop{\mathbb{E}}_{L\in\mathbb{G}(2,N-1)} \#(L \cap \mathrm{P}\Delta) = \frac{|\mathrm{P}\Delta|}{|\mathbb{R}\mathrm{P}^{N-3}|} = \frac{|\Delta|}{|S^{N-3}|} = \binom{n}{2}. \tag{3.4}$$

Remark 9. *Consequence* (3.4) *is especially interesting because it "violates" a frequent phenomenon in random algebraic geometry, which goes under the name of* square root law*: for a large class of models of random systems, often related to the so called Edelman-Kostlan-Shub-Smale models [32, 74, 33, 50, 75, 73], the average number of solutions equals (or is comparable to) the square root of the maximum number; here this is not the case. We also observe that, surprisingly enough, the* average cut *of the discriminant is an integer number (there is no reason to even expect that it should be a rational number!).*

The proof of Theorem 10 is obtained by a limiting procedure. Using the fact that the restriction of the GOE(n) measure to the unit sphere in $\mathrm{Sym}(n,\mathbb{R})$ gives the uniform measure, we will describe the volume of the ϵ-tube around Δ using the joint density of the eigenvalues of a GOE(n) matrix and then make a careful application of Weyl's tube formula to derive the asymptotic of this volume at zero (whose leading coefficient, up to a constant, equals $|\Delta|$). The main difficulties here are the explicit description of the tube, and the fact that the variety Δ is singular, which makes the application of Weyl's tube formula delicate. In this way we will prove the following result, which also includes information on the volume of the set Δ_1 of symmetric matrices whose smallest two eigenvalues are equal.

Theorem 12. *Let $\Delta_1 \subset \Delta \subset S^{N-1}$ denote the set of symmetric matrices with the smallest two eigenvalues repeated. Then we have the two following integral expressions:*

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\, n!}\binom{n}{2} \int_{\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(n-2)}\Big[\det(Q-u\mathbb{1})^2\Big]e^{-u^2}\,\mathrm{d}u, \tag{3.5}$$

$$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\, n!}\binom{n}{2} \int_{\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(n-2)}\Big[\det(Q-u\mathbb{1})^2\mathbf{1}_{\{Q-u\mathbb{1}\succ 0\}}\Big]e^{-u^2}\,\mathrm{d}u. \tag{3.6}$$

(note the appearance of the characteristic function $\mathbf{1}_{\{Q-u\mathbb{1}\succ 0\}}$ in the second integral).

The exact evaluation of the integral in (3.5) (Theorem 13 below) will take a considerable amount of work and is of independent interest. It is based on some key properties of Hermite polynomials. By contrast, we do not know whether there exists a closed form evaluation of (3.6).

Theorem 13. *For a fixed positive integer k we have*

$$\int_{\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(k)}[\det(Q-u\mathbb{1})^2]\,e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}\,\frac{(k+2)!}{2^{k+1}}.$$

The proof of Theorem 11 exploits an interesting duality that we establish between symmetric matrices with repeated eigenvalues in a 3-dimensional linear family and singularities of some algebraic surface. To be more specific, given three independent matrices $R_1, R_2, R_3 \in \mathrm{Sym}(n, \mathbb{R})$ denote by $L = \mathrm{P}(\mathrm{span}\{R_1, R_2, R_3\}) \subset \mathrm{PSym}(n, \mathbb{R})$ the projective two-plane that they generate and consider the projective *symmetroid surface*

$$\mathrm{P}\Sigma_{3,n} = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{R}\mathrm{P}^3 \mid \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\}.$$

The following result, which is a particular case of Proposition 9, describes the mentioned duality.

Proposition 4. *For generic matrices $R_1, R_2, R_3 \in \mathrm{Sym}(n, \mathbb{R})$ there is a one-to-one correspondence between singular points of the symmetroid surface $\mathrm{P}\Sigma_{3,n} \subset \mathbb{R}\mathrm{P}^2$ and symmetric matrices with repeated eigenvalues in the projective two-plane $L \simeq \mathbb{R}\mathrm{P}^2$.*

For the generic choice of R_1, R_2, R_3 the singularities of $\mathrm{P}\Sigma_{3,n}$ correspond to matrices of corank two in the linear family $x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3$. The degree of the set of symmetric matrices with corank two equals $\binom{n+1}{3}$ (see [43]), and hence Proposition 4 implies (3.3). The existence of a family attaining (3.3) is proved in Proposition 8.

3.1 The volume of the discriminant

The goal of this section is to prove Theorem 10. In fact this result will follow directly from Theorem 12 proved in Subsection 3.1.1 combined with Theorem 13 that we prove in Subsection 3.1.4.

3.1.1 Proof of Theorem 12

Let us denote by $\lambda_1(Q) \leq \cdots \leq \lambda_n(Q)$ the ordered eigenvalues of a symmetric matrix $Q \in \mathrm{Sym}(n, \mathbb{R})$ and let Δ_j, $j = 1, \ldots, n-1$ denote the set of $n \times n$ real symmetric matrices of unit norm, whose j-th and $(j+1)$-th eigenvalues are equal:

$$\Delta_j := \{Q \in S^{N-1} \mid \lambda_j(Q) = \lambda_{j+1}(Q)\}, \quad j = 1, \ldots, n-1$$

The sets $\Delta_j \subset S^{N-1}$, $j = 1, \ldots, n-1$ are semialgebraic and are of codimension two [7, 3]. The smooth locus $(\Delta_j)_{\mathrm{sm}}$ of Δ_j consists of matrices of unit norm whose j-th and $(j+1)$-th eigenvalues are equal and all other eigenvalues are of multiplicity one:

$$(\Delta_j)_{\mathrm{sm}} = \{Q \in S^{N-1} \mid \lambda_1(Q) < \cdots < \lambda_j(Q) = \lambda_{j+1}(Q) < \cdots < \lambda_n(Q)\}$$

Recall that Δ denotes the algebraic set of $n \times n$ real symmetric matrices of unit norm that have at least one repeated eigenvalue. Therefore Δ is the union of the sets Δ_j, $j = 1, \ldots, n-1$ and its smooth locus Δ_{sm} is a disjoint union of $(\Delta_j)_{\mathrm{sm}}$, $j = 1, \ldots, n-1$.

According to Proposition 5 (stated and proved in Subsection 3.1.2 below) we have

$$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{4}{Z_n}\binom{n}{2}\int_{u<\mu_1,\ldots,\mu_{n-2}} \prod_{i=1}^{n-2}(\mu_i - u)^2\, e^{-\frac{\|\mu\|^2}{2}-u^2}|\Delta(\mu)|\,\mathrm{d}(\mu,u).$$

Interpreting $\mu_1, \ldots, \mu_{n-2}$ as the eigenvalues of a $\mathrm{GOE}(n-2)$ matrix we can rewrite this as follows:

$$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{4Z_{n-2}}{Z_n}\binom{n}{2}\int_{u\in\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(n-2)}\Big[\mathbf{1}_{\{Q-u\mathbb{1}\succ 0\}}\,\det(Q-u\mathbb{1})^2\Big]e^{-u^2}\,\mathrm{d}u.$$

From (1.1) it's easy to see that $Z_n = 8\,\Gamma(\frac{n+1}{2})\Gamma(\frac{n+2}{2})\,Z_{n-2}$ or, using the duplication formula for Gamma function, $Z_n = 2^{-n+3}\sqrt{\pi}\,n!\,Z_{n-2}$. From this we get

$$\frac{|\Delta_1|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\,n!}\binom{n}{2}\int_{u\in\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(n-2)}\Big[\mathbf{1}_{\{Q-u\mathbb{1}\succ 0\}}\,\det(Q-u\mathbb{1})^2\Big]e^{-u^2}\,\mathrm{d}u. \quad (3.7)$$

which proves Theorem 12 (1).

For Theorem 12 (2) note that since $\Delta_{\mathrm{sm}} = \cup_{j=1}^{n-1}(\Delta_j)_{\mathrm{sm}}$ is a disjoint union we have that $|\Delta| = \sum_{j=1}^{n-1}|\Delta_j|$ and hence, by Proposition 5,

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{4}{Z_n}\binom{n}{2}\sum_{j=1}^{n-1}\binom{n-2}{j-1}\int_{\substack{\mu_1,\ldots,\mu_{j-1}<u\\ u<\mu_j,\ldots,\mu_{n-2}}} \prod_{i=1}^{n-2}(\mu_i - u)^2\, e^{-\frac{\|\mu\|^2}{2}-u^2}|\Delta(\mu)|\,\mathrm{d}(\mu,u),$$

This, together with the summation lemma [20, Lemma E.3.5], gives

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{4}{Z_n}\binom{n}{2}\int_{u\in\mathbb{R}}\int_{\mu\in\mathbb{R}^{n-2}} \prod_{i=1}^{n-2}(\mu_i - u)^2\, e^{-\frac{\|\mu\|^2}{2}-u^2}|\Delta(\mu)|\,\mathrm{d}\mu\mathrm{d}u,$$

Again, treating $\mu_1, \ldots, \mu_{n-2}$ as the eigenvalues of a $\mathrm{GOE}(n-2)$ matrix and then proceeding as we did to get (3.7) we obtain

$$\frac{|\Delta|}{|S^{N-3}|} = \frac{2^{n-1}}{\sqrt{\pi}\,n!}\binom{n}{2}\int_{u\in\mathbb{R}} \mathbb{E}_{\,Q\sim\mathrm{GOE}(n-2)}\Big[\det(Q-u\mathbb{1})^2\Big]e^{-u^2}\,\mathrm{d}u.$$

This proves Theorem 12 (2).

3.1.2 Volumes of Δ_j's

The following proposition describes for any $j = 1, \ldots, n-1$ the volume of the semialgebraic set $\Delta_j \subset S^{N-1}$ of symmetric matrices of unit norm whose j-th and $(j+1)$-th eigenvalues coincide.

Proposition 5. *Let $1 \leq j < n$. Then*

$$\frac{|\Delta_j|}{|S^{N-3}|} = \frac{4}{Z_n} \binom{n}{2} \binom{n-2}{j-1} \int_{\substack{\mu_1,\dots,\mu_{j-1}<u \\ u<\mu_j,\dots,\mu_{n-2}}} \prod_{i=1}^{n-2} (\mu_i - u)^2 \, e^{-\frac{\|\mu\|^2}{2} - u^2} |\Delta(\mu)| \, \mathrm{d}(\mu, u),$$

Proof. Recall that

$$(\Delta_j)_{\mathrm{sm}} = \{Q \in S^{N-1} \mid \lambda_1(Q) < \cdots < \lambda_j(Q) = \lambda_{j+1}(Q) < \cdots < \lambda_n(Q)\}.$$

In the following, we denote for brevity $\lambda_i := \lambda_i(Q)$. In order to compute $|\Delta_j| = |(\Delta_j)_{\mathrm{sm}}|$ define for $\delta > 0$

$$K_j(\delta) := \{Q \in (\Delta_j)_{\mathrm{sm}} \mid \lambda_j - \lambda_{j-1} > \delta,\ \lambda_{j+2} - \lambda_{j+1} > \delta\} \tag{3.8}$$

Then $(\Delta_j)_{\mathrm{sm}} = \bigcup_{\delta>0} K_j(\delta)$ and by continuity of the Lebesgue measure

$$|\Delta_j| = \lim_{\delta\to 0} |K_j(\delta)|. \tag{3.9}$$

For a fixed $\delta > 0$ and for any $\varepsilon > 0$ let $T^\perp(K_j(\delta), \varepsilon) \subset S^{N-1}$ denote the ε-tube around $K_j(\delta) \subset S^{N-1}$. Weyl's formula [88] gives the expansion of the volume of the ε-tube around a submanifold of the sphere. Here it is enough to have it in the following simplified form.

Theorem 14 (Weyl's tube formula for $K_j(\delta)$)**.** *For any $\varepsilon > 0$, such that the fibres of $T^\perp(K_j(\delta), \varepsilon)$ do not intersect, the volume of the ε-tube around $K_j(\delta)$ is $|T^\perp(K_j(\delta), \varepsilon)| = \pi\varepsilon^2 |K_j(\delta)| + \mathcal{O}(\varepsilon^3)$.*

In Lemma 4 (stated and proved in Subsection 3.1.3 below) we describe the ε-tube around $K_j(\delta)$ and show that for a sufficiently small $\varepsilon > 0$ its fibers do not intersect. Combining this lemma with Weyl's formula we are allowed to compute the volume of $K_j(\delta)$ as $|K_j(\delta)| = \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} |T^\perp(K_j(\delta), \varepsilon)|$ and, consequently, by (3.9):

$$|\Delta_j| = \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \, |T^\perp(K_j(\delta), \varepsilon)|. \tag{3.10}$$

To actually compute this limit, we rewrite the volume of $T^\perp(K_j(\delta), \varepsilon)$ in terms of a GOE(n) random variable as we now explain. Applying (1.2) to the measurable set $T^\perp(K_j(\delta), \varepsilon) \subset S^{N-1}$ we obtain

$$\frac{|T^\perp_{S^{N-1}}(K_j(\delta), \varepsilon)|}{|S^{N-1}|} = \mathop{\mathbb{P}}_{Q\sim \mathrm{GOE}(n)} \left\{ \begin{array}{l} \lambda_1 < \cdots < \lambda_j \leq \lambda_{j+1} < \cdots < \lambda_n, \\ \lambda_{j+1} - \lambda_j < \sqrt{2}\|Q\| \sin\varepsilon, \\ \lambda_{j+2} - \dfrac{\lambda_j + \lambda_{j+1}}{2} > \delta \|Q\| \cos\varepsilon \\ \dfrac{\lambda_j + \lambda_{j+1}}{2} - \lambda_{j-1} > \delta \|Q\| \cos\varepsilon \end{array} \right\} = (\star) \tag{3.11}$$

In the following we denote the event

$$E(\lambda) := \left\{ \begin{array}{l} \lambda_1 < \cdots < \lambda_j \le \lambda_{j+1} < \cdots < \lambda_n, \\ \lambda_{j+1} - \lambda_j < \sqrt{2}\|Q\| \sin\varepsilon, \\ \lambda_{j+2} - \dfrac{\lambda_j + \lambda_{j+1}}{2} > \delta\|Q\| \cos\varepsilon \\ \dfrac{\lambda_j + \lambda_{j+1}}{2} - \lambda_{j-1} > \delta\|Q\| \cos\varepsilon \end{array} \right\}.$$

The probability (3.11) of the event $E(\lambda)$, written in terms of the density of eigenvalues of the GOE(n) ensemble, becomes

$$(\star) = \frac{n!}{Z_n} \int_{\mathbb{R}^n} \mathbf{1}_{E(\lambda)}\, e^{-\frac{\|\lambda\|^2}{2}} |\Delta(\lambda_1, \ldots, \lambda_n)|\, \mathrm{d}\lambda,$$

where $\mathbf{1}_{E(\lambda)}$ denotes the characteristic function of $E(\lambda)$ and the factor $n!$ appears since the eigenvalues are taken to be ordered. We express now the integral in terms of the following event:

$$\tilde{E}(\lambda) := \left\{ \begin{array}{l} \lambda_1, \ldots, \lambda_{j-1} < \lambda_j, \lambda_{j+1} < \lambda_{j+1}, \ldots, \lambda_n, \\ |\lambda_{j+1} - \lambda_j| < \sqrt{2}\|\lambda\| \sin\varepsilon, \\ \lambda_i - \dfrac{\lambda_j + \lambda_{j+1}}{2} > \delta\|\lambda\| \cos\varepsilon \ \text{ for } i \ge j+2, \\ \dfrac{\lambda_j + \lambda_{j+1}}{2} - \lambda_i > \delta\|\lambda\| \cos\varepsilon \ \text{ for } i \le j-1 \end{array} \right\}.$$

There are $(j-1)!$ possibilities to arrange the first $j-1$ eigenvalues, 2 possibilities to arrange λ_j and λ_{j+1} and $(n-(j+1))!$ possibilities to arrange the last $n-(j+1)$ eigenvalues. Hence,

$$\begin{aligned} (\star) &= \frac{n!}{Z_n} \frac{1}{2(j-1)!(n-(j+1))!} \int_{\mathbb{R}^n} \mathbf{1}_{\tilde{E}(\lambda)}\, e^{-\frac{\|\lambda\|^2}{2}} |\Delta(\lambda_1, \ldots, \lambda_n)|\, \mathrm{d}\lambda \\ &= \frac{1}{Z_n} \binom{n}{2} \binom{n-2}{j-1} \int_{\mathbb{R}^n} \mathbf{1}_{\tilde{E}(\lambda)}\, e^{-\frac{\|\lambda\|^2}{2}} |\Delta(\lambda_1, \ldots, \lambda_n)|\, \mathrm{d}\lambda \end{aligned}$$

Next, we perform the following orthogonal change of variables

$$\begin{aligned} &\mu_1 := \lambda_1, \ \ldots, \ \mu_{j-1} := \lambda_{j-1}, \ \mu_j := \lambda_{j+2}, \ \ldots, \ \mu_{n-2} := \lambda_n \text{ and} \\ &x = \frac{\lambda_j + \lambda_{j+1}}{\sqrt{2}}, \ y = \frac{\lambda_{j+1} - \lambda_j}{\sqrt{2}} \end{aligned}$$

($\mu_1, \ldots, \mu_{n-2}$ now become the eigenvalues of a new GOE($n-2$) matrix and we treat the variables x, y separately). We get

$$(\star) = \tfrac{1}{Z_n}\tbinom{n}{2}\tbinom{n-2}{j-1} \int_{(\mu,x,y)\in\mathbb{R}^{n-2}\times\mathbb{R}\times\mathbb{R}} \mathbf{1}_{\widehat{E}(\mu,x,y)} g(\mu,x,y) e^{-\frac{\|\mu\|^2+y^2+x^2}{2}} |\Delta(\mu)|\, (\mu,x,y), \tag{3.12}$$

where

$$g(\mu, x, y) = \sqrt{2}|y| \prod_{i=1}^{n-2} \left(\left(\mu_i - \frac{x}{\sqrt{2}} \right)^2 - \frac{y^2}{2} \right)$$

and

$$\widehat{E}(\mu, x, y) := \left\{ \begin{array}{l} \mu_1, \dots, \mu_{j-1} < \frac{1}{\sqrt{2}}(x - y), \frac{1}{\sqrt{2}}(x + y) < \mu_j, \dots, \mu_{n-2}, \\ |y| < \|(\mu, x, y)\| \sin \varepsilon, \\ \mu_i - \dfrac{x}{\sqrt{2}} > \delta \|(\mu, x, y)\| \cos \varepsilon \;\; \text{for } i \geq j, \\ \dfrac{x}{\sqrt{2}} - \mu_i > \delta \|(\mu, x, y)\| \cos \varepsilon \;\; \text{for } i \leq j - 1 \end{array} \right\}.$$

We perform another change of varables:

$$\begin{array}{ll} t = \frac{y}{\sin \varepsilon \, \|(\mu, x, y)\|} & \mathrm{d}y = \dfrac{\sin \varepsilon \|(\mu, x)\|}{(1 - (\sin \varepsilon)^2 t^2)^{3/2}} \\ x, \mu_1, \dots, \mu_{n-2} & \text{are as before} \end{array}$$

Note that after this change a factor of $(\sin \varepsilon)^2$ appears and the function $y(t, x, \mu, \varepsilon) \to 0$ in the limits $\varepsilon \to 0$. We multiply the integral in (3.12) by $\frac{1}{\pi \varepsilon^2}$ and, thereafter, invoke the dominated convergence theorem that allows us to pass to the limit $\varepsilon \to 0$ under the integral:

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{(\mu, x, y) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}} \mathbf{1}_{\widehat{E}(\mu, x, y)} g(\mu, x, y) e^{-\frac{\|\mu\|^2 + y^2 + x^2}{2}} |\Delta(\mu)| \, \mathrm{d}(\mu, x, y) \\ =& \frac{\sqrt{2}}{\pi} \int_{(\mu, x) \in \mathbb{R}^{n-2} \times \mathbb{R}} \int_{t=-1}^{1} \mathbf{1}_{\bar{E}(\mu, x)} |t| \|\mu, x\|^2 \prod_{i=1}^{n-2} \left(\mu_i - \frac{x}{\sqrt{2}} \right)^2 e^{-\frac{\|\mu\|^2 + x^2}{2}} |\Delta(\mu)| \, \mathrm{d}t \, \mathrm{d}(\mu, x), \end{aligned}$$

where

$$\bar{E}(\mu, x) := \left\{ \begin{array}{l} \mu_1, \dots, \mu_{j-1} < \frac{x}{\sqrt{2}} < \mu_j, \dots, \mu_{n-2}, \\ \mu_i - \dfrac{x}{\sqrt{2}} > \delta \|(\mu, x)\| \;\; \text{for } i \geq j, \\ \dfrac{x}{\sqrt{2}} - \mu_i > \delta \|(\mu, x)\| \;\; \text{for } i \leq j - 1 \end{array} \right\}.$$

Using that $\int_{t=-1}^{1} |t| \, \mathrm{d}t = 1$ we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} |T^{\perp}(K_j(\delta), \varepsilon)| = \\ &\tfrac{\sqrt{2}|S^{N-1}|}{\pi Z_n} \tbinom{n}{2} \tbinom{n-2}{j-1} \textstyle\int_{\mathbb{R}^{n-2} \times \mathbb{R}} \mathbf{1}_{\bar{E}(\mu, x)} \|\mu, x\|^2 \displaystyle\prod_{i=1}^{n-2} \left(\mu_i - \tfrac{x}{\sqrt{2}} \right)^2 e^{-\frac{\|\mu\|^2 + x^2}{2}} |\Delta(\mu)| \, \mathrm{d}(\mu, x), \end{aligned}$$

Plugging this into (3.10) and again using the dominated convergence theorem we get

$$
\begin{aligned}
|\Delta_j| &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\,|T^{\perp}(K_j(\delta),\varepsilon)| \\
&= \frac{\sqrt{2}|S^{N-1}|}{\pi Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_D \|\mu,x\|^2\prod_{i=1}^{n-2}\left(\mu_i-\frac{x}{\sqrt{2}}\right)^2 e^{-\frac{\|\mu\|^2+x^2}{2}}|\Delta(\mu)|\,\mathrm{d}(\mu,x),
\end{aligned}
$$

where the region of integration is $D=\{\mu_1,\dots,\mu_{j-1}<\frac{x}{\sqrt{2}}<\mu_j,\dots,\mu_{n-2}\}$. Now for a measurable positively homogeneous function $f:\mathbb{R}^m\to\mathbb{R}$ of degree d we have by Lemma 13 from the appendix:

$$
\int_{x\in\mathbb{R}^m}\|x\|^2 f(x)e^{-\frac{\|x\|^2}{2}}\,\mathrm{d}x=(d+m)\int_{x\in\mathbb{R}^m} f(x)e^{-\frac{\|x\|^2}{2}}\,\mathrm{d}x
$$

In our case, we have $m=n-1$ and $d=2(n-2)+\frac{(n-2)(n-3)}{2}=\frac{(n-2)(n+1)}{2}$. Thus $d+m=\frac{n^2+n-4}{2}$ and

$$
|\Delta_j|=\frac{(n^2+n-4)|S^{N-1}|}{\sqrt{2}\,\pi Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_D\prod_{i=1}^{n-2}\left(\mu_i-\frac{x}{\sqrt{2}}\right)^2 e^{-\frac{\|\mu\|^2+x^2}{2}}|\Delta(\mu)|\,\mathrm{d}(\mu,x).
$$

Finally, we make a change of variables $u:=\frac{x}{\sqrt{2}}$ and use $(n^2+n-4)|S^{N-1}|=4\pi|S^{N-3}|$ to conclude that

$$
\frac{|\Delta_j|}{|S^{N-3}|}=\frac{4}{Z_n}\binom{n}{2}\binom{n-2}{j-1}\int_{\substack{\mu_1,\dots,\mu_{j-1}<u\\ u<\mu_j,\dots,\mu_{n-2}}}\prod_{i=1}^{n-2}(\mu_i-u)^2\,e^{-\frac{\|\mu\|^2}{2}-u^2}|\Delta(\mu)|\,\mathrm{d}(\mu,u).
$$

This completes the proof of Proposition 5. □

3.1.3 Description of the normal tube around $K_j(\delta)$

In the following lemma an explicit description of the normal tube around the smooth semialgebraic set $K_j(\delta)\subset S^{N-1}$ is obtained.

