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Symmetry and quantitative stability results for problems in geometric analysis and functional inequalities

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Symmetry and quantitative stability results for problems in geometric analysis and functional inequalities

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Abstract

This thesis addresses the characterization of geometric properties for problems in Partial Differential Equations (PDEs), geometric analysis and functional inequalities, with particular interest in the study of symmetry and quantitative stability issues. The thesis consists in four chapters.

Chapter 1 is about symmetry and quantitative studies for hypersurfaces embedded in space forms with some curvature close to a constant. The starting point is the well-known *Alexandrov Soap Bubble Theorem* which asserts that the distance spheres are the only embedded closed connected hypersurfaces in space forms (i.e. the Euclidean space, the hyperbolic space and the hemisphere) having constant mean curvature. Actually the theorem can be extended to more general functions of the principal curvatures $f(k_1, \ldots, k_{n-1})$ satisfying suitable conditions. The main result in Chapter 1 are sharp quantitative estimates of proximity to a single sphere for Alexandrov Soap Bubble Theorem in space forms when the curvature operator f is close to a constant. Under an assumption that prevents bubbling (the uniform touching ball condition), the proximity to a single sphere is optimally quantified in terms of the oscillation of the curvature function f. Our approach provides a unified picture of quantitative studies of the method of moving planes, i.e. the original method introduced by Alexandrov to prove its theorem, in space forms.

Chapter 2 is about symmetry results for Serrin-type overdetermined problems. *Serrin's symmetry results* asserts that if there exists a solution to an overdetermined boundary value problem associated to the equation $\Delta u = -1$ in an open domain of the Euclidean space, then the domain must be a ball.

The first result in Chapter 2 is a Serrin's symmetry result for an overdetermined boundary value problem in a particular class of Riemannian manifolds, the so-called model manifolds. We prove an Euclidean symmetry result under a suitable compatibility assumption between the solution and the geometry of the manifolds.

The second result in Chapter 2 is a Serrin's symmetry result for overdetermined boundary value problems in convex cones for (possibly) degenerate operators, such as the p-Laplace operator, in the Euclidean space as well as for a suitable generalization of the problem for convex cones in space forms. We prove symmetry results by showing that the existence of a solution implies that the domain is a spherical sector.

Chapter 3 is about symmetry results for critical anisotropic p-Laplace equations in convex cones. Given $1 < p < n$, we consider the *critical p-Laplacian equation* $\Delta_p u + u^{p^*-1} = 0$, which is related to critical points of the Sobolev inequality and to the Yamabe problem. Exploiting the moving planes method, it has been recently shown that positive and entire solutions to the ciritical p-Laplacian equation are classified and are given by the so-called Aubin-Talenti bubbles. Since the moving planes method strongly relies on the symmetries of the equation and the domain, in Chapter 3 we provide a new approach to this Liouville-type problem that allows us to give a complete classification of solutions in an anisotropic setting and in convex cones. More precisely, we characterize solutions to the critical p-Laplacian equation induced by a smooth norm inside any convex cone of the Euclidean space. One can show that the critical p-Laplace equation that we consider is related to the critical points of the anisotropic Sobolev inequality in convex cones. Since a sharp Sobolev inequality was missing in this setting, in Appendix B, we prove a general class of (weighted) anisotropic Sobolev inequalities inside arbitrary convex cones by using the optimal transport approach.

Chapter 4 is about functional inequalities on a particular class of Riemannian manifolds. In particular we consider the so-called Cartan-Hadamard manifolds, i.e. complete, simply connected, non-compact Riemannian manifolds with negative sectional curvatures everywhere. It is well-known that on every Cartan-Hadamard manifold the Sobolev inequality holds true, moreover if the sectional curvatures are bounded above by a negative constant then also the Poincaré inequality holds true. In Chapter 4 we investigate the validity, as well as the failure, of Sobolev-type inequalities on Cartan-Hadamard manifolds under suitable bounds on the sectional and the Ricci curvatures. More specifically, we prove that if the sectional curvatures are bounded from above by a negative power of the distance from a fixed pole (times a negative constant), then all the L^p-inequalities that interpolate between Poincaré and Sobolev hold in the radial setting; provided such power lies in the interval $(-2, 0)$, except the Poincaré inequality. If the power is equal to -2 then p must necessarily be strictly larger (in a quantitative way) than 2. Upon assuming similar bounds from below on the Ricci curvature, we show that the non-radial version of such inequalities fails, except for the Sobolev one. Finally, we prove optimal smoothing effects for a porous medium equation set up on the Cartan-Hadamard manifolds we are considering which follows from the Sobolev-type inequalities that we prove.

Contents

Introduction

This thesis addresses the characterization of geometric properties for solutions to problems in Partial Differential Equations (PDEs), geometric analysis and functional inequalities, with particular interest in the study of symmetry and quantitative stability issues. The thesis is divided in four parts.

Part I is about symmetry and quantitative studies for hypersurfaces embedded in space forms with some curvature close to a constant, for instance the mean curvature, and overdetermined problems for PDEs. This part collects the results obtained in [63, 65, 196].

Part II is about symmetry results for critical anisotropic p-Laplace equations in convex cones and functional inequalities on a particular class of Riemannian manifolds. Part II collects the results obtained in [57] and [174].

Part III contains the four appendixes of the thesis and Part IV collects the bibliographic references of the thesis.

This is a first introduction to the topics and results contained in this thesis. We refer to the introductions to Part I and to Part II for a more detailed description of the results contained in this thesis. We will start by describing classical results about *the isoperimetric inequality* and *the Sobolev inequality*, which motivate most of the results contained in this thesis. Roughly speaking, we can say that most of the problems that we will consider in the thesis are related to the Euler-Lagrange equations associated to these two inequalities. Moreover we will emphasize several similarities that arise when considering different proofs of these problems.

The isoperimetric problem goes back to the ancient Greece (the well-known Dido's Problem) and it has been object of studies for generations of mathematicians. The isoperimetric problem consists in minimizing the surface area among all domains having fixed volume, or equivalently maximizing the volume among all domains whose boundary surface has fixed $(n-1)$-dimensional area. It is well known that is that the unique extremal of these problems is the ball. An equivalent and more analytic formulation of the isoperimetric problem is the so-called *isoperimetric inequality*: if $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $\partial\Omega$ denotes its boundary, then

$$|\partial\Omega| \geq n\omega_n^{\frac{1}{n}} |\Omega|^{1-\frac{1}{n}} , \tag{1}$$

where ω_n denotes the volume of the unit sphere in $\mathbb{R}^n$ and $|\cdot|$ denotes either the n-dimensional (Lebesgue) measure or the $(n-1)$-dimensional surface measure of a subset of $\mathbb{R}^n$. Moreover, the equality in (1) is attained if and only if the domain is a ball (see [78] and also [179]).

The natural functional associated to (1) is the *isoperimetric functional* given by:

$$\mathcal{P}(\Omega) := \frac{|\partial\Omega|^n}{|\Omega|^{n-1}}\,. \tag{2}$$

We are interested in studying *critical points* of the functional $\mathcal{P}$. The most direct approach is to perform the methods of the calculus of variations in order to write the Euler-Lagrange equation associated to (2). Let $\Omega \subset \mathbb{R}^n$ be a critical domain for the functional (2) and assume that Ω is bounded by a smooth hypersurface S, i.e. $S = \partial\Omega$. The idea is to compute the first variation of the functional $\mathcal{P}$: we take a smooth function $h : S \to \mathbb{R}$ and we consider the normal variation of S defined by $\psi_t : S \to \mathbb{R}^n$ such that

$$\psi_t(p) = p + th(p)\nu(p)\,,$$

where ν is the unitary exterior normal field to S. We denote by S_t the hypersurface given by $\psi_t(S_t)$ and by Ω_t the domain enclosed by S_t; observe that S_t is nothing but the displacement of each point of S by the vector $th\nu$. For simplicity of notation, we set

$$\mathcal{A}(t) := |S_t| = \int_{S_t} d\sigma \qquad \text{and} \qquad \mathcal{V}(t) := |\Omega_t| = \int_{\Omega_t} dx\,.$$

Then the first variation of the functional (2) above is given by (see e.g. [161] and [208])

$$\mathcal{A}'(0) = -(n-1)\int_S hH\,d\sigma \qquad \text{and} \qquad \mathcal{V}'(0) = \int_S h\,d\sigma\,, \tag{3}$$

where H is the mean curvature of S. Since Ω is a critical domain, we have that

$$\frac{d}{dt}\Big|_{t=0}\mathcal{P}(\Omega_t) = 0\,,$$

or equivalently, from (2) and (3),

$$\int_S h\left(n|\Omega|H - |S|\right)\,d\sigma = 0\,, \quad \text{for all } h,$$

and this implies that S has constant mean curvature. Hence we prove that the Euler-Lagrange equation associated to the functional (2) (and so to the inequality (1)) is

$$H \equiv \frac{|\partial\Omega|}{n|\Omega|}\,. \tag{4}$$

Now the following characterization proved by Alexandrov in the 50's comes into play:

Theorem A ([6]): *let S be a C^2-regular, connected, closed (i.e. compact and without boundary) hypersurface embedded in the Euclidean space $\mathbb{R}^n$. Then S has constant mean curvature if and only if is a sphere.*

Putting the previous computation and Theorem A together, we prove that critical points of the functional (2) must be spheres, and in particular the equality in (1) is attained if and only if the domain is a ball.

We mention that the previous argument (in particular Alexandrov's Theorem) works if we assume to work with smooth boundaries and this is not the general situation where one wants to prove the isoperimetric inequality (we refer to the *theory of sets of finite perimeter* developed by Caccioppoli and De Giorgi in the 30-50's, see also [163] for a recent reference). So we emphasize that in this first part of the Introduction we only wanted to stress the link between the isoperimetric inequality and Alexandrov's Theorem because this is, probably, the most important motivation in order to study Alexandrov's Theorem. We mention that Theorem A is stated in its simplest formulation, indeed Alexandrov's Theorem has been widely studied and extended in several directions. Moreover, in the last years, also the quantitative version of the isoperimetric inequality has been the object of several studies. In [104] the sharp quantitative version of the isoperimetric inequality in $\mathbb{R}^n$ has been proved. Once one has a quantitative result for the minimizers of (2) it is of strong interests to understand the shape of critical points of (2) (see [58, 67]). In Chapter 1 we study a *quantitative version of Alexandrov Theorem* for hypersurfaces embedded in space forms with some curvature (not necessary the mean curvature) close to a constant.

Alexandrov Theorem is also related to an important and well-known result in the theory of elliptic PDEs, which is the following characterization of domains which support a solution to an overdetermined problem.

Theorem B ([204]): *let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2 and let ν be the outward normal to $\partial\Omega$. Then there exists a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ to*

$$\begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \,, \end{cases} \tag{5}$$

such that, for some constant c,

$$\partial_\nu u = c \qquad \text{on } \partial\Omega \,, \tag{6}$$

if and only if Ω is a ball and u is a radial function.

Theorem B was proved by Serrin in 1971 by using the so-called *method of moving planes*, which takes inspiration from *the reflection principle* introduced by Alexandrov to prove Theorem A. As we will show in Appendix D, a further link between these two theorems is that they are "equivalent", in the sense that we can prove Theorem A assuming Theorem B holds and viceversa (we mention that the technique used to prove this equivalence takes inspiration from the proof of Theorem B proposed by Weinberger in [215]).

Before giving an idea of the proofs of the two theorems, we mention that Serrin's problem (5)-(6) is called an overdetermined boundary value problem since the Dirichlet problem (5) already admits a unique solution; hence condition (6) is an additional requirement and in general the whole problem (5)-(6) may not admit a solution. Thus, the remaining data of the problem, the domain Ω, cannot be given arbitrarily, i.e. there is a requirement also on the domain: this phenomenon is called rigidity and it implies that the domain and the solution itself satisfy some symmetry.

We emphasize that overdetermined boundary value problems like (5)-(6) arise in many context. The following is one of the physical interpretations given by Serrin in [204]: consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω, then it is standard that the velocity flow u satisfies (5). In this context, Serrin's result can be formulated as follows: *the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section.*

The same differential equation and boundary condition arise in the linear theory of torsion of a solid straight bar of cross section Ω (see [204] for more details), In this context, Serrin's result states that, *when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section.* Moreover, overdetermined problems like (5)-(6) arise when one study critical domains for the torsional rigidity.

Now we describe the proofs given by Alexandrov and Serrin for Theorems A and B, respectively. The idea behind Alexandrov's proof is the following: under the assumptions of Theorem A, S is a sphere if and only if for every $\omega \in \mathbb{R}^n$ there exists a hyperplane π_ω orthogonal to ω such that the hypersurface S is symmetric about π_ω. In order to achieve this symmetry property, let $\omega \in \mathbb{R}^n$ be a fixed direction and let π_λ be a 1-parameter family of hyperplanes orthogonal to ω. We move these hyperplanes towards S until we touch S and we continue to move them until we find a critical hyperplane π^*. In order to define the critical hyperplane π^* we define: $\mathcal{T}^*$ is the halfspace such that $\partial\mathcal{T}^* = \pi^*$, $S' = S \cap \mathcal{T}^*$ is the part of S contained in $\mathcal{T}^*$ and S^* is the reflection with respect to π^* of S'. With these notations the critical hyperplane π^* is defined in the following way:

- (i) either S^* is tangent to S at a point $p \in S \setminus \pi^*$;
- (ii) or π^* is orthogonal to S at a point $q \in S \cap \pi^*$.

Since S and S^* are tangent to p (or q), we can locally write the two hypersurfaces as graphs of function u and u^*, respectively, on the tangent space at p (or q). Moreover, by construction $u - u^*$ has a sign, and u and u^* satisfy the following elliptic equation:

$$H = \frac{1}{n-1}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right), \tag{7}$$

with H constant (equal to the value of the mean curvature) and where the right-hand side is the expression of the mean curvature of the graph of a function v. At this point the PDE's tools come into our help: by using the strong maximum principle in case (i) and the Hopf's boundary point Lemma in case (ii) we conclude that $u = u^*$, which implies that the hypersurface S and its reflection S^* with respect to the plane π^* coincide. Since the direction ω is arbitrary we conclude that S must be spherically symmetric and hence S is a sphere.

The proof of Theorem B given by Serrin generalizes Alexandrov's reflection principle. Also in this case the goal is to prove that Ω is symmetric with respect to the hyperplane π_ω orthogonal to $\omega \in \mathbb{R}^n$. As before, let $\mathcal{T}^*$ be the halfspace such that $\partial\mathcal{T}^* = \pi^*$, $\Omega' = \Omega \cap \mathcal{T}^*$ is the part of Ω contained in $\mathcal{T}^*$ and Ω^* is the reflection with respect to π^* of Ω'. We consider the critical hyperplane π^* at the two critical positions (i) and (ii) with $S = \partial\Omega$. We compare the solution u to (5)-(6) to its reflection v, defined in Ω^*. Since $u - v$ is harmonic and non negative either $u - v > 0$ in Ω^* or $u - v \equiv 0$ in Ω^*. If $u - v > 0$

occurs, the Hopf's Lemma gives a contradiction in case (i). Case (ii) is more delicate and one has to use Serrin's corner Lemma (see [204, Lemma 1]) to find a contradiction. Hence $u - v \equiv 0$ in Ω^* and Ω is symmetric about π^*.

Both Alexandrov's and Serrin's results and proofs originate a great interest in geometric analysis and PDE's communities. Indeed the method of moving planes is a powerful tool which has been used to prove several results in geometric analysis, for elliptic and parabolic PDEs, Harnack's inequalities and many others results (see e.g. [9, 23, 24, 36, 37, 38, 146, 155, 168, 192, 193, 202]). Moreover, one of the most influencing application of the method of moving planes in the theory of PDEs is the approach of Gidas-Ni-Nirenberg in [109] and [110] where the authors prove symmetry results for positive solutions of elliptic PDEs.

As for Alexandrov Theorem, we mention that Theorem B is stated in its simplest version and many generalizations have been studied. In Chapter 2 we will study symmetry results in the spirit of Theorem B in more general Riemannian manifolds (see Section 2.1) and for quasilinear operators in convex cones (see Section 2.2).

As already mentioned, the inequality that motivates the study in Part II is the *Sobolev inequality* (see [210] and also [105, 177]) in $\mathbb{R}^n$, $n \geq 2$: given $1 < p < n$ there exists a constant $C_{n,p} > 0$ such that the following inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx \right)^{p/p^*} \leq C_{n,p} \int_{\mathbb{R}^n} |\nabla u|^p \, dx \tag{8}$$

holds for every function u in the homogeneous space

$$\mathcal{D}^{1,p}(\mathbb{R}^n) := \Big\{ u \in L^{p^*}(\mathbb{R}^n) \ : \ \nabla u \in L^p(\mathbb{R}^n) \Big\} \,, \tag{9}$$

where p^* is the *critical Sobolev exponent*, i.e.

$$p^* = \frac{np}{n-p} \,. \tag{10}$$

Sobolev inequality appears when one considers the embedding of the Sobolev space in some L^p-space. Moreover it naturally appears in regularity theory for PDEs and it is related to the isoperimetric inequality as well as to many other important topics in mathematical analysis.

The natural functional associated to (8) is the *Sobolev functional* given by:

$$\mathcal{J}(u) := \frac{\int_{\mathbb{R}^n} |\nabla u|^p \, dx}{\left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx \right)^{p/p^*}} \,. \tag{11}$$

We are interested in *critical points* of this functional $\mathcal{J}$ and we want to write and study the Euler-Lagrange equation associated to (11). Let u be a (positive) critical point of (11) and we compute the first variation of the functional. Let $\varepsilon > 0$ and let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a test function. Since u is a critical point we get

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{J}(u + \varepsilon\varphi) = 0 \,,$$

or equivalently, from a direct computation,

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx - \frac{\int_{\mathbb{R}^n} |\nabla u|^p \, dx}{\int_{\mathbb{R}^n} |u|^{p^*} \, dx} \int_{\mathbb{R}^n} u^{p^*-1}\varphi \, dx = 0 \, ,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. By multiplying u by an appropriate constant we may assume that u satisfies

$$\int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx - \int_{\mathbb{R}^n} u^{p^*-1}\varphi \, dx = 0 \, , \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n) \, . \tag{12}$$

Equation (12) is the weak formulation of the following quasilinear PDE:

$$\Delta_p u + u^{p^*-1} = 0 \quad \text{in } \mathbb{R}^n,$$

where $\Delta_p u$ is the usual p-Laplace operator, i.e.

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),$$

and p^* is the Sobolev critical exponent defined in (10). Hence we prove that the Euler-Lagrange equation (for positive critical point) associated to the functional (11) (and so to the inequality (8)) is the following *critical p-Laplace equation in $\mathbb{R}^n$*:

$$\begin{cases} \Delta_p u + u^{p^*-1} = 0 & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{13}$$

We observe some properties of (13) which will be useful in the following: if u is a solution to (13) then also $u(x+\alpha)$, for $\alpha \in \mathbb{R}^n$, and $\lambda^{\frac{n-p}{p}} u(\lambda x)$, for $\lambda \in \mathbb{R}$, are solutions to (13). These two properties are called *invariance under translations* and *invariance under rescaling* of (13).

As mentioned in [224], it is well known that the profile of solutions of the equation

$$\Delta_p u + |u|^{p^*-2}u = 0 \quad \text{in } \mathbb{R}^n \, ,$$

plays a central role in the blow-up theories of critical equations. Problem (13) is also interesting from the point of view of the calculus of variations. Since the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is not compact, the classical tools of the calculus of variatons (e.g. the Mountain Pass Lemma or the direct method of the calculus of variations) do not apply to

$$\mathcal{E}(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p \, dx - \frac{1}{p^*}\int_{\mathbb{R}^n} u^{p^*} \, dx \, ,$$

which is the energy functional associated to (13).

Equation (13) is also related to the *Yamabe problem*. We recall that the Yamabe problem (see [10, 201, 220, 226]) is the following: given a compact Riemannian manifold (M, g_0) of dimension $n \geq 3$, is it possible to find a metric g conformal to g_0 such that its scalar curvature R_g is equal to a prescribed constant R? One can show that this problem boils down to showing the existence of a positive solution u to the nonlinear PDE

$$\frac{4(n-1)}{n-2}\Delta_{g_0} u - R_{g_0} u + R u^{\frac{n+2}{n-2}} = 0 \, , \tag{14}$$

where Δ_{g_0} is the Laplace-Beltrami operator of the metric g_0 and R_{g_0} denotes the scalar curvature of the metric g_0. Indeed, if u solves (14) then the metric $g = u^{\frac{4}{n-2}} g_0$ is such that $R_g = R$.

When (M, g_0) is the round sphere then $R_{g_0} = n(n-1)$ and (14) can be read on $\mathbb{R}^n$ by means of the stereographic projection. More precisely, consider the inverse stereographic projection $F : \mathbb{R}^n \to \mathbb{S}^n$. Then $v : \mathbb{S}^n \to \mathbb{R}$ solves (14) if and only if the function

$$u(x) = \left(\frac{2}{1 + |x|^2} \right)^{\frac{n-2}{2}} v(F(x))$$

solves

$$\Delta u + \frac{n-2}{4(n-1)} R u^{\frac{n+2}{n-2}} = 0\,,$$

which is, up to a multiplicative constant, equation (13) with $p = 2$.

We mention that an important task related to the Sobolev inequality (8) is the following: to compute the value of the best constant in (8), i.e. to obtain the *sharp Sobolev inequality.* The value of the best constant was obtained independently by Talenti [217] and Aubin [10]; moreover they showed that equality in the sharp Sobolev inequality is achieved by the following *Aubin-Talenti radial functions* (or bubbles):

$$\mathcal{U}(x) = \frac{a}{\left(b + |x - x_0|^{\frac{p}{p-1}} \right)^{\frac{n-p}{p}}} \qquad \text{for some } a, b > 0 \text{ and } x_0 \in \mathbb{R}^n\,. \tag{15}$$

The idea to prove this result is to minimize the functional (11) in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. By using the Schwarz symmetrization technique Talenti proved that the spherically symmetric radial rearrangement of u preserves $||u||_{L^{p^*}(\mathbb{R}^n)}$ and makes $||\nabla u||_{L^p(\mathbb{R}^n)}$ smaller. Hence, minimizers are forced to be spherically symmetric radial functions and, by computing the Gateaux differential of the functional $\mathcal{J}$ for these type of functions, one obtains that minimizers of $\mathcal{J}$ (we assume for simplicity that they are positive) satisfy the following ode:

$$\frac{1}{r^{n-1}} (r^{n-1} (u')^{p-1})' + u^{p^*-1} = 0\,,$$

where $r = |x|$. This yields to the Aubin-Talenti functions (15).

We observe that the results of Aubin and Talenti classify minima of the functional (11). An interesting issue in PDE's community is to look for a characterization of critical points of the functional (11), i.e. a characterization of solutions to (13). The classification of all solution to (13) started in the seminal papers [110] (see also [178]) and [48] for $p = 2$ and it has been the object of several studies. Recently in [224] and [203], solutions to (13) belonging to the class $\mathcal{D}^{1,p}(\mathbb{R}^n)$ have been completely characterized. In particular, the following Liouville-type theorem holds:

Theorem C ([224, 203]): let $n \geq 2$, $1 < p < n$ and let u be a solution to (13) such that $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Then there exist $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$u(x) = \mathcal{U}_{\lambda, x_0}(x) = \left(\frac{\lambda^{\frac{1}{p-1}} n^{\frac{1}{p}} \left(\frac{n-p}{p-1} \right)^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}} \right)^{\frac{n-p}{p}}\,. \tag{16}$$

We mention that the previous result is not true if u may change sign (see e.g. [84, 79, 80]). Actually, in the semilinear case $p = 2$ a more general version of Theorem C is available in literature, indeed the assumption $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ can be removed by using the Kelvin tranform. The other crucial ingredient to prove Theorem C is a generalization of the method of moving planes introduced by Serrin (see e.g. [27, 109]). In the quasilinear case $p \neq 2$ the problem is more complicate because one has to tackle the nonlinear nature of the p-Laplace operator and Kelvin type transform is not available. A first version of Theorem C for $\frac{2n}{n-2} \leq p < 2$ was proved in [76]; this result was extended to the range of exponent $1 < p < 2$ in [224]. Finally, in [203] the range $2 \leq p < n$ was considered and this concludes the picture. The key point in the proof of Theorem C is to show that the solutions to (13) are radial and then, from [123] one knows that the only positive, radially symmetric solutions of (13) are of the form (16).

Motivated by recent studies on the Sobolev inequality in convex cones and on the anisotropic Sobolev inequality in Chapter 3 we prove a generalization of Theorem C for critical anisotropic p-Laplace equations in convex cones. We mention that the Kelvin transform and the method of moving planes are not helpful neither for anisotropic problems nor inside cones. For this reason, in Chapter 3, we provide a new approach to the characterization of solutions to critical p-Laplacian equations, which is based on integral identities rather than moving planes. This approach takes inspiration from [206] where non-existence results are considered and also from [29, 30, 40, 215] where classical overdetermined problems for PDEs are considered (see also [63, 180] for analogous problems in convex cones). We mention that our approach is new also in the Euclidean space with the Euclidean norm (see Appendix C)

In Chapter 4 we investigate the validity of Sobolev-type inequalities in a particular class of Riemannian manifolds. The investigation of functional inequalities on Riemannian manifolds is a very wide and active research field (see e.g. [131, 129] where the discussion is mainly devoted to the rigorous definition of *Sobolev spaces* on Riemannian manifolds and the properties of the associated embeddings or functional inequalities). As already mentioned the first result dealing with the *optimal constant* of the Euclidean Sobolev inequality is due to the celebrated papers [12] and [217]. Then, T. Aubin continued the investigation of Sobolev-type inequalities as well as related optimality issues on Riemannian manifolds: see [10] (in the case of compact manifolds with applications to the *Yamabe problem*), [11] (where higher-order inequalities are also considered) and [13] (for estimates of the best constants of subcritical Sobolev embeddings). Some of the results of [12] were then improved in [130, 132], essentially by requiring bounds on the Ricci curvature in place of the sectional curvatures.

Functional-analytic issues on the hyperbolic space $\mathbb{H}^n$ have drawn a lot of interest recently, the latter being in a sense the simplest example of a noncompact Riemannian manifold with negative curvatures. In this regard, we mention [25], where an improved version of the Poincaré (or spectral-gap) inequality is obtained with optimal remainder terms of Hardy type. In fact Hardy-type inequalities were also considered in [50], for non-standard weights satisfying suitable differential inequalities (with explicit applications to Cartan-Hadamard manifolds). Finally, in [175] the author establishes an inequality on $\mathbb{H}^n$ yielding the optimal Sobolev and Poincaré inequalities simultaneously.

For a rather complete survey dealing with connections between the Poincaré inequality, the logarithmic Sobolev inequality, measure-concentration issues and isoperimetric bounds (not only on Riemannian manifolds actually), we refer to [153], while in [18] the authors, starting from inequalities that interpolate between Poincaré and log-Sobolev, provide a method to obtain weighted inequalities of the same type for weights complying with optimal growth conditions.

Most of the results we have mentioned above are focused on proving the validity of Sobolev (or Poincaré, or Hardy) inequalities. One can also investigate topological consequences: in [51, 152] it is shown (under suitable curvature or volume-growth assumptions, respectively) that a Riemannian manifold supporting a Sobolev inequality with Euclidean constant is necessarily isometric to $\mathbb{R}^n$. For similar problems, but rather different methods of proof, see also [185].

More specifically, in Chapter 4 we study Sobolev-type inequalities on a particular class of Riemannian manifolds: the so-called *Cartan-Hadamard manifolds* (i.e. Riemannian manifolds with negative sectional curvatures) under curvature bounds. The situation is the following: given a Riemannian manifold with everywhere nonpositive sectional curvatures then the Sobolev inequality holds true. Moreover if the sectional curvatures are bounded above by a negative constant then also the Poincaré inequality holds true. Motivated by these results in Chapter 4 we study what happens in between, explicitly we will assume that the sectional curvatures of the Riemannian manifold are bounded from above by a negative power of the distance from a fixed pole (times a negative constant), then all the L^p inequalities that interpolate between Poincarè and Sobolev hold in the radial setting, provided such power lies in the interval $(-2, 0)$, except the Poincarè inequality. Moreover, if the power is equal to -2 then p must necessarily be strictly larger (in a quantitative way) than 2. Upon assuming similar bounds from below on the Ricci curvature, we show that the nonradial version of such Sobolev-type inequalities fails, except for the Sobolev one. Finally, in Section 4.6 we prove optimal smoothing effects for a porous medium equation.