Lemma 4. *For $0<\varepsilon<\arctan(\sqrt{2}\delta)$ we have*

$$
T^{\perp}(K_j(\delta),\varepsilon)=\left\{Q\in S^{N-1}\;\middle|\;\begin{array}{l}\lambda_1<\cdots<\lambda_j\le\lambda_{j+1}<\cdots<\lambda_n,\\ \lambda_{j+1}-\lambda_j<\sqrt{2}\sin\varepsilon,\\ \lambda_{j+2}-\dfrac{\lambda_j+\lambda_{j+1}}{2}>\delta\cos\varepsilon,\ \dfrac{\lambda_j+\lambda_{j+1}}{2}-\lambda_{j-1}>\delta\cos\varepsilon\end{array}\right\}
$$

and the fibers of $T^{\perp}(K_j(\delta),\varepsilon)$ do not intersect.

Proof. We can assume without loss of generality that $Q \in K_j(\delta)$ is diagonal: $Q = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\lambda_1 < \cdots < \lambda_j = \lambda_{j+1} < \cdots < \lambda_n$. Then, the fiber $N_Q \subset T_Q S^{N-1}$ of the normal bundle to $K_j(\delta) \subset S^{N-1}$ at Q is described as follows. For $a, b \in \mathbb{R}$ let $V_{a,b} = (v_{i,j}) \in \operatorname{Sym}(n, \mathbb{R})$ be the matrix that has zeros everywhere except for the following block on the diagonal: $\left(\begin{smallmatrix} v_{j,j} & v_{j,j+1} \\ v_{j+1,j} & v_{j+1,j+1} \end{smallmatrix}\right) = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$. Note that $V_{a,b} \in S^{N-1}$ if and only if $a^2 + b^2 = 1$. We claim that

$$N_Q = \{V_{a,b} \mid a, b \in \mathbb{R}\}.$$

It is easy to see that $V_{a,b}$ is orthogonal to Q, i.e., $V_{a,b} \in T_Q S^{N-1}$, and that the tangent space $T_Q K_j(\delta) \subset T_Q S^{N-1}$ to $K_j(\delta)$ at Q is spanned by the following $\binom{n+1}{2} - 3$ vectors:

$$\begin{aligned}
&\operatorname{diag}(\lambda_2 e_1 - \lambda_1 e_2),\\
&\qquad\vdots\\
&\operatorname{diag}(\lambda_{j-1} e_{j-2} - \lambda_{j-2} e_{j-1}),\\
&\operatorname{diag}(\lambda_{j+2} e_{j-1} - \lambda_{j-1} e_{j+2}),\\
&\operatorname{diag}(\lambda_{j+3} e_{j+2} - \lambda_{j+2} e_{j+3}),\\
&\qquad\vdots\\
&\operatorname{diag}(\lambda_n e_{n-1} - \lambda_{n-1} e_n)
\end{aligned}$$

and

$$\operatorname{diag}\Big(-2\lambda_j \sum_{i \neq j, j+1} \lambda_i e_i + \sum_{i \neq j, j+1} \lambda_i^2 (e_j + e_{j+1})\Big)$$

and

$$\begin{array}{c}
\begin{array}{cc} & \begin{array}{ccccc} \quad & r & & s & \end{array} \end{array}\\
\begin{array}{c} \\ r \\ \\ s \\ \\ \end{array}
\begin{pmatrix}
 & \vdots & & \vdots & \\
\dots & 0 & \dots & \lambda_s - \lambda_r & \dots \\
 & \vdots & & \vdots & \\
\dots & \lambda_s - \lambda_r & \dots & 0 & \dots \\
 & \vdots & & \vdots &
\end{pmatrix}
\end{array}, \; r, s = 1, \dots n, r \neq s, \{r, s\} \neq \{j, j+1\}.$$

It is immediate to see that these vectors are all orthogonal to $V_{a,b}$. Thus, $N_Q = \{V_{a,b} \,|\, a, b \in \mathbb{R}\}$.

Now we prove that $T^{\perp}(K_j(\delta), \varepsilon)$ has the asserted form and that the fibers of the normal ε-tube $T^{\perp}(K_j(\delta), \varepsilon)$ do not intersect provided that $\varepsilon < \arctan(\sqrt{2}\delta)$. The fibers are swept out by geodesics of length less than ε starting at Q in the direction of some $V_{a,b} \in S^{N-1}$, in formulas: $\{\cos t\, Q + \sin t\, V_{a,b} \mid 0 \le t < \varepsilon\}$. We write explicitly the matrix $\cos t\, Q + \sin t\, V_{a,b}$:

$$\begin{pmatrix}
\lambda_1 \cos t & & & & & & & \\
& \ddots & & & & & & \\
& & \lambda_{j-1} \cos t & & & & & \\
& & & \lambda_j \cos t + \frac{a}{\sqrt{2}} \sin t & \frac{b}{\sqrt{2}} \sin t & & & \\
& & & \frac{b}{\sqrt{2}} \sin t & \lambda_{j+1} \cos t - \frac{a}{\sqrt{2}} \sin t & & & \\
& & & & & \lambda_{j+2} \cos t & & \\
& & & & & & \ddots & \\
& & & & & & & \lambda_n \cos t
\end{pmatrix}$$

Provided that $\varepsilon < \arctan(\sqrt{2}\delta)$ the eigenvalues of this matrix are

$$\lambda_1 \cos t < \cdots < \lambda_{j-1} \cos t < \lambda_j \cos t \pm \frac{\sin t}{\sqrt{2}} < \lambda_{j+2} \cos t < \cdots < \lambda_n \cos t \tag{3.13}$$

since $Q \in K_j(\delta)$ (see (3.8)). Moreover, for $0 \le t < \varepsilon$ these eigenvalues satisfy the inequalities

$$\frac{(\lambda_j \cos t + \frac{\sin t}{\sqrt{2}}) + (\lambda_j \cos t - \frac{\sin t}{\sqrt{2}})}{2} - \lambda_{j-1} \cos t = (\lambda_j - \lambda_{j-1}) \cos t > \delta \cos \varepsilon,$$
$$(\lambda_j \cos t + \frac{\sin t}{\sqrt{2}}) - (\lambda_j \cos t - \frac{\sin t}{\sqrt{2}}) = \sqrt{2} \sin t < \sqrt{2} \sin \varepsilon,$$
$$\lambda_{j+2} \cos t - \frac{(\lambda_{j+1} \cos t + \frac{\sin t}{\sqrt{2}}) + (\lambda_{j+1} \cos t - \frac{\sin t}{\sqrt{2}})}{2} = (\lambda_{j+2} - \lambda_{j+1}) \cos t > \delta \cos \varepsilon$$

This shows that $T^{\perp}(K_j(\delta), \varepsilon)$ is contained in the set we claim it to be. To show the other inclusion let $A \in S^{N-1}$ be a matrix whose eigenvalues $\alpha_1 < \cdots < \alpha_{j-1} < \alpha_j \le \alpha_{j+1} < \alpha_{j+2} < \cdots < \alpha_n$ satisfy

$$\begin{aligned}
\frac{\alpha_j + \alpha_{j+1}}{2} - \alpha_{j-1} &> \delta \cos \varepsilon, \\
\alpha_{j+1} - \alpha_j &< \sqrt{2} \sin \varepsilon, \\
\text{and} \quad \alpha_{j+2} - \frac{\alpha_j + \alpha_{j+1}}{2} &> \delta \cos \varepsilon.
\end{aligned}$$

We can assume again that $A = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is diagonal. Let $0 \le t < \varepsilon$ be such that $\alpha_{j+1} - \alpha_j = \sqrt{2} \sin t$. One can easily verify that $A = \cos t\, Q + \sin t\, V_{-1,0}$ for

$$Q = \frac{1}{\cos t} \operatorname{diag}(\alpha_1, \ldots, \alpha_{j-1}, \tfrac{1}{2}(\alpha_j + \alpha_{j+1}), \tfrac{1}{2}(\alpha_j + \alpha_{j+1}), \alpha_{j+2}, \ldots, \alpha_n) \in K_j(\delta)$$

This implies that $A \in T^{\perp}(K_j(\delta), \varepsilon)$ and $T^{\perp}(K_j(\delta), \varepsilon)$ has the claimed form.

It remains to show that the fibers of the normal ε-tube $T^{\perp}(K_j(\delta), \varepsilon)$ do not intersect when $\varepsilon < \arctan(\sqrt{2}\delta)$. For this assume there is another representation $A = \cos\tilde{t}\, Q_0 + \sin\tilde{t}\, V$ of the matrix $A = \mathrm{diag}(\alpha_1, \dots, \alpha_n) \in T^{\perp}(K_j(\delta), \varepsilon)$, where $Q_0 \in K_j(\delta)$, $V \in N_{Q_0}$ and $0 \le \tilde{t} < \varepsilon$. We will prove that actually $Q_0 = Q, V = V_{-1,0}$ and $\tilde{t} = t$. To show this, we consider the diagonalization of Q_0; that is, $Q_0 = C_1^T Q_1 C_1$, where $Q_1 = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is diagonal and C_1 is orthogonal. We may assume $\lambda_1 < \cdots < \lambda_{j-1} < \lambda_j = \lambda_{j+1} < \lambda_{j+2} < \cdots < \lambda_n$. Note that the normal bundle N_{Q_0} to $K_j(\delta)$ at $Q_0 = C_1^T Q_1 C_1$ is given by $N_{C_1^T Q_1 C_1} = C_1^T N_{Q_1} C_1 = \{C_1^T V_{a,b} C_1 \mid a, b \in \mathbb{R}\}$. It follows that $V = C_1^T V_{a,b} C_1$ for some $a, b \in \mathbb{R}$ and we can write $A = C_1^T(\cos\tilde{t}\, Q_1 + \sin\tilde{t}\, V_{a,b})C_1$. Note that the eigenvalues of the inner matrix are given as in (3.13). Therefore, we can write $A = C_1^T C_2^T Q_2 C_2 C_1$, where the orthogonal matrix C_2 commutes with Q_1 and

$$Q_2 = \mathrm{diag}\left(\lambda_1 \cos\tilde{t}, \dots, \lambda_{j-1}\cos\tilde{t}, \lambda_j \cos\tilde{t} - \tfrac{\sin\tilde{t}}{\sqrt{2}}, \lambda_j \cos\tilde{t} + \tfrac{\sin\tilde{t}}{\sqrt{2}}, \lambda_{j+2}\cos\tilde{t}, \dots, \lambda_n \cos\tilde{t}\right)$$

$$= \cos\tilde{t}\, Q_1 + \sin\tilde{t}\, V_{-1,0}$$

The condition $\varepsilon < \arctan(\sqrt{2}\delta)$ together with $Q_1 \in K_j(\delta)$ ensures $\lambda_{j-1}\cos\tilde{t} < \lambda_j \cos\tilde{t} - \frac{\sin\tilde{t}}{\sqrt{2}}$ and $\lambda_j \cos\tilde{t} + \frac{\sin\tilde{t}}{\sqrt{2}} < \lambda_{j+2}\cos\tilde{t}$. Now since the diagonal matrices A and Q_2 both have ordered entries it follows that $C_2 C_1$ can be taken to be the identity matrix. Therefore $\alpha_i = \lambda_i \cos\tilde{t}$ for $i = 1, \dots, j-1, j+2, \dots, n$, and $\alpha_j = \lambda_j \cos\tilde{t} - \frac{\sin\tilde{t}}{\sqrt{2}}$ and $\alpha_{j+1} = \lambda_j \cos\tilde{t} + \frac{\sin\tilde{t}}{\sqrt{2}}$. It is straightforward now to see that $\tilde{t} = t$, $Q_0 = Q$ and $V = V_{-1,0}$. □

3.1.4 Proof of Theorem 13

In this section we give a proof of Theorem 13. But first we recall some classical facts about Hermite polynomials and prove few auxiliary results.

Lemma 5. *Let $P_m = 2^{1-m^2}\sqrt{\pi}^m \prod_{i=0}^m (2i)!$ and let Z_{2m} be the normalization constant from* (1.1)*. Then $P_m = 2^{1-2m} Z_{2m}$.*

Proof. The formula (1.1) for Z_{2m} reads

$$Z_{2m} = \sqrt{2\pi}^{2m} \prod_{i=1}^{2m} \frac{\Gamma\left(\frac{i}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} = (2\pi)^m \prod_{i=1}^{m} \frac{\Gamma\left(\frac{2i-1}{2}+1\right)\Gamma\left(\frac{2i}{2}+1\right)}{\left(\frac{\sqrt{\pi}}{2}\right)^2}.$$

Using the formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}2^{1-2z}\Gamma(2z)$ [80, 43:5:7] with $z = i + 1/2$ we obtain

$$Z_{2m} = 2^{3m} \prod_{i=1}^{m} \sqrt{\pi} 2^{1-2(i+1/2)}\Gamma(2(i+1/2)) = 2^{2m-m^2}\sqrt{\pi}^m \prod_{i=1}^{m}(2i)! = 2^{2m-1}P_m$$

$\square$

Recall now that the *(physicist's) Hermite polynomials* $H_i(x)$, $i = 0, 1, 2, \ldots$ form a family of orthogonal polynomials on the real line with respect to the measure $e^{-x^2}dx$. They are defined by

$$H_i(x) = (-1)^i e^{x^2} \frac{\mathrm{d}^i}{\mathrm{d}x^i} e^{-x^2}, \quad i \geq 0$$

and satisfy

$$\int_{u\in\mathbb{R}} H_i(u)H_j(u)e^{-u^2}\,\mathrm{d}u = \begin{cases} 2^i i!\sqrt{\pi}, & \text{if } i = j \\ 0, & \text{else.} \end{cases} \tag{3.14}$$

A Hermite polynomial is either odd (if the degree is odd) or even (if the degree is even) function:

$$H_i(-x) = (-1)^i H_i(x); \tag{3.15}$$

and its derivative satisfies

$$H_i'(x) = 2iH_{i-1}(x) \tag{3.16}$$

(see [80, (24:5:1)], [40, (8.952.1)] for these properties).

The following proposition is crucial for the proof of Theorem 13.

Proposition 6 (Expected value of the square of the characteristic polynomial)**.** *For a fixed positive integer k and a fixed $u \in \mathbb{R}$ the following holds.*

1. *If $k = 2m$ is even, then*

$$\mathop{\mathbb{E}}_{Q\sim\mathrm{GOE}(k)} \det(Q - u\mathbb{1})^2 = \frac{(2m)!}{2^{2m}} \sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \det X_j(u),$$

where

$$X_j(u) = \begin{pmatrix} H_{2j}(u) & H_{2j}'(u) \\ H_{2j+1}(u) - H_{2j}'(u) & H_{2j+1}'(u) - H_{2j}''(u) \end{pmatrix}.$$

2. *If $k = 2m+1$ is odd, then*

$$\mathop{\mathbb{E}}_{Q\sim\mathrm{GOE}(k)} \det(Q - u\mathbb{1})^2 = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \det Y_j(u),$$

where

$$Y_j(u) = \begin{pmatrix} \frac{(2j)!}{j!} & H_{2j}(u) & H_{2j}'(u) \\ 0 & H_{2j+1}(u) - H_{2j}'(u) & H_{2j+1}'(u) - H_{2j}''(u) \\ \frac{(2m+2)!}{(m+1)!} & H_{2m+2}(u) & H_{2m+2}'(u) \end{pmatrix}.$$

Proof. In Section 22 of [63] one finds two different formulas for the even $k = 2m$ and odd $k = 2m+1$ cases.

If $k = 2m$, we have by [63, (22.2.38)] that

$$\mathbb{E}\,\det(Q - u\mathbb{1})^2 = \frac{(2m)!\,P_m}{Z_{2m}} \sum_{j=0}^{m} \frac{2^{2j-1}}{(2j)!}\,\det\begin{pmatrix} R_{2j}(u) & R'_{2j}(u) \\ R_{2j+1}(u) & R'_{2j+1}(u) \end{pmatrix},$$

where $P_m = 2^{1-m^2}\sqrt{\pi}^m \prod_{i=0}^{m}(2i)!$ is as in Lemma 5, Z_{2m} is the normalization constant (1.1) and where $R_{2j}(u) = 2^{-2j}H_{2j}(u)$ and $R_{2j+1}(u) = 2^{-(2j+1)}(H_{2j+1}(u) - H'_{2j}(u))$. Using the multilinearity of the determinant we get

$$\mathbb{E}\,\det(Q - u\mathbb{1})^2 = \frac{(2m)!\,P_m}{Z_{2m}} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!}\,\det X_j(u).$$

By Lemma 5 we have $\frac{P_m}{Z_{2m}} = 2^{1-2m}$. Putting everything together yields the first claim.

In the case $k = 2m+1$ we get from [63, (22.2.39)] that

$$\mathbb{E}\,\det(Q - u\mathbb{1})^2 = \frac{(2m+1)!\,P_m}{Z_{2m+1}} \sum_{j=0}^{m} \frac{2^{2j-1}}{(2j)!}\,\det\begin{pmatrix} g_{2j} & R_{2j}(u) & R'_{2j}(u) \\ g_{2j+1} & R_{2j+1}(u) & R'_{2j+1}(u) \\ g_{2m+2} & R_{2m+2}(u) & R'_{2m+2}(u) \end{pmatrix},$$

where P_m, $R_{2j}(u), R_{2j+1}(u)$ are as above and

$$g_i = \int_{u\in\mathbb{R}} R_i(u)\exp(-\tfrac{u^2}{2})\,\mathrm{d}u.$$

Note that by (3.15) $H_{2j+1}(u)$ is an odd function. Hence, we have $g_{2j+1} = 0$. For even indices we use [40, (7.373.2)] to get $g_{2j} = 2^{-2j}\sqrt{2\pi}\frac{(2j)!}{j!}$. By the multilinearity of the determinant:

$$\mathbb{E}\,\det(Q - u\mathbb{1})^2 = \frac{\sqrt{2\pi}(2m+1)!\,P_m}{2^{2m+2}Z_{2m+1}} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!}\,\det Y_j(u). \tag{3.17}$$

From (1.1) one obtains $Z_{2m+1} = 2\sqrt{2}\,\Gamma(m+\frac{3}{2})\,Z_{2m}$, which together with Lemma 5 implies

$$\frac{P_m}{Z_{2m+1}} = \frac{2^{-2m}}{\sqrt{2}\,\Gamma(m+\frac{3}{2})}.$$

Plugging this into (3.17) we conlude that

$$\mathbb{E}\,\det(Q - u\mathbb{1})^2 = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!}\,\det Y_j(u).$$

$\square$

Everything is now ready for the proof of Theorem 13

Proof of Theorem 13. Due to the nature of Proposition 6 we also have to make a distinction for this proof.

In the case $k = 2m$ we use the formula from Proposition 6 (1) to write

$$\int_{u\in\mathbb{R}} \mathbb{E}\ \det(Q - u\mathbb{1})^2 e^{-u^2}\,\mathrm{d}u = \frac{(2m)!}{2^{2m}} \sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \int_{u\in\mathbb{R}} \det X_j(u)\,\mathrm{d}u.$$

By (3.16) we have $H_i'(u) = 2iH_{i-1}(u)$. Hence, $X_j(u)$ can be written as

$$\begin{pmatrix} H_{2j}(u) & 4jH_{2j-1}(u) \\ H_{2j+1}(u) - 4jH_{2j-1}(u) & 2(2j+1)H_{2j}(u) - 8j(2j-1)H_{2j-2}(u) \end{pmatrix}.$$

From (3.14) we can deduce that

$$\begin{aligned}\int_{u\in\mathbb{R}} \det X_j(u)\,\mathrm{d}u &= 2(2j+1)2^{2j}(2j)!\sqrt{\pi} + 16j^2 2^{2j-1}(2j-1)!\sqrt{\pi} \\ &= 2^{2j+1}(2j)!\sqrt{\pi}(4j+1).\end{aligned}$$

From this we see that

$$\sum_{j=0}^{m} \frac{2^{-2j-1}}{(2j)!} \int_{u\in\mathbb{R}} \det X_j(u)\,\mathrm{d}u = \sqrt{\pi}\sum_{j=0}^{m}(4j+1) = \sqrt{\pi}\,(m+1)(2m+1). \tag{3.18}$$

and hence,

$$\int_{u\in\mathbb{R}} \mathbb{E}\ \det(Q - u\mathbb{1})^2 e^{-u^2}\,\mathrm{d}u = \frac{(2m)!}{2^{2m}}\sqrt{\pi}\,(m+1)(2m+1) = \frac{(2m+2)!}{2^{2m+1}}\sqrt{\pi}.$$

Plugging back in $m = \frac{k}{2}$ finishes the proof of the case $k = 2m$.

In the case $k = 2m+1$ we use the formula from Proposition 6 (2) to see that

$$\int_u \mathbb{E}\ \det(Q - u\mathbb{1})^2\, e^{-u^2}\,\mathrm{d}u = \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})} \sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \int_u \det Y_j(u)\, e^{-u^2}\,\mathrm{d}u.$$

Note that the top right 2×2-submatrix of $Y_j(u)$ is $X_j(u)$, so that $\det Y_j(u)$ equals

$$\frac{(2m+2)!}{(m+1)!}\det X_j(u) + \frac{(2j)!}{j!}\det\begin{pmatrix} H_{2j+1}(u) - H_{2j}'(u) & H_{2j+1}'(u) - H_{2j}''(u) \\ H_{2m+2}(u) & H_{2m+2}'(u)\end{pmatrix} \tag{3.19}$$

Because taking derivatives of Hermite polynomials decreases the index by one (3.16) and because the integral over a product of two Hermite polynomials is only non-vanishing, if their indices agree, the integral of the determinant in (3.19) is only non-vanishing for $j = m$, in which case it is equal to

$$\int_{u\in\mathbb{R}} H_{2m+1}(u)H'_{2m+2}(u)\, e^{-u^2}\,\mathrm{d}u = 2(2m+2)2^{2m+1}(2m+1)!\,\sqrt{\pi},$$

by (3.14) and (3.16). Hence,

$$\begin{aligned}
&\int_{u\in\mathbb{R}} \det Y_j(u)\, e^{-u^2}\,\mathrm{d}u \\
=&\begin{cases} \frac{(2m+2)!}{(m+1)!} \int_{u\in\mathbb{R}} \det X_m(u)\, e^{-u^2}\,\mathrm{d}u + \frac{(2m)!}{m!}\, 2^{2m+2}(2m+2)!\sqrt{\pi}, & \text{if } j = m,\\ \frac{(2m+2)!}{(m+1)!} \int_{u\in\mathbb{R}} \det X_j(u)\, e^{-u^2}\,\mathrm{d}u, & \text{else.}\end{cases}
\end{aligned}$$

We find that

$$\begin{aligned}
&\sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \int_u \det Y_j(u)\, e^{-u^2}\,\mathrm{d}u \\
=&\frac{(2m+2)!}{m!}\sqrt{\pi} + \frac{(2m+2)!}{(m+1)!}\sum_{j=0}^{m} \frac{2^{-2j-2}}{(2j)!} \int_u \det X_j(u)\, e^{-u^2}\,\mathrm{d}u \\
=&\frac{(2m+2)!}{m!}\sqrt{\pi} + \frac{(2m+2)!}{(m+1)!}\,\frac{\sqrt{\pi}}{2}\,(m+1)(2m+1) \\
=&\frac{\sqrt{\pi}}{2}\,\frac{(2m+3)!}{m!};
\end{aligned}$$

the second-to-last line by (3.18). It follows that

$$\begin{aligned}
\int_{u\in\mathbb{R}} \mathbb{E}\,\det(Q-u\mathbb{1})^2\, e^{-u^2}\,\mathrm{d}u &= \frac{\sqrt{\pi}(2m+1)!}{2^{4m+2}\,\Gamma(m+\frac{3}{2})}\,\frac{\sqrt{\pi}}{2}\,\frac{(2m+3)!}{m!}\\
&= \frac{\pi(2m+1)!(2m+3)!}{2^{4m+3}\,\Gamma(m+\frac{3}{2})\,m!}.
\end{aligned}$$

It is not difficult to verify that the last term is $2^{-2m-2}\sqrt{\pi}\,(2m+3)!$. Substituting $2m+1 = k$ shows the assertion in this case. □

3.2 Maximal cut of the discriminant

In this section we prove the sharp upper bound (3.3) on the number $\#(L\cap \mathrm{P}\Delta)$ of matrices with repeated eigenvalues in a generic projective 2-plane $L \simeq \mathbb{R}\mathrm{P}^2 \subset \mathrm{PSym}(n,\mathbb{R})$.