Part I

Symmetry and quantitative stability results for problems in geometric analysis

Introduction to Part I

Part I of this thesis deals with quantitative studies for hypersurfaces embedded in space forms with some curvature close to a constant (in Chapter 1) and symmetry results for overdetermined problems (in Chapter 2). This part collects the results obtained in the following papers: [63, 65, 196].

In this Introduction we describe in more details the results contained in Chapters 1 and 2.

Chapter 1. The first chapter of this thesis is dedicated to quantitative studies for hypersurfaces with almost constant curvature. The starting point that motivates the study in this chapter is Alexandrov's Theorem (Theorem A in the Introduction). Alexandrov's Theorem has attracted a lot of interest in the last decades and several possible generalizations have been considered. It is well-known that Alexandrov's Theorem is in general false for non-embedded submanifolds (see e.g. [139] and [216] for classical counterexamples in higher dimension and in $\mathbb{R}^3$, respectively). However, for immersed hypersurfaces, one can add some condition in order to guarantee that S is a sphere: in particular Hopf proved in [136] that every constant mean curvature C^2-regular sphere *immersed* in the 3-dimensional Euclidean space is necessary a round sphere (see also [1, 169, 170] for a very recent generalization of Hopf's theorem to simply-connected homogeneous 3-manifolds), and Barbosa and DoCarmo [19] proved that every compact, orientable and stable hypersurface *immersed* in $\mathbb{R}^n$ is a round sphere (see also [20] for generalizations of this result). It is also well-known that there exists *non-closed* constant mean curvature hypersufaces embedded in $\mathbb{R}^3$ which are not diffeomorphic to a sphere, like for instance unduloids (see [81] and [140] for a possible generalization to higher dimensions).

Another interesting generalization was provided by Alexandrov himself. In [5] and [7] a generalization of Theorem A for hypersurfaces embedded in the hyperbolic space $\mathbb{H}^n$ and in the hemisphere $\mathbb{S}^n_+$. Here and in the following, we will indicate with $\mathbb{M}^n_+$ the three spaces: $\mathbb{R}^n$, $\mathbb{H}^n$, $\mathbb{S}^n_+$ and we call them *space forms*. Moreover the theorem holds true for a large class of curvature operators, not only the mean curvature (see [7, 52, 125, 141, 145, 158, 199, 212, 227]). This is the type of generalization that we are interested in. In order to properly describe the result, we introduce some notation.

Let S be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{M}^n_+$. Then S is always the boundary of a relatively compact connected open set $\Omega \subset \mathbb{M}^n_+$ and we orient S by using the normal vector field to S inward with respect to Ω. Let $\{k_1, \dots, k_{n-1}\}$ be the principal curvatures of S increasingly ordered.

We denote by H_S one of the following functions:

(I) the mean curvature H, with

$$H := \frac{1}{n-1} \sum_{i=1}^{n-1} k_i$$

(II) $f(k_1, \dots, k_{n-1})$, where

$$f \colon \{x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \ : \ x_1 \leq x_2 \leq \dots \leq x_{n-1}\} \to \mathbb{R} \,,$$

is a C^2-function such that

$$f(x) > 0, \text{ if } x_i > 0 \text{ for every } i = 1, \dots, n-1$$

and f is concave on the component Γ of $\{x \in \mathbb{R}^{n-1} \ : \ f(x) > 0\}$ containing $\{x \in \mathbb{R}^{n-1} \ : \ x_i > 0\}$.

For instance, if H_r denotes the r-higher order curvature of S defined as the elementary symmetric polynomial of degree r in the principal curvatures of S, then $H_r^{1/r}$ satisfies (ii). By using this notation, Alexandrov's theorem can be stated as follows:

Theorem I.A: *let S be a C^2-regular, connected, closed (i.e. compact and without boundary) hypersurface embedded in $\mathbb{M}^n_+$ and let H_S be as in (I) or (II). Then H_S is constant if and only if S is a distance sphere.*

The proof of this theorem can be done with the same technique introduced by Alexandrov, but we mention that for particular choices of the function f in (II) different proofs, based on integral inequalities and geometric identities, are avaible in literature (see e.g. [30, 127, 128, 172, 194, 198, 197]). For more recent generalizations of Alexandrov's Theorem we refer to [41], where constant mean curvature hypersurfaces are studied in rotationally symmetric Riemannian manifolds (e.g. the space forms, the Schwarzschild, the DeSitter-Schwarzschild and Anti-DeSitter-Schwarzschild manifolds), to [82] where the regularity assumptions on S are minimal and to [45, 56] where the non-local version of Alexandrov's Theorem is proved.

Theorems A, B and I.A have the same conclusion: a solution exists if and only if it is a sphere/ball. The rigidity of these problems is due to the following overdetermined conditions: the curvature H_S is constant in Alexandrov Theorems and the normal derivative $\partial_\nu u$ is constant on $\partial\Omega$ in Serrin Theorem. In view of this remark it is natural to investigate the stability of this results:

if the overdetermined condition is slightly perturbed, can we say that the domain is close to a ball? Can we quantify the closedness?

Stated like that, the answer to the questions is in general negative. Indeed, it has been showed in [39] for Serrin Theorem and in [58] for Alexandrov Theorem that if the overdetermined condition is close to a constant then bubbling phenomena can appear, i.e. the domain can be close to a bunch of balls connected by small necks. We mention that both [39] and [58] do not use the moving planes method and they perturb proofs based on integral identities. Nevertheless if some hypothesis is introduced in order to prevent bubbling phenomena then the moving planes method can be studied in a quantitative way to obtain sharp quantitative information of the proximity of the solution to a single ball. In [67, 69] one condition that can prevent bubbling phenomena is the so-called touching ball condition (we will comment a little bit on it after the theorem). Our stability result related to Theorem I.A proved in [65] can be stated as follows

Theorem I.B: *let S be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{M}^n_+$ satisfying a uniform touching ball condition of radius ρ. There exist constants ε, $C > 0$ such that if*

$$\operatorname{osc}(\mathsf{H}_{\mathrm{S}}) := \max_{\mathrm{p}\in\mathrm{S}} \mathsf{H}_{\mathrm{S}}(\mathrm{p}) - \min_{\mathrm{p}\in\mathrm{S}} \mathsf{H}_{\mathrm{S}}(\mathrm{p}) \leq \varepsilon \,, \tag{I.1}$$

then there are two concentric balls B_r and B_R in $\mathbb{M}^n_+$ such that

$$S \subseteq \overline{B}_R \setminus B_r \tag{I.2}$$

and

$$R - r \leq C \operatorname{osc}(\mathsf{H}_{\mathrm{S}}) \,. \tag{I.3}$$

The constants ε and C depend only on n, on upper bounds on ρ^{-1} and on the area of S.

The uniform touching ball condition of radius ρ in the previous theorem means that at any point of S there exist two balls of radius ρ tangent to S one from inside and one from outside. Since the constant ρ is fixed, a bubbling phenomenon can not appear as shown in [58, Theorem 1.1]. We emphasize that in Theorem I.B the stability estimate (I.3) is optimal and it is new in the general setting of Theorem I.B; moreover we mention that Theorem I.B remarkably improves the previous stability results available in literature (see Chapter 1 for a more detailed discussion).

Chapter 2. The second chapter of this thesis is dedicated to Serrin's overdetermined problem and related issues. We mention that the celebrated moving planes method introduced by Serrin can be used to prove a more general version of Serrin's Theorem involving uniformly elliptic quasilinear equation (see [204]). Moreover the moving planes method has been used to prove an analogue result in space forms (see [150, 171]) and in a nonlocal setting (see [90]). As for Alexandrov theorem, Serrin theorem can be proven by using different approaches, based on integral identities (see [215]). This allows to extend Serrin's theorem to possibly degenerate quasilinear equations and to fully nonlinear equations (see [3, 22, 29, 31, 33, 40, 42, 44, 55, 68, 70, 74, 75, 92, 102, 103, 106, 126, 133, 134, 135, 182, 190, 191, 196, 207, 209, 213]) and also for domains with Lipschitz singularities or contained in a convex cone (see [63, 180, 188]).

We now review Serrin's original proof and some alternative proofs of Serrin theorem, in particular the one due to Weinberger [215] and the more recent integral approach due to Brandolini-Nitsch-Salani-Trombetti in [40] (see also [164] and [176]).

The moving planes method. Let u be the solution to (5)-(6). For a fixed direction $\omega \in \mathbb{R}^n$, the moving plane procedure gives a critical hyperplane π^* such that

(*i*) either Ω^* is tangent to Ω at a point $p \in \partial\Omega \setminus \pi^*$,

(ii) or π^* is orthogonal to $\partial\Omega$ at some $q \in \partial\Omega \cap \pi^*$,

where we recall that Ω^* is the reflection with respect to π^* of the cap Ω'. Now we define the reflection of u with respect to π^*, i.e. $v(x) = u(x^*)$, for $x \in \Omega^*$, where x^* is the reflection of x with respect to π^*. Is it easy to show that $u - v$ satisfies the following problem:

$$\begin{cases} \Delta(u-v) = 0 & \text{in } \Omega^* \\ u - v = 0 & \text{on } \partial\Omega^* \cap \pi^* \\ u - v \geq 0 & \text{on } \partial\Omega^* \setminus \pi^* \ , \end{cases}$$

where the last inequality follows from the weak comparison principle. Then, by the strong maximum principle we have that either $u - v > 0$ in Ω^* or $u - v \equiv 0$ in Ω^*. The first case leads to a contradiction, indeed in case (i) by the Hopf Lemma we have that

$$\partial_\nu (u - v)(p) < 0 \,,$$

but from (6) we have that $\partial_\nu u(p) = \partial_\nu v(p)$. In case (ii) one shows that all the first and second derivatives of u and v coincide at q, and this is in contradiction with the so-called Serrin's corner Lemma [204, Lemma 1] (a refinement of the maximum principle).
Hence $u \equiv v$ in Ω^*, i.e. Ω is symmetric with respect to π^*. This implies that for every direction ω, Ω is symmetric with respect to the hyperplane orthogonal to ω; moreover, by construction, Ω is simply connected, and then it has to be a ball.

Weinberger's proof. Let u be the solution to (5)-(6) and consider the following function:

$$P(u(x)) := |\nabla u(x)|^2 + \frac{2}{n} u(x) \,. \tag{I.4}$$

We compute the Laplacian of this function and, by using Cauchy-Schwarz inequality, we get $\Delta P(u) \geq 0$. Since $P(u(x)) = c^2$ on $\partial\Omega$ (due to the boundary conditions) from the strong maximum principle we conclude that either

$$P(u(x)) \equiv c^2 \quad \text{in } \Omega \,, \tag{I.5}$$

or

$$P(u(x)) < c^2 \quad \text{in } \Omega \,. \tag{I.6}$$

In the second case, by integrating (I.6) over Ω and by using the divergence theorem, we get

$$\frac{n+2}{n} \int_\Omega u \, dx < c^2 |\Omega| \,. \tag{I.7}$$

The classical Pohozaev identity (see e.g. [186]) yields

$$\frac{n-2}{n} \int_\Omega |\nabla u|^2 \, dx + \frac{c^2}{2} \int_{\partial\Omega} x \cdot \nu \, d\sigma = n \int_\Omega u \, dx \,, \tag{I.8}$$

and from (5) and (6) we obtain

$$(n+2) \int_\Omega u \, dx = c^2 n |\Omega| \,, \tag{I.9}$$

which contradicts (I.7). This means that (I.5) holds true, hence $P(u)$ is a harmonic function and this implies that the equality in Cauchy-Schwarz inequality holds, i.e.

$$\nabla^2 u = -\frac{1}{n}\mathrm{Id}\,. \tag{I.10}$$

Hence

$$u(x) = \frac{1}{2n}\left(R^2 - |x - x_0|^2\right)\,, \tag{I.11}$$

for some $R > 0$ and $x_0 \in \mathbb{R}^n$; from (I.11) and since $u = 0$ on $\partial\Omega$ we immediately conclude that Ω is the ball of radius R centred at x_0.

We mention that the approach of Weinberger inspired several works in the context of elliptic PDEs (see e.g. [48, 93, 181, 211]), in particular in [92, 103, 106] Weinberger's argument is used to prove symmetry result for overdetermined problems associated to more general PDE's.

An integral approach. Given a symmetric matrix $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$, for any $k = 1, \dots, n$ we denote by $S_k(A)$ the sum of all the principal minors of A of order k. Observe that

$$S_1(A) = \mathrm{Tr}\,(A)\,, \quad S_2(A) = \sum_{1\le i_1 < i_2 \le n} \lambda_{i_1}\lambda_{i_2}\,, \quad S_n(A) = \det(A)\,, \tag{I.12}$$

where $\mathrm{Tr}\,(A)$ is the trace of A, $\det(A)$ is the determinant of A and $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A.

We are interested in S_1 and S_2: suppose $\mathrm{Tr}\,(A) \ge 0$, then the following Newton's inequality for symmetric matrices (see e.g. [124]),

$$S_2(A) \le \frac{n-1}{2n}(S_1(A))^2 \tag{I.13}$$

holds, and the equality in (I.13) holds if and only if

$$A = \frac{\mathrm{Tr}\,(A)}{n}\mathrm{Id}\,. \tag{I.14}$$

Newton's inequality (I.13) can be used to prove Serrin overdetermined Theorem. Indeed, by taking $A = \nabla^2 u$ we have that $S_1(\nabla^2 u) = \Delta u$ and a direct computation shows that $S_2(\nabla^2 u)$ can be written in the following divergence form

$$S_2(\nabla^2 u) = \frac{1}{2}\partial_i(S_{ij}^2(\nabla^2 u)\partial_j u) \qquad \text{where } S_{ij}^2(\nabla^2 u) = -\partial^2_{ij} u + \delta_{ij}\mathrm{Tr}\,(\nabla^2 u) \tag{I.15}$$

(here we are adopting the Einstein summation convention for repeated indices). Let u be the solution to (5)-(6). First we observe that, from (I.15) via an integration by parts,

$$\int_\Omega |\nabla u|^2\,dx = -2\int_\Omega |\nabla u|^2\,dx + 2(n-1)\int_\Omega H|\nabla u|^2\,dx + c^3|\partial\Omega| \tag{I.16}$$

where we used the fact that the mean curvature H of a level set of u (at a regular point) satisfies

$$-\Delta u = (n-1)H|\nabla u| - \frac{\partial^2_{ij} u\,\partial_i u\,\partial_j u}{|\nabla u|^2}\,.$$

From (I.16) and the fact that the constant c in (6) is given by $-|\Omega|/|\partial\Omega|$ we get

$$(n-1)\int_\Omega H|\nabla u|^3\,dx = \frac{3}{2}\int_\Omega |\nabla u|^2\,dx - \frac{c^2}{2}|\Omega|\,, \tag{I.17}$$

Then, from (I.9) and (I.17) we obtain

$$\int_\Omega H|\nabla u|^3\,dx = \frac{c^2|\Omega|}{n-2}\,, \tag{I.18}$$

where we used the divergence theorem and (5)-(6) to prove that

$$\int_\Omega |\nabla u|^2\,dx = \int_\Omega u\,dx\,.$$

From (I.18) we get

$$\frac{c^2|\Omega|}{n-2} = \frac{1}{n-1}\int_\Omega S^2_{ij}(\nabla^2 u)\partial_i u\partial_j u\,dx\,, \tag{I.19}$$

where we used the following pointwise identity

$$-(n-1)H = \frac{S^2_{ij}(\nabla^2 u)\partial_i u\partial_j u}{|\nabla u|^3}\,.$$

Integrating by parts and by using (I.13), (5) (I.9) we obtain

$$\frac{c^2|\Omega|}{n-2} = \frac{2}{n-1}\int_\Omega S_2(\nabla^2 u)u\,dx \le \frac{1}{n}\int_\Omega u(\Delta u)^2\,dx = \frac{c^2|\Omega|}{n-2}\,. \tag{I.20}$$

The conclusion follows because (I.20) implies that equality holds true in (I.13) so (I.10) holds true.

In Chapter 2 there are three symmetry results: the first one is contained in [196] and is related to a generalization of Weinberger's proof to a particular class of Riemannian manifolds; the remaining ones are contained in [63] and are related to a Serrin-type result for domains inside convex cones of the Euclidean space and in space forms. In what follows we will give the flavour of the results that we are going to prove in Chapter 2 and we refer to Chapter 2 for the precise definitions and statements.

Weinberger argument on Riemannian manifolds. The first symmetry result in Chapter 2 is a generalization of Weinberger's proof to the so-called *model manifolds* with non-negative Ricci curvature. Model manifolds are rotationally symmetric Riemannian manifolds $(\mathbb{M}^n_\sigma, g_{\mathbb{M}^n_\sigma})$, i.e. $\mathbb{M}^n_\sigma = [0,R)\times\mathbb{S}^{n-1}$, for some $R>0$, and the metric is

$$g_{\mathbb{M}^n_\sigma} = dr\otimes dr + \sigma^2(r)g_{\mathbb{S}^{n-1}}\,,$$

for some smooth function $\sigma:[0,R)\to[0,+\infty)$ and where $g_{\mathbb{S}^{n-1}}$ is the usual metric on the round sphere $\mathbb{S}^{n-1}$ (see Definition 2.1 for the precise definition). We mention that important examples of model manifolds are the already celebrated space-forms. In particular

- the Euclidean space $\mathbb{R}^n$ is isometric to the model manifold $\mathbb{M}^n_\sigma$ with $\sigma(r) = r : [0, +\infty) \to [0, +\infty)$;
- the hyperbolic space $\mathbb{H}^n$ is isometric to the model manifold $\mathbb{M}^n_\sigma$ with $\sigma(r) = \sinh(r) : [0, +\infty) \to [0, +\infty)$;
- the standard sphere $\mathbb{S}^n \setminus \{N\}$, where N is the north pole, is isometric to the model manifold $\mathbb{M}^n_\sigma$ with $\sigma(r) = \sin(r) : [0, \pi) \to [0, +\infty)$.

Under the assumption that the Ricci curvature of the model manifold is non-negative and under a suitable “compatibility" assumption between the solution of the overdetermined problem and the geometry of the model, in Section 2.1, we will prove the following rigidity result

Theorem I.C: *let $(\mathbb{M}^n_\sigma, g_{\mathbb{M}^n_\sigma})$ be a model manifold such that*

$$\mathrm{Ric}_{\mathbb{M}^n_\sigma} \geq 0 \,, \qquad \textit{and} \qquad \sigma' > 0 \,.$$

Let $\Omega \subset \mathbb{M}^n_\sigma$ be a smooth domain and assume that Ω supports a solution u of

$$\begin{cases} \Delta u = -1 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \\ \partial_\nu u = c & \textit{on } \partial\Omega \,. \end{cases} \tag{I.21}$$

(where Δ is the Laplace-Beltrami operator). If u satisfies the following “compatibility” condition

$$\int_\Omega \frac{(\sigma'' \sigma^{n-1})'}{\sigma^{n-1}} u^2 \geq 0 \,, \tag{I.22}$$

then Ω is a Euclidean ball.

The proof of Theorem I.C is based on the Weinberger’s approach. Explicitly we consider the P-function (I.4), and we show that it is a subharmonic function. The proof is achieved by using Bochner formula

$$\Delta \left|\nabla u\right|^2 = 2 \left|\nabla^2(u)\right|^2 + 2g(\nabla(\Delta u), \nabla u) + 2\mathrm{Ric}(\nabla u, \nabla u) \,, \tag{I.23}$$

and the inequality

$$(\Delta u)^2 \leq n \left|\nabla^2(u)\right|^2 \,, \tag{I.24}$$

which can be easily obtained by using Cauchy-Schwarz inequality. Moreover, again by Cauchy-Schwarz inequality we have that the equality sign holds if and only if

$$\nabla^2(u) = \frac{\Delta u}{n} g \,. \tag{I.25}$$

We mention that (I.23), (I.24) and (I.25) hold in every n dimensional Riemannian manifolds (M, g) and for every C^2-map $u : (M, g) \to \mathbb{R}$.

Since P is subharmonic, the strong maximum principle yields that either (I.5) or (I.6) holds. By using a suitable generalization of the Pohozaev identity (2.58) we can exclude (I.6). Hence P is constant and harmonic, and from (I.25) the conclusion follows.

Symmetry results in convex cones. The second result that we present in Chapter 2 are two symmetry results for domains in convex cones. The starting observation that motivates our study is the following. Let Σ be an open cone in $\mathbb{R}^n$ with vertex at the origin O, i.e.

$$\Sigma := \{tx \,:\, x \in \omega\,,\, t \in [0, +\infty)\}\,, \tag{I.26}$$

for some open domain $\omega \subseteq \mathbb{S}^{n-1}$. We notice that if the point x_0 in (I.11) is chosen appropriately then u given by (I.11) is still the solution to the following problem:

$$\begin{cases} \Delta u = -1 & \text{in } B_R(x_0) \cap \Sigma \\ u = 0 \text{ and } \partial_\nu u = c & \text{on } \partial B_R(x_0) \setminus \overline{\Sigma} \\ \partial_\nu u = 0 & \text{on } B_R(x_0) \cap \partial\Sigma\,. \end{cases} \tag{I.27}$$

More precisely, x_0 may coincide with O or it may be just a point of $\partial\Sigma \setminus \{O\}$ and, in this case, $B_R(x_0) \cap \Sigma$ is half a sphere lying over a at portion of $\partial\Omega$. Hence, it is natural to look for a characterization of symmetry in this direction, as done in [180]. In order to properly describe the results, we introduce some notation. Given an open cone Σ such that $\partial\Sigma \setminus \{O\}$ is smooth, we consider a bounded domain $\Omega \subset \Sigma$ and denote by Γ_0 its relative boundary, i.e.

$$\Gamma_0 = \partial\Omega \cap \Sigma\,,$$

and we set

$$\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}_0\,.$$

We assume that $\mathcal{H}_{n-1}(\Gamma_1) > 0$, $\mathcal{H}_{n-1}(\Gamma_0) > 0$ and that Γ_0 is a smooth $(n-1)$-dimensional manifold, while $\partial\Gamma_0 = \partial\Gamma_1 \subset \partial\Omega \setminus \{O\}$ is a smooth $(n-2)$-dimensional manifold. Following [180], such a domain Ω is called a *sector-like domain*. In the following, we shall write $\nu = \nu_x$ to denote the exterior unit normal to $\partial\Omega$ wherever is defined (that is for $x \in \Gamma_0 \cup \Gamma_1 \setminus \{O\}$). Under the assumption that Σ is a convex cone, in [180] it is proved that if Ω is a sector-like domain and there exists a classical solution $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\})$ to

$$\begin{cases} \Delta u = -1 & \text{in } \Omega\,, \\ u = 0 \text{ and } \partial_\nu u = c & \text{on } \Gamma_0\,, \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}\,, \end{cases} \tag{I.28}$$

and such that

$$u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)\,,$$

then

$$\Omega = B_R(x_0) \cap \Sigma$$

for some $x_0 \in \mathbb{R}^n$ and u is given by (I.11). Differently from the original paper of Serrin [204], the method of moving planes is not helpful (at least when applied in a standard way) and the rigidity result in [180] is proved by using two alternative approaches. One is based on integral identities and it is inspired from [40], the other one uses a P-function approach as in [215].

In [63], we generalize the rigidity result for Serrin's problem in [180] in two directions. The former is by considering more general operators than the Laplacian in the Euclidean space, where the operators may be of degenerate type. Here, the generalization is not trivial due to the lack of regularity of the solution (the operator may be degenerate) as well as to other technical details which are not present in the linear case. The operator that we are going to consider is the following:

$$L_f u = \operatorname{div}\left(\frac{f'(|\nabla u|)}{|\nabla u|}\nabla u\right), \tag{I.29}$$

where f is a positive, convex and super-linear function (we mention that overdetermined problems associated to the operator L_f has been considered in [92, 103, 106]).

The latter is by considering an analogous problem in space forms, i.e. the hyperbolic space and the (hemi)sphere. The operator that we consider is linear and it is interesting since it has been shown that it is a helpful generalization of the torsion problem to space forms (see e.g. [68, 189, 190]).

The first problem that we are going to consider is the following mixed boundary value problem:

$$\begin{cases} L_f u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}, \end{cases} \tag{I.30}$$

where L_f is given by (I.29), Ω is a sector like domain in $\mathbb{R}^n$ and $f : [0,+\infty) \to [0,+\infty)$ is such that

$$f \in C^1([0,\infty)) \cap C^3((0,\infty)) \text{ with } f(0) = f'(0) = 0,\ f''(s) > 0 \text{ for } s > 0$$
$$\text{and } \lim_{s\to+\infty} \frac{f(s)}{s} = +\infty\,. \tag{I.31}$$

The key observation that motivates our study is the following: we notice that the solution to $L_f u = -1$ in $B_R(x_0)$ (a ball of radius R centered at x_0) such that $u = 0$ on $\partial B_R(x_0)$ is radial and it is given by

$$u(x) = \int_{|x-x_0|}^{R} g'\left(\frac{s}{n}\right) ds\,, \tag{I.32}$$

where g denotes the Fenchel conjugate of f (see for instance [73] or [103]), i.e.

$$g = \sup\{st - f(s)\,:\, s \geq 0\}$$

(hence for us g' is the inverse function of f').

With these prelimiaries, the second rigidity result in Chapter 2 is the following

Theorem I.D: *let f satisfy* (I.31)*. Let Σ be a convex cone such that $\Sigma \setminus \{O\}$ is smooth and let Ω be a sector-like domain in Σ. If there exists a solution $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega)$ to* (I.30) *such that*

$$\partial_\nu u = -c \text{ on } \Gamma_0 \tag{I.33}$$

for some constant c, and satisfying

$$\frac{f'(|\nabla u|)}{|\nabla u|}\nabla u \in W^{1,2}(\Omega, \mathbb{R}^n)\,, \tag{I.34}$$

then there exists $x_0 \in \mathbb{R}^n$ such that $\Omega = \Sigma \cap B_R(x_0)$ with $c = g'(|\Omega|/|\Gamma_0|)$, $R = n|\Omega|/|\Gamma_0|$. Moreover u is given by (I.32)*, where x_0 is the origin or, if $\partial\Sigma$ contains flat regions, it is a point on $\partial\Sigma$.*

We refer to Section 2.2 for a detailed discussion on the hypothesis of the Theorem. The proof of Theorem I.D s based on the integral approach introduced in [40] (see also [180] and [29]). The idea is the following: we set $V(\xi) = f(|\xi|)$ and we define the matrix $W(x) = (w_{ij}(x))_{ij}$ such that

$$w_{ij}(x) = \partial_j V_{\xi_i}(\nabla u(x)) .$$

The matrix W is such that $L_f u = \mathrm{Tr}(W)$; writing the operator as a trace has the advantage that we can use Newton's inequality (I.13). By using a Pohozaev-type identity, integral inequalities and the convexity of the cone we are able to prove that the equality holds in (I.13) and this implies the rigidity result.

The second result in Section 2.2 deals with an overdetermined problem in space forms: recall that a space form is a complete simply-connected Riemannian manifold (M, g) with constant sectional curvature K. Up to homotheties we may assume $K = 0,\ 1,\ -1$: the case $K = 0$ corresponds to the Euclidean space $\mathbb{R}^n$, $K = -1$ is the hyperbolic space $\mathbb{H}^n$ and $K = 1$ is the unitary sphere with the round metric $\mathbb{S}^n$. More precisely, in the case $K = 1$ we consider the hemisphere $\mathbb{S}^n_+$. These three models can be described as warped product spaces $M = I \times \mathbb{S}^{n-1}$ equipped with the rotationally symmetric metric

$$g = dr^2 + h(r)^2\, g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the round metric on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ and

- $h(r) = r$ in the Euclidean case $(K = 0)$, with $I = [0, \infty)$;
- $h(r) = \sinh(r)$ in the hyperbolic case $(K = -1)$, with $I = [0, \infty)$;
- $h(r) = \sin(r)$ in the spherical case $(K = 1)$, with $I = [0, \pi/2)$ for $\mathbb{S}^n_+$.