Definition 7. *Let $M_0, M_1, \dots, M_\ell \in \mathrm{Sym}(n, \mathbb{R})$ be independent matrices whose linear span non-trivially intersects the cone $\mathcal{P}_n \subset \mathrm{Sym}(n, \mathbb{R})$ of positive definite symmetric matrices. Then the matrices $M_0, M_1, \dots, M_\ell$ define the (spherical)* spectrahedron

$$\mathscr{S}_{\ell,n} := \{x \in S^\ell : x_0 M_0 + x_1 M_1 + \cdots + x_\ell M_\ell \succ 0\} \tag{3.20}$$

and the real symmetroid hypersurface

$$\Sigma_{\ell,n} = \{x \in S^\ell : \det(x_0 M_0 + x_1 M_1 + \cdots + x_\ell M_\ell) = 0\} \tag{3.21}$$

Remark 10. *The spectrahedron* (3.20) *is a convex semialgebraic subset of the sphere S^ℓ and the Zariski closure of its topological boundary $\partial \mathscr{S}_{\ell,n}$ is the symmetroid hypersurface* (3.21)*.*

It is important to note that any ℓ-dimensional spectrahedron (3.20) and its symmetroid hypersurface (3.21) admit the following representations:

$$\mathscr{S}_{\ell,n} = \{x \in S^\ell : x_0 \mathbb{1} + x_1 R_1 + \cdots + x_\ell R_\ell \succ 0\} \tag{3.22}$$

$$\Sigma_{\ell,n} = \{x \in S^\ell : \det(x_0 \mathbb{1} + x_1 R_1 + \cdots + x_\ell R_\ell) = 0\} \tag{3.23}$$

Indeed, without loss of generality we can assume that the matrix $M_0 = D^t D$ is strictly positive (otherwise perform an orthogonal change of coordinates in $x \in S^\ell$ to ensure this). Setting $M_i = D^t R_i D, i = 1, \dots, \ell$ in (3.20) and (3.21) one obtains (3.22) and (3.23) respectively.

In the following proposition we prove that the semialgebraic subsets $\mathscr{S}_{\ell,n}, \Sigma_{\ell,n} \subset S^\ell$ are naturally stratified by the corank.

Proposition 7. *Let $\mathscr{S}_{\ell,n}^{(k)}$ be the set of matrices of corank k in the spectrahedron $\mathscr{S}_{\ell,n}$ and $\Sigma_{\ell,n}^{(k)}$ the set of matrices of corank k in the symmetroid hypersurface $\Sigma_{\ell,n}$. For a generic choice of $\mathcal{R} = (R_1, \dots, R_\ell) \in \mathrm{Sym}(n, \mathbb{R})^\ell$ the sets $\mathscr{S}_{\ell,n}^{(k)}, \Sigma_{\ell,n}^{(k)} \subset S^\ell$ are semialgebraic of codimension $\binom{k+1}{2}$.*

Proof. In the space $\mathrm{Sym}(n, \mathbb{R})$ consider the semialgebraic stratification given by the corank: $\mathrm{Sym}(n, \mathbb{R}) = \coprod_{k=0}^n \mathcal{Z}^{(k)}$, where $\mathcal{Z}^{(k)}$ denotes the set matrices of corank k, and the induced stratification on the cone $\mathcal{P}_n$ of positive semidefinite matrices $\mathcal{P}_n = \coprod_{k=0}^n (\mathcal{Z}^{(k)} \cap \mathcal{P}_n)$. These are Nash stratifications [4, Proposition 9] and the codimensions of both $\mathcal{Z}^{(k)}$ and $\mathcal{Z}^{(k)} \cap \mathcal{P}_n$ are equal to $\binom{k+1}{2}$.

Consider now the semialgebraic map

$$F : S^\ell \times (\mathrm{Sym}(n, \mathbb{R}))^\ell \to \mathrm{Sym}(n, \mathbb{R}), \ (x, \mathcal{R}) \mapsto x_0 \mathbb{1} + x_1 R_1 + \cdots + x_\ell R_\ell.$$

Then $\Sigma_{\ell,n}^{(k)} = \{x \in S^\ell \,|\, F(\mathcal{R}, x) \in \mathcal{Z}^{(k)}\}$ and $\mathscr{S}_{\ell,n}^{(k)} = \{x \in S^\ell \,|\, F(\mathcal{R}, x) \in \mathcal{Z}^{(k)} \cap \mathcal{P}_n\}$ and consequently they are semialgebraic.

We now prove that F is transversal to all the strata of these stratifications. Then the parametric transversality theorem [44, Chapter 3, Theorem 2.7] will imply that for a generic choice of $\mathcal{R}$ the set $\mathscr{S}_{\ell,n}$ is stratified by the $\mathscr{S}_{\ell,n}^{(k)}$ and the same for the set $\Sigma_{\ell,n}$. To see that F is transversal to all the strata of the stratifications we compute its differential. At points $(x, \mathcal{R})$ with $x \neq e_0 = (1, 0, \ldots, 0)$ we have $D_{(x,\mathcal{R})}F(0, \dot{\mathcal{R}}) = x_1\dot{R}_1 + \cdots + x_\ell \dot{R}_\ell$ and the equation $D_{(x,\mathcal{R})}F(\dot{x}, \dot{\mathcal{R}}) = P$ can be solved by taking $\dot{x} = 0$ and $\dot{\mathcal{R}} = (0, \ldots, 0, x_i^{-1}P, 0, \ldots, 0)$ where $x_i^{-1}P$ is in the i-th entry and i is such that $x_i \neq 0$ (in other words, already variations in $\mathcal{R}$ ensure surjectivity of $D_{(x,\mathcal{R})}F$). All points of the form $(e_0, \mathcal{R})$ are mapped by F to the identity matrix $\mathbb{1}$ which belongs to the open stratum $\mathcal{Z}^{(0)}$, on which transversality is automatic (because this stratum has full dimension). This concludes the proof. □

In the following proposition a sharp upper bound on the number of singular points on a generic symmetroid surface $\Sigma_{3,n}$ is given.

Proposition 8. *For generic $\mathcal{R} \in \mathrm{Sym}(n, \mathbb{R})^3$ the number of singular points ρ_n on the symmetroid $\Sigma_{3,n}$ and hence the number of singular points σ_n on $\partial\mathscr{S}_{3,n}$ is finite and satisfies*

$$\sigma_n \le \rho_n \le \frac{n(n+1)(n-1)}{3}.$$

Moreover, for any $n \ge 1$ there exists a generic symmetroid $\Sigma_{3,n}$ with $\rho_n = \frac{n(n+1)(n-1)}{3}$ singular points on it.

Proof. The fact that $\sigma_n \le \rho_n$ are generically finite follows from Proposition 7 with $k = 2$, as remarked before. Observe that ρ_n is bounded by twice (since $\Sigma_{3,n}$ is a subset of S^3) the number $\#\mathrm{Sing}(\Sigma_{3,n}^{\mathbb{C}})$ of singular points on the complex symmetroid projective surface

$$\Sigma_{3,n}^{\mathbb{C}} = \{x \in \mathbb{C}\mathrm{P}^3 |\, \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3)) = 0\}$$

Since $\mathrm{Sing}(\Sigma_{3,n}^{\mathbb{C}})$ is obtained as a linear section of the set $\mathcal{Z}_{\mathbb{C}}^{(2)}$ of $n \times n$ *complex* symmetric matrices of corank two (using similar transversality arguments as in Proposition 7) we have that generically $\#\mathrm{Sing}(\Sigma_{3,n}^{\mathbb{C}}) = \deg(\mathcal{Z}_{\mathbb{C}}^{(2)})$. The latter is equal to $\frac{n(n+1)(n-1)}{6}$; see [43].

Now comes the proof of the second claim, we are thankful to Bernd Sturmfels and Simone Naldi for helping us with this. For a generic collection of $n+1$ linear forms $L_1, \ldots, L_{n+1}$ in $\ell + 1$ variables we denote by $p(x) := L_1(x) \cdots L_{n+1}(x)$ their product and by $P = \{x \in \mathbb{R}^{\ell+1} | L_i(x) > 0, i = 1, \ldots, n+1\}$ the polyhedral cone. Let $e \in \mathrm{int}(P)$ be any interior point of P. Then [71, Thm 1.1] implies that the derivative $\langle \nabla p, e\rangle$ of p along the constant vector field $e \in \mathbb{R}^{\ell+1}$ is a hyperbolic polynomial in direction e and that the closure of the connected component of $\mathbb{R}^{\ell+1} \setminus \{\langle \nabla p, e\rangle = 0\}$ containing e is a spectrahedral cone. Let's consider the intersection of this spectrahedral cone with the generic linear 4-space $V \subset \mathbb{R}^{\ell+1}$ and denote by $\mathscr{S}_{3,n}, \Sigma_{3,n}$ the corresponding spectrahedron and its symmetroid surface respectively. It is straightforward to check that the triple intersections of the hyperplanes $L_1, \ldots, L_{n+1}$ when intersected with V produce $2\binom{n+1}{3} = \frac{(n+1)n(n-1)}{3}$ singular points on $\Sigma_{3,n}$. This completes the proof since the above number coincides with the complex bound. $\square$

Given a generic $\mathcal{R} = (R_1, R_2, R_3) \in \mathrm{Sym}(n, \mathbb{R})^3$ denote $R := \mathrm{span}\{R_1, R_2, R_3\} \simeq \mathbb{R}^3 \subset \mathrm{Sym}(n, \mathbb{R})$ and let $\mathscr{S}_{3,n}$ and $\Sigma_{3,n}$ be as in (3.22) and (3.23) respectively. In the following proposition we establish a useful identification between matrices with repeated eigenvalues in $R \cap S^{N-1}$ and singular points of $\Sigma_{3,n}$.

Proposition 9. *For generic matrices $\mathcal{R} = (R_1, R_2, R_3) \in \mathrm{Sym}(n, \mathbb{R})^3$ we have*
(i) the number $\#(R \cap \Delta_1)$ of matrices in $R \cap S^{N-1}$ whose two smallest eigenvalues coincide equals the number of singular points on the boundary $\partial\mathscr{S}_{3,n} = \{x \in \mathscr{S}_{3,n} : \det(x_0\mathbb{1} + x_1R_1 + x_2R_2 + x_3R_3) = 0\}$ of the spectrahedron $\mathscr{S}_{3,n}$, and
(ii) the number $\#(R \cap \Delta)$ of matrices with repeated eigenvalues in $R \cap S^{N-1}$ equals the number of singular points of the symmetroid surface $\Sigma_{3,n}$.

Proof. (i) By Proposition 7 for a generic choice of $\mathcal{R}$ matrices of corank 2 in $\mathscr{S}_{3,n}$ and $\Sigma_{3,n}$ constitute the singular loci of $\mathscr{S}_{3,n}$ and $\Sigma_{3,n}$ respectively, i.e.,

$$\mathscr{S}_{3,n}^{(2)} = \mathrm{Sing}(\partial\mathscr{S}_{3,n}), \quad \Sigma_{3,n}^{(2)} = \mathrm{Sing}(\Sigma_{3,n}) \tag{3.24}$$

and the sets $\mathscr{S}_{3,n}^{(2)} \subset \Sigma_{3,n}^{(2)}$ are finite. When $\mathcal{R}$ is generic the sets $R \cap \Delta_1 \subset R \cap \Delta$ are finite as well. Observe that

$$\lambda_i(x_0\mathbb{1} + R(x)) = x_0 + \lambda_i(R(x)), \; i = 1, \ldots, n, \tag{3.25}$$

where we denote $R(x) = x_1R_1 + x_2R_2 + x_3R_3$. If $R(x) \in \Delta_1$, i.e., $\lambda_1(R(x)) = \lambda_2(R(x))$, then, due to (3.24) and (3.25),

$$\frac{(-\lambda_1(R(x)), x_1, x_2, x_3)}{\sqrt{\lambda_1(R(x))^2 + x_1^2 + x_2^2 + x_3^2}} \in \mathscr{S}_{3,n}^{(2)} = \mathrm{Sing}(\partial\mathscr{S}_{3,n})$$

is a singular point of $\partial\mathscr{S}_{3,n}$. Vice versa, if $(x_0, x_1, x_2, x_3) \in \mathrm{Sing}(\mathscr{S}_{3,n})$ we have that $x = (x_1, x_2, x_3) \neq 0$, $\lambda_1(R(x)) = \lambda_2(R(x))$ (by (3.24) and (3.25)) and hence $R(x)/\|R(x)\| \in \Delta_1$. Moreover, one can easily see that the established identification is one-to-one.

The proof of (ii) is analogous. $\square$

Theorem 11 now follows immediately from the above propositions.

Proof of Theorem 11. For a generic projective 2-plane $L = \mathrm{P}R \subset \mathrm{PSym}(n,\mathbb{R})$ we have, invoking Proposition 9, that

$$\rho_n = \#(\mathrm{Sing}(\Sigma_{3,n})) = \#(R \cap \Delta) = 2\#(L \cap \mathrm{P}\Delta)$$

Proposition 8 implies the bound:

$$\#(L \cap \mathrm{P}\Delta) = \frac{1}{2}\rho_n \leq \frac{(n+1)n(n-1)}{6} = \binom{n+1}{3}$$

and it's attained for some generic $L \simeq \mathbb{R}\mathrm{P}^2 \subset \mathrm{PSym}(n,\mathbb{R})$. $\square$

Chapter 4

The condition number for polynomial eigenvalues of random matrices

Following the ideas in [73, 17], we note that many different numerical problems can be described within the following simple general framework. We consider a space of *inputs* and a space of *outputs* denoted by $\mathcal{I}$ and $\mathcal{O}$ respectively, and some equation of the form $ev(i, o) = 0$ stating when an output is a solution for a given input. Both $\mathcal{I}$ and $\mathcal{O}$, and the *solution variety*

$$\mathcal{V} = \{(i, o) \in \mathcal{I} \times \mathcal{O} : o \text{ is an output to } i\} = \{(i, o) \in \mathcal{I} \times \mathcal{O} : ev(i, o) = 0\}$$

are frequently real algebraic or just semialgebraic sets. The numerical problem to be solved can then be written as "given $i \in \mathcal{I}$, find $o \in \mathcal{O}$ such that $(i, o) \in \mathcal{V}$", or "find all $o \in \mathcal{O}$ such that $(i, o) \in \mathcal{V}$". One can have in mind the following examples:

1. Polynomial Root Finding: $\mathcal{I}$ is the set of univariate real polynomials of degree d, $\mathcal{O} = \mathbb{R}$ and $\mathcal{V} = \{(f, \zeta) : f(\zeta) = 0\}$.

2. Polynomial System Solving, which we can see as the homogeneous multivariate version of Polynomial Root Finding: $\mathcal{I}$ is the projective space of (dense or structured) systems of n real homogeneous polynomials of degrees $d_1, \ldots, d_n$ in variables $x_0, \ldots, x_n$, $\mathcal{O} = \mathbb{R}\mathrm{P}^n$ and $\mathcal{V} = \{(f, \zeta) : f(\zeta) = 0\}$.

3. EigenValue Problem: $\mathcal{I} = \mathbb{R}^{n \times n}$, $\mathcal{O} = \mathbb{R}$ and $\mathcal{V} = \{(A, \lambda) : \det(A - \lambda\,\mathrm{Id}) = 0\}$.

4. (Homogeneous) Polynomial EigenValue Problem (in the sequel called PEVP): $\mathcal{I}$ is the set of tuples of $d+1$ real $n \times n$ matrices $A = (A_0, \ldots, A_d)$, $\mathcal{O} = \mathbb{R}\mathrm{P}^1$ and $\mathcal{V} = \{(A, [\alpha : \beta]) : P(A, \alpha, \beta) = \det(\alpha^0\beta^d A_0 + \alpha^1\beta^{d-1}A_1 + \cdots + \alpha^d\beta^0 A_d) = 0\}$. One can force some of the matrices to be symmetric, a particularly important

case in applications, or consider other structured problems, see [62, 28, 41, 85]. In cases $d = 1$ and $d = 2$ polynomial eigenvalues are often referred to as *generalized eigenvalues* and *quadratic eigenvalues* respectively.

We prove a general theorem computing exactly the expected value of the condition number in a wide collection of problems, including problem 4 above.

We start by recalling the general geometric definition of the condition number, which is usually thought of as "a measure of the sensibility of the solution o under an infinitesimal perturbation of the input i". A *Finsler structure* on a differentiable manifold M is a smooth field of norms $\|\cdot\|_p : T_pM \to \mathbb{R}$, $p \in M$ on M (see [17, p. 223] for more details). In particular, a Riemannian structure $\langle \cdot, \cdot \rangle$ on M defines a Finsler structure on it by $\|\dot{p}\|_p = \sqrt{\langle \dot{p}, \dot{p} \rangle_p}$, $p \in M$, $\dot{p} \in T_pM$.

Definition 8 (Condition number in the algebraic setting)**.** *Let $\mathcal{I}, \mathcal{O}$ and $\mathcal{V}$ be real algebraic varieties such that the smooth loci of $\mathcal{I}, \mathcal{O}$ are endowed with Finsler structures and let $(i, o) \in \mathcal{V}$ be a smooth point of $\mathcal{V}$ such that $i \in \mathcal{I}, o \in \mathcal{O}$ are smooth points of $\mathcal{I}$ and $\mathcal{O}$ respectively. Moreover, assume that $D_{(i,o)}p_1 : T_{(i,o)}\mathcal{V} \to T_i\mathcal{I}$ is invertible. Then the* condition number $\mu(i, o)$ *of $(i, o) \in \mathcal{V}$ is defined as*

$$\mu(i, o) = \left\| D_{(i,o)}p_2 \circ D_{(i,o)}p_1^{-1} \right\|_{\mathrm{op}},$$

where $p_1 : \mathcal{V} \to \mathcal{I}$, $p_2 : \mathcal{V} \to \mathcal{O}$ are the projections and $\|\cdot\|_{\mathrm{op}}$ is the operator norm. For points $(i, o) \in \mathcal{V}$ not satisfying the above assumptions the condition number is set to ∞.

See [23, Sec. 14.1] for more on this geometric approach to the condition number.

Remark 11. *Definition 8 is intrinsic in $\mathcal{I}$, i.e., changing $\mathcal{I}$ to some subvariety $\mathcal{I}' \subset \mathcal{I}$ leads (in general) to different, smaller, value of the condition number, since perturbations of the input are only allowed in the direction of the tangent space to the input set. Note also that the condition number depends on choices of Finsler structures on $\mathcal{I}$ and $\mathcal{O}$.*

Example: The classical Turing's condition number $\mu(A) = \|A\|_{\mathrm{op}}\|A^{-1}\|_{\mathrm{op}}$ for matrix inversion corresponds to the following setting:

- $\mathcal{O} = \mathcal{I} = M(n, \mathbb{R})$ is the set of $n \times n$ real matrices endowed with the Finsler structure associated to relative errors in operator norm: $\|\dot{A}\|_A = \|\dot{A}\|_{\mathrm{op}}/\|A\|_{\mathrm{op}}$.
- $\mathcal{V} = \{(A, B) : AB = \mathrm{Id}\} = \{(A, B) : B = A^{-1}\}$.

In the PEVP the input space $\mathcal{I}$ is endowed with the following Riemannian structure: $\langle \dot{A}, \dot{B} \rangle_A = ((\dot{A}_0, \dot{B}_0) + \cdots + (\dot{A}_d, \dot{B}_d))/((A_0, A_0) + \cdots + (A_d, A_d))$, where $(\cdot, \cdot)$ is the Frobenius inner product, $A = (A_0, \ldots, A_d)$ and $\dot{A} = (\dot{A}_0, \ldots, \dot{A}_d)$, $\dot{B} = (\dot{B}_0, \ldots, \dot{B}_d) \in T_A\mathcal{I}$. The output space $\mathcal{O} = \mathbb{R}\mathrm{P}^1$ possesses the standard metric and the solution variety $\mathcal{V} = \{(A, [\alpha : \beta]) : P(A, \alpha, \beta) = 0\}$ is endowed with the induced product Riemannian structure. An explicit formula for the condition number for the Homogeneous PEVP was derived in [28, Th. 4.2] (we write here the relative condition number version):

$$\mu(A, (\alpha, \beta)) = \left(\sum_{k=0}^{d} \alpha^{2k} \beta^{2d-2k} \right)^{1/2} \frac{\|r\| \|\ell\|}{|\ell^t v|} \, \|A\|, \tag{4.1}$$

where $A = (A_0, \ldots, A_d)$, $(\alpha, \beta) \in \mathbb{R}^2$ is a polynomial eigenvalue of A, r and ℓ are the corresponding right and left eigenvectors and

$$v = \beta \frac{\partial}{\partial \alpha} P(A, \alpha, \beta) r - \alpha \frac{\partial}{\partial \beta} P(A, \alpha, \beta) r.$$

A given tuple A can have up to nd real isolated polynomial eigenvalues. We define the condition number of A simply as the sum of the condition numbers over all these PEVs:

$$\mu(A) = \sum_{[\alpha:\beta] \in \mathbb{R}\mathrm{P}^1 \text{is a PEV of } A} \mu(A, (\alpha, \beta)).$$

(If $A = (A_0, \ldots, A_d)$ has infinitely many polynomial eigenvalues, then we set $\mu(A) = \infty$). The most important result in this paper is a very general theorem which is designed to provide exact formulas for the expected value of the condition number in the PEVP and other problems. A simple particular case of our general theorem is as follows.

Theorem 15 (Gaussian Homogeneous PEVP are well conditioned on the average)**.** *If $A_0, \ldots, A_d \in \mathcal{N}_{M(n,\mathbb{R})}$ are independent $\mathcal{N}_{M(n,\mathbb{R})}$-distributed matrices, then*

$$\begin{aligned} \mathop{\mathbb{E}}_{A_0,\ldots,A_d \sim i.i.d. \mathcal{N}_{M(n,\mathbb{R})}} \mu(A) &= \pi \frac{\Gamma\left(\frac{(d+1)n^2}{2}\right)}{\Gamma\left(\frac{(d+1)n^2-1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{\pi}{2} \sqrt{(d+1)n^3} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \ n \to +\infty \end{aligned} \tag{4.2}$$

In Corollary 4.6 we provide an analogous formula in the case when $A_0, \ldots, A_d$ are independent $\mathrm{GOE}(n)$-distributed matrices.

Remark 12. *Recently in [6] Armentano and Beltran investigated the expectation of the squared condition number for polynomial eigenvalues of complex Gaussian* *matrices. Theorem 15 establishes the "asymptotic square root law" for the considered problem, i.e., when $n \to +\infty$ (and up to the factor $\pi/2$) our answer in* (4.2) *equals the square root of the answer in [6].*

In Section 4.1 we state our main results, of which Theorem 15 is an easy consequence. Their proofs are given in Section 4.2 and in Section 4.3, some technical results are left for the Appendix.

4.1 Statement of main results

In this section we state our most general result, from which Theorem 15 will follow. First, let us fix a general framework which analyzes the input-output problems described above in a semialgebraic context. For the rest of this paper the input and the output sets will be, respectively, the real vector space $\mathcal{I} = \mathbb{R}^m$ and the unit circle $S^1 \subset \mathbb{R}^2$ endowed with the standard Riemannian structures. The solution variety will be a semialgebraic set $\mathcal{S} \subset \mathbb{R}^m \times S^1 \subset \mathbb{R}^m \times \mathbb{R}^2$ (we change letter from $\mathcal{V}$ to $\mathcal{S}$ to remark the fact that it is semialgebraic). We denote by $\mathcal{S}_{\text{top}}$ the union of top-dimensional cells in some fixed cell decomposition of $\mathcal{S}$ (see Section 1.2 for details). Then the smooth manifold $\mathcal{S}_{\text{top}} \subset \mathbb{R}^m \times S^1$ is endowed with the induced Riemannian structure. The two projections defined on $\mathcal{S}$ are denoted by $p_1 : \mathcal{S} \to \mathbb{R}^m$, $p_2 : \mathcal{S} \to S^1$.