By using the warping structure of the manifold, we denote by O the pole of the model and it is natural to define a *cone* Σ *with vertex at* $\{O\}$ as the set

$$\Sigma = \{tx\, :\, x \in \omega,\, t \in I\}$$

for some open domain $\omega \subset \mathbb{S}^{n-1}$. Moreover, we say that Σ is a *convex cone* if the second fundamental form II is nonnegative defined at every $p \in \partial\Sigma$.

The second problem that we are going to consider is the following mixed boundary value problem

$$\begin{cases} \Delta u + nKu = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}, \end{cases} \tag{I.35}$$

where Ω is a sector-like domain in a smooth and convex cone Σ in the Euclidean space $(K = 0)$, in the hyperbolic space $(K = -1)$ or in the hemisphere $(k = 1)$.

With these preliminaries, the third rididifity result in Chapter 2 is the following

Theorem I.E: *let (M, g) be the Euclidean space, hyperbolic space or the hemisphere. Let $\Sigma \subset M$ be a convex cone such that $\Sigma \setminus \{O\}$ is smooth and let*

Ω be a sector-like domain in Σ. Assume that there exists a solution $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ to (I.35) *such that*

$$\partial_\nu u = -c \ \text{ on } \ \Gamma_0 \tag{I.36}$$

for some constant c. Then $\Omega = \Sigma \cap B_R(x_0)$ where $B_R(x_0)$ is a geodesic ball of radius R centered at x_0 and u is given by

$$u(x) = \frac{H(R) - H(d(x, x_0))}{n\dot{h}(R)} \,,$$

with

$$H(r) = \int_0^r h(s) ds$$

and where $d(x, x_0)$ denotes the distance between x and x_0.

The strategy to prove Theorem I.E follows the Weinberger approach and uses the following P-function, introduced in [68] and in [190],

$$P(x) = |\nabla u(x)|^2 + \frac{2}{n} u(x) + K u^2(x) \,.$$

One can show that, if u solves (I.35)-(I.36) then P satisfies:

$$\begin{cases} \Delta P(x) \geq 0 & \text{in } \Omega, \\ P(x) = c^2 & \text{on } \Gamma_0 \\ \partial_\nu P(x) \leq 0 & \text{on } \Gamma_1 \setminus \{O\}. \end{cases}$$

This implies that

$$P(x) \leq c^2 \quad \text{in } \Omega.$$

We can exclude the case $P(x) < c^2$ in Ω, so

$$P(x) \equiv c^2 \quad \text{in } \Omega.$$

In particular, $\Delta P(x) = 0$ in Ω and this implies that

$$\nabla^2(u) = \left(-\frac{1}{n} - K u \right) g \quad \text{in } \Omega,$$

and this equation implies the rigidity result.

Chapter 1

Quantitative stability for almost constant mean curvature hypersurfaces

The main result of this chapter is the following sharp stability estimate of proximity to a single sphere for Alexandrov's theorem.

Theorem 1.1. *Let S be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{M}^n_+$ satisfying a uniform touching ball condition of radius ρ. Let H_S given by (I) or (II). There exist constants ε, $C > 0$ such that if*

$$\mathrm{osc}(\mathsf{H}_S) \le \varepsilon, \tag{1.1}$$

then there are two concentric balls B^d_r and B^d_R of $\mathbb{M}^n_+$ such that

$$S \subset \overline{B}^d_R \setminus B^d_r, \tag{1.2}$$

and

$$R - r \le C \mathrm{osc}(\mathsf{H}_S). \tag{1.3}$$

The constants ε and C depend only on n and upper bounds on ρ^{-1} and on the area of S.

For the sake of clarity we mention that the balls in the statement of Theorem 1.1 are distance or geodesic balls in $\mathbb{M}^n_+$, i.e. if we denote with d the distance induced by the metric g of the space form then $B^d_r(p)$ denotes the ball centred at p of radius r with respect to the distance d. The center of the two balls in the statement will be specified later.

We recall that the *uniform touching ball condition* of radius ρ in Theorem 1.1 means that at any point of S there exist two balls of radius ρ both tangent to S one from inside and one from outside. As mentioned in the introduction the assumption that ρ is fixed implies that a bubbling phenomenon can not appear (see for instance [58]). As can be shown by a calculation for ellipsoids, the estimate in (1.3) is optimal and it is new in the general setting of Theorem 1.1. In the Euclidean space, quantitative results, in the spirit of Theorem 1.1, for constant mean curvature hypersurfaces are avalaible in literature only under the assumption that S bounds a convex domain. In particular, in [147, 151, 173] the case when the domain is an ovaloid is considered. Moreover in [144], where the results in [8, 200] are improved, an explicit Holder type stability estimate like (1.3) is proved. We emphasize that, in Theorem 1.1 we do not consider any convexity assumption and, as already mentioned, the estimate that we prove is optimal; hence (1.3) remarkably improves the previous stability results.

Theorem 1.1 follows the research line initiated in [67] and pursued in [69] and is contained in [65].

Theorem (1.1) has a quite interesting consequence which is a pinching result. It is well-known that, in the Euclidean space (see for instance [121]), if every principal curvature k_i of S is pinched between two positive numbers, i.e.

$$\frac{1}{r} \leq k_i \leq \frac{1+\delta}{r}, \quad \text{for } i = 1, \dots, n-1,$$

then S is close to a sphere of radius r. One can ask if this happens when only the mean curvature is pinched or, more in general if one consider the class of curvatures H_S defined in the introduction. As a consequence of Theorem 1.1 we have the following

Corollary 1.1. *Let ρ_0, $A_0 > 0$ and $n \in \mathbb{N}$ be fixed. There exists $\varepsilon > 0$, depending on n, ρ_0 and A_0, such that if S is a connected closed C^2 hypersurface embedded in $\mathbb{M}^n_+$ having area bounded by A_0, satisfying a touching ball condition of radius $\rho \geq \rho_0$, and such that*

$$\operatorname{osc}(\mathsf{H}_S) \leq \varepsilon\,, \tag{1.4}$$

then S is C^1-close to a sphere and there exists a C^2-regular map $\Psi : \partial B^d_r \to \mathbb{R}$ such that

$$F(p) = \exp_x(\Psi(p) N_p)$$

defines a C^2-diffeomorphism from B^d_r to S and

$$\|\Psi\|_{C^1(\partial B^d_r)} \leq C \operatorname{osc}(\mathsf{H}_S)^{1/2}\,, \tag{1.5}$$

where N is a normal vector field to ∂B^d_r.

Before explaining the proof of Theorem 1.1 and Corollary 1.1, we give a couple of remarks on the bounds on ρ and on the Area of S in Theorem 1.1 and Corollary 1.1. The upper bound on the Area of S is a control on the constants ε and C, which clearly change under dilatations. The upper bound on ρ^{-1} controls the C^2-regularity of the hypersurface, which is crucial for obtaining an estimate like (1.3). Indeed, if we assume that ρ is not bounded from below in the Euclidean case, it is possible to construct a family of closed surfaces embedded in $\mathbb{R}^3$, not diffeomorphic to a sphere, with osc(H) arbitrarily small and such that (1.3) fails to hold (see Figure 1.1).

As already mentioned in the introduction, we tackle the problem by doing a quantitative study of the method of the moving planes, i.e. we study the original proof of Alexandrov from a quantitative point of view. There are other possible approaches for proving the symmetry result (i.e. of Alexandrov's Theorem) which are based on integral and geometric identities. The interested reader can refer to [172, 194, 197, 198], and to [41] for a recent generalization (see also [82] for minimal assumptions on the regularity of S). The approach in [194] has

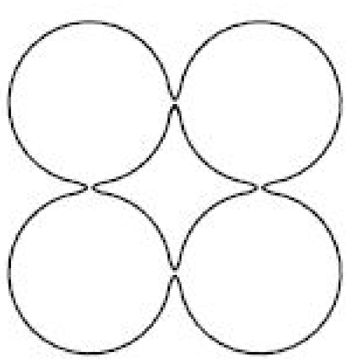

Figure 1.1: counterexample obtained by gluing pieces of suitable small perturbations of unduloids.

been exploited in [58] to tackle the problem of bubbling (see also [83] and [148]). Symmetry and quantitative studies of proximity to a single sphere by using an integral approach can be found in [30, 45, 56, 96, 127, 128, 164, 165, 166, 189]. Coming back to our approach, it is based on a quantitative analysis of the method of moving planes and uses arguments from elliptic PDEs theory. Since, as explained in the Introduction, the proof of symmetry (i.e. of Alexandrov's Theorem) is based on the maximum principle, our proof of the stability result will make use of Harnack's and Carleson's (or boundary Harnack) inequalities and Hopf Lemma, which can be considered as the quantitative counterpart of the strong and boundary maximum principles.

For the sake of completeness we have to say that a quatitative study of the method of moving planes was first performed in [2], where the authors obtained a stability result for Serrin theorem (see the introduction), and it has been used in the following series of papers [60, 61, 62] to study the stability of radial symmetry for Serrin's and other overdetermined problems (see also the survey [64]).

In order to prove Theorem 1.1 we follow the approach of [2], but in our case the setting is complicated by the fact that we have to deal with Riemannian manifolds. As we will show in Section 1.4 the main task is to prove the approximate symmetry result for one direction. To prove this result we apply the method of moving planes and show that the union of the maximal cap and of its reflection provides a set that fits S well. This is done in Theorem 1.4; by the method of moving planes we know that the hypersurface and the reflected cap are tangent (internally or at the boundary) at some point p_0. We write the two hypersurfaces as graphs of two functions in neighbourhood of p_0. We set w the difference of these two functions and we have that it satisfies an elliptic equation

$$Lw = f\,,$$

where $||f||_\infty$ is bounded by $\mathrm{osc}(\mathsf{H}_S)$. Then via Harnack's inequality and interior regularity estimates we are able to prove a bound on the C^1 norm of w, which implies that the two surfaces are close in the following sense: not only the distance between points is controlled by a constant times $\mathrm{osc}(\mathsf{H}_S)$, but also the two corresponding Gauss map are close (always by a constant times $\operatorname{osc}(\mathsf{H}_S)$) in a neighbourhood of p_0. To propagate this estimate we connect any point of the reflected cap with p_0 via a chain of balls and by using careful estimates and interior or boundary Harnack inequality we propagate the estimate (assuming that $\operatorname{osc}(\mathsf{H}_S)$ is small).

The chapter is organized as follows. In Section 1.1 we review the method of the moving planes in space forms and set up some notation. In Section 1.2 we give technical local quantitative estimates in space forms. In Section 1.3 we find estimates on curvatures of projected surfaces in conformally Euclidean spaces. In Section 1.4 we prove the approximate symmetry in one direction. In Section 1.5 we show how to prove global approximate symmetry result by using the approximate symmetry in any direction. Finally, in Section 1.6 we complete the proof of main results.

We will use the following models of space forms:

- $\mathbb{H}^n$ is the half-space $\{x \in \mathbb{R}^n \,:\, x_n > 0\}$ with the Riemannian metric
$$g_x = \tfrac{1}{x_n}\langle \cdot,\cdot\rangle\,, \quad \text{for every } x \in \mathbb{R}^n \tag{1.6}$$
where $\langle \cdot,\cdot\rangle$ is the Euclidean product on $\mathbb{R}^n$; for simplicity of notation, we will indicate it also with " $\cdot$ ";

- $\mathbb{S}^n$ is the n-dimensional unitary sphere $\{x \in \mathbb{R}^{n+1} \,:\, |x| = 1\}$ with the round metric g induced by the Euclidean metric in $\mathbb{R}^{n+1}$. Here we recall that if we consider the stereographic projection $\mathbb{S}^n \backslash\{\text{one point}\} \to \mathbb{R}^n$, then g is projected to the Riemann metric
$$g_x = \frac{4}{(1+|x|^2)^2}\langle \cdot,\cdot\rangle\,, \quad \text{for every } x \in \mathbb{R}^n\,. \tag{1.7}$$

We recall that $\mathbb{M}^n$ denotes the space forms, i.e. the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ and the sphere $\mathbb{S}^n$; while $\mathbb{M}^n_+$ denotes the Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ and the hemisphere $\mathbb{S}^n_+$. Moreover, in all the chapter S denotes a C^2-regular, connected, closed hypersurface embedded in $\mathbb{M}^n_+$ and $\Omega \subset \mathbb{M}^n_+$ denotes a relatively compact connected open set such that $\partial\Omega = S$ (such an Ω exists since S is an embedded hypersurface).

1.1 The method of the moving planes

In this preliminary section we recall the method of the moving planes in $\mathbb{M}^n_+$.

We begin by recalling the definition of *center of mass* in the context of Riemannian geometry (see e.g. [142] for more details).

Let (M,g) be an oriented complete Riemannian manifold and let Ω be a bounded domain (i.e. a bounded connected open set). Let $P_\Omega \colon M \to \mathbb{R}$ be the function
$$P_\Omega(x) = \frac{1}{2\,|\Omega|_g}\int_\Omega d(x,a)^2\,da\,,$$

where $|\Omega|_g$ is the volume of Ω with respect to g. Then the gradient of P_Ω takes the following expression

$$\nabla P_\Omega(x) = -\frac{1}{|\Omega|_g}\int_\Omega \exp_x^{-1}(a)\,da\,, \tag{1.8}$$

where $\exp_x : T_xM \to M$ denotes the exponential map of the Riemannian manifold (M,g) at $x \in M$ (T_xM denotes the tangent space to the manifold M at $x \in M$) and $\exp_x^{-1}$ denotes its inverse. In some cases P_Ω is a convex map and it attains the minimum at only one point $\mathcal{O}$, which is usually called the *center of mass* of Ω. For instance this occurs in the following cases:

- all the sectional curvatures of M are non-positive;
- Ω is contained in a geodesic ball of radius $r < \frac{1}{2}\min\left\{\operatorname{inj} M, \frac{\pi}{2\sqrt{K}}\right\}$, where K is an upper bound on the sectional curvatures of M and $\operatorname{inj} M$ is the injectivity radius of M;
- $M = \mathbb{S}^n$ and Ω is contained in $\mathbb{S}^n_+$.

We further recall that a Riemannian manifold (M,g) is a symmetric space if for every $p \in M$ there exists an isometry $f\colon M \to M$ such that $f(p) = p$ and $f_{*|p} = -\mathrm{Id}$.

Lemma 1.1. *Let (M,g) be a symmetric space, Ω a bounded domain in M and $x \in M$ be such that $\nabla P_\Omega(x) = 0$. Assume that for every hyperplane π in M not containing x there exists a hyperplane π_1 passing through x and such that $\pi \cap \pi_1 \cap \Omega = \emptyset$. Then every hyperplane of symmetry for Ω contains x.*

Proof. Assume by contradiction that there exists a hyperplane π of symmetry for Ω not containing x. Let π_1 be a hyperplane passing through x and disjoint from π inside Ω, i.e. $\pi \cap \pi_1 \cap \Omega = \emptyset$. Since π_1 and π are disjoint, they subdivide Ω in three disjoint subsets Ω_1, Ω_2 and Ω_3, with $|\Omega_i|_g > 0$, $i = 1, 2, 3$. Since Ω is symmetric about π, we have that

$$|\Omega_1|_g + |\Omega_2|_g = |\Omega_3|_g.$$

Moreover formula (1.8) implies

$$\int_{\Omega_1} \exp_x^{-1}(a)\,da = -\int_{\Omega_2\cup\Omega_3} \exp_x^{-1}(a)\,da\,. \tag{1.9}$$

Let $f\colon M \to M$ be the isometry such that $f(x) = x$ and $f_{|*x} = -\mathrm{Id}$. Then

$$\exp_x^{-1} f(a) = -\exp_x^{-1}(a)$$

and

$$-\int_{\Omega_2\cup\Omega_3} \exp_x^{-1}(a)\,da = \int_{f(\Omega_2\cup\Omega_3)} \exp_x^{-1}(a)\,da\,.$$

Therefore (1.9) implies

$$|\Omega_1|_g = |\Omega_2|_g + |\Omega_3|_g.$$

which gives a contradiction since $|\Omega_2|_g > 0$. □

Lemma 1.1 can be in particular applied in space forms $\mathbb{M}^n_+$, where we have the uniqueness of the center of mass, to prove the following result

Proposition 1.1 (Characterization of the distance balls in $\mathbb{M}^n_+$)**.** *Let $S = \partial\Omega$ be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{M}^n_+$, where Ω is a relatively compact domain. Assume that for every geodesic path $\gamma\colon \mathbb{R} \to \mathbb{M}^n$ there exists a hyperplane π orthogonal to γ such that S is symmetric about π. Then S is a distance sphere about $\mathcal{O}$.*

The previous result is a generalization of the following characterization theorem, due to H. Hopf, of the sphere in the Euclidean space (see also [136, Chapter VII, Lemma 2.2])

Theorem 1.2. *Let $S = \partial\Omega$ be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{R}^n$, where Ω is a relatively compact domain. If for every direction $\omega \in \mathbb{R}^n$ there exists a hyperplane of symmetry of Ω orthogonal to ω, then S is a round sphere.*

Proof of Proposition 1.1. In view of Lemma 1.1 any hyperplane of symmetry of S contains the point $\mathcal{O}$. Therefore S is invariant by reflections about every hyperplane passing through $\mathcal{O}$. For all the three possible cases of space forms we can choose a model where $\mathbb{M}^n_+$ is a ball in $\mathbb{R}^n$, $\mathcal{O}$ is the origin of $\mathbb{R}^n$ and the hyperplanes passing through $\mathcal{O}$ are Euclidean hyperplanes. In such a model S is invariant by reflections about every Euclidean hyperplane passing through the origin and hence it is an Euclidean ball, since every rotation in the Euclidean space can be obtained as the composition of two reflections about hyperplanes. Hence S is a round sphere in $\mathbb{M}^n_+$ as required. □

We describe the method of the moving planes in $\mathbb{M}^n_+$ and we introduce some notation. The method consists in moving hyperplanes along a geodesic path orthogonal to a fixed direction and it is similar in $\mathbb{R}^n$, $\mathbb{H}^n$ and $\mathbb{S}^n_+$. The method can be described in terms of a point o that we fix. Since $\mathbb{M}^n_+$ is a homogeneous space, the construction does not depend on the choice of the point we fix. In particular we choose o to be:

- the origin in $\mathbb{R}^n$;
- e_n in $\mathbb{H}^n$;
- the north pole in $\mathbb{S}^n_+$.

For every direction $v \in T_{\mathsf{o}}\mathbb{M}^n_+$, we consider the geodesic path $\gamma_v\colon I \to \mathbb{M}^n_+$ satisfying $\gamma_v(0) = \mathsf{o}$ and $\dot\gamma_v(0) = v$. The domain of γ_v is $I = \mathbb{R}$ in $\mathbb{R}^n$ and $\mathbb{H}^n$ and $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ in $\mathbb{S}^n_+$. For any $s \in I$ we denote by $\pi_{v,s}$ the hyperplane passing through $\gamma_v(s)$ and orthogonal to $\dot\gamma_v(s)$, and we define

$$S_{v,s} = \{p \in S \ : \ p \in \pi_{v,t} \text{ for some } t > s\}\,.$$

We denote by $S^\pi_{v,s}$ be the reflection of $S_{v,s}$ about $\pi_{v,s}$. Note that

- $S_{v,s} = \{p \in S \ : \ p \cdot v > s\}$, if $\mathbb{M}^n_+ = \mathbb{R}^n$;
- $S_{v,s} = \{p \in S \ : \ p \cdot \dot\gamma_v(s) > 0\}$, if $\mathbb{M}^n_+ = \mathbb{S}^n_+$.

In the hyperbolic case, giving an explicit description of $S_{v,s}$ is more complicated, but it can be simplified by assuming $v = e_1$. This assumption is not restrictive since we can always rotate every direction v in e_1 by using an isometry of $\mathbb{H}^n$. In this case we have

- $S_{e_1,s} = \{p \in S \ : \ p \cdot e_1 > s\}$, if $\mathbb{M}^n_+ = \mathbb{H}^n$.

Let

$$m_v = \inf\{s \in I \ : \ S^\pi_{v,t} \subset \Omega \text{ for every } t > s\}\,.$$

The hyperplane $\pi_v := \pi_{v,m_v}$ is called the *critical hyperplane*. By construction S^π_{v,m_v} is contained in $\overline{\Omega}$ and S and S^π_{v,m_v} are tangent at some point p_0 which can be either interior to S^π_{v,m_v}, or on $\partial S^\pi_{v,m_v}$ (and in this last case $p_0 \in \pi_v$).

Now for the sake of completeness we recall the generalized version of Alexandrov's theorem (Alexandrov theorem II in the introduction) that we study in this chapter and its proof in $\mathbb{M}^n_+$ (see [5, 145, 172, 194, 197, 198]).

Theorem 1.3. *The only closed C^2-regular connected hypersurfaces embedded in $\mathbb{M}^n_+$ and such that H_S is constant are the distance spheres.*

Proof. The proof consists in showing that for every unitary vector $v \in T_{\mathsf{o}}\mathbb{M}^n_+$, S is symmetric about π_v. This is obtained by showing that $S \cap S^\pi_{v,m_v}$ is open and closed in S^π_{v,m_v}. Note that $S \cap S^\pi_{v,m_v}$ is not empty since S^π_{v,m_v} is tangent to S at some point and $S \cap S^\pi_{v,m_v}$ is closed in S^π_{v,m_v}. The only nontrivial step is that $S \cap S^\pi_{v,m_v}$ is open, which is obtained by using maximum principles for solutions to elliptic equations.

Let $p_0 \in S \cap S^\pi_{v,m_v}$. By construction we have that

$$T_{p_0}S = T_{p_0}S^\pi\,,$$

where S^π is the reflection of S about π_v. From the implicit function theorem, S and S^π are locally the Euclidean graph of C^2-regular functions u and $\hat{u}$, respectively, defined in a (Euclidean) ball B_r of radius r centered at the origin O in $T_{p_0}S$. The functions u and $\hat{u}$ satisfy the elliptic equation $Lu(x) = \mathsf{H}_S(x, u(x))$ for $x \in B_r$; here the operator L is the mean curvature operator or, more generally, a fully nonlinear operator. The ellipticity of L is standard in the case of the mean curvature operator and it follows from [145] for the other cases considered.

Since H_S is constant, the difference $u - \hat{u}$ satisfies an elliptic equation of the form $\mathcal{L}(u - \hat{u}) = 0$ with $u(O) - \hat{u}(O) = 0$.

If p_0 is interior to S^π_{v,m_v} then we can choose r sufficiently small such that $u - \hat{u} \geq 0$ in B_r and by the strong maximum principle we obtain that $u - \hat{u} \equiv 0$ in B_r.

If p_0 is on the boundary of S^π_{v,m_v}, then by construction $u - \hat{u} \geq 0$ in a half ball B^+_r, $u(O) - \hat{u}(O) = 0$ and $\nabla u(O) = \nabla u(O) = 0$. By applying Hopf's boundary point lemma at the point O we obtain that $u - \hat{u} \equiv 0$ in B^+_r.

Hence, we have proved that $S \cap S^\pi_{v,m_v}$ is open, and the conclusion follows. □

1.2 Local quantitative estimates in space forms

In this section we prove some preliminary estimates which we will use in the proof of Theorem 1.1. We have the following preliminary lemma about the local equivalences of distances (we will denote the Euclidean norm with $|\cdot|$).

Lemma 1.2. • *Let d be the distance induced by the hyperbolic metric* (1.6) *in* $\mathbb{H}^n$ *and let q be such that $d(q, e_n) \le R$; then*

$$c|q - e_n| \le d(q, e_n) \le C|q - e_n| \,, \tag{1.10}$$

for some positive constants c and C depending only on R.

• *Let d be the distance induced by the round metric* (1.7) *in* $\mathbb{R}^n$ *and let p, q in $\mathbb{R}^n$ be such that $|p|, |q| \le R$. Then*

$$\frac{2}{1+R^2}|p - q| \le d(p, q) \le \pi|p - q| \,. \tag{1.11}$$

Proof. • We recall that the hyperbolic distance induced by the hyperbolic metric (1.6) is given, in terms of the Euclidean distance, by the following expression

$$d(p, q) = \operatorname{arccosh}\left(1 + \frac{|p-q|^2}{2p_nq_n}\right) .$$

In particular

$$d(e_n, te_n) = |\log(t)| \,, \qquad \text{for any } t \in (0, \infty).$$

This expression and the fact that $d(q, e_n) \le R$ imply that

$$e^{-R} \le q_n \le e^R \,. \tag{1.12}$$

For simplicity, we set

$$t = 1 + \frac{|q - e_n|^2}{2q_n}$$

hence, from (1.12),

$$1 + \frac{e^{-R}}{2}|q - e_n|^2 \le t \le 1 + \frac{e^R}{2}|q - e_n|^2 \,, \tag{1.13}$$

and, since $|q - e_n| \le R(e^R - 1)$, then

$$t \le A \,, \tag{1.14}$$

where A is a constant which depends only on R. Now we set $\phi(t) = \operatorname{arccosh}(t)$, for $t \in [1, +\infty)$. Since, from (1.14), $1 \le t \le A$ we have

$$\frac{1}{\sqrt{A+1}}\frac{1}{\sqrt{t-1}} \le \phi'(t) \le \frac{1}{\sqrt{t-1}}$$

where we used the fact that $\phi'(t) = (t^2 - 1)^{-1/2}$, hence

$$\frac{1}{2\sqrt{A+1}}\sqrt{t-1} \le \phi(t) \le \frac{1}{2}\sqrt{t-1} \qquad \text{for every } t \in [1, A]. \tag{1.15}$$

The conclusion (1.10) follows combining (1.15) with

$$\frac{e^{-\frac{R}{2}}}{\sqrt{2}}|q - e_n| \le \sqrt{t-1} \le \frac{e^{\frac{R}{2}}}{\sqrt{2}}|q - e_n|$$

which follows from (1.13).

• We recall that the spherical distance induced by the round metric (1.7) is given, in terms of the Euclidean distance, by the following expression

$$d(p,q) = 2\arcsin\left(\frac{|p-q|}{\sqrt{(1+|p|^2)(1+|q|^2)}}\right).$$

Since $|p|$,$|q| \leq R$ and from the previous expression we get

$$2\arcsin\left(\frac{|p-q|}{1+R^2}\right) \leq d(p,q) \leq 2\arcsin(|p-q|)$$

i.e.

$$\frac{|p-q|}{1+R^2} \leq \sin\left(\frac{d(p,q)}{2}\right) \leq |p-q|\,. \tag{1.16}$$

The conclusion (1.11) follows combining (1.16) with the following trivial property of the sine function:

$$\frac{2t}{\pi} \leq \sin(t) \leq t\,, \qquad \text{for } 0 \leq t \leq \frac{\pi}{2};$$

recall that, being on the sphere $d(p,q) \leq \pi$. □

Let us consider now, as in Alexandrov theorem, a C^2-regular, connected, closed hypersurface $S = \partial\Omega$ embedded in $\mathbb{M}^n_+$, where Ω is a relatively compact domain and denote by N the inward normal vector filed (inward with respect to Ω). For $p \in S$ we denote by $\varphi_p \colon \mathbb{M}^n_+ \to \mathbb{R}^n$ the following function whose definition depends on the geometry of $\mathbb{M}^n$:

- if $\mathbb{M}^n$ is $\mathbb{R}^n$, $\varphi_p \in \mathrm{SO}(n) \rtimes \mathbb{R}^n$ and it is such that $\varphi_p(p) = 0$ and $\varphi_{p*|p}(T_pS) = \{x_n = 0\}$;
- if $\mathbb{M}^n$ is $\mathbb{H}^n$, φ_p is an orientation preserving isometry of $\mathbb{H}^n$ such that $\varphi_p(p) = e_n$ and $\varphi_{p*|p}(T_pS) = \{x_n = 0\}$;
- if $\mathbb{M}^n$ is $\mathbb{S}^n$, φ_p is the stereographic projection form the antipodal point to p restricted to $\mathbb{S}^n_+$ composed with a rotation of $\mathbb{R}^n$ in order to have $\varphi_{p*|p}(T_pS) = \{x_n = 0\}$.

Note that in all the three cases we have that $\varphi_p(S)$ is a hypersurface embedded in $\mathbb{R}^n$ and

$$\varphi_{p*|p}(T_pS) = \{x_n = 0\}\,.$$

For $r > 0$, we denote by $\mathcal{U}_r(p)$ the open neighbourhood of p in S such that $\varphi_p(\mathcal{U}_r(p))$ is the (Euclidean) graph of a C^2-function $u \colon B_r \to \mathbb{R}$ defined in the ball of radius r of $\mathbb{R}^{n-1}$ centred at the origin. Even if we don't have a canonical choice of φ_p, the subsets $\mathcal{U}_r(p)$ do not depend on the choice of φ_p. Moreover, the implicit function theorem implies that any $p \in S$ has a neighbourhood $\mathcal{U}_r(p)$ for r sufficiently small. In order to establish the quantitative estimates we need in the proof of the main Theorem, we have to show that r can be uniformly bounded from below with a bound depending only on ρ. The following lemma also introduces the quantity ρ_1 which will be largely used in what follows.

Lemma 1.3. *Let S be a C^2-regular closed hypersurface embedded in $\mathbb{M}^n_+$ and satisfying a touching ball condition of radius ρ and let ρ_1 be defined in the following way:*

- $\rho_1 = \rho$, *if* $\mathbb{M}^n = \mathbb{R}^n$;
- $\rho_1 = (1 - \mathrm{e}^{-\rho} \sinh \rho)\mathrm{e}^{-\rho} \sinh \rho$, *if* $\mathbb{M}^n = \mathbb{H}^n$;
- $\rho_1 = \dfrac{\rho}{\pi}$, *if* $\mathbb{M}^n = \mathbb{S}^n$.

Then

(i) *any point $p \in S$ admits a neighbourhood $\mathcal{U}_{\rho_1}(p)$ and $\varphi_p(\mathcal{U}_{\rho_1}(p))$ is the graph of a C^2-function $u\colon B_{\rho_1} \to \mathbb{R}$ satisfying*

$$|u(x) - u(O)| \leq \rho_1 - \sqrt{\rho_1^2 - |x|^2}\,, \tag{1.17}$$

and

$$|\nabla u(x)| \leq \frac{|x|}{\sqrt{\rho_1^2 - |x|^2}}\,; \tag{1.18}$$

where B_{ρ_1} denotes the Euclidean ball of radius ρ_1 centred at the origin O of $\mathbb{R}^{n-1}$.

(ii) *There exists a universal constant C such that for any $0 < \alpha < \frac{1}{2}\min(1, \rho_1^{-1})$ and q in $\mathcal{U}_{\alpha\rho_1}(p)$ we have*

$$d_S(p, q) \leq \alpha C \rho_1\,, \tag{1.19}$$

where d_S is the geodesic distance on S.

Proof. We analyse each case separately and for every case we show that (i) holds true.

- $\rho_1 = \rho$, if $\mathbb{M}^n = \mathbb{R}^n$. By the implicit function theorem, we have that there exist $r > 0$, $u\colon B_r \to \mathbb{R}$ and $\mathcal{U}_r(p)$ as in (i), i.e.

$$\varphi_p(\mathcal{U}_r(p)) = \{p + x + u(x)\nu_p \,:\, x \in B_r\}$$

where ν_p denotes the (Euclidean) inward unitary normal vector at p to S. Moreover we may assume $r \leq \rho$. The bound (1.17) in B_r follows easily from the definition of the interior and exterior touching balls at p. We now prove the estimate (1.18) in B_r, which allows us to enlarge the domain of u to B_ρ. Let $q \in \varphi_p(\mathcal{U}_r(p))$, then $q = p + x + u(x)\nu_p$, with $|x| < r$. Since $B_\rho(p + \rho\nu_p) \cap B_\rho(q - \rho\nu_q) = \emptyset$, then

$$|p + \rho\nu_p - q + \rho\nu_q| \geq 2\rho\,.$$

Analogously, since $B_\rho(p - \rho\nu_p) \cap B_\rho(q + \rho\nu_q) = \emptyset$, then

$$|q + \rho\nu_q - p + \rho\nu_p| \geq 2\rho\,.$$

By adding the squares of the last two inequalities we get

$$|p - q|^2 + 2\rho^2 \nu_p \cdot \nu_q \geq 2\rho^2\,,$$

and from (1.17) we get

$$\nu_p \cdot \nu_q \geq \frac{1}{\rho}\sqrt{\rho^2 - |x|^2} \quad \text{and} \quad |\nu_p - \nu_q| \leq \sqrt{2}\frac{|x|}{\rho}\,. \tag{1.20}$$

Moreover, since $q = p + x + u(x)\nu_p$ then

$$\nu_q = \frac{\nu_p - \nabla u(x)}{\sqrt{1 + |\nabla u(x)|}}\,,$$

and from (1.20) we get (1.18) in B_r. Since $|\nabla u|$ is bounded in $\overline{B}_r$, we can extend u to a larger ball where (1.18) is still satisfied. It is clear that we can choose $r = \rho$ and (1.17) and (1.18) hold.

- $\rho_1 = (1 - \mathrm{e}^{-\rho}\sinh\rho)\mathrm{e}^{-\rho}\sinh\rho$, if $\mathbb{M}^n = \mathbb{H}^n$. It is enough to observe that $\varphi_p(S)$ satisfies an Euclidean touching ball condition of radius ρ_1 (this motivates the choice of ρ_1). This can be easily deduced by using Lemma 1.2. Hence the statement follows from the Euclidean case.
- $\rho_1 = \dfrac{\rho}{\pi}$, if $\mathbb{M}^n = \mathbb{S}^n$. Also in this case it is enough to observe that $\varphi_p(S)$ satisfies an Euclidean touching ball condition of radius ρ_1 (by using Lemma 1.2). Hence, as before, the statement follows from the Euclidean case.

Now we show that (ii) holds true. Let $q \in \mathcal{U}_{\rho_1}(p)$. Then $\varphi_p(q) = (x, v(x))$ for some $|x| < \rho_1$. Let $\gamma : [0,1] \to \varphi_p(S)$ be the curve joining $\varphi_p(p)$ to $\varphi_p(q)$ defined as $\gamma(t) = (tx, v(tx))$. Then

$$\dot\gamma(t) = (x, \nabla v(tx) \cdot x)$$

and the Cauchy-Schwarz inequality implies

$$|\dot\gamma(t)| \leq |x|\sqrt{1 + |\nabla v(tx)|^2}\,.$$

Then (1.17) yields

$$|\dot\gamma(t)| \leq \frac{\rho_1|x|}{\sqrt{\rho_1^2 - t^2|x|^2}} \leq \frac{|x|}{\sqrt{1-\alpha^2}} \leq \frac{2}{\sqrt{3}}|x|,$$

for $0 \leq |x| \leq \alpha\rho_1$. Since

$$d_S(p,q) \leq l(\gamma)$$

and

$$l(\gamma) = \int_0^1 |\dot\gamma|\,dt\,, \text{ in the Euclidean case,}$$

$$l(\gamma) = \int_0^1 \frac{|\dot\gamma|}{v(tx)}\,dt\,, \text{ in the hyperbolic case,}$$

$$l(\gamma) = \int_0^1 \frac{2}{1+|\gamma|^2}|\dot\gamma|\,dt\,, \text{ in the spherical case,}$$

we obtain that

$$d_S(p,q) \leq C|x|$$

for a universal constant C and (1.19) follows.

□

From Lemma 1.3 it follows the following

Corollary 1.2. *Let S be a compact C^2-regular embedded hypersurfaces in $\mathbb{M}^n_+$ satisfying a touching ball condition of radius ρ. Let $q \in \mathcal{U}_{\alpha\rho_1}(p)$, with $0 < \alpha < \frac{1}{2}\min(1, \rho_1^{-1})$. Then*

$$d_S(p, q) \leq C d(p, q)\,,$$

where C depends only on ρ. In particular for every $p \in S$ the geodesic ball $\mathcal{B}_r(p)$ centred at p and with radius $r < \frac{1}{2}\min(1, \rho_1^{-1})$ satisfies

$$\mathrm{Area}(\mathcal{B}_r(p)) \geq c r^{n-1}\,, \tag{1.21}$$

and c is a constant depending only on n.

1.2.1 Quantitative stability of the parallel transport

In this section we study quantitative estimates involving the parallel transport which will be useful in the proof of the main result. We recall that the parallel transport along a smooth curve $\alpha : [t_0, t_1] \to \mathbb{M}^n$ is the linear map $\tau : T_{\gamma(t_0)}\mathbb{M}^n \to T_{\gamma(t_1)}\mathbb{M}^n$ given by

$$\tau(v) = X(t_1)$$

where $X : [t_0, t_1] \to \mathbb{M}^n$ is the solution to the following linear ODE:

$$\begin{cases} \dot{X}_k + \sum_{i,j=1}^n X_j \dot{\alpha}_j \Gamma_{ij}^k(\alpha) = 0 & k = 1, \dots, n \\ X_k(t_0) = v_k & k = 1, \dots, n\,, \end{cases}$$

where $\dot{X}(t) = (\dot{X}_1(t), \dots, \dot{X}_n(t))$, $\dot{\alpha}(t) = (\dot{\alpha}_1(t), \dots, \dot{\alpha}_n(t))$, $v = (v_1, \dots, v_n)$ and Γ_{ij}^k are the Christoffel symbol in $\mathbb{M}^n$. In what follows we adopt the following notation: given $p, q \in \mathbb{M}^n_+$ with $p \neq q$, we denote by

$$\tau_p^q : T_p\mathbb{M}^n \to T_q\mathbb{M}^n$$

the parallel transport along the unique geodesic path σ connecting p to q. We further assume that τ_p^p is the identity of $T_p\mathbb{M}^n$. A direct computation shows that:

- if $\mathbb{M}^n$ is $\mathbb{R}^n$, then τ_p^q is the identity for every p, q;
- if $\mathbb{M}^n = \mathbb{H}^n$ and if p and q belongs to the same vertical line, then we have

  $$\tau_p^q(v) = \frac{q_n}{p_n} v\,;$$

  moreover, if p and q do not belong to the same vertical line, then in the following lemma we provide the expression of the parallel transport in this general case (for simplicity we assume $p = e_n$).

  Lemma 1.4. *Let $q \in \mathbb{H}^n$ be such that $q \in \mathrm{Span}\,\{e_{n-1}, e_n\}$ and let $v \in \mathbb{R}^n$. Assume that $q_{n-1} \neq 0$, then*

  $$\tau_q^{e_n}(v) = \frac{1}{q_n}(v_1, \dots, v_{n-2}, \tilde{v}_{n-1}, \tilde{v}_n)\,,$$

where

$$\begin{pmatrix} \tilde{v}_{n-1} \\ \tilde{v}_n \end{pmatrix} = \frac{1}{1+a^2} \begin{pmatrix} a(a-q_{n-1})+q_n & a-q_{n-1}-aq_n \\ -(a-q_{n-1}-aq_n) & a(a-q_{n-1})+q_n \end{pmatrix} \begin{pmatrix} v_{n-1} \\ v_n \end{pmatrix}$$

and

$$a = \frac{|q|^2-1}{2q_{n-1}} .$$

Proof. Let $\alpha : [t_0, t_1] \to \mathbb{H}^n$ be defined in the following way

$$\alpha(t) = (\sqrt{1+a^2}\cos(t)+a)e_{n-1} + \sqrt{1+a^2}\sin(t)e_n ,$$

such that $\alpha(t_0) = q$ and $\alpha(t_1) = e_n$. Then, up to reparametrization, α is a geodesic path connecting q to e_n. The parallel transport equation along α yields

$$(\tau_q^{e_n}(v))_k = v_k , \quad k = 1, \dots, n-2 ,$$

while

$$(\tau_q^{e_n}(v))_{n-1} = X_{n-1}(t_1) \quad \text{and} \quad (\tau_q^{e_n}(v))_n = X_n(t_1) ,$$

where the pair (X_{n-1}, X_n) solves the following system

$$\begin{pmatrix} \dot{X}_{n-1} \\ \dot{X}_n \end{pmatrix} = \begin{pmatrix} \operatorname{cotan}(t) & -1 \\ 1 & \operatorname{cotan}(t) \end{pmatrix} \begin{pmatrix} X_{n-1} \\ X_n \end{pmatrix}$$

and

$$\begin{pmatrix} X_{n-1}(t_0) \\ X_n(t_0) \end{pmatrix} = \begin{pmatrix} v_{n-1} \\ v_n \end{pmatrix} .$$

Therefore,

$$\begin{pmatrix} X_{n-1}(t) \\ X_n(t) \end{pmatrix} = A(t)A(t_0)^{-1} \begin{pmatrix} v_{n-1} \\ v_n \end{pmatrix}$$

where

$$A(t) := \begin{pmatrix} \cos(t)\sin(t) & -\sin^2(t) \\ \sin^2(t) & \cos(t)\sin(t) \end{pmatrix} ;$$

and the claim follows. □

- if $\mathbb{M}^n$ is $\mathbb{S}^n_+$, then $\tau_p^q = (P_q \circ R_\alpha)_{|T_p\mathbb{S}^n}$, where R_α acts as the rotation of the angle $\alpha = d(p,q)$ in the plane π containing σ and as the identity in the orthogonal complement of π and P_q is the orthogonal projection onto $T_q\mathbb{S}^n$.

Moreover, we introduce the following notation:

- given $p \in \mathbb{M}^n$ and $v \in T_p\mathbb{M}^n$, $|v|_p := g_p(v,v)^{1/2}$;
- if $S = \partial\Omega$ is a compact C^2-regular embedded hypersurface in $\mathbb{M}^n_+$, where Ω is a relatively compact domain in $\mathbb{M}^n_+$, N is the inward unitary normal vector field on S.

The first result of this section is the following

Proposition 1.2. *Let S be a compact C^2-regular embedded hypersurface in $\mathbb{M}^n_+$ satisfying a touching ball condition of radius ρ. There exists $\delta_0 = \delta_0(\rho)$ such that if $p, q \in S$ with*

$$d_S(p,q) \leq \delta_0 \,,$$

then

$$g_p(N_p, \tau_q^p(N_q)) \geq \sqrt{1 - C^2 d_S(p,q)^2} \,, \tag{1.22}$$

and

$$|N_p - \tau_q^p(N_q)|_p \leq C d_S(p,q) \,, \tag{1.23}$$

where C is a constant depending only on ρ.

Proof. We divide the proof in three parts where we consider the Euclidean, the hyperbolic and the spherical case. Let $\delta_0 = \min(\rho_1, \frac{1}{C})$, where C will be specified later.

- If $\mathbb{M}^n_+ = \mathbb{R}^n$ then (1.22) (1.23) follow immediately from (1.20).
- If $\mathbb{M}^n_+ = \mathbb{H}^n$, after applying the isometry φ_p we may assume that $p = e_n$ and $q = te_n$. As already observed, from the definition of ρ_1, we have that S (actually $\varphi_p(S)$) satisfies an Euclidean touching ball condition of radius ρ_1. Let $\delta_0 = \min(\rho_2, \frac{1}{C})$, where ρ_2 and C will be specified later. From Lemma (1.2) we get that there exists $0 < r_2 < r_2(\rho)$ such that if $d(e_n, q) \leq r_2$ then $|e_n - q| \leq \rho_1/2$; this implies that, being

  $$d(p,q) \leq d_S(p,q) \leq r_2 \,,$$

  we have

  $$|1 - t| = |p - q| \leq \frac{\rho_1}{2} \,.$$

  Hence we can apply the Euclidean estimates (1.20) (with r_1 in place of ρ) and we get

  $$\nu_p \cdot \nu_q \geq \sqrt{1 - \frac{|p-q|^2}{\rho_1^2}} \,.$$

  where, we recall that, ν denotes the Euclidean inward normal vector field on S. Since $d(p,q) \leq \rho$, from (1.10) we have that $|p - q| \leq C_1 d(p,q) \leq C_1 d_S(p,q)$ for some constant $C_1 = C_1(\rho)$, and hence

  $$\nu_p \cdot \nu_q \geq \sqrt{1 - C^2 d_S(p,q)^2} \,, \tag{1.24}$$

  where C depends only ρ (actually $C = C_1/\rho_1$) and provided that $d_S(p,q) \leq \frac{1}{C}$ (observe that this implies that $d(p,q) \leq \frac{1}{C}$). Moreover the inward g-unitary normal vector field N to S satisfies

  $$N_p = \nu_p \,, \quad \nu_q = \frac{1}{t} N_q = \tau_q^p(N_q) \,,$$

  and (1.24) implies

  $$g_p(N_p, \tau_q^p(N_q)) \geq \frac{1}{2}\sqrt{1 - C^2 d_S(p,q)^2} \,,$$

  which is (1.22). Inequality (1.23) follows by a direct computation.

- If $\mathbb{M}^n_+ = \mathbb{S}^n_+$ It is convenient to regard S as a hypersurface of $\mathbb{R}^n$ equipped with the spherical metric (1.7). We may further assume that p is the origin O of $\mathbb{R}^n$ and q belongs to a straight line passing through O. Let $\delta_0 = \min(\rho_1, \frac{1}{C})$, where C will be specified later. If $d(p,q) \leq \rho_1 = \frac{\rho}{\pi}$, then we can apply the Euclidean estimates (1.20) and obtain

$$\nu_p \cdot \nu_q \geq \sqrt{1 - \frac{|p-q|^2}{\rho_1^2}}\,,$$

where, we recall that, ν denotes the Euclidean inward normal vector field on S. Since, from

$$d(p,q) \leq \pi |p-q|\,,$$

we have

$$\nu_p \cdot \nu_q \geq \sqrt{1 - C^2 d_S(p,q)^2}\,, \tag{1.25}$$

where C depends only ρ. Moreover the inward g-unitary normal vector field N to S satisfies

$$N_p = \frac{1}{2}\nu_p\,, \qquad \frac{2}{1+|q|^2} N_q = \nu_q = 2\tau_q^p(N_q)\,,$$

and (1.25) implies

$$g_p(N_p, \tau_q^p(N_q)) \geq \frac{1}{2}\sqrt{1 - C^2 d_S(p,q)^2}\,,$$

which is (1.22). Inequality (1.23) follows by a direct computation and the claim follows.

□

The last result of this section will be used several times in the proof of the main result, for the sake of clarity we split the result in two lemmas: in the first one we state and prove the result in the hyperbolic space while in the second one we state its "spherical counterpart"; in the Euclidean space the result is trivial.

Lemma 1.5. *Let Σ and $\hat\Sigma$ be two compact embedded hypersurfaces in $\mathbb{H}^n$ satisfying both a touching ball condition of radius ρ. Assume that $e_n \in \Sigma$ and $T_{e_n}\Sigma = \{x_n = 0\}$ and that there exist two local parametrizations $u, \hat u : B_r \to \mathbb{R}$ of Σ and $\hat\Sigma$, respectively, with $0 < r \leq \rho_1$ and such that $u - \hat u \geq 0$.*

Let $p_1 = (x_1, u(x_1))$ and $\hat p_1^* = (x_1, \hat u(x_1))$, with $x_1 \in \partial B_{r/4}$, and denote by γ the geodesic path starting from p_1 and tangent to $-\nu_{p_1}$ at p_1. Assume that

$$d(p_1, \hat p_1^*) + |\nu_{p_1} - \nu_{\hat p_1^*}| \leq \theta\,, \tag{1.26}$$

for some $\theta \in [0, 1/2]$, where ν is the Euclidean unitary normal vector field to Σ.

There exists $\bar r$ depending only on ρ such that if $r \leq \bar r$ we have that $\gamma \cap \hat\Sigma \neq \emptyset$ and, if we denote by $\hat p_1$ the first intersection point between γ and $\hat\Sigma$, then

$$d(p_1, \hat p_1) + |N_{p_1} - \tau_{\hat p_1}^{p_1}(N_{\hat p_1})|_{p_1} \leq C\theta\,,$$

where C is a constant depending only on n and ρ, and provided that $C\theta < 1/2$.

Proof. We first notice that, by choosing r small enough in terms of ρ, from Lemma 1.3 we have that $|\nu_{p_1} - e_n| \leq 1/4$. Let B^+ and B^- be the exterior and interior touching balls of $\hat{\Sigma}$ at $\hat{p}_0 = (0, \hat{u}(0))$, respectively. A standard geometrical argument shows that it is possible to choose $\bar{r}$ small enough in terms of ρ such that γ intersects B^+ and B^- at points which are distant from the origin less than $\bar{r}$. This implies the existence of the point $\hat{p}_1$ in the assertion for any $r \leq \bar{r}$.

Now we estimate the distance between p_1 and $\hat{p}_1$ as follows. Let q be the unique point having distance 2ε from p_1 and lying on the geodesic path containing p_1 and $\hat{p}_1^*$. Let T be the geodesic right-angle triangle having vertices p_1 and q and hypotenuse contained in the geodesic passing through p_1 and $\hat{p}_1$ (see Figure 1.2). Since the angle α at the vertex p_1 is such that $|\sin \alpha| \leq 1/4$, then from the sine rule for hyperbolic triangles we have that

$$d(p_1, \hat{p}_1) \leq C\theta \,. \tag{1.27}$$

Moreover, the cosine law in hyperbolic space gives that

$$d(\hat{p}_1^*, \hat{p}_1) \leq C\theta \tag{1.28}$$

for some constant C, and from Proposition 1.2 we obtain that

$$|N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq |N_{p_1} - \tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*})|_{p_1} + |\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \,. \tag{1.29}$$

Since p_1 and $\hat{p}_1^*$ are on the same vertical line, we have that

$$|N_{p_1} - \tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*})|_{p_1} = |\nu_{p_1} - \nu_{\hat{p}_1^*}| \leq C\theta \,. \tag{1.30}$$

where the last inequality follows from (1.26). Now we show that

$$|\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C\theta \,. \tag{1.31}$$

We obtain (1.31) by showing that if q, $\hat{q}$ and z in $\mathbb{H}^n$ are such that q and $\hat{q}$ belong to the Euclidean ball centred at z of radius R, then

$$|\tau_q^z(v) - \tau_{\hat{q}}^z(w)|_z \leq C\left(d(z,q) + d(z,\hat{q}) + d(q,\hat{q}) + |v - \tau_{\hat{q}}^q(w)|_q\right) \,, \tag{1.32}$$

for every v, $w \in \mathbb{R}^n$ with $|v|_q = |w|_{\hat{q}} = 1$, where the constant C depends only on R. In order to prove (1.32) we may assume, up to apply the isometry φ_z, that $z = e_n$, q and $\hat{q}$ belong to the same vertical line and z, q and $\hat{q}$ belong to the same plane generated by e_{n-1} and e_n. Note that, by construction, $q_{n-1} = \hat{q}_{n-1} \neq 0$. Taking into account that

$$|v| = q_n \,, \quad |w| = \hat{q}_n \,, \quad |\tau_q^z(v)| = 1 \,, \quad |\tau_{\hat{q}}^z(w)| = 1 \,,$$

where we used Lemma 1.4, and that

$$|v - \tau_{\hat{q}}^q(w)|_q = \left| v - \frac{\hat{q}_n}{q_n} w \right|_q = \frac{1}{q_n} \left| v - \frac{q_n}{\hat{q}_n} w \right| = \left| \frac{1}{q_n} v - \frac{1}{\hat{q}_n} w \right| \,,$$

where we used the fact that q and $\hat{q}$ belong to the same vertical line, we have

$$\begin{aligned}|\tau^z_q(v)-\tau^z_{\hat q}(w)|_z \leq & \left|1-\frac{1}{q_n}\right||\tau^z_q(v)|+\left|\tau^z_q\left(\frac{1}{q_n}v-\frac{1}{\hat q_n}w\right)\right| \\ &+\frac{1}{\hat q_n}|\tau^z_q(w)-\tau^z_{\hat q}(w)|+\left|\frac{1}{\hat q_n}-1\right||\tau^z_{\hat q}(w)| \\ =&\frac{1}{q_n}\left(|q_n-1|+\left|\frac{1}{q_n}v-\frac{1}{\hat q_n}w\right|\right)+\frac{1}{\hat q_n}|\tau^z_q(w)-\tau^z_{\hat q}(w)| \\ &+\frac{|\hat q_n-1|}{\hat q_n} \\ =&\frac{1}{q_n}\left(|q_n-1|+|v-\tau^q_{\hat q}(w)|_q\right)+\frac{1}{\hat q_n}|\tau^z_q(w)-\tau^z_{\hat q}(w)| \\ &+\frac{|\hat q_n-1|}{\hat q_n}\,.\end{aligned}$$

Moreover, from Lemma 1.4 we have that

$$\frac{1}{\hat q_n}|\tau^z_q(w)-\tau^z_{\hat q}(w)|\leq Cd(q,\hat q)\,,$$

where C is a constant depending only on R, and from Lemma 1.2 we get (1.32).

Now we are ready to prove (1.31), indeed from (1.32) we get

$$\begin{aligned}&|\tau^{p_1}_{\hat p_1^*}(N_{\hat p_1^*})-\tau^{p_1}_{\hat p_1}(N_{\hat p_1})|_{p_1} \\ &\qquad\leq C\left(d(p_1,\hat p_1)+d(p_1,\hat p_1^*)+d(\hat p_1,\hat p_1^*)+|N_{\hat p_1}-\tau^{\hat p_1}_{\hat p_1^*}(N_{\hat p_1^*})|_{\hat p_1}\right)\,,\end{aligned}$$

and from (1.26), (1.27), (1.28) and (1.30) we obtain (1.31). By using (1.31) and (1.30) in (1.29) we get

$$|N_{p_1}-\tau^{p_1}_{\hat p_1}(N_{\hat p_1})|_{p_1}\leq C\theta\,,$$

and therefore, from (1.27) we conclude. □

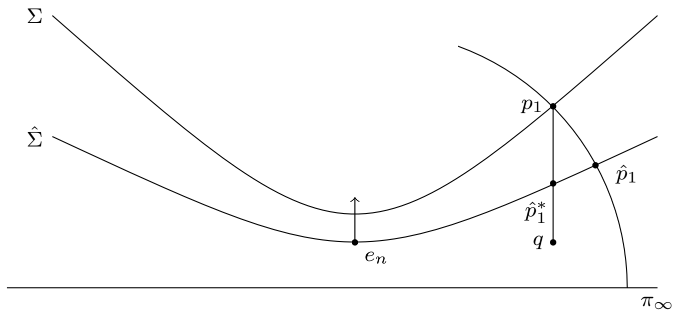

Figure 1.2: Proof of Lemma 1.5.

Lemma 1.6. *Let Σ and $\hat\Sigma$ be two compact embedded hypersurfaces in $\mathbb{R}^n$ with the round metric* (1.7) *satisfying a touching ball condition of radius ρ. Assume*

$O \in \Sigma$ and $T_O\Sigma = \{x_n = 0\}$, where O is the origin of $\mathbb{R}^n$, and that there exist two local parametrizations $u, \hat{u} : B_r \to \mathbb{R}$ of Σ and $\hat{\Sigma}$, respectively, with $0 < r \leq \rho_1$ and such that $u - \hat{u} \geq 0$. Let $p_1 = (x_1, u(x_1))$ and $\hat{p}_1^* = (x_1, \hat{u}(x_1))$, with $x_1 \in \partial B_{r/4}$, and denote by γ the geodesic path starting from p_1 and tangent to $-\nu_{p_1}$ at p_1. Assume that

$$d(p_1, \hat{p}_1^*) + |\nu_{p_1} - \nu_{\hat{p}_1^*}| \leq \theta\,. \tag{1.33}$$

for some $\theta \in [0, 1/2]$. where ν is the Euclidean unitary normal vector field to Σ. There exists $\bar{r}$ depending only on ρ such that if $r \leq \bar{r}$ we have that $\gamma \cap \hat{\Sigma} \neq \emptyset$ and, if we denote by $\hat{p}_1$ the first intersection point between γ and $\hat{\Sigma}$, then

$$d(p_1, \hat{p}_1) + |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C\theta\,,$$

where C is a constant depending only on n and ρ, and provided that $C\theta < 1/2$.