Definition 9 (Condition number in the semialgebraic setting)**.** *Near a regular point $(a, x) \in \mathcal{S}_{\text{top}}$ the first projection $p_1 : \mathcal{S}_{\text{top}} \to \mathbb{R}^m$ is locally invertible, i.e., there exists a neighbourhood $U \subset \mathbb{R}^m$ of $a \in U$ and a unique smooth map $p_1^{-1} : U \to \mathcal{S}_{\text{top}}$ such that $p_1^{-1}(a) = (a, x)$ and $p_1 \circ p_1^{-1} = \mathrm{id}_U$. In this case the* local relative condition number *$\mu(a, x)$ is defined as*

$$\mu(a, x) := \|a\| \sup_{\dot{a} \in \mathbb{R}^m \setminus \{0\}} \frac{\|D_a(p_2 \circ p_1^{-1})(\dot{a})\|}{\|\dot{a}\|}$$

For points $(a, x) \in \mathcal{S}_{\text{low}} = \mathcal{S} \setminus \mathcal{S}_{\text{top}}$ in the cells of lower dimension in the decomposition of $\mathcal{S}$ as well as for critical points $(a, x) \in \mathcal{S}_{\text{top}}$ of $p_1 : \mathcal{S}_{\text{top}} \to \mathbb{R}^m$ we set $\mu(a, x) := \infty$.

The relative condition number *$\mu(a)$ of $a \in \mathbb{R}^m$ is defined to be the sum of all local relative condition numbers $\mu(a, x)$:*

$$\mu(a) := \sum_{x \in S^1 : \, (a,x) \in \mathcal{S}} \mu(a, x)$$

Remark 13. *Note that Definition 9 agrees with Definition 8 if we endow the input space $\mathcal{I} = \mathbb{R}^m$ with the Riemannian structure associated to relative errors, that is $\langle \dot{a}, \dot{b} \rangle_a = (\dot{b}^t \dot{a})/\|a\|^2$, $a \in \mathbb{R}^m$.*

To simplify terminology, throughout the rest of the paper, we omit the word "relative" when refering to (local) relative condition number.

We deal with a large class of semialgebraic subsets of $\mathbb{R}^m \times S^1$ that we define next.

Definition 10. *We say that the semialgebraic set $\mathcal{S} \subset \mathbb{R}^m \times S^1$ is* regular *if the following conditions are satisfied:*

1. *for any $x \in S^1$ the fiber $p_2^{-1}(x)$ is of dimension $m-1$,*
2. *the semialgebraic set $\Sigma_1 \subset \mathcal{S}_{\text{top}}$ of critical points of $p_1 : \mathcal{S}_{\text{top}} \to \mathbb{R}^m$ is at most $(m-1)$-dimensional. In Proposition 10 we show that this condition is equivalent to the following one:*

2′. *there exists a semialgebraic subset $B \subset \mathbb{R}^m$ of dimension at most $m-2$ such that for any $a \notin B$ the fiber $p_1^{-1}(a)$ is finite.*

The first condition in Definition 10 implies that $\mathcal{S}$ is m-dimensional (see Lemma 6). To perform our probabilistic study we take the input variables $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ to be independent standard gaussians: $a \sim N(0,1)$. In the following theorem we establish a general formula for the expectation of the condition number $\mu(a)$ of a randomly chosen $a \in \mathbb{R}^m$:

Theorem 16. *If $\mathcal{S} \subset \mathbb{R}^m \times S^1$ is a regular semialgebraic set, then*

$$\mathbb{E}_{a\sim N(0,1)} \left(\sum_{x\in S^1 :\, (a,x)\in\mathcal{S}} \mu(a,x) \right) = \frac{1}{\sqrt{2\pi}^m} \int\limits_{x\in S^1} \int\limits_{a\in p_2^{-1}(x)} \|a\| e^{-\frac{\|a\|^2}{2}} \, da \, dx. \quad (4.3)$$

If, moreover, $\mathcal{S}$ is scale-invariant with respect to the first m variables, i.e., $(a,x) \in \mathcal{S}$ if and only if $(ta,x) \in \mathcal{S}$ for any $t > 0$, then

$$\mathbb{E}_{a\sim N(0,1)} \left(\sum_{x\in S^1 :\, (a,x)\in\mathcal{S}} \mu(a,x) \right) = \frac{\Gamma\left(\frac{m}{2}\right)}{2\sqrt{\pi}^m} \int\limits_{x\in S^1} |p_2^{-1}(x) \cap S^{m-1}| \, dx,$$

where $|p_2^{-1}(x) \cap S^{m-1}|$ denotes the volume of the $(m-2)$-dimensional semialgebraic spherical set $p_2^{-1}(x) \cap S^{m-1}$.

The following form of Theorem 16 for sets in $\mathbb{R}^m \times \mathbb{R}\mathrm{P}^1$ better fits our purposes.

Corollary 3. *Let $\mathcal{S} \subset \mathbb{R}^m \times S^1$ be a regular semialgebraic set that is scale-invariant with respect to the first m variables and suppose that $\mathcal{S}$ is invariant under the map*

$(a,x) \mapsto (a,-x)$, $(a,x) \in \mathbb{R}^m \times S^1$. *Then* $\mu(a,x) = \mu(a,-x)$, $(a,x) \in \mathcal{S}$, *the fibers* $p_2^{-1}(x), p_2^{-1}(-x)$ *are isometric and*

$$\mathbb{E}_{a\sim N(0,1)}\left(\sum_{[x]\in\mathbb{R}\mathrm{P}^1:\,(a,x)\in\mathcal{S}} \mu(a,x)\right) = \frac{\Gamma\left(\frac{m}{2}\right)}{2\sqrt{\pi}^m} \int\limits_{[x]\in\mathbb{R}\mathrm{P}^1} |p_2^{-1}(x)\cap S^{m-1}|\, d[x],$$

Note that Corollary 3 is a "projective" version of the second part of Theorem 16.

As pointed out in the introduction, we are specifically interested in the *polynomial eigenvalue problem.* Given $d+1$ matrices $A_0,\ldots,A_d \in M(n,\mathbb{R})$ a point $[x] = [\alpha:\beta] \in \mathbb{R}\mathrm{P}^1$ is a (real) *polynomial eigenvalue* (PEV) of $A = (A_0,\ldots,A_d)$ if

$$\det(\alpha^0\beta^d A_0 + \cdots + \alpha^d\beta^0 A_d) = 0.$$

The space $M(n,\mathbb{R})$ of $n\times n$ real matrices is endowed with the Frobenius inner product and the associated norm:

$$(A,B) = \mathrm{tr}(A^tB), \quad \|A\|^2 = (A,A), \quad A,B \in M(n,\mathbb{R}).$$

Then a k-dimensional vector subspace $V \subset M(n,\mathbb{R})$ is endowed with the standard normal probability distribution $\mathcal{N}_V$:

$$\mathrm{P}_{\mathcal{N}_V}(U) = \frac{1}{\sqrt{2\pi}^k}\int\limits_U e^{-\frac{\|v\|^2}{2}} dv,$$

where dv is the Lebesgue measure on $(V,(\cdot,\cdot))$ and $U \subset V$ is a measurable subset. Let us also denote by $\Sigma_V = \{A \in V : \det A = 0\} \subset V$ the variety of singular matrices in V.

The condition number for polynomial eigenvalues of $A = (A_0,\ldots,A_d) \in V^{d+1}$ is defined via

$$\mu(A) := \sum_{[x]\in\mathbb{R}\mathrm{P}^1 \text{is a PEV of } A} \mu(A,x),$$

where $\mu(A,x)$ is as in Definition 9 with $\mathbb{R}^m = (V,(\cdot,\cdot))^{d+1}$ and

$$\mathcal{S} = \{(A,x) = ((A_0,\ldots,A_d),(\alpha,\beta)) \in V^{d+1}\times S^1 : \det(\alpha^0\beta^dA_0 + \cdots + \alpha^d\beta^0A_d) = 0\}$$

As proved in [28], in the case $V = M(n,\mathbb{R})$ this definition for $\mu(A,x)$ is equivalent to (4.1). In the following theorem we investigate the expected condition number for polynomial eigenvalues of independent $\mathcal{N}_V$-distributed matrices $A_0,\ldots,A_d \in V$.

Theorem 17. *If $\Sigma_V \subset V$ is of codimension one, then*

$$\underset{A_0,\dots,A_d \sim i.i.d.\, \mathcal{N}_V}{\mathbb{E}} \mu(A) = \sqrt{\pi} \frac{\Gamma\left(\frac{(d+1)k}{2}\right)}{\Gamma\left(\frac{(d+1)k-1}{2}\right)} \frac{|\Sigma_V \cap S^{k-1}|}{|S^{k-2}|}. \tag{4.4}$$

Poincaré formula (Corollary 2 from Section 1.3) allows to derive the following universal upper bound.

Corollary 4. *If $\Sigma_V \subset V$ is of codimension one, then*

$$\underset{A_0,\dots,A_d \sim i.i.d.\, \mathcal{N}_V}{\mathbb{E}} \mu(A) \leq \sqrt{\pi} n \frac{\Gamma\left(\frac{(d+1)k}{2}\right)}{\Gamma\left(\frac{(d+1)k-1}{2}\right)} \tag{4.5}$$

In case $V = M(n, \mathbb{R})$ of all square matrices we provide an explicit formula for the expected condition number, that is the claim of our Theorem 15 above.

We give an explicit answer also in the case $V = \mathrm{Sym}(n, \mathbb{R})$ of symmetric matrices. In this case the probability space $(\mathrm{Sym}(n, \mathbb{R}), \mathcal{N}_{\mathrm{Sym}(n,\mathbb{R})})$ is called *the Gaussian Orthogonal Ensemble* (GOE).

Corollary 5. *If $A_0, \dots, A_d \in \mathrm{Sym}(n, \mathbb{R})$ are independent $\mathrm{GOE}(n)$-matrices and n is even, then*

$$\begin{aligned}\underset{A_0,\dots,A_d \sim i.i.d.\, \mathrm{GOE}(n)}{\mathbb{E}} \mu(A) &= \sqrt{2} n \frac{\Gamma\left(\frac{(d+1)n(n+1)}{4}\right)}{\Gamma\left(\frac{(d+1)n(n+1)-2}{4}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \\ &= \sqrt{(d+1)n^3}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right), \ n \to +\infty\end{aligned} \tag{4.6}$$

If n is odd the explicit formula is more complicated and is given in the proof of the corollary. However, the above asymptotic formula is valid for both even and odd n.

4.2 Proof of main results

In this section we prove our main results, Theorems 16 and 17.

First we fix some notations that are used in the rest of this chapter: for a regular subset $\mathcal{S} \subset \mathbb{R}^m \times S^1$ by $\Sigma_1, \Sigma_2 \subset \mathcal{S}_{\mathrm{top}}$ we denote the semialgebraic sets of critical points of $p_1 : \mathcal{S}_{\mathrm{top}} \to \mathbb{R}^m$ and $p_2 : \mathcal{S}_{\mathrm{top}} \to S^1$ respectively, the corresponding semialgebraic sets of critical values are denoted by $\sigma_1 = p_1(\Sigma_1) \subset \mathbb{R}^m$ and $\sigma_2 = p_2(\Sigma_2) \subset S^1$.

The following proposition establishes equivalence of the conditions (2) and (2′) in Definition 10 of a regular semialgebraic subset $\mathcal{S} \subset \mathbb{R}^m \times S^1$.

Proposition 10. *Let $\mathcal{S} \subset \mathbb{R}^m \times S^1$ be a semialgebraic subset of dimension m. Then*

(2) *the semialgebraic set $\Sigma_1 \subset \mathcal{S}_{\text{top}}$ of critical points of the first projection $p_1 : \mathcal{S}_{\text{top}} \to \mathbb{R}^m$ is at most $(m-1)$-dimensional if and only if*

($2'$) *there exists a semialgebraic subset $B \subset \mathbb{R}^m$ of dimension at most $m-2$ such that for any $a \notin B$ the fiber $p_1^{-1}(a)$ is finite.*

Proof. (2) $\Rightarrow$ ($2'$) By Sard's theorem the semialgebraic set $\sigma_1 = p_1(\Sigma_1) \subset \mathbb{R}^m$ of critical values of $p_1 : \mathcal{S}_{\text{top}} \to \mathbb{R}^m$ is of dimension $\leq m-1$. The set $p_1^{-1}(\sigma_1) \subset \mathcal{S}$ of critical fibers is also of dimension $\leq m-1$. Indeed, if it was m-dimensional there would exist a nonempty open set $U \subset p_1^{-1}(\sigma_1) \setminus (\Sigma_1 \cup \mathcal{S}_{\text{low}})$ of regular points of p_1. The image $p_1(U) \subset \sigma_1$ of U is open in $\mathbb{R}^m$ which contradicts to $\dim(\sigma_1) \leq m-1$.

For the map $p_1 : p_1^{-1}(\sigma_1) \to \sigma_1$ define $B_1 := \{a \in \sigma_1 : \dim(p_1^{-1}(a)) = 1\}$, the semialgebraic set of points in σ_1 for which the fiber $p_1^{-1}(a)$ is infinite. Since $\dim(p_1^{-1}(\sigma_1)) \leq m-1$ Corollary 2 implies that $\dim(B_1) \leq m-2$.

Similarly, for the map $p_1 : \mathcal{S}_{\text{low}} \to p_1(\mathcal{S}_{\text{low}})$ let us define $B_2 := \{a \in p_1(\mathcal{S}_{\text{low}}) : \dim(p_1^{-1}(a) \cap \mathcal{S}_{\text{low}}) = 1\}$, the semialgebraic set of points in $p_1(\mathcal{S}_{\text{low}})$ for which the fiber $p_1^{-1}(a) \cap \mathcal{S}_{\text{low}}$ is infinite. Since $\dim(\mathcal{S}_{\text{low}}) \leq m-1$ Corollary 2 gives $\dim(B_2) \leq m-2$.

Take now any $a \notin B_1 \cup B_2$. If $a \in \sigma_1$ the fiber $p_1^{-1}(a)$ is finite since $a \notin B_1$. If $a \notin \sigma_1$ it's a regular point of the map $p_1 : \mathcal{S}_{\text{top}} \to \mathbb{R}^m$ between two m-dimensional manifolds. Therefore the semialgebraic set $p_1^{-1}(a) \cap \mathcal{S}_{\text{top}}$ is zero-dimensional manifold and hence it's finite. The set $p_1^{-1}(a) \cap \mathcal{S}_{\text{low}}$ is finite because $a \notin B_2$. Consequently, the fiber $p_1^{-1}(a) = (p_1^{-1}(a) \cap \mathcal{S}_{\text{top}}) \cup (p_1^{-1}(a) \cap \mathcal{S}_{\text{low}})$ is finite for any point $a \notin B$ out of the at most $(m-2)$-dimensional semialgebraic subset $B := B_1 \cup B_2 \subset \mathbb{R}^m$.

(2) $\Leftarrow$ ($2'$) Recall that $\dim(\sigma_1) \leq m-1$ and let us consider the map $p_1 : \Sigma_1 \to \sigma_1$. If Σ_1 was m-dimensional the semialgebraic set $B := \{a \in \sigma_1 : \dim(p_1^{-1}(a) \cap \Sigma_1) = 1\}$, by Corollary 2, would be $(m-1)$-dimensional, which would contradict to ($2'$). $\square$

4.2.1 Proof of Theorem 16

In this subsection $\mathcal{S}$ denotes a regular semialgebraic subset of $\mathbb{R}^m \times S^1$. For the proof of Theorem 16 we need few technical lemmas which we state and prove below.

Lemma 6. *The semialgebraic sets $\mathcal{S} \subset \mathbb{R}^m \times S^1$ and $p_1(\mathcal{S}) \subset \mathbb{R}^m$ are of dimension m.*

Proof. Since $\mathcal{S}$ is regular, for every $x \in S^1$ the fiber $p_2^{-1}(x)$ is $(m-1)$-dimensional. From Theorem 2 it follows that for some $x \in S^1$ we have $\dim(\mathcal{S}) = \dim(p_2^{-1}(x)) + \dim(S^1) = (m-1) + 1 = m$.

The map $p_1 : \mathcal{S}_{\text{top}} \to \mathbb{R}^m$ has a regular point $(a, x) \in \mathcal{S}_{\text{top}} \setminus \Sigma_1$ since $\mathcal{S}$ is m-dimensional and the set Σ_1 of critical points of p_1 is at most $(m-1)$-dimensional.

The image $p_1(U)$ of a small open neighbourhood $U \subset \mathcal{S}_{\text{top}} \setminus \Sigma_1$ of $(a, x) \in U$ is open in $\mathbb{R}^m$ and hence $\dim(p_1(\mathcal{S})) = m$. $\square$

Lemma 7. *There exists an open semialgebraic subset $M \subset \mathcal{S}_{\text{top}}$ such that $p_1(M)$ is open in $\mathbb{R}^m$, $M = p_1^{-1}(p_1(M))$, the restriction $p_1 : M \to p_1(M)$ is a submersion and* $\dim(\mathcal{S} \setminus M) \leq m - 1$.

Proof. Define $M := p_1^{-1}(\mathbb{R}^m \setminus N) = \mathcal{S}_{\text{top}} \setminus p_1^{-1}(N)$, where $N := \overline{p_1(\mathcal{S}_{\text{low}} \cup \Sigma_1)}$ and the bar stands for the euclidean closure of a set. Note that M is an open subset of $\mathcal{S}_{\text{top}}$ and $M = p_1^{-1}(p_1(M))$. Moreover M consists of regular points of the projection $p_1 : \mathcal{S}_{\text{top}} \to \mathbb{R}^m$, which implies that $p_1(M)$ is an open subset of $\mathbb{R}^m$ and $p_1 : M \to p_1(M)$ is a submersion of smooth manifolds. Indeed, for $a \in p_1(M)$ and $(a, x) \in M$ the image $p_1(U)$ of a small open neighborhood $U \subset M$ of $(a, x) \in U$ is open in $\mathbb{R}^m$ and $a \in p_1(U)$.

We now prove that $\mathcal{S} \setminus M = p_1^{-1}(N)$ is at most $(m-1)$-dimensional. Since $\mathcal{S}$ is regular there exists a semialgebraic set $B \subset \mathbb{R}^m$ with $\dim(B) \leq m - 2$ such that $p_1^{-1}(a)$ is finite for $a \notin B$. We decompose the semialgebraic set $N = (N \cap B) \cup (N \setminus B)$. From Theorem 2 it follows that there exists some $a \in N \cap B$ such that $\dim(p_1^{-1}(N \cap B)) \leq \dim(p_1^{-1}(a)) + \dim(N \cap B) \leq 1 + (m-2) = m - 1$. For $a \in N \setminus B$ the fiber $p_1^{-1}(a)$ is discrete, which together with the non-degeneracy of $\mathcal{S}$ and Theoren 2 implies $\dim(p_1^{-1}(N \setminus B)) \leq \dim(p_1^{-1}(a)) + \dim(N \setminus B) \leq \dim(\mathcal{S}_{\text{low}} \cup \Sigma_1) \leq m - 1$. Therefore, $\dim(\mathcal{S} \setminus M) = \dim(p_1^{-1}(N)) = \dim(p_1^{-1}(N \cap B) \cup p_1^{-1}(N \setminus B)) \leq m - 1$. $\square$

Lemma 8. *There exists an open semialgebraic subset $R \subset \mathcal{S}_{\text{top}}$ such that $S^1 \setminus p_2(R)$ is finite, $p_2 : R \to p_2(R)$ is a submersion,* $\dim(\mathcal{S} \setminus R) \leq m-1$ *and* $\dim(p_2^{-1}(x) \setminus R) \leq m - 2$ *for $x \in p_2(R)$.*

Proof. Since $\mathcal{S}$ is regular every fiber $p_2^{-1}(x), x \in S^1$ is $(m-1)$-dimensional.

Note that the set $S^1 \setminus p_2(\mathcal{S}_{\text{top}})$ is semialgebraic and zero-dimensional, thus finite. Indeed, if it was one-dimensional Theorem 2 together with $\dim(p_2^{-1}(x)) = m - 1, x \in S^1$ would imply that $p_2^{-1}(S^1 \setminus p_2(\mathcal{S}_{\text{top}})) \subset \mathcal{S} \setminus \mathcal{S}_{\text{top}}$ is m-dimensional which would contradict to $\dim(\mathcal{S} \setminus \mathcal{S}_{\text{top}}) \leq m - 1$.

The semialgebraic set $\sigma_2 = p_2(\Sigma_2) \subset S^1$ of critical values of $p_2 : \mathcal{S}_{\text{top}} \to S^1$ has measure zero by Sard's theorem. Hence $\sigma_2 \subset S^1$ consists of a finite number of points.

Applying Corollary 1 to the map $p_2 : \mathcal{S}_{\text{low}} \to S^1$ we have that $C := \{x \in S^1 : \dim(p_2^{-1}(x) \cap \mathcal{S}_{\text{low}}) = m - 1\}$ is a semialgebraic subset of S^1 and $\dim(C) \leq \dim(\mathcal{S}_{\text{low}}) - (m-1) \leq 0$. Thus C is a (possibly empty) finite set.

Set now $R := \mathcal{S}_{\text{top}} \setminus p_2^{-1}(\sigma_2 \cup C)$. Note that R is an open semialgebraic subset of $\mathcal{S}_{\text{top}}$ and $S^1 \setminus p_2(R) = \sigma_2 \cup C \cup (S^1 \setminus p_2(\mathcal{S}_{\text{top}}))$ is finite by the above arguments. Since R consists of regular points of $p_2 : \mathcal{S}_{\text{top}} \to S^1$ the map $p_2 : R \to p_2(R)$ is a submersion. Since $\dim(\mathcal{S}_{\text{low}}) \leq m-1$ and $p_2^{-1}(\sigma_2 \cup C)$ is a finite collection of $(m-1)$-dimensional fibers we have that $\dim(\mathcal{S} \setminus R = \mathcal{S}_{\text{low}} \cup p_2^{-1}(\sigma_2 \cup C)) \leq m-1$. Finally, $\dim(p_2^{-1}(x) \setminus R = p_2^{-1}(x) \cap \mathcal{S}_{\text{low}}) \leq m-2$ for $x \in p_2(R)$ because $p_2(R) \cap C = \varnothing$. $\square$

Lemma 9. *For any measurable function $f : \mathcal{S} \to [0, +\infty)$ we have*

$$\int\limits_{a \in \mathbb{R}^m} \sum_{x \in S^1 \colon (a,x) \in \mathcal{S}} f(a,x)\, da = \int\limits_{x \in S^1} \int\limits_{a \in p_2^{-1}(x)} \frac{NJ_{(a,x)}p_1}{NJ_{(a,x)}p_2} f(a,x)\, da\, dx$$

Proof. Let $M \subset \mathcal{S}_{\text{top}}$ be as in Lemma 7. The smooth coarea formula [45, (A-2)] applied to the measurable function $f : M \to [0, +\infty)$ and to the submersion $p_1 : M \to p_1(M)$ reads

$$\int\limits_{(a,x) \in M} NJ_{(a,x)}p_1\, f(a,x)\, d(a,x) = \int\limits_{a \in p_1(M)} \sum_{x \in S^1 \colon (a,x) \in \mathcal{S}} f(a,x)\, da, \tag{4.7}$$

where we used that $M = p_1^{-1}(p_1(M))$ (Lemma 7) to be able to sum over the whole fiber $p_1^{-1}(a) = \{(a,x) \in \mathcal{S}\}$, $a \in p_1(M)$. By Lemma 7 we have $\dim(\mathcal{S} \setminus M) \leq m-1$ and hence $\dim(p_1(\mathcal{S}) \setminus p_1(M) = p_1(\mathcal{S} \setminus M)) \leq \dim(\mathcal{S} \setminus M) \leq m-1$. Thus we extend the integrations in (4.7) over $\mathcal{S}$ and $p_1(\mathcal{S})$ respectively without changing the result. Moreover the integration over $p_1(\mathcal{S})$ can be further extended to the whole space $\mathbb{R}^m$ since for a point $a \in \mathbb{R}^m \setminus p_1(\mathcal{S})$ the summation $\sum_{x \in S^1 : (a,x) \in \mathcal{S}} f(a,x)$ is performed over the empty set $p_1^{-1}(a)$ in which case the sum is conventionally set to 0. All together the above arguments imply

$$\int\limits_{(a,x) \in \mathcal{S}} NJ_{(a,x)}p_1\, f(a,x)\, d(a,x) = \int\limits_{a \in \mathbb{R}^m} \sum_{x \in S^1 \colon (a,x) \in \mathcal{S}} f(a,x)\, da, \tag{4.8}$$

Let $R \subset \mathcal{S}_{\text{top}}$ be as in Lemma 8. Applying the smooth coarea formula [45, (A-2)] to the measurable function $\frac{NJp_1}{NJp_2} f : R \to [0, +\infty)$ and to the submersion $p_2 : R \to p_2(R)$ we obtain

$$\int\limits_{(a,x) \in R} NJ_{(a,x)}p_1\, f(a,x)\, d(a,x) = \int\limits_{x \in p_2(R)} \int\limits_{a \in p_2^{-1}(x) \cap R} \frac{NJ_{(a,x)}p_1}{NJ_{(a,x)}p_2} f(a,x)\, da\, dx \tag{4.9}$$

By Lemma 8 $\dim(\mathcal{S} \setminus R) \leq m-1$, $S^1 \setminus p_2(R)$ is finite, and $\dim(p_2^{-1}(x) \setminus R) \leq m-2$ for $x \in p_2(R)$. Thus the integrations in (4.9) can be extended over $\mathcal{S}, S^1$ and $p_2^{-1}(x)$ respectively leading to

$$\int\limits_{(a,x) \in \mathcal{S}} NJ_{(a,x)}p_1\, f(a,x)\, d(a,x) = \int\limits_{x \in S^1} \int\limits_{a \in p_2^{-1}(x)} \frac{NJ_{(a,x)}p_1}{NJ_{(a,x)}p_2} f(a,x)\, da\, dx \tag{4.10}$$

Combining (4.10) with (4.8) we finish the proof. $\square$

Now comes the proof of Theorem 16.