Proof. We first notice that, by choosing r small enough in terms of ρ, from Lemma 1.3 we have that $|\nu_{p_1} - e_n| \leq 1/4$. We observe that the geodesic γ is almost flat, i.e., viewed as an Euclidean circle its radius R satisfies

$$R = O\left(\frac{1}{|x_1|^2}\right) \text{ as } |x_1| \to 0\,. \tag{1.34}$$

Indeed, up to apply a rotation, we may assume that both p_1 and ν_{p_1} belong to the plane π_1 spanned by $\{e_1, e_2\}$. In this way, the geodesic path γ belongs to the plane π_1 and we can work in a "bidimensional way". We can write $p_1 = (x_1, y_1)$ and $\nu_{p_1} = (\nu_1, \nu_2)$ and we compute the geodesic γ passing through p_1 and tangent to ν_{p_1}. We solve

$$\begin{cases} (x_1 + a)^2 + (y_1 + b)^2 = 1 + a^2 + b^2 \\ (x_1 + a, y_1 + b) \cdot (\nu_1, \nu_2) = 0 \end{cases}$$

and we find

$$a = \frac{(1 - |p_1|^2)\nu_2 + 2y_1(p_1 \cdot \nu)}{2(x_1\nu_2 - y_1\nu_1)}\,, \qquad b = \frac{-(1 - |p_1|^2)\nu_1 - 2x_1(p_1 \cdot \nu)}{2(x_1\nu_2 - y_1\nu_1)}.$$

If $|x_1| \to 0$, according to Lemma 1.3, we have that $y_1 = O(x_1^2)$, $\nu_1 = O(x_1)$ and $\nu_2 = 1 + o(1)$; so we get that $a \sim \frac{1}{2x_1}$, b is bounded and (1.34) follows.

Let B^+ and B^- be the exterior and interior touching balls of $\hat{\Sigma}$ at $\hat{p}_0 = (0, \hat{u}(0))$, respectively. A standard geometrical argument shows that it is possible to choose $\bar{r}$ small enough in terms of ρ such that γ intersects B^+ and B^- at points which are distant from the origin less than $\bar{r}$. This implies the existence of the point $\hat{p}_1$ in the assertion for any $r \leq \bar{r}$.

Now we estimate the distance between p_1 and $\hat{p}_1$ as follows. Let q be the unique point having distance 2ε from p_1 and lying on the geodesic path containing p_1 and $\hat{p}_1^*$. Let T be the geodesic right-angle triangle having vertices p_1 and q and hypotenuse contained in the geodesic passing through p_1 and $\hat{p}_1$ (see Figure 1.3 and recall (1.34)). Since the angle α at the vertex p_1 is such that $|\sin\alpha| \leq 1/4$, then from the cosine rule for spherical triangles we have that

$$d(p_1, \hat{p}_1) \leq C\theta\,. \tag{1.35}$$

Moreover, the triangle inequality gives that

$$d(\hat{p}_1^*, \hat{p}_1) \le C\theta \tag{1.36}$$

for some constant C, and from (1.22) we obtain that

$$|N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \le |N_{p_1} - \tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*})|_{p_1} + |\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1}\,. \tag{1.37}$$

Since p_1 and $\hat{p}_1^*$ are on the same vertical line (1.33) implies

$$|N_{p_1} - \tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*})|_{p_1} = |\nu_{p_1} - \nu_{\hat{p}_1^*}| \le C\theta\,. \tag{1.38}$$

As next step we show that

$$|\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \le C\theta\,. \tag{1.39}$$

We obtain (1.39) by showing that if $p, q \in \mathbb{R}^n$ belong to the Euclidean ball centred at the origin and having radius s, then

$$\begin{aligned} 2|\tau_p^O(v) - \tau_q^O(w)| \le & \frac{(1+s)^2}{4}\left(d(p,O)^2 + d(q,O)^2\right) \\ & + |v - \tau_q^p(w)|_p + \frac{9}{2}d(p,q)\,. \end{aligned} \tag{1.40}$$

for every $v, w \in \mathbb{R}^n$, $|v|_q = |w|_{\hat{q}} = 1$. We have

$$\tau_p^O(v) = \frac{1}{1+|p|^2}v\,, \quad \tau_q^O(w) = \frac{1}{1+|q|^2}w.$$

and using Cauchy-Schwarz inequality and taking into account Lemma 1.2 we have

$$||q|^2 - |p|^2| = |(q-p)\cdot(q+p)| \le |q-p||q+p| \le s(1+s^2)\,d(p,q)$$

since

$$|w| = \frac{1+|\hat{q}|^2}{2}$$

we have

$$\begin{aligned} 2\left|\frac{1}{1+|q|^2}\tau_p^O(w) - \frac{1}{1+|q|^2}\tau_q^O(w)\right| &= 2\left|\frac{|q|^2 - |p|^2}{(1+|p|^2)(1+|q|^2)}\right| \\ &\le 2s(1+s^2)\,d(p,q)\,. \end{aligned}$$

Now using

$$2\left|\tau_p^O(v)\right| = 1\,, \quad 2\left|\tau_q^O(w)\right| = 1;$$

and

$$|v| = \frac{1+|p|^2}{2}, \quad |w| = \frac{1+|q|^2}{2}\,,$$

we compute

$$\begin{aligned}
2|\tau_p^O(v) - \tau_q^O(w)| \le & 2\left|\tau_p^O(v) - \frac{1}{1+|p|^2}\tau_p^O(v)\right| + 2\left|\frac{1}{1+|p|^2}\tau_p^O(v) - \tau_q^O(w)\right| \\
\le & 2\left|1 - \frac{1}{1+|p|^2}\right| \left|\tau_p^O(v)\right| + 2\left|\frac{1}{1+|p|^2}\tau_q^O(v) - \frac{1}{1+|q|^2}\tau_p^O(w)\right| \\
& + 2\left|\frac{1}{1+|q|^2}\tau_p^O(w) - \frac{1}{1+|q|^2}\tau_q^O(w)\right| + 2\left|1 - \frac{1}{1+|q|^2}\right| |\tau_q^O(w)| \\
\le & \left|1 - \frac{1}{1+|p|^2}\right| + 2\left|\frac{1}{1+|p|^2}\tau_q^O(v) - \frac{1}{1+|q|^2}\tau_p^O(w)\right| \\
& + 4d(p,q) + \left|1 - \frac{1}{1+|q|^2}\right| \\
\le & |p|^2 + 2\left|\frac{1}{1+|p|^2}\tau_q^O(v) - \frac{1}{1+|q|^2}\tau_p^O(w)\right| + 4d(p,q) + |q|^2 \\
\le & \frac{(1+s)^2}{4}\left(d(p,O)^2 + d(q,O)^2\right) \\
& + 2\left|\frac{1}{1+|p|^2}\tau_p^O(v) - \frac{1}{1+|q|^2}\tau_p^O(w)\right| + 4d(p,q)\,.
\end{aligned}$$

Now we show that

$$2\left|\frac{1}{1+|p|^2}\tau_p^O(v) - \frac{1}{1+|q|^2}\tau_p^O(w)\right| \le |v - \tau_q^p(w)|_p + \frac{1}{2}d(p,q)\,.$$

Let σ be the geodesic path connecting p with q. Then σ is contained in a circle of $\mathbb{R}^n$ and denotes by C its center and by α the angle between $p-C$ and $q-C$. Then

$$\frac{1+|p|^2}{1+|q|^2}w = R_\alpha \tau_q^p w\,, \text{ for every } w \in \mathbb{R}^n\,,$$

where R_α is the rotation (clockwise or anti-clockwise) about α in the plane containing C and is the identity in the complement. Therefore we have

$$\begin{aligned}
2\left|\frac{1}{1+|p|^2}\tau_p^O(v) - \frac{1}{1+|q|^2}\tau_p^O(w)\right| &= \left|\frac{1}{1+|p|^2}v - \frac{1}{1+|q|^2}w\right|_p \\
&\le \left|v - \frac{1+|p|^2}{1+|q|^2}w\right|_p \\
&= \left|v - R_\alpha\tau_q^p w\right|_p \\
&\le \left|v - \tau_q^p w\right|_p + \left|\tau_q^p w - R_\alpha \tau_q^p w\right|_p
\end{aligned}$$

and, consequently, we deduce,

$$\left|\tau_q^p w - R_\alpha\tau_q^p w\right|_p \le |\alpha| \le \frac{1}{2}d(p,q)$$

which implies (1.40).

Therefore, by applying (1.40) to $|\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1}$, we have

$$\begin{aligned} |\tau_{\hat{p}_1^*}^{p_1}(N_{\hat{p}_1^*}) - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq & \frac{(1+C\theta)^2}{4}\left(d(p_1,\hat{p}_1^*)^2 + d(p_1,\hat{p}_1)^2\right) \\ & + |N_{\hat{p}_1^*} - \tau_{\hat{p}_1}^{\hat{p}_1^*}(N_{\hat{p}_1^*})|_{\hat{p}_1^*} \\ & + \frac{9}{2} d(\hat{p}_1,\hat{p}_1^*) \\ \leq & C\theta\,, \end{aligned}$$

where the last inequality follows from (1.33),(1.35),(1.36) and (1.22). This last inequality, (1.37) and (1.38) imply that

$$|N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C\theta\,,$$

and therefore from (1.35) we conclude. □

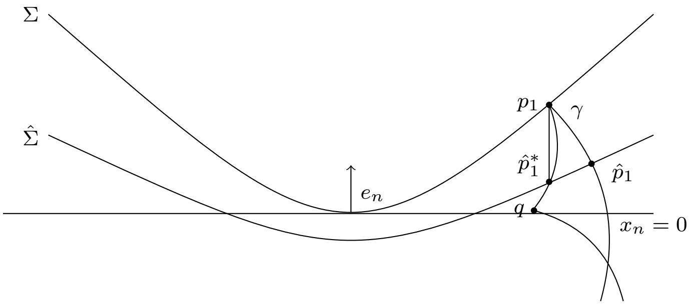

Figure 1.3: Proof of Lemma 1.6.

1.3 Curvatures of projected surfaces in conformally Euclidean spaces

In this section we consider a connected open set Ω in $\mathbb{R}^n$ equipped with a metric $g(\cdot,\cdot) = h^2\langle\cdot,\cdot\rangle$ conformal to the Euclidean metric. We further assume the existence of an Euclidean hyperplane π of $\mathbb{R}^n$ such that $\Omega \cap \pi$ is a totally geodesic hypersurface in Ω. This setting includes the Euclidean space, the hyperbolic space and $\mathbb{R}^n$ with the round metric (1.7). For instance in the half-space model of the hyperbolic space we can take as π any vertical Euclidean hyperplane; in the spherical case we can consider Euclidean hyperplanes passing through the origin.

For our purposes, we consider a hypersurface U of class C^2 embedded in Ω which intersects π transversally. The implicit function theorem implies that $U' = U \cap \pi$ is a C^2-submanifold of π. Furthermore if ν_q is an Euclidean unit normal vector field to U and w is a unit normal vector to π, we have that

$$\nu'_q = \frac{(-1)^n * (*(\nu_q \wedge w) \wedge w)}{|*(*(\nu_q \wedge w) \wedge w)|}\,, \quad q \in U',$$

is an Euclidean unitary normal vector field to U' in π, where $*$ is the Euclidean Hodge star operator in $\mathbb{R}^n$. In particular U' is orientable in π. Let

$$N_q = \frac{1}{h(q)}\nu_q\,, \quad N'_q = \frac{1}{h(q)}\nu'_q$$

be the normal vectors with respect to the metric g and

$$\omega_p = \frac{1}{h(p)}w\,, \quad p \in \Omega\,.$$

Proposition 1.3. *Let κ_j, $j = 1, \dots, n-1$ be the principal curvatures of U with respect to metric g and to the orientation induced by N. Then the principal curvatures κ'_i of U' (viewed as submanifold of π) with respect to the orientation induced by N' satisfy*

$$\frac{1}{\sqrt{1-g_q(\omega_q, N_q)^2}}\kappa_1(q) \le \kappa'_i(q) \le \frac{1}{\sqrt{1-g_q(\omega_q, N_q)^2}}\kappa_{n-1}(q)\,, \tag{1.41}$$

for every $q \in U'$. Moreover, the principal curvatures $\breve{\kappa}'_i$ of U' seen as a hypersurface of U satisfy

$$|\breve{\kappa}'_i(q)| \le \frac{|g_q(\omega_q, N_q)|}{\sqrt{1-g_q(\omega_q, N_q)^2}}\ \max\{|\kappa_1(q)|, |\kappa_{n-1}(q)|\}\,, \tag{1.42}$$

for every $q \in U'$.

Proof. Let $v \in T_qU'$ satisfy $|v|_q = 1$ and

$$\kappa_q(v) = g_q(\nabla_v \tilde{N}, v)\,,$$

where $\tilde{N}$ denotes an extension of N in Ω and ∇ is the Levi-Civita connection of g. For $p \in U'$, N_q is orthogonal to T_qU' and consequently it lies on the plane spanned by w and N'_q and hence

$$N = a\,w + bN'\,,$$

where a is a function on U' and

$$b = g(N, N')\,.$$

If $\tilde{a}$, $\tilde{b}$ and $\tilde{N}'$ are extensions of a, b and N' in Ω,

$$\tilde{N} = \tilde{a}\,w + \tilde{b}\,\tilde{N}'$$

defines an extension of N and a direct computation yields

$$\kappa_p(v) = a(p)\,g_p(\nabla_v w, v) + b(p)\,g_p(\nabla_v \tilde{N}', v) = b(p)\,g_p(\nabla_v \tilde{N}', v)\,,$$

where we used that $\pi \cap \Omega$ is totally geodesic. Therefore

$$\frac{1}{g_q(N_q, N'_q)}\kappa_q(v) = g_q(\nabla_v \tilde{N}', v)$$

and consequently

$$\frac{1}{g_q(N_q, N'_q)}\kappa_1(q) \le \kappa'_i(q) \le \frac{1}{g_q(N_q, N'_q)}\kappa_{n-1}(q)$$

for every $q \in U'$ and $i = 1, \dots, n-2$.

Now we show

$$g_q(N_q, N'_q) = \sqrt{1 - g_q(\omega_q, N_q)^2}\,, \tag{1.43}$$

which implies (1.41). We have

$$\begin{aligned} \nu_q \cdot \nu'_q =& \frac{(-1)^n * (*(\nu_q \wedge w) \wedge w)}{| * (*(\nu_q \wedge w) \wedge w)|} \cdot \nu_q \\ =& \frac{1}{| * (*(\nu_q \wedge w) \wedge w)|}(-1)^n * (\nu_q \wedge w) \wedge w \cdot *\nu_q\,. \end{aligned} \tag{1.44}$$

Let $\{\nu_q, e_1, \dots, e_{n-1}\}$ be a positive-oriented orthonormal basis of $\mathbb{R}^n$ such that

- $\{e_1, \dots, e_{n-1}\}$ is a positive-oriented Euclidean-orthonormal basis of T_qU;
- $\{e_2, \dots, e_{n-1}\}$ is a basis of T_qU'.

In this way $w \in \text{span}\{\nu_q, e_1\}$,

$$*(\nu_q \wedge w) = (w \cdot e_1)\, e_2 \wedge \cdots \wedge e_{n-1}\,, \quad *\nu_q = e_1 \wedge \cdots \wedge e_{n-1},$$

and

$$| * (*(\nu_q \wedge w) \wedge w)| = | * (\nu_q \wedge w) \wedge w| = w \cdot e_1\,.$$

So, from (1.44) we get

$$\begin{aligned} \nu_q \cdot \nu'_q =& \frac{(-1)^n}{\omega \cdot e_1} * (\nu_q \wedge w) \wedge w \cdot *\nu_q \\ =& \frac{(-1)^n}{\omega \cdot e_1}(\omega \cdot e_1)\, e_2 \wedge \cdots \wedge e_{n-1} \wedge w \cdot e_1 \cdots \wedge e_{n-1} \\ =& (-1)^n(\omega \cdot e_1)\, e_2 \wedge \cdots \wedge e_{n-1} \wedge e_1 \cdot e_1 \wedge \cdots \wedge e_{n-1} \\ =& (\omega \cdot e_1)\, e_1 \wedge \cdots \wedge e_{n-1} \cdot e_1 \wedge \cdots \wedge e_{n-1} \\ =& \omega \cdot e_1\,. \end{aligned}$$

Since $|w| = 1$, we have $w \cdot e_1 = \sqrt{1 - (w \cdot \nu_q)^2}$ and so

$$\nu_q \cdot \nu'_q = \sqrt{1 - (w \cdot \nu_q)^2}\,. \tag{1.45}$$

Since

$$\nu_q \cdot \nu'_q = g_q(N_q, N'_q)\,, \quad \text{and} \quad w \cdot \nu_q = g_q(\omega_q, N_q)\,,$$

(1.43) follows.

Now we prove (1.42). In this case we regard U' as a submanifold of U. Let $q \in U'$, $v \in T_qU'$ such that $|v|_q = 1$ and let $\alpha\colon (-\delta, \delta) \to S$ be a unitary speed curve satisfying $\alpha(0) = q$ and $\dot\alpha(0) = v$. Let $\tilde{N}'$ be a unitary normal vector field of U' in U near q. We may complete v with an orthonormal basis $\{v, v_2, \dots v_{n-2}\}$ of T_qU' such that

$$\check{N}'_q = *_q(N_q \wedge v \wedge v_2 \wedge \cdots \wedge v_{n-2})\,,$$

where $*_q$ is the Hodge star operator at q in Ω with respect to g and to the standard orientation. Let

$$\breve{\kappa}'_q(v) = g_q(*_q(\check{N}_q \wedge v \wedge v_2 \wedge \cdots \wedge v_{n-2}), D_t\dot{\alpha}_{|t=0}),$$

where D_t is the covariant derivative in (Ω, g). Since $D_t\dot{\alpha}_{|t=0} \in \pi$, we have

$$\breve{\kappa}'_q(v) = g_q(N_q, \omega_q) g_q(*_q(\omega_q \wedge v \wedge v_2 \wedge \cdots \wedge v_{n-2}), D_t\dot{\alpha}_{|t=0}).$$

Now, $*_q(\omega_q \wedge v \wedge v_2 \wedge \cdots \wedge v_{n-2})$ is a normal vector to T_qU' in π and so

$$\breve{\kappa}'_q(v) = g_q(N_q, \omega_q) g_q(\nabla_v \tilde{N}, v),$$

where $\tilde{N}$ is an arbitrary extension of N in a neighbourhood of q. From (1.41) we obtain

$$|\breve{\kappa}'_q(v)| \leq \frac{|g_q(N_q, \omega_q)|}{\sqrt{1 - g_q(\omega_q, N_q)^2}} \max\{|\kappa_1(q)|, |\kappa_{n-1}(q)|\},$$

as required. □

Remark 1.1. It may be convenient to explain the meaning of (1.45) when $n = 3$. In this case $*(v \wedge w)$ is the vector product $v \times w$, so

$$\nu'_q = -\frac{(\nu_q \times w) \times w}{|(\nu_q \times w) \times w|} \quad \text{and} \quad |(\nu_q \times w) \times w| = \sqrt{|\nu_q|^2|w|^2 - (\nu_q \cdot w)^2}.$$

So

$$\begin{aligned}
(\nu_q \cdot \nu'_q) &= -\frac{1}{\sqrt{|\nu_q|^2|w|^2 - (\nu_q \cdot w)^2}} (\nu_q \times w) \times w \cdot \nu_q \\
&= -\frac{1}{\sqrt{|\nu_q|^2|w|^2 - (\nu_q \cdot w)^2}} (\nu_q \times w) \cdot (w \times \nu_q) \\
&= \frac{|\nu_q \times w|^2}{\sqrt{|\nu_q|^2|w|^2 - (\nu_q \cdot w)^2}} \\
&= \frac{|\nu_q|^2|w|^2 - (\nu_q \cdot w)^2}{\sqrt{|\nu_q|^2|w|^2 - (\nu_q \cdot w)^2}} \\
&= \sqrt{1 - (\nu_q \cdot w)^2}.
\end{aligned}$$

Now we focus in a different setting. Let $\bar{\Omega}$ be the projection of Ω onto $\{x_n = 0\}$ and let $\pi \subseteq \Omega$ be the graph of a C^2 function $F\colon A \to \mathbb{R}$, where $A \subseteq \bar{\Omega}$ is a open subset.

Proposition 1.4. *Let U' be a C^2 regular oriented hypersurface of π and let U'' be the orthogonal projection of U' onto $\{x_n = 0\}$. Then the principal curvatures of U'' satisfy*

$$\begin{aligned}
|\kappa''_i(\bar{q})| \leq \frac{h(q)}{\sqrt{1 + |\nabla F(\bar{q})|^2}} &\left((\nu'_q \cdot e_n)^2 + \frac{1}{1 + |\nabla F(\bar{q})|^2}\right)^{-3/2} \times \\
&\left(\max\{|\kappa'_1(q)|, |\kappa'_{n-1}(q)|\} + 4\frac{|\nabla h(q)|}{h(q)^2}\right),
\end{aligned} \tag{1.46}$$

for every $i = 1, \ldots, n-2$, where $\{\kappa''_i\}$ are the principal curvatures of U'' with respect to the Euclidean metric, $q \equiv (\bar{q}, q_n) \in U'$ and $\nu'_q = h(q)N'_q$.

Proof. If X is a local positive oriented parametrization of U', then

$$\bar{X} = X - (X \cdot e_n)e_n$$

is a local parametrization of U'', and we can orient U'' with

$$\nu'' \circ \bar{X} := \mathrm{vers}(*(\bar{X}_1 \wedge \bar{X}_2 \wedge \cdots \wedge \bar{X}_{n-2} \wedge e_n))\,, \tag{1.47}$$

where $\bar{X}_k$ is the k^{th} derivative of $\bar{X}$ with respect to the coordinates of its domain and $*$ is the Hodge star operator in $\mathbb{R}^n$ with respect to the the Euclidean metric and the standard orientation.

Now we prove inequalities (1.46). Fix a point $q = (\bar{q}, q_n) \in U'$ and $\bar{v} \in T_{\bar{q}}U'$ be nonzero. Let $\beta\colon (-\delta, \delta) \to U''$ be an arbitrary regular curve contained in U'' such that

$$\beta(0) = \bar{q}\,, \quad \dot{\beta}(0) = \bar{v}\,.$$

Then

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{1}{|\bar{v}|^2}\,\nu''_{\bar{q}} \cdot \ddot{\beta}(0)$$

is the normal curvature of U'' at $(\bar{q}, \bar{v})$, viewed as hypersurface of $\{x_n = 0\}$ with the Euclidean metric. We can write

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{1}{|\bar{v}|^2}\,\nu''_{\bar{q}} \cdot \ddot{\alpha}(0)$$

where $\alpha = (\beta, \alpha_n)$ whose projection onto U' is β. From

$$\bar{X}_k = X_k - (X_k \cdot e_n)e_n\,,$$

and the definition of ν'' (1.47) we have

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{*(X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n) \cdot \ddot{\alpha}(0)}{|\dot{\beta}(0)|^2 |X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n|}\,.$$

We may assume that $\{X_1(q), \ldots, X_{n-2}(q)\}$ is an orthonormal basis of T_qU' with respect to the Euclidean metric. Let

$$\mathsf{N_q} = \frac{(-\nabla F(\bar{q}), 1)}{\sqrt{1 + |\nabla F(\bar{q})|^2}}$$

be the Euclidean normal vector to π at q and let

$$a = \sqrt{1 + |\nabla F(\bar{q})|^2}\,.$$

Therefore $\{X_1(q), \ldots, X_{n-2}(q), \nu'_q, \mathsf{N}_q\}$ is an Euclidean orthonormal basis of $\mathbb{R}^n$ and we can split $\mathbb{R}^n$ in

$$\mathbb{R}^n = T_qU' \oplus \langle \nu'_q \rangle \oplus \langle \mathsf{N}_q \rangle\,, \tag{1.48}$$

and e_n splits accordingly into

$$e_n = e'_n + e''_n + e'''_n\,.$$

Therefore

$$*(X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n) \cdot \ddot{\alpha}(0) = *(X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e'''_n) \cdot \ddot{\alpha}(0)\,,$$

i.e.

$$*(X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n) \cdot \ddot{\alpha}(0) = \frac{1}{a} * (X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \mathsf{N}_q) \cdot \ddot{\alpha}(0) .$$

Since

$$\nu'_q = * (X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \mathsf{N}_q)$$

we obtain

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{1}{a|\dot{\beta}(0)|^2} \frac{\nu'_q \cdot \ddot{\alpha}(0)}{|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n|} .$$

We may assume that α is parametrized by arc length with respect to the metric g, i.e.

$$|\dot{\alpha}|^2 = h(\alpha)^{-2}$$

and so

$$|\dot{\beta}|^2 = h(\alpha)^{-2} - \dot{\alpha}_n^2 ,$$

which implies

$$\kappa''_{\bar{q}}(\bar{v}) = \frac{1}{a(h(q)^{-2} - v_n^2)} \frac{\nu'_q \cdot \ddot{\alpha}(0)}{|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n|} . \tag{1.49}$$

Since

$$X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n = X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e''_n + X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e'''_n$$

and

$$X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e''_n = (\nu'_q \cdot e_n)\, X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \nu'_q ,$$
$$X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e'''_n = \frac{1}{a} X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge \mathsf{N}_q ,$$

we obtain

$$|X_1(q) \wedge \cdots \wedge X_{n-2}(q) \wedge e_n| = \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2} \right)^{1/2} .$$

On the other hand

$$\kappa'_q(v) = g_q(N'_q, D_t \dot{\alpha}_{|t=0})$$

where D_t is the covariant derivative in π. It is well-known that the Christoffel symbols of g are given by

$$\Gamma^k_{ij} = \delta^k_i \partial_j f + \delta^k_j \partial_i f - \delta^j_i \partial_k f ,$$

where $f = \log h$. We have

$$\begin{aligned} D_t \dot{\alpha} &= \ddot{\alpha} + \sum_{i,j,k=1}^{n} \Gamma^k_{ij}(\alpha) \dot{\alpha}_i \dot{\alpha}_j\, e_k \\ &= \ddot{\alpha} + \sum_{i,j,k=1}^{n} (\delta^k_i \partial_j f(\alpha) + \delta^k_j \partial_i f(\alpha) - \delta^j_i \partial_k f(\alpha)) \dot{\alpha}_i \dot{\alpha}_j\, e_k \\ &= \ddot{\alpha} + \sum_{i,k=1}^{n} (2\partial_i f(\alpha) \dot{\alpha}_i \dot{\alpha}_k - \dot{\alpha}_i^2 \partial_k f(\alpha))\, e_k + \sum_{k=1}^{n} \partial_k f(\alpha) \dot{\alpha}_k^2\, e_k \end{aligned}$$

and

$$D_t\dot{\alpha}_{|t=0} = \ddot{\alpha}(0) + \sum_{i,k=1}^{n} \left(2\partial_i f(q) v_i v_k - v_i^2 \partial_k f(q)\right) e_k + \sum_{k=1}^{n} \partial_k f(q) v_k^2 \, e_k \,.$$

Therefore

$$\begin{aligned}
\kappa'_q(v) &= g_q\left(N'_q, \ddot{\alpha}(0) + \sum_{i,k=1}^{n} \left(2\partial_i f(q) v_i v_k - v_i^2 \partial_k f(q)\right) e_k + \sum_{k=1}^{n} \partial_k f(q) v_k^2 \, e_k\right) \\
&= h(q)\nu'_q \cdot \ddot{\alpha}(0) + h(q) \sum_{i,k=1}^{n} \left(2\partial_i f(q) v_i v_k - v_i^2 \partial_k f(q)\right) e_k \cdot \nu'_q \\
&\quad + h(q) \sum_{k=1}^{n} \partial_k f(q) v_k^2 \, \nu'_q \cdot e_k \\
&= h(q)\nu'_q \cdot \ddot{\alpha}(0) + \sum_{i,k=1}^{n} \left(2\partial_i h(q) v_i v_k - v_i^2 \partial_k h(q)\right) e_k \cdot \nu'_q \\
&\quad + \sum_{k=1}^{n} \partial_k h(q) v_k^2 \nu'_q \cdot e_k \,,
\end{aligned}$$

and we get

$$\begin{aligned}
\nu'_q \cdot \ddot{\alpha}(0) =& \frac{\kappa'_q(v)}{h(q)} - \frac{1}{h(q)} \sum_{i,k=1}^{n} \left(2\partial_i h(q) v_i v_k - v_i^2 \partial_k h(q)\right) e_k \cdot \nu'_q \\
& - \frac{1}{h(q)} \sum_{k=1}^{n} \partial_k h(q) v_k^2 \, \nu'_q \cdot e_k \,.
\end{aligned}$$