Proof of Theorem 16. The following identity is the key point of the proof:

$$\mu(a,x) = \|a\| \frac{NJ_{(a,x)}p_2}{NJ_{(a,x)}p_1}, \quad (a,x) \in M \cap R \subset \mathcal{S}_{\text{top}} \tag{4.11}$$

where $M \subset \mathcal{S}_{\text{top}}$ and $R \subset \mathcal{S}_{\text{top}}$ are as in Lemma 7 and Lemma 8 respectively and $\mu(a,x)$, the local condition number of $(a,x) \in \mathcal{S}$, is defined in Definition 9. The proof of the identity comes after we derive the statement of Theorem 16.

Applying Lemma 9 to the measurable function $f(a,x) = \mu(a,x)e^{-\|a\|^2/2}/\sqrt{2\pi}^m$, $(a,x) \in \mathcal{S}$, and using (4.11) we obtain:

$$\begin{aligned}
\mathbb{E}_{a\sim N(0,1)}\left(\sum_{x\in S^1:\,(a,x)\in\mathcal{S}} \mu(a,x)\right) &= \frac{1}{\sqrt{2\pi}^m} \int\limits_{a\in\mathbb{R}^m} \left(\sum_{x\in S^1:\,(a,x)\in\mathcal{S}} \mu(a,x)\right) e^{-\frac{\|a\|^2}{2}} da \\
&= \frac{1}{\sqrt{2\pi}^m} \int\limits_{x\in S^1} \int\limits_{a\in p_2^{-1}(x)} \|a\| e^{-\frac{\|a\|^2}{2}} da\, dx = (*),
\end{aligned}$$

which gives the claimed formula (4.3). If $\mathcal{S}$ is scale-invariant with respect to $a \in \mathbb{R}^m$ by Lemma 13 we have

$$(*) = \frac{\Gamma\left(\frac{m}{2}\right)}{2\sqrt{\pi}^m} \int\limits_{x\in S^1} |p_2^{-1}(x) \cap S^{m-1}|\, dx.$$

Now we turn to the proof of (4.11).

For $(a,x) \in M \cap R \subset \mathcal{S}_{\text{top}}$ let $(\dot a_0, \dot x_0), (\dot a_1, 0), \ldots, (\dot a_{m-1}, 0)$ be an orthonormal basis of $T_{(a,x)}R$ with $(\dot a_j, 0) \in \ker D_{(a,x)}p_2$, $j = 1, \ldots, m-1$. Note that $\dot a_0 \in \mathbb{R}^m, \dot x_0 \in T_xS^1$ are non-zero since $p_1 : M \to p_1(M), p_2 : R \to p_2(R)$ are submersions and $\dot a_0 \in \mathbb{R}^m$ is orthogonal to $\dot a_j \in \mathbb{R}^m$, $j = 1, \ldots, m-1$. We compute the normal Jacobians $NJ_{(a,x)}p_1$ and $NJ_{(a,x)}p_2$ using the following orthonormal bases:

$$\begin{aligned}
\{(\dot a_0, \dot x_0), (\dot a_1, 0), \ldots, (\dot a_{m-1}, 0)\} &\subset T_{(a,x)}\mathcal{S}_{\text{top}} \\
\left\{\frac{\dot a_0}{\|\dot a_0\|}, \dot a_1, \ldots, \dot a_{m-1}\right\} &\subset T_a\mathbb{R}^m \\
\left\{\frac{\dot x_0}{\|\dot x_0\|}\right\} &\subset T_xS^1
\end{aligned}$$

It is straightforward to see that $NJ_{(a,x)}p_1 = \|\dot a_0\|$ and $NJ_{(a,x)}p_2 = \|\dot x_0\|$ and hence

$$\frac{NJ_{(a,x)}p_2}{NJ_{(a,x)}p_1} = \frac{\|\dot x_0\|}{\|\dot a_0\|}$$

Since $D_a(p_2 \circ p_1^{-1})(\dot{a}_j) = D_{(a,x)}p_2 \circ D_a p_1^{-1}(\dot{a}_j) = 0$ for $j = 1, \ldots, m-1$ and since $D_a(p_2 \circ p_1^{-1})(\dot{a}_0) = D_{(a,x)}p_2 \circ D_a p_1^{-1}(\dot{a}_0) = \dot{x}_0$ we obtain

$$\mu(a, x) = \|a\| \sup_{\dot{a} \in \mathbb{R}^m \setminus \{0\}} \frac{\|D_a(p_2 \circ p_1^{-1})(\dot{a})\|}{\|\dot{a}\|} = \|a\| \frac{\|\dot{x}_0\|}{\|\dot{a}_0\|}$$

This together with (5) implies the claimed identity (4.11). $\square$

4.2.2 Proof of Theorem 17

For a k-dimensional vector subspace $V \subset M(n, \mathbb{R})$ and for a basis $f = (f_0(\alpha, \beta), \ldots, f_d(\alpha, \beta))$ of the space $P_{d,2}$ of binary forms of degree $d \geq 1$ let us define the algebraic variety

$$\mathcal{S}(V, f) := \{(A, x) \in V^{d+1} \times S^1 : \det(A_0\, f_0(\alpha, \beta) + \cdots + A_d\, f_d(\alpha, \beta)) = 0\}$$

Theorem 17 follows from the following more general result.

Theorem 18. *If $\Sigma_V \subset V$ is of codimension one and f is any basis of $P_{d,2}$, then $\mathcal{S}(V, f)$ is regular and*

$$\mathop{\mathbb{E}}_{A_0, \ldots, A_d \sim i.i.d.\, \mathcal{N}_V} \left(\sum_{[x] \in \mathbb{RP}^1 :\, (A,x) \in \mathcal{S}(V,f)} \mu(A, x) \right) = \sqrt{\pi} \frac{\Gamma\left(\frac{(d+1)k}{2}\right)}{\Gamma\left(\frac{(d+1)k-1}{2}\right)} \frac{|\Sigma_V \cap S^{k-1}|}{|S^{k-2}|},$$

Proof. Observe first that for any $x = (\alpha, \beta) \in S^1$ the vector $f(x) = (f_0(\alpha, \beta), \ldots, f_d(\alpha, \beta))$ is non-zero. For any such fixed x, let $g = (g_{ij}) \in O(d+1)$ be an orthogonal matrix that sends $f(x)$ to $(c, 0, \ldots, 0) \in \mathbb{R}^{d+1} \setminus \{0\}$, where $c \neq 0$ is some constant, i.e.,

$$\sum_{j=0}^{d} g_{ij} f_j(\alpha, \beta) = \begin{cases} c, i = 0, \\ 0, i = 1, \ldots, d \end{cases}$$

It is easy to verify that the linear change of coordinates $A_j = \sum_{i=0}^{d} g_{ij} \tilde{A}_i$, $j = 0, \ldots, d$ is an isometry of the product space $(V, (\cdot, \cdot))^{d+1}$ and

$$\sum_{j=0}^{d} f_j(\alpha, \beta) A_j = \sum_{j=0}^{d} f_j(\alpha, \beta) \left(\sum_{i=0}^{d} g_{ij} \tilde{A}_i \right) = \sum_{i=0}^{d} \left(\sum_{j=0}^{d} g_{ij} f_j(\alpha, \beta) \right) \tilde{A}_i = c\, \tilde{A}_0$$

Therefore, for $x = (\alpha, \beta) \in S^1$ there is a global isometry $\mathcal{I}_x : (V, (\cdot, \cdot))^{d+1} \to (V, (\cdot, \cdot))^{d+1}$ that sends the fiber $p_2^{-1}(x) = \{A \in V^{d+1} : \det(A_0\, f_0(\alpha, \beta) + \cdots + A_d\, f_d(\alpha, \beta)) = 0\}$ to $\{\tilde{A} \in V^{d+1} : \det(\tilde{A}_0) = 0\} = \Sigma_V \times V^d$. In particular, under the assumption $\dim(\Sigma_V) = k-1$ we have that $p_2^{-1}(x)$ is of codimension one in V^{d+1} and hence condition (1) in Definition 10 is satisfied.

Since both $f_0(\alpha,\beta),\ldots,f_d(\alpha,\beta)$ and $\alpha^0\beta^d,\ldots,\alpha^d\beta^0$ are bases of $P_{d,2}$ for some $h=(h_{ij})\in GL(d+1)$ we have $\alpha^i\beta^{d-i} = \sum_{j=0}^d h_{ij}f_j(\alpha,\beta)$, $i=0,\ldots,d$. Let us define $B = \{A\in V^{d+1} : A_j = \sum\limits_{i=0}^{d} h_{ij}\tilde{A}_i,\ j=0,\ldots,d,\ \det(\tilde{A}_0) = \det(\tilde{A}_d) = 0\}$. Since $\dim(\Sigma_V) = k-1$ and since h is a non-singular linear transformation the algebraic subset $B\subset V^{d+1}$ has codimension 2. For $A\notin B$ the matrix

$$\sum_{j=0}^d f_j(\alpha,\beta)A_j = \sum_{j=0}^d f_j(\alpha,\beta)\left(\sum_{i=0}^d h_{ij}\tilde{A}_i\right) = \sum_{i=0}^d\left(\sum_{j=0}^d h_{ij}f_j(\alpha,\beta)\right)\tilde{A}_i = \sum_{i=0}^d \alpha^i\beta^{d-i}\tilde{A}_i$$

is non-singular at $(\alpha:\beta)=(0,1)$ (at $(\alpha,\beta)=(1,0)$) if $\det(\tilde{A}_0)\neq 0$ (if $\det(\tilde{A}_d)\neq 0$, respectively) and hence the binary form $\det(A_0f_0(\alpha,\beta)+\cdots+A_df_d(\alpha,\beta))$ is non-zero. Consequently, the fiber $p_1^{-1}(A) = \{x\in S^1 : \det(A_0f_0(\alpha,\beta)+\cdots+A_df_d(\alpha,\beta)) = 0\}$ is finite for any $A\notin B$ and condition (2′) in Definition 10 is satisfied. Applying Corollary 3 to $\mathcal{S}(V,f)\subset\mathbb{R}^{(d+1)k}\times S^1$, $\mathbb{R}^{(d+1)k}\simeq V^{d+1}$ we obtain

$$\mathop{\mathbb{E}}_{A_0,\ldots,A_d\sim i.i.d.\mathcal{N}_V}\left(\sum_{[x]\in\mathbb{R}\mathrm{P}^1:\,(A,x)\in\mathcal{S}(V,f)}\mu(A,x)\right) = \frac{1}{\sqrt{2\pi}^{(d+1)k}}\int_{[x]\in\mathbb{R}\mathrm{P}^1}\int_{A\in p_2^{-1}(x)}\|A\|e^{-\frac{\|A\|^2}{2}}dA\,dx,$$

where $\|A\|^2 = \|A_0\|^2+\cdots+\|A_d\|^2$. Since each fiber $p_2^{-1}(x), x\in S^1$ is an algebraic subset of $\mathbb{R}^{(d+1)k}$ of codimension one we have by Lemma 13

$$\int_{A\in p_2^{-1}(x)}\|A\|e^{-\frac{\|A\|^2}{2}}dA = \sqrt{2}\,\frac{\Gamma\left(\frac{(d+1)k}{2}\right)}{\Gamma\left(\frac{(d+1)k-1}{2}\right)}\int_{A\in p_2^{-1}(x)}e^{-\frac{\|A\|^2}{2}}dA$$

Performing the isometric change of coordinates $\mathcal{I}_x : (V,(\cdot,\cdot))^{d+1}\to(V,(\cdot,\cdot))^{d+1}$ that was constructed above we write the last integral as follows:

$$\begin{aligned}\int_{A\in p_2^{-1}(x)}e^{-\frac{\|A\|^2}{2}}dA &= \int_{\{\tilde{A}\in V^{d+1}:\,\det(\tilde{A}_0)=0\}}e^{-\frac{\|\tilde{A}_0\|^2}{2}}e^{-\frac{\|\tilde{A}_1\|^2}{2}}\ldots e^{-\frac{\|\tilde{A}_d\|^2}{2}}d\tilde{A}_0d\tilde{A}_1\ldots d\tilde{A}_d\\ &= \sqrt{2\pi}^{dk}\int_{\tilde{A}_0\in\Sigma_V}e^{-\frac{\|\tilde{A}_0\|^2}{2}}d\tilde{A}_0 = \sqrt{2\pi}^{dk}\sqrt{2}^{k-3}\Gamma\left(\frac{k-1}{2}\right)|\Sigma_V\cap S^{k-1}|,\end{aligned}$$

where in the last step Lemma 13 has been used. Collecting everything together we write

$$\mathop{\mathbb{E}}_{A_0,\ldots,A_d\sim i.i.d.\,\mathcal{N}_V}\left(\sum_{[x]\in\mathbb{R}\mathrm{P}^1:\,(A,x)\in\mathcal{S}(V,f)}\mu(A,x)\right)=\sqrt{\pi}\frac{\Gamma\left(\frac{(d+1)k}{2}\right)}{\Gamma\left(\frac{(d+1)k-1}{2}\right)}\frac{\Gamma\left(\frac{k-1}{2}\right)}{2\sqrt{\pi}^{k-1}}|\Sigma_V\cap S^{k-1}|$$

$$=\sqrt{\pi}\frac{\Gamma\left(\frac{(d+1)k}{2}\right)}{\Gamma\left(\frac{(d+1)k-1}{2}\right)}\frac{|\Sigma_V\cap S^{k-1}|}{|S^{k-2}|},$$

since $|S^{k-2}|=2\sqrt{\pi}^{k-1}/\Gamma\left(\frac{k-1}{2}\right)$. This completes the proof. □

Proof of Theorem 17. Taking $f_i(\alpha,\beta)=\alpha^i\beta^{d-i}$, $i=0,\ldots,d$ in Theorem 18 we obtain the claim of Theorem 17. □

4.3 Applications of main results

In this section we derive Theorem 15 and Corollaries 4, 5.

Proof of Corollary 4. By Poincaré formula (Corollary 2 from Section 1.3) applied to the projective hypersurface $\mathrm{P}\Sigma_V\subset \mathrm{P}V\simeq\mathbb{R}\mathrm{P}^{k-1}$ we have

$$\frac{|\Sigma_V\cap S^{k-1}|}{|S^{k-2}|}=\frac{|\mathrm{P}\Sigma_V|}{|\mathbb{R}\mathrm{P}^{k-2}|}=\mathop{\mathbb{E}}_{\ell\in\mathbb{G}(1,k-1)}\#(\mathrm{P}\Sigma_V\cap\ell)\le\deg(\mathrm{P}\Sigma_V)=n$$

which together with (4.4) implies the claimed bound (4.5). □

In case of any particular space $V\subset M(n,\mathbb{R})$ satisfying $\dim(\Sigma_V)=k-1=\dim(V)-1$ by Theorem 17 explicit computation of the expected condition number for polynomial eigenvalues amounts to computing the volume of the hypersurface $\Sigma_V\cap S^{k-1}$. In cases $V=M(n,\mathbb{R})$ and $V=\mathrm{Sym}(n,\mathbb{R})$ formulas for the volume of $\Sigma_V\cap S^{k-1}$ were found in [33] and [55] respectively.

Proof of Theorem 15. Formula from [33] reads

$$\frac{|\Sigma_{M(n,\mathbb{R})}\cap S^{n^2-1}|}{|S^{n^2-2}|}=\sqrt{\pi}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

Plugging it in (4.4) for $V=M(n,\mathbb{R}),k=\dim(V)=n^2$ leads to

$$\mathop{\mathbb{E}}_{A_0,\ldots,A_d\sim i.i.d.\,\mathcal{N}_{M(n,\mathbb{R})}}\mu(A)=\pi\frac{\Gamma\left(\frac{(d+1)n^2}{2}\right)}{\Gamma\left(\frac{(d+1)n^2-1}{2}\right)}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$=\frac{\pi}{2}\sqrt{(d+1)n^3}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),\ n\to+\infty,$$

where the asymptotic is obtained using formula (1) from [86]. □

Proof of Corollary 5. In [55] it was proved that

$$\frac{|\Sigma_{\mathrm{Sym}(n,\mathbb{R})} \cap S^{\frac{n(n+1)}{2}-1}|}{|S^{\frac{n(n+1)}{2}-2}|} = \sqrt{\frac{2}{\pi}}\, n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \tag{4.12}$$

for even n and

$$\frac{|\Sigma_{\mathrm{Sym}(n,\mathbb{R})} \cap S^{\frac{n(n+1)}{2}-1}|}{|S^{\frac{n(n+1)}{2}-2}|} = \frac{(-1)^m \sqrt{\pi} n!}{2^n m! \Gamma\left(\frac{n+2}{2}\right)} \left(1 - \frac{4\sqrt{2}}{\sqrt{\pi}} \sum_{i=0}^{m-1} (-1)^i \frac{\Gamma\left(\frac{2i+3}{2}\right)}{i!}\right) \tag{4.13}$$

for odd $n = 2m+1$. Plugging (4.12) and (4.13) in (4.4) for $V = \mathrm{Sym}(n,\mathbb{R}), k = \frac{n(n+1)}{2}$ leads to explicit formulas for the expected condition number (see (4.6) in case of even n). In [55, Remark 3] it was shown that

$$\frac{|\Sigma_{\mathrm{Sym}(n,\mathbb{R})} \cap S^{\frac{n(n+1)}{2}-1}|}{|S^{\frac{n(n+1)}{2}-2}|} = \frac{2\sqrt{n}}{\sqrt{\pi}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right),\ n \to +\infty$$

regardless parity of n. This leads to the asymptotic

$$\begin{aligned}\mathop{\mathbb{E}}_{A_0,\ldots,A_d \sim i.i.d.\, GOE(n)} \mu(A) &= \sqrt{\pi} \frac{\Gamma\left(\frac{(d+1)n(n+1)}{4}\right)}{\Gamma\left(\frac{(d+1)n(n+1)-2}{4}\right)} \frac{|\Sigma_{\mathrm{Sym}(n,\mathbb{R})} \cap S^{\frac{n(n+1)}{2}-1}|}{|S^{\frac{n(n+1)}{2}-2}|} \\ &= \sqrt{(d+1)n^3} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)\right),\ n \to +\infty,\end{aligned}$$

where we again used formula (1) from [86] for the asymptotic of the ratio of two Gamma functions. $\square$

Chapter 5

On the number of flats tangent to convex hypersurfaces in random position

Flats simultaneously tangent to several hypersurfaces

Given $d_{k,n} = (k+1)(n-k)$ projective hypersurfaces $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ a classical problem in enumerative geometry is to determine how many k-dimensional projective subspaces of $\mathbb{R}\mathrm{P}^n$ (called *k-flats*) are simultaneously tangent to $X_1, \dots, X_{d_{k,n}}$.

Geometrically we can formulate this problem as follows. Let $\mathbb{G}(k,n)$ denote the Grassmannian of k-dimensional projective subspaces of $\mathbb{R}\mathrm{P}^n$ (note that $d_{k,n} = \dim \mathbb{G}(k,n)$). If $X \subset \mathbb{R}\mathrm{P}^n$ is a smooth hypersurface, we denote by $\Omega_k(X) \subset \mathbb{G}(k,n)$ the variety of k-tangents to X, i.e. the set of k-flats that are tangent to X at some point. The number of k-flats simultaneously tangent to hypersurfaces $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ equals

$$\# \, \Omega_k(X_1) \cap \dots \cap \Omega_k(X_{d_{k,n}}).$$

Of course this number depends on the mutual position of the hypersurfaces $X_1, \dots, X_{d_{k,n}}$ in the projective space $\mathbb{R}\mathrm{P}^n$.

In [77] F.Sottile and T.Theobald proved that there are at most $3 \cdot 2^{n-1}$ real lines tangent to $2n-2$ general spheres in $\mathbb{R}^n$ and they found a configuration of spheres with $3 \cdot 2^{n-1}$ common tangent lines. They also studied [78] the problem of k-flats tangent to $d_{k,n}$ many general quadrics in $\mathbb{R}\mathrm{P}^n$ and proved that the "complex bound" $2^{d_{k,n}} \cdot \deg(\mathbb{G}_{\mathbb{C}}(k,n))$ can be attained by real quadrics. See also [19, 60, 61, 79] for other interesting results on real enumerative geometry of tangents.

An exciting point of view comes by adopting a random approach: one asks for the *expected value* for the number of tangents to hypersurfaces in *random position*. We say that the hypersurfaces $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ are in *random position* if each one of them is randomly translated by elements $g_1, \dots, g_{d_{k,n}}$ sampled independently from the orthogonal group $O(n+1)$ endowed with the uniform distribution. The average number $\tau_k(X_1, \dots, X_{d_{k,n}})$ of k-flats tangent to $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ in random position is then given by

$$\tau_k(X_1, \dots, X_{d_{k,n}}) := \mathbb{E}_{\,g_1,\dots,g_{d_{k,n}} \in O(n+1)} \#\, \Omega_k(g_1 X_1) \cap \dots \cap \Omega_k(g_{d_{k,n}} X_{d_{k,n}}).$$

The computation and study of properties of this number is precisely the goal of this chapter.

A special feature of our study is that we concentrate on the case when the hypersurfaces (not necessarily algebraic) are boundaries of convex sets. Part of the results we present, however, hold in higher generality as we discuss in Section 5.5.

Definition 11 (Convex hypersurface)**.** *A subset C of $\mathbb{R}\mathrm{P}^n$ is called (strictly) convex if C does not intersect some hyperplane L and it is (strictly) convex in the affine chart $\mathbb{R}\mathrm{P}^n \setminus L \simeq \mathbb{R}^n$. A smooth hypersurface $X \subset \mathbb{R}\mathrm{P}^n$ is said to be* convex *if it bounds a strictly convex open set of $\mathbb{R}\mathrm{P}^n$.*

Remark 14 (Spherical versus projective geometry)**.** *Our considerations in projective spaces run parallel to what happens on spheres, with just small adaptations. A set $C \subset S^n$ is called (strictly) convex if it is the intersection of a (strictly) convex cone $K \subset \mathbb{R}^{n+1}$ with S^n. A smooth hypersurface $X \subset S^n$ is said to be* convex *if it bounds a strictly convex open set of S^n. For the purposes of enumerative geometry, the notion of flats should be replaced with the one of plane sections of S^n. Computations involving volumes and the generalized integral geometry formula also require very small modifications (mostly multiplications by a factor of two) and we leave them to the reader.*

Probabilistic enumerative geometry

Recently, Bürgisser and Lerario [24] have studied the similar problem of determining the average number of k-flats that simultaneously intersect $d_{k,n}$ many $(n-k-1)$-flats in random position in $\mathbb{R}\mathrm{P}^n$. They have called this number the *expected degree* of the real Grassmannian $\mathbb{G}(k,n)$, here denoted by $\delta_{k,n}$, and have claimed that this is the key quantity governing questions in random enumerative geometry of flats. (The name comes from the fact that the number of solutions of the analogous problem over the complex numbers coincides with the degree of $\mathbb{G}_{\mathbb{C}}(k,n)$ in the Plücker embedding. Note however that the notion of expected degree is intrinsic and does not require any embedding.)

For reasons that will become more clear later, it is convenient to introduce the *special Schubert variety*[1] $\mathrm{Sch}(k,n) \subset \mathbb{G}(k,n)$ consisting of k-flats in $\mathbb{R}\mathrm{P}^n$ intersecting

[1]Note that in the notation of [24] $\mathrm{Sch}(k,n) = \Sigma(k+1, n+1)$ and $\delta_{k,n} = \mathrm{edeg}\, G(k+1, n+1)$.

a fixed $(n-k-1)$-flat. The volume of the special Schubert variety is computed in [24, Theorem 4.2] and equals

$$|\mathrm{Sch}(k,n)| = |\mathbb{G}(k,n)| \cdot \frac{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)},$$

where $|\mathbb{G}(k,n)|$ denotes the volume of the Grassmannian (see Subection 1.3.1). The following theorem relates our main problem to the expected degree (this is Theorem 20 from Section 5.3).

Theorem (Probabilistic enumerative geometry)**.** *The average number of k-flats in $\mathbb{R}\mathrm{P}^n$ simultaneously tangent to convex hypersurfaces $X_1, \ldots, X_{d_{k,n}}$ in random position equals*

$$\tau_k(X_1, \ldots, X_{d_{k,n}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|\mathrm{Sch}(k,n)|},$$

where $|\Omega_k(X)|$ denotes the volume of the manifold of k-tangents to X.

The number $\delta_{k,n}$ equals (up to a multiple) the volume of a convex body for which the authors of [24] coined the name *Segre zonoid.* Except for $\delta_{0,n} = \delta_{n-1,n} = 1$, the exact value of this quantity is not known, but it is possible to compute its asymptotic as $n \to \infty$ for fixed k. For example, in the case of the Grassmannian of lines in $\mathbb{R}\mathrm{P}^n$ one has [24, Theorem 6.8]

$$\delta_{1,n} = \frac{8}{3\pi^{5/2}} \cdot \frac{1}{\sqrt{n}} \cdot \left(\frac{\pi^2}{4}\right)^n \cdot \left(1 + \mathcal{O}(n^{-1})\right). \tag{5.1}$$

The number $\delta_{1,3}$ (the average number of lines meeting four random lines in $\mathbb{R}\mathrm{P}^3$) can be written as an integral [24, Proposition 6.7], whose numerical approximation is $\delta_{1,3} = 1.7262...$. It is an open problem whether this quantity has a closed formula (possibly in terms of special functions).

The above theorem reduces our study to the investigation of the geometry of the manifold of tangents, for which we prove the following result (Proposition 11 and Corollary 6 below).

Proposition (The volume of the manifold of k-tangents)**.** *For a convex hypersurface $X \subset \mathbb{R}\mathrm{P}^n$ we have*

$$\frac{|\Omega_k(X)|}{|\mathrm{Sch}(k,n)|} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_X \sigma_k(x) dV_X.$$

where $\sigma_k : X \to \mathbb{R}$ is the k-th elementary symmetric polynomial of the principal curvatures of the embedding $X \hookrightarrow \mathbb{R}\mathrm{P}^n$.

Remark 15. *After this result was obtained P. Bürgisser has pointed out to us it can be also derived using a limiting argument from [5], where the tube neighborhood around $\Omega_k(X)$ is described.*

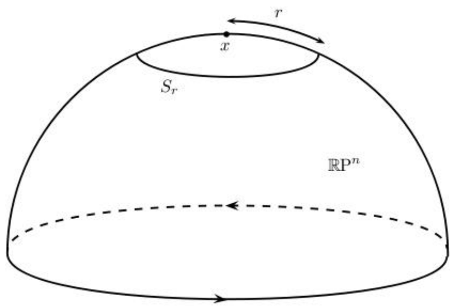

Figure 5.1: The equation $x_1^2 + \cdots + x_n^2 = (\tan r)^2 x_0^2$ defines in $\mathbb{RP}^n$ a metric sphere of radius r, i.e. the set of all points at distance r from a fixed point.

Example 1 (Spheres in projective space)**.** *Let $S_{r_i} = \{x_1^2 + \cdots + x_n^2 = (\tan r_i)^2\, x_0^2\} \subset \mathbb{RP}^n$ be a metric sphere in $\mathbb{RP}^n$ of radius $r_i \in (0, \pi/2)$, $i = 1, \ldots, d_{k,n}$ (see Figure 5.1). Since all principal curvatures of S_{r_i} are constants equal to $\cot r_i$ and since $|S_{r_i}| = \frac{2\sqrt{\pi}^n}{\Gamma(\frac{n}{2})}(\sin r_i)^{n-1}$ Corollary 6 gives*

$$\frac{|\Omega_k(S_r)|}{|Sch(k,n)|} = \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)} \cdot (\cos r_i)^k (\sin r_i)^{n-k-1},$$

Combining this into Theorem 20 we obtain

$$\tau_k(S_{r_1}, \ldots, S_{r_{d_{k,n}}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \left(\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)} \cdot (\cos r_i)^k (\sin r_i)^{n-k-1} \right) \tag{5.2}$$

For a fixed k it is natural to find the maximum of the expectation in the case when all the hypersurfaces are spheres. For example, when $k = 1$ one can easily see that $\cos r_i (\sin r_i)^{n-2}$ is maximized at $r_i = \arccos \frac{1}{\sqrt{n-1}} = \frac{\pi}{2} - \frac{1}{n^{1/2}} + O(n^{-1/2})$, which is just a bit smaller than $\frac{\pi}{2}$. Therefore,

$$\begin{aligned} \max_{r \in (0,\pi/2)} \frac{|\Omega_k(S_r)|}{|Sch(k,n)|} &= \frac{4}{\sqrt{\pi}} \cdot \frac{\left(\frac{n-2}{n-1}\right)^{\frac{n-2}{2}}}{(n-1)^{\frac{1}{2}}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \\ &= \left(\frac{8}{e\pi}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2n} + \mathcal{O}(n^{-2})\right) \end{aligned}$$

and, together with (5.1) *and* (5.2), *this gives*

$$\begin{aligned}\max_{r_1,\dots,r_{2n-2}\in(0,\frac{\pi}{2})} \tau_1(S_{r_1},\dots,S_{r_{2n-2}}) &= \delta_{1,n}\cdot\left(\left(\frac{8}{e\pi}\right)^{\frac{1}{2}}\left(1+\frac{1}{2n}+\mathcal{O}(n^{-2})\right)\right)^{2n-2} \\ &= \frac{e^2}{3\pi^{\frac{3}{2}}}\cdot\frac{1}{\sqrt{n}}\cdot\left(\frac{2\pi}{e}\right)^n\cdot\left(1+\mathcal{O}(n^{-1})\right).\end{aligned} \tag{5.3}$$

Observe that a hypersurface $S_{y,r}$ which is a sphere in some affine chart $U\simeq\mathbb{R}^n$, i.e. $S_{y,r}=\{x\in\mathbb{R}^n:\sum_{i=1}^n(x_i-y_i)^2=r^2\}$, is a convex hypersurface in $\mathbb{R}\mathrm{P}^n$, but it is not a sphere with respect to the projective metric unless it's centered at the origin ($y=0$); and, viceversa, a metric sphere in $\mathbb{R}\mathrm{P}^n$ needs not be a sphere in an affine chart. In fact, (5.3) *tells that Sottile and Theobald's upper bound $3\cdot 2^{n-1}$ for the number of lines tangent to $d_{1,n}$ affine spheres in $\mathbb{R}^n$* does not *apply to the case of spheres in $\mathbb{R}\mathrm{P}^n$: since $\frac{2\pi}{e}>2$, when n is large* (5.3) *is larger than $3\cdot 2^{n-1}$; as a consequence there must be a configuration of $d_{1,n}$ projective spheres in $\mathbb{R}\mathrm{P}^n$ with (exponentially) more common tangent lines.*

Remark 16 (The semialgebraic case)**.** *The theorem above remains true in the case of semialgebraic hypersurfaces $X_1,\dots,X_{d_{k,n}}\subset\mathbb{R}\mathrm{P}^n$ satisfying some mild non-degeneracy conditions (see Section 5.5 for more details). Specifically it still holds true that*

$$\tau_k(X_1,\dots,X_{d_{k,n}})=\delta_{k,n}\cdot\prod_{i=1}^{d_{k,n}}\frac{|\Omega_k(X_i)|}{|Sch(k,n)|},$$

but the volume of the manifold of k-tangents has a more complicated description:

$$\frac{|\Omega_k(X)|}{|Sch(k,n)|}=\frac{\binom{n-1}{k}\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{\pi^{\frac{n+1}{2}}}\int_X\mathbb{E}_{\Lambda\in Gr_k(T_xX)}|B_x(\Lambda)|dV_X,$$

where $|B_x(\Lambda)|$ denotes the absolute value of the determinant of the matrix of the second fundamental form of $X\hookrightarrow\mathbb{R}\mathrm{P}^n$ restricted to $\Lambda\in Gr_k(T_xX)$ and written in an orthonormal basis of Λ (see Subsection 5.2.1), and the expectation is taken with respect to the uniform distribution on $Gr_k(T_xX)\simeq Gr(k,n-1)$.

Relation with intrinsic volumes

The quantities $|\Omega_k(X)|$ offer an alternative interesting interpretation of the classical notion of *intrinsic volumes.* Recall that if C is a convex set in $\mathbb{R}\mathrm{P}^n$, the spherical Steiner's formula [38, (9)] allows to write the volume of the ϵ-neighborhood $\mathcal{U}_{\mathbb{R}\mathrm{P}^n}(C,\epsilon)$ of C in $\mathbb{R}\mathrm{P}^n$ as

$$|\mathcal{U}_{\mathbb{R}\mathrm{P}^n}(C,\epsilon)|=|C|+\sum_{k=0}^{n-1}f_k(\epsilon)|S^k||S^{n-k-1}|V_k(C),$$

where

$$f_k(\epsilon) = \int_0^{\epsilon} (\cos t)^k (\sin t)^{n-1-k} dt. \tag{5.4}$$

The quantities $V_0(C), \ldots, V_{n-1}(C)$ are called intrinsic volumes of C. What is remarkable is that when C is smooth and strictly convex, $|\Omega_k(\partial C)|$ coincides, up to a constant depending on k and n only, with the $(n-k-1)$-th intrinsic volume of C (again this property can be derived by a limiting argument from the results in [5]).

Proposition (The manifold of k-tangents and intrinsic volumes)**.** *Let $C \subset \mathbb{R}\mathrm{P}^n$ be a smooth strictly convex set. Then*

$$|V_{n-k-1}(C)| = \frac{1}{4} \cdot \frac{|\Omega_k(\partial C)|}{|\mathrm{Sch}(k,n)|}, \quad k = 0, \ldots, n-1.$$

This interpretation offers possible new directions of investigation and allows to prove the following upper bound (see Corollary 9)

$$\tau_k(X_1, \ldots, X_{d_{k,n}}) \leq \delta_{k,n} \cdot 4^{d_{k,n}},$$

where the right-hand side depends only on k and n. However, already for $n = 3$, as observed by T. Theobald [84] there is no upper bound on the number of lines that can be simultaneously tangent to four convex hypersurfaces in $\mathbb{R}\mathrm{P}^3$ in general position (see Section 5.4 for a proof of this fact).

Related work

Enumerative geometry over the field of complex numbers is classical. Over the Reals it is a much harder subject, due to the nonexistence of generic configurations. From the deterministic point of view we mention, among others, the papers that are closest to our work and that gave a motivation for it: [19, 60, 61, 77, 78, 79]. The probabilistic approach to real enumerative geometry was initiated in [24] for what concerns Schubert calculus, and in [13] for the study of the number of real lines on random hypersurfaces.

5.1 Preliminaries

By $\mathbb{G}(k,n) \simeq Gr(k+1, n+1)$ we denote the Grassmannian of $(k+1)$-planes in $\mathbb{R}^{n+1}$ (or, equivalently, the set of projective k-flats in $\mathbb{R}\mathrm{P}^n$). Both notations are used throughout this chapter. The dimension of $\mathbb{G}(k,n)$ is denoted by $d_{k,n} := \dim \mathbb{G}(k,n) = (k+1)(n-k)$. The Grassmannian $\mathbb{G}(k,n)$ admits an embedding into $\mathbb{R}^{(n+1)^2}$ as a non-singular real algebraic subset [18, Theorem 3.4.4]. Below, when applying classical results on semialgebraic sets and mappings from Section 1.2 to Grassmannians and, in particular, to projective spaces, we implicitly refer to the mentioned real algebraic embedding.

5.1.1 Integral geometry of coisotropic hypersurfaces of Grassmannian

A smooth (respectively semialgebraic) hypersurface $\mathcal{H}$ of $\mathbb{G}(k,n)$ is said to be *coisotropic* if for any $\Lambda \in \mathcal{H}$ (respectively for any $\Lambda \in \mathcal{H}_{\text{top}}$) the normal space $N_\Lambda \mathcal{H} \subset T_\Lambda \mathbb{G}(k,n) \simeq \mathrm{Hom}(\Lambda, \Lambda^\perp)$ is spanned by a rank one operator.
For $k,m \geq 1$ let $u_j \in S(\mathbb{R}^k), v_j \in S(\mathbb{R}^m)$, $j = 1,\dots,km$ be unit independent random vectors. Then the *average scaling factor* $\alpha(k,m)$ is defined as

$$\alpha(k,m) := \mathbb{E}\,\|(u_1 \otimes v_1) \wedge \cdots \wedge (u_{km} \otimes v_{km})\|$$

where the norm $\|\cdot\|$ is induced from the standard scalar product on $\mathbb{R}^k \otimes \mathbb{R}^m$: $(u_1 \otimes v_1, u_2 \otimes v_2) := (u_1,u_2)(v_1,v_2)$. We will use the generalized Poincaré formula for coisotropic hypersurfaces of $\mathbb{G}(k,n)$ proved in [24, Thm. 3.19]:

Theorem 19. *Let $\mathcal{H}_1, \dots, \mathcal{H}_{d_{k,n}}$ be coisotropic hypersurfaces of $\mathbb{G}(k,n)$. Then*

$$\mathbb{E}\,\#(g_1\mathcal{H}_1 \cap \cdots \cap g_{d_{k,n}}\mathcal{H}_{d_{k,n}}) = \alpha(k+1, n-k)\,|\mathbb{G}(k,n)| \prod_{i=1}^{d_{k,n}} \frac{|\mathcal{H}_i|}{|\mathbb{G}(k,n)|}$$

where $g_1, \dots, g_{d_{k,n}} \in O(n+1)$ are independent randomly chosen orthogonal transformations.

Remark 17. *This theorem expresses the average number of points in the intersection of $d_{k,n}$ many hypersurfaces of $\mathbb{G}(k,n)$ in* random position *in terms of the volumes of the hypersurfaces and the average scaling factor $\alpha(k+1,n-k)$, which only depends on the pair (k,n).*

5.1.2 Intersection of special real Schubert varieties

A *special real Schubert variety* $\mathrm{Sch}(k,n)$ consists of all projective k-flats in $\mathbb{R}\mathrm{P}^n$ that intersect a fixed projective $(n-k-1)$-flat Π:

$$\mathrm{Sch}(k,n) = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap \Pi \neq \varnothing\}$$

It is a coisotropic algebraic hypersurface of $\mathbb{G}(k,n)$. In [24] P. Bürgisser and the second author of the current article had introduced a notion of *expected degree $\delta_{k,n}$* of the Grassmannian $\mathbb{G}(k,n)$. It is defined as the average number of projective k-flats in $\mathbb{R}\mathrm{P}^n$ simultaneously intersecting $d_{k,n}$ many random projective $(n-k-1)$-flats independently chosen in $\mathbb{G}(n-k-1,n)$. In other words,

$$\delta_{k,n} := \mathbb{E}\,\#(g_1\mathrm{Sch}(k,n) \cap \cdots \cap g_{d_{k,n}}\mathrm{Sch}(k,n)).$$

Using the formula [24, Thm. 4.2]:

$$|\mathrm{Sch}(k,n)| = |\mathbb{G}(k,n)| \frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+1}{2})} \frac{\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{n-k}{2})}$$

for the volume of $\mathrm{Sch}(k,n)$ and Theorem 19 one can express

$$\delta_{k,n} = \alpha(k+1,n-k)\,|\mathbb{G}(k,n)| \left(\frac{\Gamma(\frac{k+2}{2})}{\Gamma(\frac{k+1}{2})} \frac{\Gamma(\frac{n-k+1}{2})}{\Gamma(\frac{n-k}{2})} \right)^{d_{k,n}}$$

Remark 18. *The exact value of $\delta_{k,n}$ (equivalently $\alpha(k+1,n-k)$) remains unknown for $0 < k < (n-1)$. See [24, Sec. 6] for various asymptotics of $\delta_{k,n}$.*

Remark 19. *Note that one can define a notion of "expected degree" even over the complex numbers, by sampling complex projective subspaces uniformly from the complex Grassmannian. Denoting by $c_{k,n} \in H^2(\mathbb{G}^{\mathbb{C}}(k,n);\mathbb{Z})$ the first Chern class of the tautological bundle and by $[\mathbb{G}^{\mathbb{C}}(k,n)] \in H_{2d_{k,n}}(\mathbb{G}^{\mathbb{C}}(k,n);\mathbb{Z})$ the fundamental class we have that*

$$\textit{the expected degree over the complex numbers} = \left\langle (c_{k,n})^{d_{k,n}}, [\mathbb{G}^{\mathbb{C}}(k,n)] \right\rangle$$

The resulting number also equals the degree of $\mathbb{G}^{\mathbb{C}}(k,n)$ in the Plücker embedding.

5.2 The manifold of tangents

5.2.1 The volume of the manifold of tangents to a convex hypersurface

Let $X = \partial C$ be a convex hypersurface of $\mathbb{R}\mathrm{P}^n$ (bounding the strictly convex open set $C \subset \mathbb{R}\mathrm{P}^n$) and let $p : Gr_k(X) \to X$ be the Grassmannian bundle of k-planes of X (this is a smooth fiber bundle over X whose fiber $p^{-1}(x)$ is the Grassmannian $Gr_k(T_xX) \simeq Gr(k,n-1)$). Define the *kth Gauss map*

$$\begin{aligned} \psi :\ & Gr_k(X) \to \mathbb{G}(k,n) \\ & (x,\Lambda) \mapsto \mathrm{P}(\mathrm{Span}\{x,\Lambda\}) \end{aligned}$$

here we identify the tangent space $T_x\mathbb{R}\mathrm{P}^n$ with the hyperplane $x^\perp \subset \mathbb{R}^{n+1}$ and thus Λ and x (a line in $\mathbb{R}^{n+1}$) are both subspaces of $\mathbb{R}^{n+1}$.

With this notation we observe that ψ is a smooth embedding and that $\Omega_k(X)$, the set of all k-flats tangent to X, coincides, by definition, with $\mathrm{im}(\psi)$.

Let's choose a unit normal vector field ν to $X \subset \mathbb{R}\mathrm{P}^n$ pointing inside the convex region C. Then the second fundamental form B of X is positive definite everywhere. For $(x, \Lambda) \in Gr_k(X)$ and an orthonormal basis $v_1, \ldots, v_k$ of Λ let's denote by $B_x(\Lambda) = \det(B(v_i, v_j))$ the determinant of the $k \times k$ matrix $\{B(v_i, v_j)\}$. Note that $B_x(\Lambda)$ does not depend on the choice of $v_1, \ldots, v_k$. Using the smooth coarea formula we prove the following proposition.

Proposition 11. *If $X \subset \mathbb{R}\mathrm{P}^n$ is a convex hypersurface, then*

$$|\Omega_k(X)| = \frac{|Gr(k, n-1)|}{\binom{n-1}{k}} \int_X \sigma_k(x) dV_X \tag{5.5}$$

where $\sigma_k : X \to \mathbb{R}$ is the k-th elementary symmetric polynomial of the principal curvatures of the embedding $X \hookrightarrow \mathbb{R}\mathrm{P}^n$.

Proof. The $O(n+1)$-invariant metric g on $\mathbb{G}(k, n)$ induces a Riemannian metric $\psi^* g$ on $Gr_k(X)$ through the embedding ψ. Note that the restriction of $\psi^* g$ to the fibers $Gr_k(T_x X)$ is $O(T_x X) \simeq O(n-1)$-invariant. We apply the smooth coarea formula [45, (A-2)] to $p : (Gr_k(X), \psi^* g) \to (X, g_X)$, where g_X denotes the induced metric on $X \hookrightarrow \mathbb{R}\mathrm{P}^n$. We obtain:

$$|\Omega_k(X)| = \int_{Gr_k(X)} dV_{Gr_k(X)} = \int_X \int_{Gr_k(T_x X)} \left(NJ_{(x,\Lambda)} p\right)^{-1} dV_{Gr_k(T_x X)}\, dV_X.$$

We show that the normal Jacobian $NJ_{(x,\Lambda)} p$ equals $|B_x(\Lambda)|^{-1} = |\det(B(v_i, v_j))|^{-1}$.

Given a point $x \in X$, a unit normal $\nu \in T_x \mathbb{R}\mathrm{P}^n$ to $T_x X$ and an orthonormal basis $v_1, \ldots, v_k \in T_x X$ of $\Lambda \in Gr_k(T_x X)$ let's complete them to an orthonormal basis $x, \nu, v_1, \ldots, v_k, v_{k+1}, \ldots, v_{n-1}$ of $\mathbb{R}^{n+1}$. Using these vectors we describe the tangent space to $Gr_k(X)$ at (x, Λ).