From (1.49) we deduce

$$\begin{aligned}
\kappa''_{\bar{q}}(\bar{v}) &= \frac{1}{a(h(q)^{-2} - v_n^2)} \left(\langle \nu'_q, e_n \rangle^2 + \frac{1}{a^2}\right)^{-1/2} \times \\
&\left(\frac{\kappa'_q(v)}{h(q)} - \frac{1}{h(q)} \sum_{i,k=1}^{n} \left(2\partial_i h(q) v_i v_k - v_i^2 \partial_k h(q)\right) e_k \cdot \nu'_q - \frac{1}{h(q)} \sum_{k=1}^{n} \partial_k h(q) v_k^2 \nu'_q \cdot e_k\right) \,,
\end{aligned}$$

for every $v \in T_q U'$ such that $g_q(v, v) = 1$. Therefore

$$\begin{aligned}
\kappa''_1(\bar{q}) &= \frac{1}{ah(q)} \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2}\right)^{-1/2} \inf_{v \in \mathbb{S}_q^{n-2}} A_q(v) \,, \\
\kappa''_{n-2}(\bar{q}) &= \frac{1}{ah(q)} \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2}\right)^{-1/2} \sup_{v \in \mathbb{S}_q^{n-2}} A_q(v) \,,
\end{aligned}$$

where

$$\begin{aligned}
A_q(v) &= \frac{1}{h(q)^{-2} - v_n^2} \times \\
&\left(\kappa'_q(v) - \sum_{i,k=1}^{n} \left(2\partial_i h(q) v_i v_k - v_i^2 \partial_k h(q)\right) e_k \cdot \nu'_q - \sum_{k=1}^{n} \partial_k h(q) v_k^2 \nu'_q \cdot e_k\right)
\end{aligned}$$

and $\mathbb{S}_q^{n-2} = \{v \in T_qU' \ : \ |v|_q = 1\}$. Since $|v|_q^2 = 1$, then $|v|^2 = h(q)^{-2}$ and we can rewrite $A_q(v)$ as

$$A_q(v) = \frac{1}{h(q)^{-2} - v_n^2} \times \left(\kappa'_q(v) - \nu'_q \cdot \left(2(\nabla h(q) \cdot v)v - h(q)^{-2}\nabla h(q) + \sum_{k=1}^{n} \partial_k h(q) v_k^2 e_k\right)\right).$$

Since $|\kappa''_i(\bar{q})| \le \max\{|\kappa''_1(\bar{q})|, |\kappa''_{n-2}(\bar{q})|\}$, $i = 1, \dots, n-2$, we obtain

$$|\kappa''_i(\bar{q})| \le \frac{1}{ah(q)} \left((\nu'_q \cdot e_n)^2 + \frac{1}{a^2}\right)^{-1/2} \sup_{v \in \mathbb{S}_q^{n-2}} |A_q(v)|. \tag{1.50}$$

We have

$$\begin{aligned}
|A_q(v)| =& \frac{1}{h(q)^{-2} - v_n^2} \times \\
& \left|\kappa'_q(v) - \nu'_q \cdot \left(2(\nabla h(q) \cdot v)\, v - \frac{1}{h(q)^2}\nabla h(q) + \sum_{k=1}^{n} \partial_k h(q) v_k^2 e_k\right)\right| \\
\le& \frac{1}{h(q)^{-2} - v_n^2} \times \\
& \left(|\kappa'_q(v)| + \left|2(\nabla h(q) \cdot v)\, v - \frac{1}{h(q)^2}\nabla h(q) + \sum_{k=1}^{n} \partial_k h(q) v_k^2 e_k\right|\right) \\
\le& \frac{1}{h(q)^{-2} - v_n^2} \times \\
& \left(|\kappa'_q(v)| + 2\,|(\nabla h(q) \cdot v)\, v| + \frac{1}{h(q)^2}\,|\nabla h(q)| + \left|\sum_{k=1}^{n} \partial_k h(q) v_k^2 e_k\right|\right) \\
\le& \frac{1}{h(q)^{-2} - v_n^2} \times \\
& \left(|\kappa'_q(v)| + \frac{2}{h(q)^2}\,|\nabla h(q)| + \frac{1}{h(q)^2}\,|\nabla h(q)| + \frac{1}{h(q)^2}\,|\nabla h(q)|\right)
\end{aligned}$$

i.e.,

$$|A_q(v)| \le \frac{1}{h(q)^{-2} - v_n^2} \left(|\kappa'_q(v)| + \frac{4}{h(q)^2}\,|\nabla h(q)|\right).$$

Since $\mathbb{R}^n = T_qU' \oplus \langle \mathsf{N}_q \rangle \oplus \langle \nu'_q \rangle$, we can write

$$e_n = e''_n + \frac{1}{a} + (\nu_q \cdot e_n)\, \nu'_q\,,$$

where $\tilde{e}_n$ is the orthogonal projection of e_n onto T_qU'. Therefore

$$1 - |e'_n|^2 = \frac{1}{a^2} + (\nu'_q \cdot e_n)^2\,.$$

Since $h(q)v$ lies in T_qU' and it has unitary Euclidean norm, we have

$$|e'_n|^2 \ge h(q)^2(e_n \cdot v)^2 = h(q)^2 v_n^2$$

and so

$$1 - h(q)^2 v_n^2 \geq \frac{1}{a^2} + (\nu_q' \cdot e_n)^2 ,$$

i.e.

$$h(q)^{-2} - v_n^2 \geq \left(\frac{1}{a^2} + (\nu_q' \cdot e_n)^2 \right) h(q)^{-2} .$$

Hence

$$|A_q(v)| \leq (|\kappa_q'(v)| + 4h(q)^{-2}|\nabla h(q)|) \left(\frac{1}{a^2} + (\nu_q' \cdot e_n)^2 \right)^{-1} h(q)^2 ,$$

which yields

$$|\kappa_i''(\bar{q})| \leq \frac{h(q)}{a} \left((\nu_q' \cdot e_n)^2 + \frac{1}{a^2} \right)^{-3/2} \sup_{v \in \mathbb{S}_q^{n-2}} (|\kappa_q'(v)| + 4h(q)^{-2}|\nabla h(q)|) , \quad (1.51)$$

which implies (1.46). □

Now we use (1.46) in space forms.
In the *Euclidean space* we have $\Omega = \mathbb{R}^n$ and $h(q) = 1$ and (1.46) reduces to

$$|\kappa_i''(\bar{q})| \leq \frac{1}{\sqrt{1 + |\nabla F(\bar{q})|^2}} \frac{\max\{|\kappa_1'(q)|, |\kappa_{n-1}'(q)|\}}{\left(\langle \nu_q', e_n \rangle^2 + \frac{1}{1+|\nabla F(\bar{q})|^2} \right)^{3/2}} . \quad (1.52)$$

In particular, if π is an hyperplane and if we set $\omega_1 = \frac{(\nabla F(\bar{q}), -1)}{\sqrt{1+|\nabla F(\bar{q})|^2}}$ and $\omega_2 = e_n$ then we have

$$|\kappa_i''(\bar{q})| \leq \frac{|\omega_1 \cdot \omega_2|}{\left((\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu_q') \right)^{3/2}} \max\{|\kappa_1'(q)|, |\kappa_{n-1}'(q)|\} , \quad (1.53)$$

for every $i = 1, \ldots, n-2$.

In the *Hyperbolic space* we have $\Omega = \{q_n > 0\}$ and $h(q) = \frac{1}{q_n}$ and (1.46) reduces to

$$|\kappa_i''(\bar{q})| \leq \frac{1}{q_n \sqrt{1 + |\nabla F(\bar{q})|^2}} \frac{\left(\max\{|\kappa_1'(q)|, |\kappa_{n-1}'(q)|\} + 4 \right)}{\left((\nu_q' \cdot e_n)^2 + \frac{1}{1+|\nabla F(\bar{q})|^2} \right)^{3/2}} , \quad (1.54)$$

In particular, if π is a half-sphere of radius R with center in π_∞, then we have

$$|\kappa_i''(\bar{q})| \leq \frac{1}{R} \left((\nu_q' \cdot e_n)^2 + \frac{q_n^2}{R^2} \right)^{-\frac{3}{2}} \left(\max\{|\kappa_1'(q)|, |\kappa_{n-1}'(q)|\} + 4 \right) . \quad (1.55)$$

for every $i = 1, \ldots, n-2$.

Now we focus on $\mathbb{R}^n$ equipped with the spherical metric. In this case $h(q) = \frac{2}{1+|q|^2}$ and (1.46) gives

$$|\kappa_i''(\bar{q})| \leq \frac{2}{(1 + |q|^2)\sqrt{1 + |\nabla F(\bar{q})|^2}} \frac{\left(\max\{|\kappa_1'(q)|, |\kappa_{n-1}'(q)|\} + 4|q| \right)}{\left((\nu_q' \cdot e_n)^2 + \frac{1}{1+|\nabla F(\bar{q})|^2} \right)^{3/2}} .$$

In particular if π is the hemisphere of some hyperplane which does not contain the origin, then we have

$$|\kappa_i''(\bar{q})| \leq \frac{h(q)|(q-O_\pi)_n|}{R} \frac{\left(\max\{|\kappa_1'(q)|, |\kappa_{n-1}'(q)|\} + 4|q|\right),}{\left((\nu_q' \cdot e_n)^2 + \frac{(q-O_\pi)_n^2}{R^2}\right)^{3/2}}, \tag{1.56}$$

for every $i = 1, \ldots, n-2$, where O_π and R are the center and the radius of π, respectively.

1.4 Approximate symmetry in one direction

We consider the following set-up: let $S = \partial\Omega$ be a C^2-regular connected closed hypersurface embedded in $\mathbb{M}_+^n$, where Ω is a bounded domain. Assume that S satisfies a uniform touching ball condition of radius $\rho > 0$. We fix a direction v in $T_{\mathrm{o}}\mathbb{M}^n$ and we apply the method of the moving planes as described in Section 1.1. Let $\pi = \pi_{v,m_v}$ be the critical hyperplane and in order to simplify the notation we set

$$S_+ = \{p \in S \ : \ p \in \pi_{v,t} \text{ for some } t > m_v\},$$
$$S_- = \{p \in S \ : \ p \in \pi_{v,t} \text{ for some } t < m_v\}.$$

From the method of the moving planes we have that the reflection S_+^π of S_+ with respect to π is contained in Ω and it is tangent to S_- at a point p_0 (internally or at the boundary). Let Σ and $\hat{\Sigma}$ be the connected components of S_+^π and S_- containing p_0, respectively.

The main result in this section is the following

Theorem 1.4. *There exists $\varepsilon > 0$ such that if*

$$\mathrm{osc}(\mathsf{H}_S) \leq \varepsilon,$$

then for any $p \in \Sigma$ there exists $\hat{p} \in \hat{\Sigma}$ such that

$$d(p, \hat{p}) + |N_p - \tau_{\hat{p}}^p(N_{\hat{p}})|_p \leq C\, \mathrm{osc}(\mathsf{H}_S).$$

Here, the constants ε and C depend only on n, ρ and the area of S. In particular ε and C do not depend on the direction v.

Moreover, Ω is contained in a neighbourhood of radius $C\mathrm{osc}(\mathsf{H}_S)$ of $\Sigma \cup \Sigma^\pi$ (Σ^π is the reflection of Σ about π), i.e.

$$d(p, \Sigma \cup \Sigma^\pi) \leq C\mathrm{osc}(\mathsf{H}_S),$$

for every $p \in \Omega$.

Before giving the proof of Theorem 1.4, we provide two preliminary results about the geometry of Σ. For $t > 0$ we set

$$\Sigma_t = \{p \in \Sigma : \ d_\Sigma(p, \partial\Sigma) > t\}.$$

The following lemmas quantitatively show that Σ_t is connected for t small enough.

Here we use the results in Section (1.3) and we consider the unitary normal vector field ω to π directed as the geodesic γ in $\mathbb{M}_+^n$ satisfying $\dot{\gamma}(0) = v$.

Lemma 1.7. *Assume*

$$g_p(N_p, \omega_p) \leq \mu \tag{1.57}$$

for every p on the boundary of Σ, for some $\mu \leq 1/2$, and let $t_0 = \rho\sqrt{1-\mu^2}$. Then Σ_t is connected for any $0 < t < t_0$.

Proof. We can work in $\mathbb{R}^n$ for every space form considered, and we may assume that π is an Euclidean hyperplane of $\mathbb{R}^n$ (in the spherical case we can consider the projection from a point antipodal to a point inside π).

Let Σ' be the subset of π obtained by projecting Σ onto π (for any point $p \in \Sigma$ we define the projection of p onto π as the point on π which realizes the distance d of p from π). Σ' is an open set of π with $\partial\Sigma' = \partial\Sigma$. Proposition 1.3 gives

$$|\kappa_i'(p)| \leq \frac{1}{\sqrt{1-(g_p(N_p,\omega_p))^2}} \max\{|\kappa_1(p)|, |\kappa_{n-1}(p)|\},$$

for any $p \in \partial\Sigma$ and $i = 1, \dots, n-1$, where κ_i' are the principal curvatures of $\partial\Sigma$ viewed as a hypersurface of π. Since S satisfies a touching ball condition of radius ρ, we have

$$\max\{|\kappa_1(p)|, |\kappa_{n-1}(p)|\} \leq \frac{1}{\rho}$$

and, consequently,

$$|\kappa_i'(p)| \leq \frac{1}{\rho\sqrt{1-(g_p(N_p,\omega_p))^2}}, \tag{1.58}$$

for $i = 1, \dots, n-1$. From (1.57) and (1.58) we have that $\partial\Sigma'$ satisfies a touching ball condition of radius

$$\rho' \geq \rho\sqrt{1-(g_p(N_p,\omega_p))^2} \geq t_0 .$$

Therefore if $s < t_0$,

$$\mathcal{C}_s = \{z \in \pi : \ d(z, \partial\Sigma) < s\}$$

is a collar neighbourhood of $\partial\Sigma$ in Σ' of radius s. Since π is a critical hyperplane in the method of moving planes, if p belongs to the maximal cap S_+ then any point on the geodesic path connecting p to its projection onto π is contained in the closure of Ω. It follows that the preimage of $\mathcal{C}_s$ via the projection contains a collar neighbourhood of $\partial\Sigma$ of radius s in Σ. This implies that Σ can be retracted in Σ_t for any $t \leq s$ which completes the proof. □

Lemma 1.8. *There exists $\bar\delta > 0$ depending only on ρ with the following property. Assume that there exists a connected component Γ_δ of Σ_δ, for some $0 < \delta \leq \bar\delta$, such that one of the following two assumptions is fulfilled:*

i) $0 \leq g_q(N_q, \omega_q) \leq \frac{1}{8}$ *for any $q \in \partial\Gamma_\delta$,*

ii) for any $q \in \partial\Gamma_\delta$ there exists $\hat q \in \hat\Sigma$ such that

$$d(q, \hat q) + |N_q - \tau_{\hat q}^q(N_{\hat q})|_q \leq \delta .$$

Then

$$0 \leq g_q(N_q, \omega_q) \leq \frac{1}{4} \tag{1.59}$$

for any $q \in \partial\Sigma$ and Σ_δ is connected.

Proof. Case i). The crucial observation is that we can choose $\bar{\delta}$ small enough such that $\bar{\delta} \leq \delta_0$, where δ_0 is the bound appearing in Proposition 1.2, and the set $\Sigma \setminus \Gamma_\delta$ is enclosed by π and the set obtained as the union of all the exterior and interior touching balls to the reflection of S about π, S^π. This implies that for any $p \in \Sigma \setminus \Gamma_\delta$ there exists $q \in \partial\Gamma_\delta$ such that $d_\Sigma(p,q) \leq \delta$ and we can apply the estimates in proposition 1.2. Indeed from (1.22) and (1.23) we have that

$$|N_p - \tau_q^p(N_q)|_p \leq C\delta\,, \qquad \text{and} \qquad g_p(N_p, \tau_q^p(N_q)) \geq \sqrt{1 - C^2\delta^2}\,,$$

where $C = C(\rho)$. Therefore

$$\begin{aligned} g_p(N_p, \omega_p) &= g_p(N_p - \tau_q^p(N_q), \omega_p) + g_p(\tau_q^p(N_q), \omega_p) \\ &\leq C\delta + g_p(\tau_q^p(N_q), \omega_p) \end{aligned}$$

and by using

$$g_p(\tau_q^p(N_q), \omega_p) = g_q(N_q, \tau_p^q(\omega_q))$$

we obtain

$$\begin{aligned} g_p(N_p, \omega_p) &\leq C\delta + g_q(N_q, \omega_q) + g_q(N_q, \tau_p^q(\omega_p) - \omega_q) \\ &\leq C\delta + g_q(N_q, \omega_q) + |\tau_p^q(\omega_p) - \omega_q|_q\,. \end{aligned}$$

Since

$$|\tau_p^q(\omega_p) - \omega_q|_q = 0\,,$$

we deduce

$$g_p(N_p, \omega_p) \leq C\delta + g_q(N_q, \omega_q)\,.$$

This last bound holds for every $p \in \partial\Sigma$ and by choosing δ small enough in terms of ρ we obtain (1.59), as required.

Case ii): Γ_δ satisfies ii). Let $q \in \partial\Gamma_\delta$. By construction of the method of moving planes, $g_q(N_q, \omega_q) \geq 0$. We denote by q^π the reflection of q about π and we have

$$d(q^\pi, \hat{q}) \leq d(q^\pi, q) + d(q, \hat{q}) \leq 3\delta\,.$$

Up to consider a smaller δ in terms of ρ, from Corollary 1.2 we find $C = C(\rho)$ such that $d_S(q^\pi, \hat{q}) \leq C\delta$ and $q^\pi \in \mathcal{U}_{\rho_1}(\hat{q})$. Hence we can apply (1.22) and obtain

$$g_{\hat{q}}(N_{\hat{q}}, \tau_{q^\pi}^{\hat{q}}(N_{q^\pi})) \geq \sqrt{1 - C^2\delta^2} \qquad \text{and} \qquad |N_{\hat{q}} - \tau_{q^\pi}^{\hat{q}}(N_{q^\pi})|_{\hat{q}} \leq C\delta\,.$$

Since N_{q^π} and q^π are the reflection of N_q and q about π, respectively, we have that

$$g_q(N_q, \omega_q) = -g(\tau_{q^\pi}^q(N_{q^\pi}), \omega_q)\,,$$

and hence

$$\begin{aligned} 2g_q(N_q, \omega_q) &= g_q(N_q - \tau_{q^\pi}^q(N_{q^\pi}), \omega_q) \\ &= g_q(N_q - \tau_{\hat{q}}^q(N_{\hat{q}}), \omega_q) + g_q(\tau_{\hat{q}}^q(N_{\hat{q}}) - \tau_{q^\pi}^q(N_{q^\pi}), \omega_q)\,. \end{aligned}$$

This implies that

$$0 \leq 2g_q(N_q, \omega_q) \leq |N_q - \tau_{\hat{q}}^q(N_{\hat{q}})|_q + |\tau_{\hat{q}}^q(N_{\hat{q}}) - \tau_{q^\pi}^q(N_{q^\pi})|_q\,.$$

Next we observe that

$$|\tau^q_{\hat{q}}(N_{\hat{q}}) - \tau^q_{q^\pi}(N_{q^\pi})|_q = |N_{\hat{q}} - \tau^{\hat{q}}_q \tau^q_{q^\pi}(N_{q^\pi})|_{\hat{q}} \leq c(\delta)|N_{\hat{q}} - \tau^{\hat{q}}_{q^\pi}(N_{q^\pi})|_{\hat{q}}$$

where $c(\delta) \to 0$ when $\delta \to 0$. Hence for a suitable choice of $\bar{\delta}$ we get

$$0 \leq 2g_q(N_q, \omega_q) \leq \frac{1}{8}\,, \tag{1.60}$$

and the claim follows from case i). □

Now we can focus on the proof of the first part of Theorem 1.4, and show that there exist constants ε and C, depending only on n, ρ and $|S|_g$ (the area of S with respect to g), such that if

$$\operatorname{osc}(\mathsf{H}_S) \leq \varepsilon,$$

then for any p in Σ there exists $\hat{p}$ in $\hat{\Sigma}$ satisfying

$$d(p, \hat{p}) + |N_p - \tau^p_{\hat{p}}(N_{\hat{p}})|_p \leq C \operatorname{osc}(\mathsf{H}_S)\,. \tag{1.61}$$

In the proof of Theorem 1.4 we are going to choose a number $\delta > 0$ sufficiently small in terms of ρ, n and $|S|_g$. A first requirement on δ is that the assumptions of Lemmas 1.7 and 1.8 are satisfied. Other restrictions on the value of δ will be done in the development of the proof. We subdivide the proof of the first part of the statement in four cases depending on the whether the distances of p_0 and p from $\partial\Sigma$ are greater or less than δ.

Case 1: $d_\Sigma(p_0, \partial\Sigma) > \delta$ and $d_\Sigma(p, \partial\Sigma) \geq \delta$.

In this first case we assume that p_0 and p are interior points of Σ, which are far from $\partial\Sigma$ more than δ. We first assume that p_0 and p are in the same connected component of Σ_δ; then, Lemma 1.8 will be used in order to show that Σ_δ is in fact connected.

Let $r_0 > 0$ be such that $\mathcal{U}_{r_0}(p_i) \subset \Sigma$ for every $p_i \in \Sigma_\delta$. The value of r_0 follows from (ii), Lemma 1.3 by letting

$$r_0 = \min(\bar{r}, \alpha\rho_1)\,, \tag{1.62}$$

where $\bar{r}$ is given by Lemmas 1.5 and 1.6, $\alpha \in (0, \frac{1}{2}\min(1, \rho_1^{-1}))$ is such that $\alpha C \rho_1 \leq \frac{\delta}{4}$, and C is the constant appearing in (1.19).

Lemma 1.9. *Let $\varepsilon_0 \in [0, 1/2]$, p_0 and p be in a connected component of Σ_δ and $r_i = (1 - \varepsilon_0^2)^i r_0$. There exist an integer $J \leq J_\delta$, where*

$$J_\delta := \max\left(4, \frac{2^{n-1}|S|_g}{\delta^{n-1}}\right)\,, \tag{1.63}$$

and a sequence of points $\{p_1, \ldots, p_J\}$ in $\Sigma_{\delta/2}$ such that

$$\begin{aligned}
&p_0, p \in \bigcup_{i=0}^{J} \overline{\mathcal{U}}_{r_i/4}(p_i)\,,\\
&\mathcal{U}_{r_0}(p_i) \subseteq \Sigma, \quad i = 0, \ldots, J\,,\\
&p_{i+1} \in \overline{\mathcal{U}}_{r_i/4}(p_i), \quad i = 0, \ldots, J-1\,.
\end{aligned}$$

Proof. In view of Corollary 1.2, for every z in Σ and $r \leq \rho_0$, the geodesic ball $\mathcal{B}_r(z)$ in Σ satisfies

$$\text{Area}(\mathcal{B}_r(z)) \geq c r^{n-1}$$

where c is a constant depending only on n. A general result for Riemannian manifolds with boundary (see Proposition A.1) implies that there exists a piecewise geodesic path parametrized by arc length $\gamma\colon [0, L] \to \Sigma_{\delta/2}$ connecting p_0 to p and of length L bounded by δJ_δ, where J_δ is given by (1.63).

We define $p_i = \gamma(r_i/4)$, for $i = 1, \dots, J-1$ and $p_J = p$. Our choice of r_0 guarantees that $\mathcal{U}_{r_0}(p_i) \subset \Sigma$, for every $i = 0, \dots, J$, and the other required properties are satisfied by construction. □

Since p and p_0 are in a connected component of Σ_δ, there exists a sequence of points $p_1, \dots, p_J$ in the connected component of $\Sigma_{\delta/2}$ containing p_0, with $J \in \mathbb{N}$ and $p_J = p$, and a chain of subsets $\{\mathcal{U}_{r_0}(p_i)\}_{\{i=0,\dots,J\}}$ of Σ as in Lemma 1.9. We notice that Σ and $\hat{\Sigma}$ are tangent at p_0 and that in particular the two normal vectors to Σ and $\hat{\Sigma}$ at p_0 coincide. Now we apply the map φ_{p_0} (see section 1.2). Then $\varphi_{p_0}(\Sigma)$ and $\varphi_{p_0}(\hat{\Sigma})$ can be locally parametrized near $\varphi_{p_0}(p_0)$ as graphs of two functions u_0, $\hat{u}_0\colon B_{r_0} \subset \{x_n = 0\} \to \mathbb{R}$. Lemma 1.3 implies that $|\nabla u_0|, |\nabla \hat{u}_0| \leq M$ in B_{r_0}, where M is some constant which depends only on r_0, i.e. only on ρ. Hence the difference $u_0 - \hat{u}_0$ solves a second-order linear uniformly elliptic equation of the form

$$\mathcal{L}(u_0 - \hat{u}_0)(x) = \mathsf{H}_{\text{Graph}(u_0)}(x, u_0(x)) - \mathsf{H}_{\text{Graph}(\hat{u}_0)}(x, \hat{u}_0(x))$$

with ellipticity constants uniformly bounded by a constant depending only on n and ρ. Since $u_0(0) = \hat{u}_0(0)$ and $u_0 \geq \hat{u}_0$, Harnack's inequality (see Theorems 8.17 and 8.18 in [112]) yields

$$\sup_{B_{r_0/2}} (u_0 - \hat{u}_0) \leq C \operatorname{osc}(\mathsf{H}_S)\,,$$

and from interior regularity estimates (see e.g. [112, Theorem 8.32]) we obtain

$$\|u_0 - \hat{u}_0\|_{C^1(B_{r_0/4})} \leq C \operatorname{osc}(\mathsf{H}_S), \tag{1.64}$$

where C depends only on ρ and n. Now we use Lemmas 1.5 and 1.6. Since $p_1 \in \partial \mathcal{U}_{r_0/4}(p_0)$, we can write $\varphi_{p_0}(p_1) = (x_1, u_0(x_1))$, with $x_1 \in \partial B_{r_0/4}$. Let $\hat{p}_1^* \in \hat{\Sigma}$ be such that

$$\varphi_{p_0}(\hat{p}_1^*) = (x_1, \hat{u}(x_1))\,,$$

and let $\hat{p}_1$ be the first intersection point between $\hat{\Sigma}$ and the geodesic path γ starting from p_1 and tangent to $-N_{p_1}$ at p_1. From (1.64) we have

$$d(\varphi_{p_0}(p_1), \varphi_{p_0}(\hat{p}_1^*)) + |\nu_{\varphi_{p_0}(p_1)} - \nu_{\varphi_{p_0}(\hat{p}_1^*)}| \leq C \operatorname{osc}(\mathsf{H}_S)\,, \tag{1.65}$$

which implies that the assumptions in Lemmas 1.5 and 1.6 are fulfilled, and we obtain

$$d(p_1, \hat{p}_1) + |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1} \leq C \operatorname{osc}(\mathsf{H}_S)\,, \tag{1.66}$$

where C depends only on n and ρ.