For $i = 1, \ldots, n-1$ and $j = 1, \ldots, k$ let $x_i = x_i(t)$ be a small curve through x in the direction v_i and let $v_j^i = v_j^i(t)$ be the parallel transport of v_j along x_i, i.e. the vector field solving $\nabla^X_{\dot{x}_i} v_j^i = 0, v_j^i(0) = v_j$. Note that for any time t the vectors $v_1^i(t), \ldots, v_k^i(t) \in T_{x_i(t)} X$ remain pairwise orthonormal. Consider now curves in $Gr_k(X)$ and their tangents produced by these vectors:

$$\begin{aligned} \widetilde{\gamma}_i(t) &= (x_i(t), v_1^i(t) \wedge \cdots \wedge v_k^i(t)) \\ \widetilde{\Gamma}_i &:= \dot{\widetilde{\gamma}}_i(0) = (v_i, \sum_{j=1}^k v_1 \wedge \cdots \wedge \dot{v}_j^i(0) \wedge \cdots \wedge v_k) \end{aligned}$$

Observe that

$$\dot{v}^i_j(0) = \nabla^{\mathbb{R}^{n+1}}_{v_i} v^i_j = \underbrace{\nabla^X_{v_i} v^i_j}_{=0} + a_{ij}\, x + b_{ij}\, \nu = a_{ij}\, x + b_{ij}\, \nu$$

Since the standard scalar product on $\mathbb{R}^{n+1}$ (here denoted by a dot) induces the metric on $T_x\mathbb{R}\mathrm{P}^n = T_x S^n = x^\perp$ and since the second fundamental form of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ coincides with the metric tensor we have

$$\begin{aligned} a_{ij} &= (\nabla^{\mathbb{R}^{n+1}}_{v_i} v^i_j) \cdot x = \delta_{ij} \\ b_{ij} &= (\nabla^{\mathbb{R}^{n+1}}_{v_i} v^i_j) \cdot \nu = (\nabla^{\mathbb{R}\mathrm{P}^n}_{v_i} v^i_j + \delta_{ij}\, x) \cdot \nu = (\nabla^{\mathbb{R}\mathrm{P}^n}_{v_i} v^i_j) \cdot \nu = B(v_i, v_j) \end{aligned} \tag{5.6}$$

The tangent space to the fiber $T_{(x,\Lambda)}Gr_k(T_xX) = \ker(p_*)$ is spanned by the following $k(n-1-k)$ vectors:

$$\begin{aligned} \widetilde{\theta}_{ij}(t) &= (x, v_1 \wedge \cdots \wedge \underset{i}{(v_i\, \cos t + v_j\, \sin t)} \wedge \cdots \wedge v_k),\ i = 1, \ldots, k \\ \widetilde{\Theta}_{ij} :&= \dot{\widetilde{\theta}}_{ij}(0) = (0, v_1 \wedge \cdots \wedge \underset{i}{v_j} \wedge \cdots \wedge v_k),\ j = k+1, \ldots, n-1 \end{aligned}$$

We work with the images $\Gamma_i, \Theta_{ij} \in T_{\mathrm{Span}\{x,\Lambda\}}\mathbb{G}(k,n)$ of $\widetilde{\Gamma}_i$ and $\widetilde{\Theta}_{ij}$ under ψ_*. It is easy to see that

$$\Gamma_i = \psi_*\widetilde{\Gamma}_i = v_i \wedge v_1 \wedge \cdots \wedge v_k + \sum_{j=1}^{k} b_{ij}\, x \wedge v_1 \wedge \cdots \wedge \underset{j}{\nu} \wedge \cdots \wedge v_k, \quad 1 \le i \le n-1$$

$$\Theta_{ij} = \psi_*\widetilde{\Theta}_{ij} = x \wedge v_1 \wedge \cdots \wedge \underset{i}{v_j} \wedge \cdots \wedge v_k, \quad 1 \le i \le k,\ k+1 \le j \le n-1$$

and Γ_i's are orthogonal to Θ_{ij}'s, but Γ_i's are not in general orthonormal vectors. Therefore, since $p_*\widetilde{\Gamma}_i = v_i$ and the v_i's form an orthonormal basis for T_xX in order to compute the normal Jacobian $NJ_{(x,\Lambda)}p$ we need to find a change of basis matrix from $\{\Gamma_i\}_{1\le i\le n-1}$ to some orthonormal basis of $\mathrm{Span}\{\Gamma_i\}_{1\le i\le n-1} = \ker(p_* \circ \psi_*^{-1})^\perp$. For this purpose let's note that for the orthonormal vectors

$$\begin{aligned} S_j &= x \wedge v_1 \wedge \cdots \wedge \underset{j}{\nu} \wedge \cdots \wedge v_k, \qquad && 1 \le j \le k \\ P_i &= v_i \wedge v_1 \wedge \cdots \wedge v_k, && k+1 \le i \le n-1 \end{aligned}$$

we have

$$\begin{pmatrix} \Gamma_1 \\ \vdots \\ \Gamma_k \\ \Gamma_{k+1} \\ \vdots \\ \Gamma_{n-1} \end{pmatrix} = \begin{pmatrix} b & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} S \\ R \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & b_{1k} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1} & \dots & b_{kk} & 0 & 0 & \dots & 0 \\ b_{k+1,1} & \dots & b_{k+1,k} & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n-1,1} & \dots & b_{n-1,k} & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_1 \\ \vdots \\ S_k \\ P_{k+1} \\ \vdots \\ P_{n-1} \end{pmatrix}$$

where $b = \{b_{ij}\}_{1\le i,j\le k} = \{B(v_i, v_j)\}_{1\le i,j\le k}$ by (5.6). Note that

$$\psi_* \text{ is injective iff } b \text{ is invertible iff } B|_\Lambda \text{ is non-degenerate.} \tag{5.7}$$

Then since B is positive definite everywhere b is invertible and

$$\begin{pmatrix} S_1 \\ \vdots \\ S_k \\ P_{k+1} \\ \vdots \\ P_{n-1} \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \vdots \\ \Gamma_k \\ \Gamma_{k+1} \\ \vdots \\ \Gamma_{n-1} \end{pmatrix}$$

Applying $p_* \circ \psi_*^{-1}$ to the S_j, P_i's we obtain that

$$NJ_{(x,\Lambda)}p = |\det(b^{-1})| = |B_x(\Lambda)|^{-1}$$

and thus

$$|\Omega_k(X)| = \int\limits_X \int\limits_{Gr_k(T_xX)} |B_x(\Lambda)|\, dV_{Gr_k(T_xX)} dV_X. \tag{5.8}$$

Since the fibers $Gr_k(T_xX)$ are endowed with $O(n-1) \simeq O(T_xX) \simeq O(\{x,\nu_x\}^\perp)$-invariant metric we may rewrite the inner integral as

$$\int\limits_{Gr_k(T_xX)} |B_x(\Lambda)|\, dV_{Gr_k(T_xX)} = |Gr(k,n-1)|\, \mathbb{E}_{\,\Lambda\in Gr(k,n-1)} |B_x(\Lambda)| \tag{5.9}$$

Since the restriction $B|_\Lambda$ of a positive definite form B is also positive definite, we have $B_x(\Lambda) > 0$ and hence

$$\mathbb{E}_{\,\Lambda\in Gr(k,n-1)} |B_x(\Lambda)| = \mathbb{E}_{\,\Lambda\in Gr(k,n-1)} B_x(\Lambda).$$

We prove that

$$\mathbb{E}_{\Lambda\in Gr(k,n-1)}B_x(\Lambda)=\binom{n-1}{k}^{-1}s_k(d_1(x),\dots,d_{n-1}(x))$$

where the $d_i(x)$'s are the principal curvatures of $X\subset\mathbb{RP}^n$ at the point x and s_k is the k-th elementary symmetric polynomial. Now let's choose an o.n.b. $e=\{\delta_1,\dots,\delta_{n-1}\}$ of T_xX in which the second fundamental form B is diagonal $D=\operatorname{diag}\{d_1,\dots,d_{n-1}\}$. For vectors v_i we denote by the same letters their coordinate representation in the basis e. Let V and E be $(n-1)\times k$ matrices with columns $\{v_i\}_{1\le i\le k}$ and $\{\delta_i\}_{1\le i\le k}$ respectively:

$$V=\begin{pmatrix} | & & | \\ v_1 & \dots & v_k \\ | & & | \end{pmatrix}\qquad E=\begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

There exists an orthogonal matrix $g\in O(n-1)$ s.t. $V=g\cdot E$ and then $b=\{B(v_i,v_j)\}_{1\le i,j\le k}$ can be written as $b=V^tDV=E^tg^tDgE$. In this view $B_x(\Lambda)=\det(b)=\det(E^tg^tDgE)$ is just the leading principal minor of g^tDg of order k. Note that $B_x(\Lambda)$ does not depend on the choice of g, namely it's invariant under the action of $\mathrm{Stab}_{\mathrm{Span}\{\delta_1,\dots,\delta_k\}}\simeq O(k)\times O(n-1-k)\subset O(n-1)$. Using this and the fact that the induced metric on the fibers $Gr_k(T_xX)\simeq Gr(k,n-1)$ is the standard $O(n-1)$-invariant metric we obtain

$$\begin{aligned}\mathbb{E}_{\Lambda\in Gr(k,n-1)}B_x(\Lambda)&=\frac{1}{|Gr(k,n-1)|}\int\limits_{Gr(k,n-1)}B_x(\Lambda)\,dV_{Gr(k,n-1)}\\&=\frac{1}{|Gr(k,n-1)|\cdot|O(k)|\cdot|O(n-1-k)|}\int\limits_{O(n-1)}\det(E^tg^tDgE)\,dg\\&=\frac{1}{|O(n-1)|}\int\limits_{O(n-1)}\det(E^tg^tDgE)\,dg\end{aligned}$$

where $dg=dV_{O(n-1)}$ is the invariant Haar measure on $O(n-1)$.

Now for any k-subset $I=\{i_1,\dots,i_k\}\subset\{1,\dots,n-1\}$ denote by E_I the $(n-1)\times k$ matrix with columns $\delta_{i_1},\dots,\delta_{i_k}$. E_I can be obtained as a left multiplication of E by the permutation matrix M_{σ_I}: $E_I=M_{\sigma_I}\cdot E$, where σ_I is any permutation that sends $1, \dots, k$ into $i_1, \dots, i_k$ respectively. Using invariance of dg we get

$$\int\limits_{O(n-1)} \det(E_I^t g^t D g E_I)\, dg = \int\limits_{O(n-1)} \det(E^t (g M_{\sigma_I})^t D (g M_{\sigma_I}) E)\, dg = \int\limits_{O(n-1)} \det(E^t g^t D g E)\, dg$$

Consequently we can express $\mathbb{E}_{\Lambda \in Gr(k,n-1)} B_x(\Lambda)$ as a sum over all k-subsets $I \subset \{1, \dots, n-1\}$ divided by $\binom{n-1}{k}$:

$$\mathbb{E}_{\Lambda \in Gr(k,n-1)} B_x(\Lambda) = \binom{n-1}{k}^{-1} \frac{1}{|O(n-1)|} \int\limits_{O(n-1)} \sum_{\substack{I \subset \{1,\dots,n-1\},\\ |I|=k}} \det(E_I^t g^t D g E_I)\, dg$$

The integrand here is the sum of all principal minors of $g^t D g$ of order k and thus does not depend on g and is equal to the k-th elementary symmetric polynomial $s_k(d_1, \dots, d_{n-1})$ of $d_1 \dots, d_{n-1}$. Combining this with (5.8) and (5.9) we end the proof. $\square$

In particular we can derive the following corollary.

Corollary 6. *If $X \subset \mathbb{R}\mathrm{P}^n$ is a convex hypersurface, then*

$$\frac{|\Omega_k(X)|}{|\mathrm{Sch}(k,n)|} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_X \sigma_k(x) dV_X.$$

Proof. We first observe that

$$\frac{|Gr(k,n-1)|}{|\mathbb{G}(k,n)|} = \frac{1}{\pi^{n/2}} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}$$

and, recalling [24, Theorem 4.2],

$$\frac{|\mathrm{Sch}(k,n)|}{|\mathbb{G}(k,n)|} = \frac{|\Sigma(k+1,n+1)|}{|Gr(k+1,n+1)|} = \frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)} \cdot \frac{\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)}.$$

Substituting into (5.5) we obtain

$$\begin{aligned}
\frac{|\Omega_k(X)|}{|\mathrm{Sch}(k,n)|} &= \frac{|Gr(k,n-1)|}{|\mathbb{G}(k,n)|}\cdot\frac{|\mathbb{G}(k,n)|}{|\mathrm{Sch}(k,n)|}\cdot\frac{1}{\binom{n-1}{k}}\int_X \sigma_k(x)dV_X\\
&= \frac{1}{\pi^{n/2}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}\cdot\frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}\cdot\frac{1}{\binom{n-1}{k}}\int_X \sigma_k(x)dV_X\\
&= \frac{1}{\pi^{n/2}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}\cdot\frac{1}{\binom{n-1}{k}}\int_X \sigma_k(x)dV_X\\
&= \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{\pi^{\frac{n+1}{2}}}\int_X \sigma_k(x)dV_X.
\end{aligned}$$

□

5.2.2 Intrinsic volumes

Recall that the intrinsic volumes $V_0(C),\ldots,V_{n-1}(C)$ of a convex set $C\subset\mathbb{R}\mathrm{P}^n$ are characterized by Steiner's formula, which gives the exact expansion (for small $\epsilon>0$) of the volume of the ϵ-neighbourhood of C:

$$|\mathcal{U}_{\mathbb{R}\mathrm{P}^n}(C,\epsilon)| = |C| + \sum_{k=0}^{n-1} f_k(\epsilon)|S^k||S^{n-k-1}|V_k(C) \tag{5.10}$$

(the functions f_k are defined in (5.4)). The formula (5.10) is obtained from the spherical Steiner's formula [38, (9)] as follows. For a convex set $C\subset\mathbb{R}\mathrm{P}^n$ denote by $\tilde{C}\subset S^n$ any of the two components of $p^{-1}(C)$, where $p: S^n\to\mathbb{R}\mathrm{P}^n$ is the double covering. Under p an open hemisphere in S^n maps isometrically onto $\mathbb{R}\mathrm{P}^n$ minus a hyperplane. Therefore, for small $\varepsilon>0$ we have $|\mathcal{U}_{\mathbb{R}\mathrm{P}^n}(\tilde{C},\epsilon)| = |\mathcal{U}_{\mathbb{R}\mathrm{P}^n}(C,\epsilon)|$ and $V_j(\tilde{C}) = V_j(C), j=0,\ldots,n-1$. As a consequence we obtain.

Corollary 7 (The manifold of k-tangents and intrinsic volumes)**.** *Let $C\subset\mathbb{R}\mathrm{P}^n$ be a strictly convex set with the smooth boundary ∂C. Then*

$$4\cdot V_{n-k-1}(C) = \frac{|\Omega_k(\partial C)|}{|\mathrm{Sch}(k,n)|},\quad k=0,\ldots,n-1.$$

Proof. From [38, (10)] and Corollary 6 it follows that

$$\begin{aligned} V_{n-k-1}(C) &= \frac{1}{|S^k||S^{n-k-1}|}\int_{\partial C}\sigma_k(x)dV_{\partial C} \\ &= \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}{4\pi^{\frac{n+1}{2}}}\int_{\partial C}\sigma_k(x)dV_{\partial C} \\ &= \frac{1}{4}\cdot\frac{|\Omega_k(\partial C)|}{|\mathrm{Sch}(k,n)|}. \end{aligned}$$

□

This together with [38, (15)] implies the following interesting corollary.

Corollary 8. *Let $C \subset \mathbb{R}\mathrm{P}^n$ be a strictly convex set with the smooth boundary ∂C and let C° be the polar set of $\tilde{C} \subset S^n$. Then*

$$\frac{4|C|}{|S^n|} + \frac{4|C^\circ|}{|S^n|} + \sum_{k=0}^{n-1}\frac{|\Omega_k(\partial C)|}{|\mathrm{Sch}(k,n)|} = 4.$$

In particular, for every $k = 0, \ldots, n-1$ we have

$$\frac{|\Omega_k(\partial C)|}{|\mathrm{Sch}(k,n)|} \leq 4. \tag{5.11}$$

5.3 Hypersurfaces in random position

Theorem 20. *The average number of k-planes in $\mathbb{R}\mathrm{P}^n$ simultaneously tangent to convex hypersurfaces $X_1, \ldots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ in random position equals*

$$\tau_k(X_1, \ldots, X_{d_{k,n}}) = \delta_{k,n}\cdot\prod_{i=1}^{d_{k,n}}\frac{|\Omega_k(X_i)|}{|\mathrm{Sch}(k,n)|}. \tag{5.12}$$

Proof. We use the generalized kinematic formula for coisotropic hypersurfaces of $\mathbb{G}(k,n)$ proved in [24] (Theorem 19 above).

In order to apply Theorem 19 to the case $\mathcal{H}_i = \Omega_k(X_i), i = 1, \ldots, d_{k,n}$, we need to prove that each $\Omega_k(X_i)$ is a coisotropic hypersurface of $\mathbb{G}(k,n)$. Given $(x, \Lambda) \in Gr_k(X_i)$ as in the proof of Proposition 11 let's consider an orthonormal basis $v_1, \ldots, v_{n-1}$ of T_xX_i such that $\Lambda = \mathrm{span}\{v_1, \ldots, v_k\}$ and a unit normal $\nu \in T_x\mathbb{R}\mathrm{P}^n$ to T_xX_i. For a curve $x_\nu(t) \subset \mathbb{R}\mathrm{P}^n$ through x in the direction ν we consider the parallel transports $v_1^\nu(t), \ldots, v_k^\nu(t) \in T_{x_\nu(t)}\mathbb{R}\mathrm{P}^n$ of $v_1, \ldots, v_k$ along $x_\nu(t)$. We claim that the tangent vector to the curve $\gamma(t) = x_\nu(t) \wedge v_1^\nu(t) \wedge \cdots \wedge v_k^\nu(t) \in \mathbb{G}(k, n)$ is normal to $T_{x\wedge v_1 \wedge \cdots \wedge v_k}\Omega_k(X_i)$. Indeed,

$$\dot\gamma(0) = \nu \wedge v_1 \wedge \cdots \wedge v_k + \sum_{j=1}^{k} x \wedge v_1 \wedge \cdots \wedge \dot v_j^\nu(0) \wedge \cdots \wedge v_k = \nu \wedge v_1 \wedge \cdots \wedge v_k$$

since $\dot v_j^\nu(0) = \nabla_\nu^{\mathbb{R}\mathrm{P}^n} v_j^\nu + a_j x = 0 + a_j x$ is proportional to x. Now it is elementary to verify that $\dot\gamma(0)$ is orthogonal to the tangent space $T_{x\wedge v_1\wedge\cdots\wedge v_k}\Omega_k(X_i)$ described in (11). Seen as an operator $\dot\gamma(0)$ sends x to ν and all vectors in Λ to 0. Hence $\Omega_k(X_i)$ is coisotropic.

Applying now Theorem 19 we deduce

$$\tau_k(X_1, \dots, X_{d_{k,n}}) = \alpha(k+1, n-k)\, |\mathbb{G}(k, n)| \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|\mathbb{G}(k, n)|}. \tag{5.13}$$

Note that applying Theorem 19 to the special real Schubert variety $\mathrm{Sch}(k, n)$ we obtain

$$\begin{aligned} \delta_{k,n} &= \mathbb{E}\,\#(g_1\mathrm{Sch}(k, n) \cap \cdots \cap g_{d_{k,n}}\mathrm{Sch}(k, n)) \\ &= \alpha(k+1, n-k)\, |\mathbb{G}(k, n)| \left(\frac{|\mathrm{Sch}(k, n)|}{|\mathbb{G}(k, n)|}\right)^{d_{k,n}}. \end{aligned}$$

This gives an expression for $\alpha(k+1, n-k)$, which substituted into (5.13) gives (5.12). □

As a consequence we derive the following corollary, which gives a universal upper bound to our random enumerative problem.

Corollary 9. *If $X_1, \dots, X_{d_{k,n}} \subset \mathbb{R}\mathrm{P}^n$ are convex hypersurfaces, then*

$$\tau_k(X_1, \dots, X_{d_{k,n}}) \le \delta_{k,n} \cdot 4^{d_{k,n}}.$$

Proof. This follows immediately from (5.12) and (5.11). □

5.4 Convex bodies with many common tangents

In this section we show that for every $m > 0$ there exist convex surfaces $X_1, \dots, X_4 \subset \mathbb{R}\mathrm{P}^3$ in general position such that the intersection $\Omega_1(X_1) \cap \cdots \cap \Omega_1(X_4) \subset \mathbb{G}(1, 3)$ is transverse and consists of at least m points. We owe the main idea for this to T. Theobald.

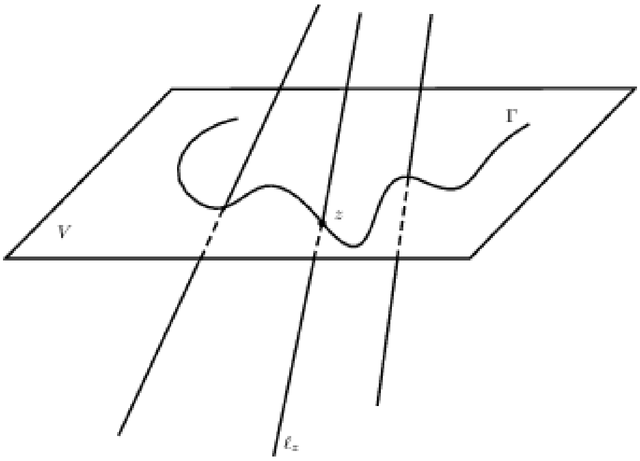

Figure 5.2: The construction of the coordinate system.

5.4.1 A coordinate system

Let $X_1, X_2, X_3 \subset \mathbb{R}\mathrm{P}^3$ be smooth convex semialgebraic surfaces such that the intersection $Z = \Omega_1(X_1) \cap \Omega_1(X_2) \cap \Omega_1(X_3)$ is transverse (hence Z is a smooth curve in $\mathbb{G}(1,3)$). Let

$$P = \{(\Lambda, [v]) : \Lambda \in Z, [v] \in \Lambda \simeq \mathbb{R}\mathrm{P}^1\}$$

be the projectivized tautological bundle over Z and consider the tautological map

$$\begin{aligned} \eta : P &\to \mathbb{R}\mathrm{P}^3 \\ (\Lambda, [v]) &\mapsto [v] \end{aligned}$$

We determine points where η is an immersion.

Lemma 10. *$\eta_* : T_{(\Lambda,[v])}P \to T_{[v]}\mathbb{R}\mathrm{P}^3 \simeq v^\perp$ is injective if and only if v is not annihilated by the generator of $T_\Lambda Z \subset \mathrm{Hom}(\Lambda, \Lambda^\perp)$.*

Proof. Let $\Lambda(t) = v(t) \wedge u(t)$ be a local parametrization of Z near $\Lambda = \Lambda(0)$, where $\{u(t), v(t)\}$ is an orthonormal basis of $\Lambda(t)$ and $v = v(0), u = u(0)$. The tangent vectors to the curves $\gamma_1(t) = (\Lambda, [\cos t\, v + \sin t\, u])$, $\gamma_2(t) = (\Lambda(t), [v(t)])$ at $t = 0$ span the tangent space $T_{(\Lambda,[v])}P$ and $\eta_*(\dot\gamma_1(0)) = [u]$, $\eta_*(\dot\gamma_2(0)) = [\dot v(0)]$. Any generator of the one-dimensional space $T_\Lambda Z \subset \mathrm{Hom}(\Lambda, \Lambda^\perp)$ sends $v \in \Lambda$ to $\dot v(0) \in \Lambda^\perp \subset v^\perp$. The assertion follows. $\square$

Let $(\Lambda, [v]) \in P$ be a point where η is an immersion (by the above lemma such $(\Lambda, [v]) \in P$ exists for any $\Lambda \in Z$) and let $V \simeq \mathbb{R}\mathrm{P}^2 \subset \mathbb{R}\mathrm{P}^3$ be a plane through $[v] = \eta((\Lambda, [v])) \in \mathbb{R}\mathrm{P}^3$ that is transversal to the line $\ell_{[v]} := \eta((\Lambda, \Lambda))$. The map

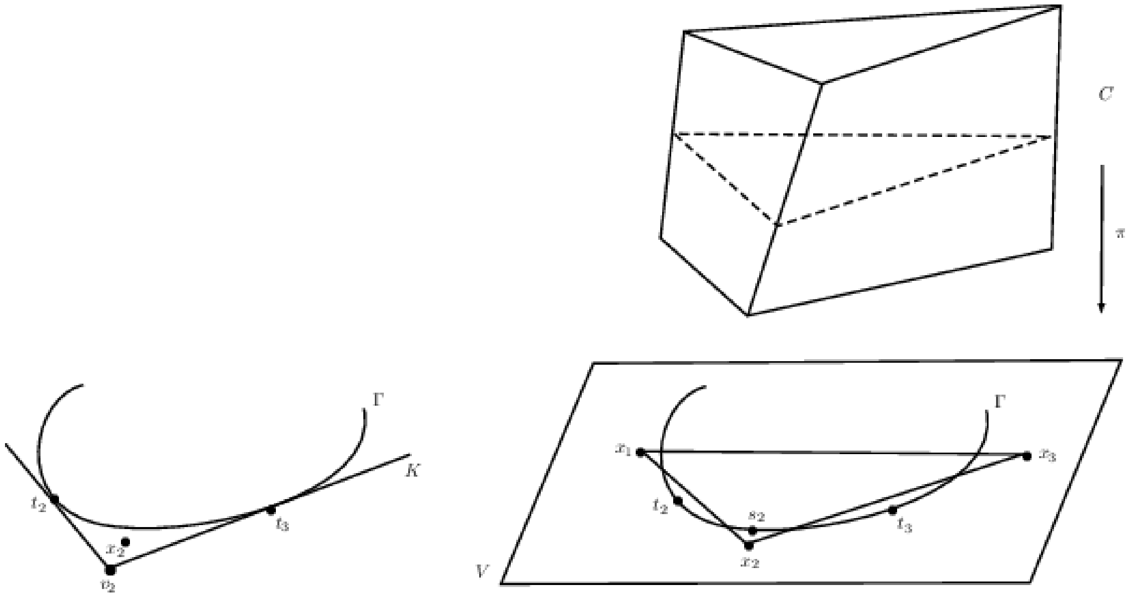

Figure 5.3: The convex body C.

η is an embedding locally near $(\Lambda, [v])$. Therefore the image under η of a small neighbourhood of $(\Lambda, [v])$ intersects V along a smooth curve which we denote by Γ. Moreover, the images of the fibers of P define a smooth field of directions $\{\ell_z : z \in \Gamma\}$ on Γ (see Figure 5.2) which can be smoothly extended to a field of directions $\{\ell_z : z \in U\}$ on a neighbourhood $U \subset V$ of Γ.

As a consequence there exists a neighborhood $W \subset \mathbb{R}\mathrm{P}^3$ of $[v]$ of the form

$$W = \coprod_{z \in U} \ell_z \cap W \simeq U \times (-1, 1)$$

On this neighbourhood we have a smooth map (the projection on the first factor):

$$\pi : W \to U.$$

This map has the following property:

Lemma 11. *If $B \subset W$ is a smooth strictly convex subset in $\mathbb{R}\mathrm{P}^3$ and $z \in U$ is a critical value for $\pi|_{\partial B}$, then ℓ_z is tangent to ∂B.*

Proof. In fact if $\#\{\ell_z \cap \partial B\} = 2$ then the line ℓ_z would be trasversal to ∂B and z would be a regular value for $\pi|_{\partial B}$. $\square$

5.4.2 The construction

Using strict convexity of X_1, X_2, X_3 it is easy to show that for a generic choice of the plane V a small arc of the curve Γ is strictly convex. Let's use the same letter

Γ to denote such an arc. For a given number $m > 0$ pick $n = m + 1$ distinct points $t_1, \ldots, t_n$ on Γ and consider an $n-$polygonal arc K tangent to Γ at the points $t_1, \ldots, t_n$. Call $v_1 \ldots, v_{n-1}$ the ordered vertices of K and for every (curvilater) triangle $t_i v_i t_{i+1}$ pick a point x_i in its interior (see left picture in Figure 5.3).

Let now $C \subset W$ be the convex body in $\mathbb{R}\mathrm{P}^3$ defined as the convex hull of the segments in $W \simeq U \times (-1, 1)$:

$$C = \mathrm{conv}(\{x_1\} \times (-\delta, \delta), \ldots, \{x_{n-1}\} \times (-\delta, \delta)),$$

where $\delta > 0$ is chosen small enough such that none of $t_1, \ldots, t_n$ belongs to $\pi(C)$. Note that the polygon $x_1 \cdots x_{n-1}$ is a subset of $\pi(C) \subset C$. As a consequence, there exist points $s_1, \ldots, s_{n-1}$ on Γ, interlacing $t_1, \ldots, t_n$ such that they all belong to $\mathrm{im}(\pi|_{\mathrm{int}(C)})$. (See the right picture in Figure 5.3.)

Let now $C_\epsilon \subset W$ be a smooth, strictly convex semialgebraic approximation of C such that:

(1) $s_1, \ldots, s_n \in \pi|_{\mathrm{int}(C_\epsilon)}$;

(2) $t_1, \ldots, t_n \notin \pi(C_\epsilon)$;

(3) the intersection $\Omega_1(C_\epsilon) \cap \Omega_1(X_1) \cap \Omega_1(X_2) \cap \Omega_1(X_3)$ is transverse.

The conditions (1) and (2) imply that $\pi(\partial C_\epsilon) \cap \Gamma$ (a semialgebraic subset of Γ) consists of intervals:

$$\pi(\partial C_\epsilon) \cap \Gamma = [a_1, b_1] \cup \cdots \cup [a_N, b_N]$$

possibly reduced to points and $N \geq n - 1$. Now each a_i (and b_i) is critical for $\pi|_{\partial C_\epsilon}$: otherwise the image of $\pi|_{\partial C_\epsilon}$ near a_i would contain an open set and a_i would not be a boundary point of the intersection $\pi(\partial C_\epsilon) \cap \Gamma$. By Lemma 11 this implies that each line ℓ_{a_i} is tangent to ∂C_ϵ and condition (3) implies that the transverse intersection $\Omega_1(C_\epsilon) \cap \Omega_1(X_1) \cap \Omega_1(X_2) \cap \Omega_1(X_3)$ (which is finite) contains more than $n - 1 = m$ lines.

5.5 The semialgebraic case

In this section we discuss a generalization of some of the results of Section 5.2 and Section 5.3 to the case of semialgebraic hypersurfaces satisfying some nondegeneracy conditions.

Let X be a smooth closed semialgebraic hypersurface in $\mathbb{R}\mathrm{P}^n$. As in Section 5.2.1 we define the Grassmannian bundle of k-planes over X:

$$\begin{aligned} p : Gr_k(X) &\to X \\ (x, \Lambda) &\mapsto x \\ Gr_k(X) :&= \{(x, \Lambda) : x \in X, \Lambda \in Gr_k(T_x X) \simeq Gr(k, n-1)\} \end{aligned}$$

The variety $\Omega_k(X)$ of k-tangents to X coincides with the image $\mathrm{im}(\psi)$ of the *kth Gauss map*:

$$\begin{aligned}\psi : Gr_k(X) &\to \mathbb{G}(k,n)\\ (x,\Lambda) &\mapsto \mathrm{P}(\mathrm{Span}\{x,\Lambda\})\end{aligned}$$

but now, unlike to the case of a convex hypersurface, $\Omega_k(X)$ is in general singular.

It is convenient to identify the smooth manifold $Gr_k(X)$ with its image in $X \times \mathbb{G}(k,n)$ under the map $\mathrm{id} \times \psi$:

$$\begin{aligned}Gr_k(X) \simeq (\mathrm{id} \times \psi)(Gr_k(X)) &= \{(x,\Lambda) \in X \times \mathbb{G}(k,n) : T_x\Lambda \subset T_xX\}\\ \mathrm{id} \times \psi : Gr_k(X) &\to X \times \mathbb{G}(k,n)\\ (x,\Lambda) &\mapsto (x, \mathrm{P}(\mathrm{Span}\{x,\Lambda\}))\end{aligned}$$

Note that $Gr_k(X)$ is a smooth semialgebraic subvariety of $X \times \mathbb{G}(k,n)$ and the variety of tangents $\Omega_k(X)$ is obtained by projecting it onto the second factor.

For a point $x \in X$ let's denote by B the second fundamental form of X defined locally near x using any of the two local coorientations of X. For $(x,\Lambda) \in Gr_k(X)$ and an orthonormal basis $v_1, \dots, v_k$ of Λ denote by $B_x(\Lambda) = \det(B(v_i, v_j))$ the determinant of the $k \times k$ matrix $\{B(v_i, v_j)\}$. Notice that $|B_x(\Lambda)|$ does not depend on the choice of $v_1, \dots, v_k$ and the local coorientation of X near x.

Definition 12. *We say that $X \subset \mathbb{R}\mathrm{P}^n$ is* k-non-degenerate *if*

1. *the semialgebraic set*

$$D := \{\Lambda \in \mathbb{G}(k,n) : \#(\psi^{-1}(\Lambda)) > 1\} \subset \Omega_k(X)$$

of k-flats that are tangent to X at more than one point has codimension at least one in $\Omega_k(X)$ and

2. *the semialgebraic set*

$$S := \{(x,\Lambda) \in Gr_k(X) : B|_{T_x\Lambda} \textit{ is degenerate}\}$$

has codimension at least one in the semialgebraic variety $Gr_k(X)$.

Remark 20. *Note that the sets D and S are closed in $\Omega_k(X)$ and $Gr_k(X)$ respectively and, by the same reasoning as in the proof of Proposition 11 (up to* (5.7)*), the set S consists of such $(x,\Lambda) \in Gr_k(X)$ where $\pi_2 : Gr_k(x) \to \mathbb{G}(k,n)$ is not an immersion.*

A convex semialgebraic hypersurface is k-non-degenerate for any $k = 0, \dots, n-1$ since in this case the sets D, S from Definition 12 are empty. The following lemma shows that a generic algebraic surface in $\mathbb{R}\mathrm{P}^3$ of sufficiently high degree is 1-non-degenerate.

Lemma 12. *Let $X^{\mathbb{C}} \subset \mathbb{C}\mathrm{P}^3$ be an irreducible smooth surface of degree $d \geq 4$ which does not contain any lines and such that $X = \mathbb{R}X^{\mathbb{C}} \subset \mathbb{R}\mathrm{P}^3$ is of dimension 2. Then X is 1-non-degenerate.*

Proof. Theorem 4.1 in [49] asserts that under the assumptions of the current lemma the singular locus $\Sigma^{\mathbb{C}} := \mathrm{Sing}(\Omega_1(X^{\mathbb{C}}))$ of the variety $\Omega_1(X^{\mathbb{C}}) \subset \mathbb{G}^{\mathbb{C}}(1,3)$ of complex lines tangent to the complex surface $X^{\mathbb{C}} \subset \mathbb{C}\mathrm{P}^3$ is described as follows:

$$\Sigma^{\mathbb{C}} = D^{\mathbb{C}} \cup I^{\mathbb{C}},$$

where $D^{\mathbb{C}}$ consists of lines that are tangent to $X^{\mathbb{C}}$ at more than one point and $I^{\mathbb{C}}$ consists of lines intersecting $X^{\mathbb{C}}$ at some point with multiplicity at least 3.

We now show that the singular locus $\Sigma := \mathrm{Sing}(\Omega_1(X)) = \Omega_1(X) \cap \mathrm{Sing}(\Omega_1^{\mathbb{C}}(X))$ of $\Omega_1(X)$ is of dimension at most 2. There are two cases: either (1) there exists $\Lambda \in \Sigma$ which is smooth for both Σ and $\Sigma^{\mathbb{C}}$ or (2) any smooth point $\Lambda \in \Sigma$ of Σ is singular for $\Sigma^{\mathbb{C}}$. In the case (1) we have $\dim_{\mathbb{R}}(\Sigma) = \dim_{\mathbb{R}}(T_\Lambda \Sigma) = \dim_{\mathbb{C}}(T_\Lambda \Sigma^{\mathbb{C}}) = \dim_{\mathbb{C}}(\Sigma^{\mathbb{C}}) < \dim_{\mathbb{C}}(\Omega_1(X^{\mathbb{C}})) = 3$ and therefore $\dim_{\mathbb{R}}(\Sigma) \leq 2$. In the case (2) we have $\dim_{\mathbb{R}}(\Sigma) = \dim_{\mathbb{R}}(T_\Lambda \Sigma) \leq \dim_{\mathbb{C}}(\mathrm{Sing}(\Sigma^{\mathbb{C}})) < \dim_{\mathbb{C}}(\Sigma^{\mathbb{C}}) < \dim_{\mathbb{C}}(\Omega_1(X^{\mathbb{C}})) = 3$ and hence $\dim_{\mathbb{R}}(\Sigma) \leq 1$.

For the complex surface $X^{\mathbb{C}} \subset \mathbb{R}\mathrm{P}^3$ let $Gr_1(X^{\mathbb{C}}) = \{(x,\Lambda) \in X^{\mathbb{C}} \times \mathbb{G}^{\mathbb{C}}(1,3) : T_x\Lambda \subset T_x X^{\mathbb{C}}\}$ be the Grassmannian bundle of complex lines over $X^{\mathbb{C}}$. In the proof of [49, Thm. 4.1] it is shown that a line $\Lambda \in I^{\mathbb{C}}$ intersects $X^{\mathbb{C}} \subset \mathbb{C}\mathrm{P}^3$ at a point $x \in X^{\mathbb{C}}$ with multiplicity at least 3 if and only if the differential $(\pi_2)_* : T_{(x,\Lambda)} Gr_1(X^{\mathbb{C}}) \to T_\Lambda \mathbb{G}^{\mathbb{C}}(1,3)$ is not injective. By (5.7) for $(x,\Lambda) \in S$ the differential $(\pi_2)_* : T_{(x,\Lambda)} Gr_1(X) \to T_\Lambda \mathbb{G}(1,3)$ (and hence also $(\pi_2)_* : T_{(x,\Lambda)} Gr_1(X^{\mathbb{C}}) \to T_\Lambda \mathbb{G}^{\mathbb{C}}(1,3)$) is not injective. In particular, $\pi_2(S) \subset \Omega_1(X) \cap I^{\mathbb{C}}$. Now, if $X^{\mathbb{C}}$ does not contain any lines, the fibers of the projection $\pi_2 : Gr_1(X) \to \mathbb{G}(1,3)$ are finite and hence $\dim(\pi_2(S)) = \dim S$. On the other hand, since $\pi_2(S) \subset \Sigma$, the above arguments show that $\dim(\pi_2(S)) \leq \dim(\Sigma) \leq 2$ and consequently $\dim(S) \leq 2 < 3 = \dim(Gr_1(X))$. Moreover, this together with (5.7) imply that there exists a point in $Gr_1(X)$ at which $\pi_2 : Gr_1(X) \to \mathbb{G}(1,3)$ is an immersion and hence $\Omega_1(X) = \pi_2(Gr_1(X))$ is of dimension 3.

Observe finally that $D \subset \Omega_1(X) \cap D^{\mathbb{C}} \subset \Sigma$ and the above arguments imply that $\dim(D) \leq \dim(\Sigma) \leq 2 < 3 = \dim(\Omega_1(X))$. This finishes the proof. $\square$

Remark 21. *The above lemma implies that a generic algebraic surface $X \subset \mathbb{R}\mathrm{P}^3$ of high enough degree is 1-non-degenerate.*

In the following proposition we provide a formula for the volume of $\Omega_k(X)$.

Proposition 12. *Let X be a k-non-degenerate semialgebraic hypersurface in $\mathbb{R}\mathrm{P}^n$. Then*

$$|\Omega_k(X)| = |Gr(k,n-1)| \int_X \mathbb{E}_{\Lambda \in Gr(k,n-1)} |B_x(\Lambda)| \, dV_X \tag{5.14}$$

Proof. The complement

$$R := Gr_k(X) \setminus S = \{(x, \Lambda) \in Gr_k(X) : B|_{T_x\Lambda} \text{ is non-degenerate}\}$$

of S is an open dense semialgebraic subset of $Gr_k(X)$. Let's pull back the metric from $\mathbb{G}(k, n)$ to R through the immersion $\pi_2|_R$. Then repeating the proof of Proposition 11 up to the point (5.8) we get

$$\int_R dV_R = \int_{X_R} \int_{\Lambda \in \pi_1^{-1}(x) \cap R} |B_x(\Lambda)| \, dV_{\pi_1^{-1}(x) \cap R} \, dV_{X_R} \tag{5.15}$$

where $X_R := \pi_1(R) \subset X$ is the projection of $R \subset X \times \mathbb{G}(k, n)$ onto the first factor and the fiber $\pi_1^{-1}(x) = Gr_k(T_xX) \simeq Gr(k, n-1) \subset Gr_k(X)$ is endowed with the uniform distribution. Note that since $B_x(\Lambda) = 0$ precisely for $\Lambda \in \pi_1^{-1}(x) \setminus R$ we can extend the integration over the whole fiber $\pi_1^{-1}(x)$ in (5.15). Moreover, since $X_R = \pi_1(R)$ is open and dense in X (being the image of an open and dense set under the projection π_1) and since the function

$$x \mapsto \int_{\Lambda \in \pi_1^{-1}(x)} |B_x(\Lambda)| \, dV_{\pi_1^{-1}(x)}$$

is continuous (5.15) becomes

$$\int_R dV_R = \int_X \int_{\Lambda \in \pi_1^{-1}(x)} |B_x(\Lambda)| \, dV_{\pi_1^{-1}(x)} \, dV_X = |Gr(k, n-1)| \int_X \mathbb{E}_{\Lambda \in Gr(k, n-1)} |B_x(\Lambda)| \, dV_X$$

It remains to prove that $|\Omega_k(X)| = \int_R dV_R$. For this let's consider the set

$$\tilde{D} := \pi_2^{-1}(D) = \{(x, \Lambda) \in Gr_k(X) : \#(\pi_2^{-1}(\Lambda)) > 1\}$$

Note that $\tilde{D}$ is a closed semialgebraic subset of $Gr_k(X)$ and from Definition 12 it follows that $\tilde{D} \subset Gr_k(X)$ is of codimension at least one. As a consequence, the semialgebraic set $R \setminus \tilde{D}$ is open and dense in $Gr_k(X)$ (and hence also in R) and therefore its projection $\pi_2(R \setminus \tilde{D})$ is open and dense in $\Omega_k(X)$. In particular,

$$|\Omega_k(X)| = |\pi_2(R \setminus \tilde{D})| = \int_{R \setminus \tilde{D}} dV_{R \setminus \tilde{D}} = \int_R dV_R$$

□

Remark 22. *Using, for example, the Cauchy-Binet theorem it is easy to derive the inequality*

$$|\Omega_k(X)| \leq \frac{|Gr(k,n-1)|}{\binom{n-1}{k}} \int_X s_k(|d_1(x)|,\dots,|d_{n-1}(x)|)\, dV_X$$

where $s_k(|d_1(x)|,\dots,|d_{n-1}(x)|)$ *is the* k*th elementary symmetric poynomial of the absolute principal curvatures at* $x \in X$. *Unfortunately, we do not have a clear geometric interpretation of the right-hand side of the above inequality.*

In the case of lines tangent to a surface in $\mathbb{R}\mathrm{P}^3$ we can refine the formula (5.14) as follows.

Corollary 10. *If* $X \subset \mathbb{R}\mathrm{P}^3$ *is a smooth* 1*-non-degenerate surface then*

$$|\Omega_1(X)| = \int\limits_X h(d_1(x), d_2(x))\, dV_X$$

where

$$h(d_1,d_2) = \begin{cases} \dfrac{\pi}{2}\,|d_1+d_2|, & \text{if } d_1 d_2 \geq 0 \\ 2\sqrt{-d_1 d_2} + 2|d_1+d_2| \cdot \left|\arctan\sqrt{-\dfrac{d_1}{d_2}} - \dfrac{\pi}{4}\right|, & \text{if } d_1 d_2 < 0 \end{cases}$$

and $d_1(x), d_2(x)$ *are the principal curvatures of* X *at the point* x.

Proof. The formula (5.14) reads

$$|\Omega_1(X)| = \pi \int\limits_X \mathbb{E}_{\,\Lambda\in Gr(1,2)} |B_x(\Lambda)|\, dV_X$$

In coordinates in which the second fundamental form B_x of $X \subset \mathbb{R}\mathrm{P}^3$ at the point $x \in X$ is diagonal with values d_1, d_2 we have

$$\pi\,\mathbb{E}_{\,\Lambda\in Gr(1,2)}|B_x(\Lambda)| = \pi\,\mathbb{E}_{\,v\in S^1}|B_x(v,v)| = \int_{-\pi/2}^{\pi/2} |d_1\cos^2\varphi + d_2 \sin^2\varphi| d\varphi$$

The last integral can be evaluated by elementary integration methods giving $\frac{\pi}{2}|d_1+d_2|$ in case $d_1 d_2 \geq 0$ and

$$2\sqrt{-d_1 d_2} + 2|d_1+d_2| \cdot \left|\arctan\sqrt{-\frac{d_1}{d_2}} - \frac{\pi}{4}\right|$$

in case $d_1 d_2 < 0$. $\square$

Finally we prove an analog of Theorem 20 for k-non-degenerate semialgebraic hypersurfaces.

Theorem 21. *The average number of k-flats in $\mathbb{RP}^n$ simultaneously tangent to k-non-degenerate semialgebraic hypersurfaces $X_1, \dots, X_{d_{k,n}}$ in random position equals*

$$\tau_k(X_1, \dots, X_{d_{k,n}}) = \delta_{k,n} \cdot \prod_{i=1}^{d_{k,n}} \frac{|\Omega_k(X_i)|}{|\mathrm{Sch}(k,n)|}.$$

Proof. Exactly in the same way as in the proof of Theorem 20 one can show that the union $\Omega_k(X_i)_{\mathrm{top}}$ of all top-dimensional cells of some fixed decomposition of $\Omega_k(X_i)$ (see Section 1.2 for details) is a coisotropic hypersurface of $\mathbb{G}(k,n)$. Since $\Omega_k(X_i) \setminus \Omega_k(X_i)_{\mathrm{top}}$ has codimension ≥ 2 in $\mathbb{G}(k,n)$ by standard transversality arguments we have that

$$g_1\Omega_k(X_1) \cap \cdots \cap g_{d_{k,n}}\Omega_k(X_{d_{k,n}}) = g_1\Omega_k(X_1)_{\mathrm{top}} \cap \cdots \cap g_{d_{k,n}}\Omega_k(X_{d_{k,n}})_{\mathrm{top}}$$

for a generic choice of $g_1, \dots, g_{d_{k,n}} \in O(n+1)$.

The claim follows by applying the integral geometry formula (Theorem 19) to the coisotropic semialgebraic hypersurfaces $\Omega_k(X_1)_{\mathrm{top}}, \dots, \Omega_k(X_{d_{k,n}})_{\mathrm{top}}$ as in the proof of Theorem 20. □

Remark 23. *(Random invariant hypersurfaces) The previous Theorem can be used for computing the expectation of the number of k-flats tangent to random Kostlan hypersurfaces of degree $m_1, \dots, m_{d_{k,n}}$ in $\mathbb{RP}^n$ – notice that here the randomness comes directly from the hypersurfaces! Let us discuss the case $n=3$, $k=1$.*

Let $f_1, \dots, f_4 \in \mathbb{R}[x_1, \dots, x_4]$ be random, independent, $O(4)$-invariant polynomials of degrees $m_1, \dots, m_4 \geq 4$ and denote by $X(f_i) = \{f_i = 0\} \subset \mathbb{RP}^3$, $i = 1, \dots, 4$ the corresponding projective hypersurfaces. We are interested in computing

$$(*) = \mathbb{E}_{f_1,\dots,f_4} \# \Omega_1(X(f_1)) \cap \cdots \cap \Omega_1(X(f_4)).$$

We use the fact that the polynomials are invariant and write:

$$\begin{aligned}(*) &= \mathbb{E}_{g_1,\dots,g_4} \mathbb{E}_{f_1,\dots,f_4} \# \Omega_1(g_1X(f_1)) \cap \cdots \cap \Omega_1(g_4X(f_4)) \\ &= \mathbb{E}_{f_1,\dots,f_4} \mathbb{E}_{g_1,\dots,g_4} \# \Omega_1(g_1X(f_1)) \cap \cdots \cap \Omega_1(g_4X(f_4)).\end{aligned}$$

For $i = 1, \dots, 4$ with probability one $X(f_i)$ is irreducible and there are no lines on it; hence by Lemma 12 with probability one each $X(f_i)$ is 1-non-degenerate. Applying Theorem 21 we have

$$\mathbb{E}_{f_1,\dots,f_4} \# \Omega_1(X(f_1)) \cap \cdots \cap \Omega_1(X(f_4)) = \delta_{1,3} \cdot \prod_{i=1}^{4} \frac{\mathbb{E}_{f_i} |\Omega_1(X(f_i))|}{|\mathrm{Sch}(1,3)|}.$$

Computation of the expected volume $\mathbb{E}_f |\Omega_1(X(f))|$ of the variety of tangent lines to a random invariant hypersurface is a difficult task though.

Appendix

The following elementary lemma is used in Proposition 5 and throughout Section 4.2.

Lemma 13. *If $X \subset (\mathbb{R}^m, \|\cdot\|)$ is a scale-invariant semialgebraic set of dimension $p \leq m$, $f : X \to \mathbb{R}$ is a measurable positively homogeneous function of degree $d \geq 0$ and $q > 0$, then*

$$\int\limits_{a\in X} f(a)\|a\|^q\, e^{-\frac{\|a\|^2}{2}} dV_X = \sqrt{2}^q \frac{\Gamma\left(\frac{d+p+q}{2}\right)}{\Gamma\left(\frac{d+p}{2}\right)} \int\limits_{a\in X} f(a) e^{-\frac{\|a\|^2}{2}} dV_X, \tag{5.16}$$

where $|X \cap S^{m-1}|$ denotes the volume of the $(r-1)$-dimensional semialgebraic spherical set $X \cap S^{m-1}$. If $f = 1$ on X, then

$$\int\limits_{a\in X} \|a\|^q\, e^{-\frac{\|a\|^2}{2}} dV_X = \sqrt{2}^{p+q-2} \Gamma\left(\frac{p+q}{2}\right) |X \cap S^{m-1}| \tag{5.17}$$

Proof. By the smooth coarea formula (Theorem 6) applied to the submersion $\pi : X_{\text{top}} \to X_{\text{top}} \cap S^{m-1}$, $\pi(a) = a/\|a\|$ whose Normal Jacobian is $NJ_a\pi = 1/\|a\|^{p-1}$ we have:

$$\begin{aligned}
\int\limits_{a\in X} f(a)\|a\|^q e^{-\frac{\|a\|^2}{2}} dV_X &= \int\limits_0^{+\infty} r^{d+p+q-1} e^{-\frac{r^2}{2}} dr \int\limits_{a\in X\cap S^{m-1}} f(a)\, dV_{X\cap S^{m-1}} \\
&= \sqrt{2}^{d+p+q-2} \Gamma\left(\frac{d+p+q}{2}\right) \int\limits_{a\in X\cap S^{m-1}} f(a)\, dV_{X\cap S^{m-1}}
\end{aligned}$$

Combining this with the same formula for $q = 0$ we obtain (5.16).

If $f = 1$ on X we have

$$\int\limits_{a\in X\cap S^{m-1}} f(a)\, dV_{X\cap S^{m-1}} = |X \cap S^{m-1}|$$

and (5.17) follows. □

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