Now we apply φ_{p_1}. By definition of φ_{p_1}, we have $\varphi_{p_1}(\hat{p}_1) = t e_n$ for some $t \in \mathbb{R}$ (t depends on the geometry of the ambient space). A standard computation yields

$$|\nu_{\varphi_{p_1}(p_1)} - \nu_{\varphi_{p_1}(\hat{p}_1)}| = |N_{p_1} - \tau_{\hat{p}_1}^{p_1}(N_{\hat{p}_1})|_{p_1}$$

which in view of (1.66) implies

$$|\nu_{p_1} - \nu_{\hat{p}_1}| \leq C\mathrm{osc}(\mathsf{H}_\mathrm{S})\,,$$

where C is a constant that depends only on ρ and n. Since $\mathrm{osc}(\mathsf{H}_\mathrm{S}) \leq \varepsilon$ then $|\nu_{p_1} - \nu_{\hat{p}_1}| < C\varepsilon$ and by choosing ε such that $C\varepsilon < 1$, we can use the implicit function theorem and obtain that Σ and $\hat{\Sigma}$ are locally graphs of two functions

$$u_1, \hat{u}_1 : B_{r_1} \to \mathbb{R}^+\,,$$

such that $u_1(0) = \varphi_{p_1}(p_1)$ and $\hat{u}_1(0) = \varphi_{p_1}(\hat{p}_1)$. Since both Σ and $\hat{\Sigma}$ satisfy a touching ball condition of radius ρ_1 then $r_1 < \rho_1$; in particular one can show that

$$r_1 = (1 - C^2\varepsilon^2)\rho < \rho_1\,.$$

Now, we can iterate the argument we did before. Indeed, since

$$0 \leq \inf_{B_{r_1/2}} (u_1 - \hat{u}_1) \leq u_1(0) - \hat{u}_1(0) \leq C\mathrm{osc}(\mathsf{H}_\mathrm{S})\,,$$

by applying Harnack's inequality we obtain that

$$\sup_{B_{r_1/2}} (u_1 - \hat{u}_1) \leq C\,\mathrm{osc}(\mathsf{H}_\mathrm{S})$$

and from interior regularity estimates we find

$$\|u_1 - \hat{u}_1\|_{C^1(B_{r_1/4})} \leq C\,\mathrm{osc}(\mathsf{H}_\mathrm{S})\,, \tag{1.67}$$

where C depends only on ρ and n. Hence, (1.67) is the analogue of (1.64), and we can iterate the argument. The iteration goes on until we arrive at $p_J = p$ and obtain a point $\hat{p}_J \in \hat{\Sigma}$ such that

$$d(p, \hat{p}_J) + |N_p - \tau^p_{\hat{p}_J}(N_{\hat{p}_J})|_p \leq C\mathrm{osc}(\mathsf{H}_\mathrm{S})\,.$$

In view of Lemma 1.8 we have that Σ_δ is connected and the claim follows.

Case 2: $d_\Sigma(p_0, \partial\Sigma) \geq \delta$ and $d_\Sigma(p, \partial\Sigma) < \delta$.

We extend the estimates found in case 1 to a point p which is far less than δ from the boundary of Σ. Let $q \in \Sigma$ and $p_{min} \in \partial\Sigma$ be such that

$$d_\Sigma(q, \partial\Sigma) = \delta\,, \quad d_\Sigma(p, q) + d_\Sigma(p, \partial\Sigma) = \delta\,, \quad \text{and} \quad d_\Sigma(p, p_{min}) = d_\Sigma(p, \partial\Sigma)\,.$$

From case 1 we have that there exists $\hat{q}$ in $\hat{\Sigma}$ such that

$$d(q, \hat{q}) + |N_q - \tau^q_{\hat{q}}(N_{\hat{q}})|_q \leq C\,\mathrm{osc}(\mathsf{H}_S)\,.$$

Lemma 1.8 (case (ii)) yields that

$$0 \leq g_z(N_z, \omega_z) \leq \tfrac{1}{4}, \tag{1.68}$$

for any $z \in \partial\Sigma$ and Σ_δ is connected.

Let $q^\pi \in S$ be the reflection of q about π and fix $r \leq \rho_1$ in order to define $\mathcal{U}_r(q^\pi)$. We denote by $U_r(q)$ the reflection of $\mathcal{U}_r(q^\pi) \cap S$ about π and $U' =$

$\mathcal{U}_r(q^\pi) \cap \pi$. Proposition 1.3 implies that U' is a hypersurface of π with an induced orientation and its principal curvatures κ'_i satisfy the following bounds

$$\frac{1}{\sqrt{1-g_z(N_z,\omega_z)^2}}\kappa_1(z) \leq \kappa'_i(z) \leq \frac{1}{\sqrt{1-g_z(N_z,\omega_z)^2}}\kappa_{n-1}(z)\,,$$

for every $z \in U'$ and $i = 1, \dots, n-1$. From (1.68) and since $|\kappa_i(z)| \leq \rho^{-1}$ for any $z \in S$ (this follows from the touching ball condition), we have

$$|\kappa'_i(z)| \leq \frac{2}{\rho}\,, \tag{1.69}$$

for any $z \in U'$. Let U'' be the Euclidean orthogonal projection of $\varphi_q(U')$ onto $\{x_n = 0\}$. In order to apply Carleson estimates in [26, Theorem 1.3], we need to prove the following

Lemma 1.10. *Let $\{\kappa''_1, \dots, \kappa''_{n-2}\}$ be the Euclidean principal curvature of U'' viewed as a hypersurface of $\mathbb{R}^{n-1}$. Then*

$$\|\kappa''_i\|_\infty \leq C\,, \quad i = 1, \dots, n-2\,, \tag{1.70}$$

for some constant $C = C(\rho)$.

Proof. Here we use the same notation as in Section 1.3 and we analyse each case separately.

- $M^n_+ = \mathbb{R}^n$. Up to apply φ_q we may assume that $\varphi_q(q) = 0$ and that the normal vector to $\varphi_q(S)$ at $\varphi_q(q)$ is e_n. Hence Proposition 1.4 (actually formula (1.53)) and (1.69) yield, for every $i = 1, \dots, n-2$,

$$|\kappa''_i(\bar z)| \leq \frac{2}{\rho}\frac{|\omega \cdot e_n|}{[(\omega\cdot e_n)^2 + (e_n\cdot\nu'_z)^2]^{3/2}} \leq \frac{2}{\rho}\frac{|\omega\cdot e_n|}{(e_n\cdot\nu'_z)^3}\,, \tag{1.71}$$

  for every $z = (\bar z, z_n) \in \varphi_q(U')$. Now we estimate $e_n \cdot \nu'_z$ in the following way: by writing

$$e_n \cdot \nu'_z = (e_n - \nu_z)\cdot\nu'_z + \nu_z\cdot\nu'_z\,,$$

  exploiting (1.45) and from (1.59) we get

$$\nu_z\cdot\nu'_z = \sqrt{1-(\omega\cdot\nu_z)^2} \geq \frac{1}{2}\,. \tag{1.72}$$

  Moreover, from (1.20) we get

$$|e_n - \nu_z| \leq \frac{1}{4}\,, \tag{1.73}$$

  where we use the fact that $\bar z \in U''$ where U'' is the orthogonal projection of $\varphi_q(U')$ and $U' = \mathcal{U}_r(q^\pi)\cap\pi$ with $r \leq \rho_1 = \rho$. So, from (1.72) and (1.73) we obtain

$$e_n\cdot\nu'_z \geq \frac{1}{2}\,. \tag{1.74}$$

  The conclusion now follows from (1.74), (1.68) and (1.71).

- $M^n_+ = \mathbb{H}^n$. Up to apply the isometry φ_q we may assume that $\varphi_q(q) = e_n$ and the normal vector to $\varphi_q(S)$ at $\varphi_q(q)$ is e_n. It is clear that $\varphi_q(\pi)$ is either a vertical plane or a half-sphere intersecting $\varphi_q(S)$. In the first case we immediately conclude since the curvatures of U'' vanish. Thus, we may assume that $\varphi_q(\pi)$ is a half-sphere. A straightforward computation yields that the radius of $\varphi_q(\pi)$ is

$$R = \frac{q_n(\Theta^2 + 1)}{2|\Theta||a\Theta + q_n|},$$

where

$$\Theta = -\frac{\sin(\theta)}{1 + \cos(\theta)}, \quad \cos(\theta) = \nu_q \cdot e_n$$

and a is the Euclidean distance of q from π. It is easy to see that

$$a \leq q_n \sinh(\delta)$$

and so

$$\frac{1}{R} \leq \frac{2|\Theta|(\sinh(\delta)|\Theta| + 1)}{\Theta^2 + 1}$$

which implies

$$\frac{1}{R} \leq 1 + 2\sinh(\delta). \tag{1.75}$$

Now we show (1.70). Proposition 1.4 (actually formula (1.55)), (1.69) (1.75) imply

$$\begin{aligned} |\kappa_i''(\bar{z})| &\leq \frac{1}{R}\left((\nu_z' \cdot e_n)^2 + \frac{z_n^2}{R^2}\right)^{-3/2} \frac{4(1+\rho)}{\rho} \\ &\leq \frac{4(1+\rho)(1+2\sinh(\delta))}{\rho|\nu_z' \cdot e_n|^3}, \end{aligned} \tag{1.76}$$

for every $z = (\bar{z}, z_n) \in \varphi_q(U')$ and $i = 1, \dots, n-2$. Now we show a lower bound on $\nu_z' \cdot e_n$. We write

$$\nu_z' \cdot e_n = \nu_z' \cdot (e_n - \nu_z) + \nu_z' \cdot \nu_z = \nu_z' \cdot (e_n - \nu_z) + g_z(N_z', N_z). \tag{1.77}$$

From (1.43) we get

$$g_z(N_z', N_z) = \sqrt{1 - g_z(\omega_z, N_z)^2}$$

and from (1.59) we obtain

$$g_z(N_z', N_z) \geq \frac{1}{2}.$$

Since $\varphi_{q*|q}(\nu_q) = e_n$ from (1.20) we have that $|e_n - \nu_z| \leq 1/4$, by choosing r small enough in terms of ρ_1. Hence

$$\nu_z' \cdot e_n = \nu_z' \cdot (e_n - \nu_z) + g_z(N_z', N_z) \geq \frac{1}{2} - |\nu_z' \cdot (e_n - \nu_z)| \geq \frac{1}{4}. \tag{1.78}$$

Therefore (1.70) follows from (1.76) and (1.78).

- $M^n_+ = \mathbb{S}^n_+$. If we apply the stereographic projection φ_q then it is clear that $\varphi_q(\pi)$ is a hemisphere in $\mathbb{R}^n$ of center O_π and radius R. We notice that the radius of $\varphi_q(\pi)$ is given by

$$R = \tfrac{1}{2a} + \tfrac{a}{2}\,,$$

where a is the Euclidean distance between $\varphi_q(\pi)$ and the origin of $\mathbb{R}^n$. This follows from the proof of lemma 1.6: indeed, up to apply a rotation, we may assume that the point on π having minimal Euclidean distance from the origin is $(a, 0, \ldots, 0)$, with $a > 0$, and that the normal to π in $(a, 0, \ldots, 0)$ is e_1. From a straightforward calculation we obtain the value of R. Since $d(q, \pi) < \delta$ we have $a < \delta$ which implies

$$R > \frac{1}{2\delta}\,. \tag{1.79}$$

Since $U' \subset \mathcal{U}_r(q)$ and $\varphi_q(q) = O$, our choice of r implies that for any $z \in \varphi_q(U')$ we have

$$|z| \leq \rho_1 = \frac{\rho}{\pi} \tag{1.80}$$

and

$$|z - O_\pi| \leq |z - q| + |q - O_\pi| \leq \delta + R \leq 2R\,. \tag{1.81}$$

Now we show (1.70). Proposition 1.4 (actually formula (1.56)), (1.79), (1.80) and (1.81) yield

$$|\kappa''_i(\bar{z})| \leq 8(\nu'_z \cdot e_n)^{-3} \left(\max\{|\kappa'_1(z)|, |\kappa'_{n-1}(z)|\} + 4\rho_1\right)\,, \tag{1.82}$$

for every $z = (\bar{z}, z_n) \in \varphi_q(U')$ and for every $i = 1, \ldots, n-2$.

Now, as in the hyperbolic case, we give a lower bound of $\nu'_z \cdot e_n$. We can write (1.77) and arguing as before we obtain (1.78) (here we can change configuration by considering the stereographic projection from the antipodal point to $\varphi_q^{-1}(z)$ in order to regard π as a vertical hyperplane of $\mathbb{R}^n$). Also in this case (1.70) follows from (1.82), (1.78) and (1.69).

□

We denote by x and E_r the projections of $\varphi_q(p_{min})$ and $\varphi_q(U_r(q))$ onto $\{x_n = 0\}$, respectively. The Euclidean distance of x from ∂E_r is less than $C\delta$ where C depends only on ρ and, up to chose a smaller δ in terms of ρ, the projection of p_{min} stays close to $U'' \subset \partial E_r$ and we can apply Theorem 1.3 in [26], Corollary 8.36 in [112] and Harnack's inequality (see e.g. [112, Corollary 8.36]) to obtain

$$\sup_{B_{2\delta C}(x)\cap E_r} (u - \hat{u}) \leq C(u - \hat{u})(z) + \mathrm{osc}(\mathsf{H}_S) \tag{1.83}$$

with $z = x + 4C\delta\nu''_x$, where ν''_x is the interior normal to U'' at x. Thanks to (1.70) and by choosing δ small enough in terms of ρ, from (1.83) and Harnack's inequality we obtain

$$0 \leq \|u - \hat{u}\|_{C^1(B_{C\delta}(x)\cap E_r)} \leq C((u(0) - \hat{u}(0)) + \mathrm{osc}(\mathsf{H}_S))\,. \tag{1.84}$$

Since $d_\Sigma(q,\partial\Sigma)=\delta$, from Case 1 we know that

$$d(q,\hat q)+|N_q-\tau^q_{\hat q}(N_{\hat q})|_q\le C\mathrm{osc}(\mathsf{H}_\mathrm{S})\,,$$

and from (1.84) we obtain that

$$0\le\|u-\hat u\|_{C^1(B_{C\delta}(x)\cap E_r)}\le C\mathrm{osc}(\mathsf{H}_\mathrm{S})\,. \tag{1.85}$$

From Lemmas 1.5 and 1.6 we deduce

$$d(p,\hat p)+|N_p-\tau^p_{\hat p}(N_{\hat p})|_p\le C\mathrm{osc}(\mathsf{H}_\mathrm{S})\,,$$

as required.

Case 3: $0<d_\Sigma(p_0,\partial\Sigma)<\delta$**.**

We first prove the following preliminary lemma which implies via Lemma 1.7 that Σ is connected.

Lemma 1.11. *By choosing δ small enough in terms of ρ, the following inequality holds*

$$0\le g_{p_0}(N_{p_0},\omega_{p_0})\le\frac{1}{4}\,. \tag{1.86}$$

Proof. We first prove the statement in the Euclidean case: in this setting (1.86) reads as follows

$$0\le\nu_{p_0}\cdot\omega\le\frac{1}{4}\,. \tag{1.87}$$

Since p_0 is the tangency point, it is easy to show that the center of the interior touching sphere of radius ρ to S at p_0 lies in the half-space $\{q\in\mathbb{R}^n\,:\,q\cdot\omega\le m\}$ (see e.g. [61, Lemma 2.1]). From this and since

$$|p_0\cdot\omega-m|\le d_\Sigma(p_0,\partial\Sigma)<\delta$$

we obtain (1.87). Indeed let p_0^π be the reflection of p_0 about π and let

$$t=\nu_{p_0}\cdot\omega\,.$$

By construction of the moving planes, it is clear that $t\ge0$ and the first inequality in (1.87) follows. We denote by $\nu_{p_0^\pi}$ the inner normal vector to S at p_0^π. Since $\nu_{p_0}\cdot\omega=-\nu_{p_0^\pi}\cdot\omega$ and $\nu_{p_0}-\nu_{p_0^\pi}=2t\omega$, we have

$$\nu_{p_0}\cdot\nu_{p_0^\pi}=1-2t^2\,.$$

We notice that p_0^π and p_0 both lie in S and $|p_0^\pi-p_0|\le2\delta$, which implies that $p_0\in\mathcal{U}_\rho(p_0^\pi)$, provided that $2\delta<\rho$. Hence, (1.20) yields

$$\nu_{p_0}\cdot\nu_{p_0^\pi}\ge\sqrt{1-\frac{4\delta^2}{\rho^2}}\,,$$

i.e.

$$1-2t^2\ge\sqrt{1-\frac{4\delta^2}{\rho^2}}\,;$$

and by choosing δ small enough in terms of ρ we obtain the second inequality in (1.87).

Now we show how to deduce, from the Eulidean case, the claim in the hyperbolic and in the spherical case. We first consider the Hyperbolic case. Up to apply an isometry we can assume that $p_0 = e_n$ and $\pi = \{x_1 = 0\}$. Our assumptions on S imply that its diameter is bounded in terms of ρ and $|S|_g$ (see Proposition A.2 in Appendix A). Therefore S is contained in an Euclidean ball about the origin and of radius depending only on ρ and $|S|_g$. Up to choose δ small enough in terms of ρ, we have that (1.87) holds, i.e.

$$0 \le \nu_{p_0} \cdot e_1 \le \frac{1}{4} .$$

Since $\nu_{p_0} \cdot e_1 = g_{p_0}(N_{p_0}, \omega_{p_0})$, the claim follows. In the spherical case the proof is analogue once the setting is modified as follows: we work in $(\mathbb{R}^n, g)$, where g is the round metric (1.7), assuming that $p_0 = O$ and π is an Euclidean hyperplane. □

Then we prove the existence of a point $q \in \Sigma$ such that

$$\begin{cases} d(q,\hat q) + |N_q - \tau^q_{\hat q}(N_{\hat q})|_q \le C \mathrm{osc}(\mathsf{H}_S) \\ d_\Sigma(q, \partial\Sigma) \ge \delta \end{cases} \tag{1.88}$$

and we apply cases 1 and 2 to conclude.

In the same fashion as in case 2, we can locally write $\varphi_{p_0}(\Sigma)$ and $\varphi_{p_0}(\hat\Sigma)$ as graphs of function $u, \hat u \colon E_r \to \mathbb{R}$ near $\varphi_{p_0}(p_0)$, respectively. Without loss of generality we can assume $r < 1$ (indeed r must be chosen small enough in terms of δ). Let $U'' \subset \partial E_r$ be the projection of $\varphi_{p_0}(U_r(p_0) \cap \pi)$ onto $\{x_n = 0\}$. Analogously to case 2, the Euclidean principal curvatures of U'' are bounded by a constant $\mathcal{K}$ depending only on ρ. Then let $\bar x \in U''$ be a point such that

$$|\bar x| = \min_{x \in U''} |x| .$$

Notice that $|\bar x| \le c_* d_\Sigma(p_0, \partial\Sigma) < c_*\delta$, where c_* is 1 in the Euclidean and in the spherical case, while it is the constant c appearing in (1.10) in the hyperbolic case. Let $\nu''_{\bar x}$ be the interior normal to U'' at $\bar x$, and set

$$y = \bar x + 2c_*\delta\nu''_{\bar x}$$

(see Figure 1.4) By choosing δ sufficiently small in terms of ρ, we have $2c_*\delta \le \mathcal{K}^{-1}$ and the ball $B_{2c_*\delta}(y)$ is contained in E_r and it is tangent to U'' at $\bar x$, with $\nu_{\bar x} = -\bar x/|\bar x|$. Since $u(0) = \hat u(0)$ from Harnack's inequality and from interior regularity estimates we find that

$$||u - \hat u||_{C^1(B_{C\delta/2}(y))} \le C \mathrm{osc}(\mathsf{H}_S) ,$$

which implies that

$$d(w, \hat w^*) + |\nu_w - \nu_{\hat w^*}| \le C \mathrm{osc}(\mathsf{H}_S) ,$$

where $w = (y, u(y))$ and $\hat w^* = (y, \hat u(y))$. Up to choose a smaller δ, we can assume that $2c_*\delta \le \bar r$, so that Lemmas 1.5 and 1.6 yield

$$d(q,\hat q) + |N_q - \tau^q_{\hat q}(N_{\hat q})|_q \le C \mathrm{osc}(\mathsf{H}_S) ,$$

where $q = \varphi_{p_0}^{-1}(w)$, $\hat{q}^* = \varphi_{p_0}^{-1}(\hat{w}^*)$ and $\hat{q}$ the first intersection point between $\hat{\Sigma}$ and the geodesic path starting from q and tangent to $-N_q$ at q.

Let z be a point on $\partial U_r(p_0)$ realizing $d(q, \partial U_r(p_0))$. By construction and from Lemma 1.2 we have

$$d_\Sigma(q, \partial\Sigma) \geq d(q,z) \geq c_*|\varphi_{p_0}(q) - \varphi_{p_0}(z)| \geq 2\delta\,.$$

Since $d_\Sigma(q, \partial\Sigma) \geq \delta$ then q satisfies (1.88) and the claim follows.

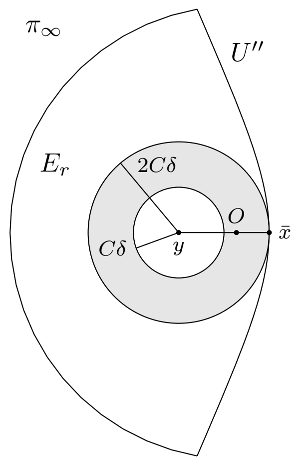

Figure 1.4: Case 3 in the proof of Theorem 1.4.

Case 4: $p_0 \in \partial\Sigma$**.**

This is the limit configuration of case 3 when $d_\Sigma(p_0, \partial\Sigma) \to 0$. Indeed, here E_r is a half-ball in $\mathbb{R}^n$ and the argument used in case 3 can be easily adapted. This completes the proof of the first part of Theorem 1.4.

Last step: $d(x, \Sigma \cup \Sigma^\pi) \leq C\text{osc}(\mathsf{H}_\text{S})$ **for every** $x \in \Omega$**.**

Assume by contradiction that

$$d(x, \Sigma \cup \Sigma^\pi) > C\text{osc}(\mathsf{H}_\text{S})$$

for some x in Ω. Since Ω is connected, it is possible to find $y \in \Omega$, such that

$$y \in \Omega_- \quad \text{and} \quad C\text{osc}(\mathsf{H}_\text{S}) < \text{d}(\text{y}, \Sigma) \leq 2\text{Cosc}(\mathsf{H}_\text{S})\,,$$

where

$$\Omega_+ = \{p \in \Omega \ : \ p \in \pi_{v,t} \text{ for some } t > m_v\}\,,$$
$$\Omega_- = \{p \in \Omega \ : \ p \in \pi_{v,t} \text{ for some } t < m_v\}\,.$$

Let p be a projection of y over $\Sigma \cup \Sigma^\pi$. If $p \in \Omega_+$, then y belongs to the exterior touching ball of S at p, which gives a contradiction. The same contradiction is obtained when $p \in \pi$ since, in that case $g_p(N_p, \omega_p) \leq 1/4$. If $p \in \Omega_-$, we can find a point $\hat{p} \in S$ such that $\hat{p}$ and p lies on the geodesic γ starting from p and orthogonal to Σ^π and such that

$$d(p, \hat{p}) + |N_p - \tau_{\hat{p}}^p(N_{\hat{p}})| \leq C \mathrm{osc}(\mathsf{H}_\mathrm{S})\,.$$

By the smallness of $\mathrm{osc}(\mathsf{H}_\mathrm{S})$ we obtain that y belongs to the exterior touching ball of S at p, which is a contradiction.

□

1.5 Global approximate symmetry

From the previous section we have that if a C^2-regular closed hypersurface $S = \partial\Omega$ embedded in $\mathbb{M}^n_+$ satisfies the assumptions of Theorem 1.4 then it is almost symmetric with respect to any direction, with the almost symmetry quantified by the deficit $\mathrm{osc}(\mathsf{H}_S)$. In this section we show how this result leads to the almost radial symmetry of S. Such procedure is not peculiar of the kind of deficit considered, but it can be applied whenever one has the approximate symmetry in any direction with respect to some deficit. More precisely we consider the following

Definition 1.1. Let Υ be the space of open sets Ω in $\mathbb{M}^n_+$ whose boundary is a C^2-regular connected closed embedded hypersurface, with the topology induced by the Hausdorff distance. A *deficit function* is any continuous function $\mathrm{def}(\Omega) : \Upsilon \to [0, +\infty)$ such that $\mathrm{def}(\Omega) = 0$ if and only if Ω is a ball.

Form now on we fix a deficit function def.

Definition 1.2. We say that a bounded open set Ω satisfies the *approximate symmetry property (ASP)* if there exists a constant $\mathcal{K} > 0$ satisfying the following condition: for every direction v there exists a connected component Σ of the maximal cap in the direction v such that

$$d(p, \Sigma \cup \Sigma^{\pi_v}) \leq \mathcal{K}\, \mathrm{def}(\Omega)\,,$$

for every $p \in \Omega$.

The main theorem in this section is the following

Theorem 1.5. *Let $S = \partial\Omega$ be a C^2-regular closed hypersurface embedded in $\mathbb{M}^n_+$, with Ω satisfying (ASP) and*

$$\mathrm{def}(\Omega) \leq \frac{|\Omega|_g}{4\mathcal{K}}\,. \tag{1.89}$$

There exist $\mathcal{O}$ in $\mathbb{M}^n_+$ and two balls $B_r^d(\mathcal{O})$ and $B_R^d(\mathcal{O})$ centred at $\mathcal{O}$ of radius r and R, respectively, with $r \leq R$, such that

$$B_r^d(\mathcal{O}) \subseteq \Omega \subseteq B_R^d(\mathcal{O})$$

and

$$R - r \leq C\, \mathrm{def}(\Omega)\,, \tag{1.90}$$

where C depends on $n, \rho, |S|_g$ and $\mathcal{K}$.

The following lemma is needed in order to prove Theorem 1.5; here we show that if $\mathrm{def}(\Omega)$ is small enough, then π_v is close to a point $\mathcal{O}$ (we will call it the *approximate center of symmetry*), for every direction v.

Lemma 1.12. *Let $S = \partial\Omega$ be a C^2-regular closed hypersurface embedded in $\mathbb{M}^n_+$, with Ω satisfying (ASP) and (1.89). Then there exists $\mathcal{O}$ in $\mathbb{M}^n_+$ such that*

$$d(\mathcal{O}, \pi_v) \leq C\mathrm{def}(\Omega)\,, \tag{1.91}$$

for every direction v in $T_{\mathsf{o}}\mathbb{M}^n$, where C depends on $n, \rho, |S|_g$ and $\mathcal{K}$.

Before proving it, we observe that: given a direction v, let Ω_v be the corresponding maximal cap. If Ω satisfies (ASP) then we have

$$|\Omega_v|_g \geq \frac{|\Omega|_g}{2} - C\mathrm{def}(\Omega)\,, \tag{1.92}$$

for some constant C depending only on $n, \rho, |S|_g$ and $\mathcal{K}$. Moreover we have

$$|\Omega \triangle \Omega^\pi|_g = 2(|\Omega|_g - 2|\Omega_v|_g) \leq 4C\mathrm{def}(\Omega)\,, \tag{1.93}$$

where $\Omega\triangle\Omega^\pi$ denotes the symmetric difference between Ω and Ω^π and, we recall that Ω^π denotes the reflection of Ω about the critical hyperplane π in the direction v.

Proof of Lemma 1.12. We fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space at the "origin" o and we consider the corresponding critical hyperplanes π_{e_i}. We define an approximate center of symmetry $\mathcal{O}$ as follows:

$$\mathcal{O} := \bigcap_{i=1}^{n} \pi_{e_i}\,.$$

We notice that in the Euclidean case $\mathcal{O}$ is well-defined. In $\mathbb{S}^n_+$ the existence of $\mathcal{O}$ is always guaranteed. Indeed every π_{e_i} is given by the intersection of a plane Π_{e_i} of $\mathbb{R}^{n+1}$ with $\mathbb{S}^n_+$ and the intersection of all the Π_{e_i}'s is a straight line r which, by construction, can not lie in the plane $\{x_{n+1} = 0\}$; hence $\mathcal{O} = r \cap \mathbb{S}^n_+ \neq \emptyset$. Although in the hyperbolic space n orthogonal hyperplanes do not always intersect, but we show that (1.89) implies the existence of $\mathcal{O}$. Indeed it is enough to show that

$$\pi_{e_i} \cap \pi_{e_j} \neq \emptyset \qquad \text{for every } i, j = 1, \ldots, n. \tag{1.94}$$

For simplicity, we may assume that $e_n \in S$. To simplify the notation we set, accordingly to the notation introduced in Section 1.1,

$$\pi_k^s = \pi_{e_k, m_{e_k}+s} \qquad \text{for } k \in \{1, \ldots, n\} \text{ and } s \in \mathbb{R}\,,$$

so that the critical hyperplane in the direction e_k is denoted by π_k^0. We prove (1.94) by contradiction. Assume that $\pi_i^0 \cap \pi_j^0 = \emptyset$ for some $i \neq j$. Then π_i^0 and π_j^0 divide Ω in three disjoint sets which we denote by Ω_1, Ω_2 and Ω_3 and we may assume that Ω_1 is the maximal cap in the direction e_i and $\Omega_1 \cup \Omega_2$ is the maximal cap in the direction e_j (see figure 1.5).

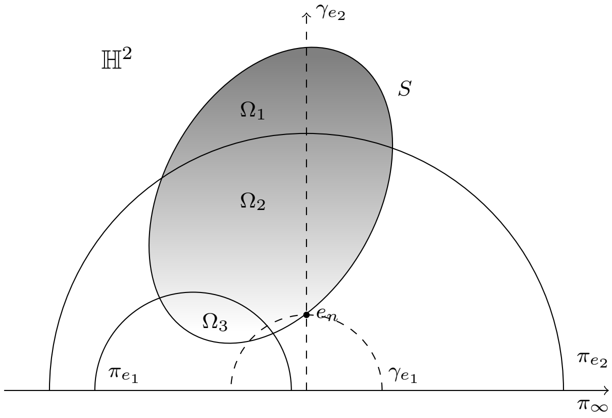

Figure 1.5: a picture of the proof of (1.94) in $\mathbb{H}^2$. Here $e_j = e_1$ and $e_i = e_2$.

Moreover, in view of (1.92) we have that

$$|\Omega_1|_g \geq \frac{|\Omega|_g}{2} - C\mathrm{def}(\Omega)\,, \qquad \text{and} \qquad |\Omega_1|_g + |\Omega_2|_g \geq \frac{|\Omega|_g}{2} - C\mathrm{def}(\Omega)\,.$$

From this, and since the reflection Ω_1 about π_i^0 is contained in $\Omega_2 \cup \Omega_3$ and the reflection of $\Omega_1 \cup \Omega_2$ about π_j^0 is contained in Ω_3 we have that

$$|\Omega_2|_g \leq 2C\mathrm{def}(\Omega)\,.$$

We notice that for every $k = 1, \ldots, n$, we have that π_k^{s+t} and π_k^{s-t} are two connected components of the set of points which are far t from π_k^t. We define

$$\ell := \min\{d(\pi_i^0 \cap \Omega, \pi_j^0 \cap \Omega) \,:\, i, j = 1, \ldots, n \text{ and } i \neq j\}\,.$$

Since π_i^0 and π_j^0 do not intersect $S \subset B^d_{\mathrm{diam}\,(S)}(e_n)$, we have that $\ell > 0$ and Proposition A.2 implies that ℓ depends only on n, ρ and $|S|_g$. Therefore

$$\Omega_2 \supseteq \mathcal{E}_1 := \bigcup_{s \in (0,l)} \Omega \cap \pi_j^s\,,$$

and hence $|\mathcal{E}_1|_g \leq 2C\mathrm{def}(\Omega)$. By reflecting $\mathcal{E}_1$ about π_i^0 we obtain that most of the mass of Ω_1 must be at distance more than ℓ from π_i^0, i.e.

$$|\Omega_{e_i,\ell}|_g \geq \frac{|\Omega|_g}{2} - C\mathrm{def}(\Omega)\,,$$

where

$$\Omega_{e_i,\ell} := \bigcup_{s \in (\ell,+\infty)} \Omega \cap \pi_i^s\,.$$

Since $d(\Omega_{e_i,\ell}, \pi_j^0 \cap \Omega) \geq 2\ell$, we have that most of the mass of Ω_3 is at distance 2ℓ from π_j^0. This implies that the set

$$\mathcal{E}_2 := \bigcup_{s \in (-2\ell,\ell)} \Omega \cap \pi_i^s\,,$$

is such that $|\mathcal{E}_2|_g \leq 4C\mathrm{def}(\Omega)$. By iterating the argument above we find $m \in \mathbb{N}$ such that $m\ell > \mathrm{diam}\,(S)$ and

$$0 = |\Omega_{e_i,m\ell}|_g \geq \frac{|\Omega|_g}{2} - (m+1)C\mathrm{def}(\Omega)\,.$$

This leads to a contradiction provided that $C\mathrm{def}(\Omega)$ is small in terms of n, ρ and $|S|_g$. Therefore (1.94) holds true and the point $\mathcal{O}$ is well-defined also in the hyperbolic case.

Now we prove (1.91). Let $\mathcal{R}$ be the reflection about $\mathcal{O}$. Note that

$$\mathcal{R}(p) = \pi_{e_1} \circ \cdots \circ \pi_{e_n}(p)\,,$$

where we identify π_{e_i} with the reflection about the corresponding hyperplane. Next we work as in [56, Lemma 4.1]. Here we only sketch the argument referring to [56] for details.

Without lost of generalities, we may assume $\mathcal{O} \in \pi_{v,m_v-\mu}$, for some $\mu > 0$ and for $k \in \mathbb{N}$ we define

$$\mu_k = \big|\{p \in \Omega \cap \pi_{v,s}:\; m_v + (k-1)\mu < s < m_v + k\mu\}\big|_g\,,$$

here we use the notation introduced in Section 1.1. By construction μ_k is decreasing and, in particular,

$$\mu_k \leq \mu_0 := \big|\{\Omega \cap \pi_{v,s}:\; m_v - \mu < s < m_v\}\big|_g\,.$$

Moreover, μ_0 is bounded by $C\mathrm{def}(\Omega)$. Indeed, formula (1.92) yields

$$|\Omega \triangle \mathcal{R}(\Omega)|_g \leq C\mathrm{def}(\Omega)\,,$$

and then we obtain

$$|\Omega \cap \mathcal{R}(\Omega_v)|_g \geq |\Omega_v|_g - |\Omega \triangle \mathcal{R}(\Omega)|_g \geq \frac{|\Omega|_g}{2} - C\mathrm{def}(\Omega)\,.$$

Since

$$\mathcal{R}(\Omega_v) \subset \bigcup_{s<0} \pi_{v,m_v-s}\,,$$

we obtain that

$$\mu_0 := \big|\{\Omega \cap \pi_{v,s}:\; m_v - \mu < s < m_v\}\big|_g \leq C\mathrm{def}(\Omega)\,.$$

Therefore

$$\mu_k \leq C\mathrm{def}(\Omega) \tag{1.95}$$

for every k in $\mathbb{N}$. Again from (1.92) we get

$$\begin{aligned}\frac{|\Omega|_g}{2} - C\mathrm{def}(\Omega) &\leq |\Omega_v|_g\\ &\leq \sum_{k=0}^{k_0} \mu_k\\ &\leq k_0\mu_0\\ &\leq \frac{\mathrm{diam}\,(\Omega)}{m_v} C\mathrm{def}(\Omega)\,,\end{aligned}$$

where k_0 is the integer part of $\frac{\operatorname{diam}(\Omega)}{m_v}$. From Proposition A.1 we have

$$m_v \leq C \operatorname{def}(\Omega)\,,$$

where C depends only on n, ρ, $|S|_g$ and $\mathcal{K}$, as required. □

Proof of Theorem 1.5. Let $\mathcal{O}$ be as in Lemma 1.12 and define

$$r = \sup\{s > 0:\ B^d_s(\mathcal{O}) \subset \Omega\} \quad \text{and} \quad R = \inf\{s > 0:\ B^d_s(\mathcal{O}) \supset \Omega\}\,,$$

so that

$$B^d_r(\mathcal{O}) \subseteq \Omega \subseteq B^d_R(\mathcal{O})\,.$$

Let $p, q \in S$ be such that $d(p, \mathcal{O}) = r$ and $d(q, \mathcal{O}) = R$. We can assume that $p \neq q$ (otherwise $r = R$ and S is a round sphere). Let $v \in T_{\mathsf{o}}\mathbb{M}^n$ be the direction

$$v := \frac{1}{d(p,q)} \tau_p^{\mathsf{o}}(\exp_p^{-1}(q))$$

and π_v the critical hyperplane in the v-direction. We denote by γ the geodesic path passing through p and q and let s_p and s_q in $\mathbb{R}$ be such that

$$\gamma(s_p) = p \text{ and } \gamma(s_q) = q\,.$$

Let $z \in \pi_v$ be such that $d(z, \mathcal{O}) = d(\mathcal{O}, \pi_v)$. We have

$$p \in \pi_{v,s_p}\,, \quad q \in \pi_{v,s_q}\,, \quad s_q = s_p + t\,;$$

see Section 1.1 for the definition of π_{v,s_p} and π_{v,s_q}. We first show that $d(q, z) \leq d(p, z)$. Assume by contradiction that $d(q, z) > d(p, z)$. Since q and p belong to a geodesic orthogonal to the hyperplanes $\pi_{v,s}$ and $s_p < s_q$, then $s_q > m_v$. Since $\pi_v = \pi_{v,m_v}$ corresponds to the critical position of the method of the moving planes in the direction v, we have that $\gamma(s) \in \Omega$ for any $s \in (m_v, s_q)$. Since $s_p < s_q$ we have that $|s_p - m_v| \geq |s_q - m_v|$ and since γ is orthogonal to π_v we obtain $d(q, z) \leq d(p, z)$, which gives a contradiction. Since $d(q, z) \leq d(p, z)$ we have

$$r \geq R - d(\mathcal{O}, z) = R - d(\mathcal{O}, \pi_v)$$

and Lemma 1.12 implies (1.90). □

1.6 Proof of Theorem 1.1 and Corollary 1.1

We have all the ingredients to prove Theorem 1.1 and Corollary 1.1.

Proof of Theorem 1.1. Let $S = \partial\Omega$ be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{M}^n_+$ satisfying a uniform touching ball condition of radius ρ, where Ω is a relatively compact domain. Theorem 1.4 implies that there exit ϵ and C positive such that if

$$\operatorname{osc}(\mathsf{H}_S) \leq \varepsilon,$$

then

$$d(p, \Sigma \cup \Sigma^{\pi_v}) \leq C \operatorname{osc}(\mathsf{H}_S)\,,$$

for every $p \in \Omega$. □

Proof of Corollary 1.1. The proof consists in one more application of the method of the moving planes and it is in the spirit of [56, Theorems 1.2 and 1.5]. Let $B_r^d(\mathcal{O})$ and $B_R^d(\mathcal{O})$ be as in Theorem 1.5 and let $0 < t < r - C\mathrm{def}(\Omega)$. We aim at proving that for any $p \in S$, there exist two cones with vertex at p and of fixed aperture, one contained in Ω and one contained in the complementary of Ω. The first cone $C^-(p)$, is obtained by considering all the geodesic path connecting p to the boundary of $B_t^d(\mathcal{O})$ tangentially. The second cone $C^+(p)$ is the reflection of $\mathcal{C}^-(p)$ with respect to p. We show that $C^-(p)$ is contained in Ω and an analogous argument shows that $\mathcal{C}^+(p)$ is contained in the complementary of Ω. We assume, by contradiction, that $p \notin B_r^d(\mathcal{O})$ (otherwise the claim is trivial) and that there exists a point $q \in \mathcal{C}^-(p) \cap \partial B_t^d(\mathcal{O})$ such that the geodesic path γ connecting q to p is not contained in Ω. We apply the method of the moving planes in the direction v defined by

$$v := \frac{1}{d(p,q)} \tau_q^{\circ}(\exp_q^{-1}(p)) \,.$$

Since γ is not contained in Ω, the method of the moving planes stops before reaching q and one can prove that

$$d(\mathcal{O}, \pi_\omega) \geq r - t \,.$$

Since $0 < t < r - C\mathrm{def}(\Omega)$, from Lemma 1.12, we obtain

$$C\mathrm{def}(\Omega) < r - t \leq d(\mathcal{O}, \pi_\omega) \leq C\mathrm{def}(\Omega) \,,$$

which gives a contradiction. The argument above shows also that for any $p \in S$ the geodesic path connecting p to $\mathcal{O}$ is contained in Ω. This implies that there exists a C^2-regular map $\Psi : \partial B_r^d(\mathcal{O}) \to \mathbb{R}$ such that

$$F(p) = \exp_x(\Psi(p) N_p) \,,$$

defines a C^2-diffeomorphism from $B_r^d(\mathcal{O})$ to S. By choosing $t = r - \sqrt{C\mathrm{def}(\Omega)}$ we have that for any $p \in S$ there exists a uniform cone of opening $\pi - \sqrt{C\mathrm{def}(\Omega)}$ with vertex at p and axis on the geodesic connecting p to $\mathcal{O}$. This implies that Ψ is locally Lipschitz and the bound (1.5) on $\|\Psi\|_{C^1}$ follows (see also [56, Theorem 1.2]). □

Remark 1.2. We observe that if $\mathsf{H}_S = H$ is the mean curvature of $\partial\Omega$, then (1.5) can be improved and we can obtain the optimal linear bound

$$\|\Psi\|_{C^{1,\alpha}} \leq C\mathrm{osc}(\mathrm{H})$$

by using elliptic regularity. Indeed, let $\phi\colon U \to \partial B_r^d(\mathcal{O})$ be a local parametrization of $\partial B_r^d(\mathcal{O})$, where U is an open set of $\mathbb{R}^{n-1}$. From the proof of Corollary 1.1, $F \circ \phi$ gives a local parametrization of S. A standard computation yields that

$$L(\Psi \circ \phi) = H(F \circ \phi) - H_{\mathsf{B}_r}$$

where H_{B_r} is the mean curvature of $\partial \mathsf{B}_r$ and L is an elliptic operator which, thanks to the bounds on Ψ above, can be seen as a second order linear operator acting on $\Psi \circ \phi$. Then [112, Theorem 8.32] implies the bound on the $C^{1,\alpha}$-norm of Ψ, as required.

Chapter 2

Symmetry results for Serrin-type overdetermined problems

In this chapter we consider symmetry results for Serrin-type overdetermined problems, in particular in Section 2.1 we prove a Serrin-type result for domains inside the so-called model manifolds by using a generalization of Weinberger's proof; in Section 2.2 we prove a Serrin-type result for domains inside convex cones od the Euclidean space and in space forms.

2.1 A symmetry result in model manifolds

As already mentioned, in [204], J. Serrin proved the following celebrated result: if there exists a positive solution $u \in C^2(\bar{\Omega})$ to the following semilinear problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class C^2 and f is a function of class C^1, such that

$$\partial_\nu u = c \quad \text{on } \partial\Omega, \tag{2.2}$$

for some constant c, where ν denotes the outward unit normal to $\partial\Omega$. Then Ω must be a ball and u radially symmetric. We have also already mentioned that, in [215], H. Weinberger provided a simpler proof in the case $\Delta u = -1$ based on what are nowadays called P-function and using integral identities.

In literature there are generalizations of Serrin's result for domains in the so-called space forms, i.e. complete, simply connected Riemannian manifolds with constant sectional curvature. Thanks to the Killing-Hopf theorem (see [137, 143]) it is well-known that space forms are isometric to the Euclidean space $\mathbb{R}^n$, to the hyperbolic space $\mathbb{H}^n$ or to the sphere $\mathbb{S}^n$. In particular in [150] and [171] the moving planes method is used to prove the analogue of Serrin's result for the problem (2.1)-(2.2) for domains in $\mathbb{H}^n$ and in $\mathbb{S}^n_+$ (we mention that in $\mathbb{S}^n$ the theorem is not true, see e.g. [91]). In [196] we extend Weinberger's approach to a particular class of Riemannian manifolds: the already cited model manifolds, i.e. rotationally symmetric Riemannian manifolds. We recall the precise definition of model manifolds:

Definition 2.1. A n-dimensional Riemannian manifold $(\mathbb{M}^n_\sigma, g_{\mathbb{M}^n_\sigma})$ is called a model manifold if

$$\mathbb{M}^n_\sigma := \frac{[0,R)\times\mathbb{S}^{n-1}}{\sim} \quad \text{and} \quad g_{\mathbb{M}^n_\sigma} := dr\otimes dr + \sigma^2(r) g_{\mathbb{S}^{n-1}}\,,$$

where $R\in(0,+\infty]$, $\sim$ is the relation that identifies all the points of $\{0\}\times\mathbb{S}^{n-1}$ and $\sigma:[0,R)\to[0,+\infty)$ is a smooth function such that:

$$\sigma(r)>0\,,\ \text{for all } r>0\,, \quad \sigma^{(2k)}(0)=0\,,\ \text{for all } k=0,1,2,\ldots\,, \quad \sigma'(0)=1\,.$$

The point $o\in\mathbb{M}^n_\sigma$ corresponding to $r=0$ is called the *pole* of the model and σ is called the warping function.

The importance and the convenience of the model manifolds lies in the fact that their geometry and some natural differential operators (such as the Laplacian, see formula (2.8) below) have a particularly simple and explicit description. In particular we will use the following explicit expression of the Ricci curvature (see e.g. [184]): given $x\in\mathbb{M}^n_\sigma$ and $X\in\nabla r(x)^\perp$ in $T_x\mathbb{M}^n_\sigma$ a unit vector we have

$$\mathrm{Ric}_{\mathbb{M}^n_\sigma}(X,X) = (n-2)\frac{1-(\sigma')^2}{\sigma^2} - \frac{\sigma''}{\sigma},$$

and

$$\mathrm{Ric}_{\mathbb{M}^n_\sigma}(\nabla r,\nabla r) = -(n-1)\frac{\sigma''}{\sigma}.$$

With these preliminaries, the main Theorem of the first section is the following (see also Theorem I.C in the introduction to Part I)

Theorem 2.1. *Let $\Omega\subset\mathbb{M}^n_\sigma$ be a smooth domain with $o\in\Omega$. Assume that $\Omega\Subset B_{\tilde R}(o)$ where the ray $\tilde R>0$ is such that the following conditions on σ are satisfied on the interval $[0,\tilde R)$:*

(*a*) $\mathrm{Ric}_{\mathbb{M}^n_\sigma}\geq 0$, *i.e.* $\sigma''\leq 0$ *and* $(n-2)\big(1-(\sigma')^2\big)-\sigma\,\sigma''\geq 0$;

(*b*) $\sigma'>0$.

If Ω supports a solution $u\in C^2(\Omega)\cap C^1(\bar\Omega)$ of

$$\begin{cases}\Delta u = -1 & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\,,\end{cases} \tag{2.3}$$

such that, for some constant c,

$$\partial_\nu u = c \quad \text{on } \partial\Omega\,, \tag{2.4}$$

and u satisfies the following "compatibility" condition

$$\int_\Omega \frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}}u^2\geq 0 \tag{2.5}$$

then we have that Ω is a Euclidean ball of radius ρ centred in the pole o of the model and u has the specific form:

$$u(r) = \frac{1}{2n}(\rho^2 - r^2) \tag{2.6}$$

where $r(x) = dist(x, o)$.

We briefly analyse the hypothesis of the Theorem: the condition $\sigma' > 0$ appears in other articles on the subject (see e.g. [66]). The "compatibility" condition (2.5) describes a property of the solution in relation to the geometry of the model. It is automatically satisfied by any solution of (2.3)-(2.4) in the case of the Euclidean space and it can not be reduced to a simple condition on the model, like

$$(\sigma''\sigma^{n-1})' \geq 0.$$

Indeed, in this case, the three conditions are compatible only with the flat case: consider $f(r) := \sigma''(r)\sigma^{n-1}(r)$. Then $f(0) = 0$ and if $f'(r) \geq 0$, i.e. $f(r)$ is non-decreasing, so $f(r) \geq 0$ for $r > 0$. But $\sigma''(r) \leq 0$ according to (a), so we have that $\sigma''(r) = 0$. In this case the result is well known and is presented in Weinberger's article. Moreover, an analogue condition can be found in [4] where they consider a symmetry result for a overdetermined problem and they assume a "compatibility" condition as an integral on the boundary of the domain involving the solution and its gradient.

We mention that also in [189] and [68] a P-function approach is used to prove a symmetry result in space forms for the following equation

$$\Delta u + nKu = -1 \tag{2.7}$$

where $K = 0$ in $\mathbb{R}^n$, $K = 1$ in $\mathbb{S}^n_+$ and $K = -1$ in $\mathbb{H}^n$. The big difference between the conclusion of Theorem 2.1 and the results in [189] and [68] is that in Theorem 2.1 the domain is a Euclidean ball, while in [189] and [68] the domain is a geodesic ball.

2.1.1 Explicit computations towards the proof of Theorem 2.1

The Laplace-Beltrami operator Δ of $\mathbb{M}^n_\sigma$ acts on C^2-functions $u : \mathbb{M}^n_\sigma \to \mathbb{R}$ as follows:

$$\Delta u = \partial_r^2 u + (n-1)\frac{\sigma'}{\sigma}\partial_r u + \frac{1}{\sigma^2}\bar{\Delta} u = \frac{\partial_r(\sigma^{n-1}\partial_r u)}{\sigma^{n-1}} + \frac{1}{\sigma^2}\bar{\Delta} u\,, \tag{2.8}$$

where $\bar{\Delta}$ denotes the Laplacian on the standard sphere $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$. Using this expression we obtain:

Lemma 2.1. *The following general formula holds:*

$$\Delta(\sigma\,\partial_r u) = \sigma\,\partial_r\Delta u + 2\sigma'\,\Delta u + (2-n)\sigma''\,\partial_r u. \tag{2.9}$$

Proof. We compute

$$\begin{aligned}\sigma\partial_r(\Delta u) =&\sigma\left\{\partial_r^3 u+(n-1)\frac{\sigma''\sigma-(\sigma')^2}{\sigma^2}\partial_r u+(n-1)\frac{\sigma'}{\sigma}\partial_r^2 u-2\frac{\sigma'}{\sigma^3}\bar{\Delta}u+\frac{1}{\sigma^2}\partial_r(\bar{\Delta}u)\right\}\\ =&\sigma\partial_r^3 u+(n-2)\sigma''\partial_r u+\sigma''\partial_r u-2(n-1)\frac{(\sigma')^2}{\sigma}\partial_r u+(n-1)\frac{(\sigma')^2}{\sigma}\partial_r u\\ &+(n+1)\sigma'\partial_r^2 u-2\sigma'\partial_r^2 u-2\frac{\sigma'}{\sigma^2}\bar{\Delta}u+\frac{1}{\sigma}\partial_r(\bar{\Delta}u)+\frac{1}{\sigma^2}\bar{\Delta}(\sigma\partial_r u)\\ &-\frac{1}{\sigma^2}\bar{\Delta}(\sigma\partial_r u)\\ =&\Delta(\sigma\partial_r u)+(n-2)\sigma''\partial_r u-2\sigma'\Delta u+\frac{1}{\sigma}\partial_r(\bar{\Delta}u)-\frac{1}{\sigma^2}\bar{\Delta}(\sigma\partial_r u),\end{aligned}$$

i.e.

$$\Delta(\sigma\partial_r u)=\sigma\partial_r(\Delta u)+(2-n)\sigma''\partial_r u+2\sigma'\Delta u.$$

□

Now we focus on the solution u of (2.3)-(2.4) and we show the following

Lemma 2.2. *Let* Ω *and* u *as in Theorem 2.1. Then:*

$$(n+2)\int_\Omega u\,\sigma'=nc^2\int_\Omega\sigma'+\frac{(n-2)}{2}\int_\Omega\frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}}u^2. \tag{2.10}$$

Remark 2.1. In particular, if $\sigma(r)=r$ and, hence, $\mathbb{M}^n_\sigma=\mathbb{R}^m$, we obtain

$$(n+2)\int_\Omega u=nc^2|\Omega|, \tag{2.11}$$

which is (I.9).

Proof. First of all we observe that, in this setting, formula (2.9) becomes

$$\Delta(\sigma\,\partial_r u)=-2\sigma'+(2-n)\sigma''\,\partial_r u.$$

So by Green's Theorem

$$\begin{aligned}\int_\Omega[-2\sigma'\,u+(2-n)\sigma''\,\partial_r u\,u+\sigma\,\partial_r u]&=\int_\Omega[\Delta(\sigma\,\partial_r u)u-\sigma\,\partial_r u\,\Delta u]\\ &=\int_{\partial\Omega}[\partial_\nu(\sigma\,\partial_r u)u-\sigma\,\partial_r u\,\partial_\nu u]\\ &=-\int_{\partial\Omega}\sigma\,(\partial_\nu u)^2\partial_\nu r\\ &=-c^2\int_{\partial\Omega}\sigma\,\partial_\nu r\\ &=-c^2\int_\Omega\left[\sigma\,\Delta r+g_{\mathbb{M}^n_\sigma}(\nabla r,\nabla\sigma)\right]\\ &=-c^2\int_\Omega\left[\sigma\,(n-1)\frac{\sigma'}{\sigma}+\sigma'\right]\\ &=-c^2 n\int_\Omega\sigma',\end{aligned}$$

where we have used the fact that $u = 0$ on $\partial\Omega$ and that $\partial_\nu u = c$ on $\partial\Omega$.
Now note that

$$\begin{aligned}\int_\Omega \sigma \partial_r u &= \int_\Omega g_{\mathbb{M}^n_\sigma}(\nabla u, \nabla(\int_0^r \sigma(s)ds)) \\ &= -\int_\Omega u\Delta(\int_0^r \sigma(s)ds) \\ &= -n\int_\Omega u\sigma'.\end{aligned}$$

Using this and the previous computation we have

$$(n+2)\int_\Omega u\,\sigma' = nc^2\int_\Omega \sigma' + (2-n)\int_\Omega \sigma''\,u\partial_r u. \tag{2.12}$$

Finally we observe that

$$\begin{aligned}\int_\Omega \sigma''\,u\partial_r u &= \int_\Omega g_{\mathbb{M}^n_\sigma}(\nabla\sigma', \nabla(\frac{1}{2}u^2)) \\ &= -\frac{1}{2}\int_\Omega \Delta\sigma'\,u^2 \\ &= -\frac{1}{2}\int_\Omega \frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}}u^2,\end{aligned} \tag{2.13}$$

where the second and the third equations are obtained using the condition $u = 0$ on $\partial\Omega$ and the expression (2.8), respectively.

□

Remark 2.2. Observe that, by the Strong Maximum Principle, a solution u of (2.3) is positive in Ω. Moreover since $\partial_\nu u = c \neq 0$ on Ω we obtain that $|\nabla u| \neq 0$ on $\partial\Omega$ and the smooth hypersurface $\partial\Omega = \{u = 0\}$ has exterior normal given by

$$\nu = -\frac{\nabla u}{|\nabla u|}\,|_{\partial\Omega}\,.$$

This implies that

$$\partial_\nu u = -\,|\nabla u| \ \text{ on } \partial\Omega.$$

2.1.2 Proof of Theorem 2.1

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let u and Ω as in the statement of Theorem 2.1; by the Bochner formula (I.23) and the Cauchy-Schwarz inequality (I.24) we get

$$\begin{aligned}\Delta\left(|\nabla u|^2 + \frac{2}{n}u\right) &= 2|\nabla^2(u)|^2 + 2\mathrm{Ric}_{\mathbb{M}^n_\sigma}(\nabla u, \nabla u) + \frac{2}{n}\Delta u \\ &\geq 2\left(|\nabla^2 u|^2 + \frac{1}{n}\Delta u\right) \\ &= 2\left(|\nabla^2 u|^2 - \frac{1}{n}(\Delta u)^2\right) \\ &\geq 0 \quad \text{ on } \Omega,\end{aligned} \tag{2.14}$$

and the equality holds if and only if

$$\nabla^2(u) = \frac{\Delta u}{n} g_{\mathbb{M}^n_\sigma}$$

and

$$\mathrm{Ric}_{\mathbb{M}^n_\sigma}(\nabla u, \nabla u) = 0.$$

Since, according to Remark 2.2,

$$|\nabla u|^2 + \frac{2}{n} u = c^2 \text{ on } \partial\Omega, \tag{2.15}$$

we conclude from the Strong Maximum Principle that either

$$|\nabla u|^2 + \frac{2}{n} u < c^2 \text{ on } \Omega \tag{2.16}$$

or

$$|\nabla u|^2 + \frac{2}{n} u \equiv c^2 \text{ on } \Omega. \tag{2.17}$$

By contradiction assume that condition (2.16) is satisfied. According to (b) we can multiply both the members of (2.16) by σ' and integrate in order to obtain

$$n \int_\Omega |\nabla u|^2 \sigma' + 2 \int_\Omega u\, \sigma' < nc^2 \int_\Omega \sigma'. \tag{2.18}$$

Now we use the identity (2.10) to deal with the second term i.e.

$$2 \int_\Omega u\, \sigma' = nc^2 \int_\Omega \sigma' + \frac{(n-2)}{2} \int_\Omega \frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}} u^2 - n \int_\Omega u\sigma'. \tag{2.19}$$

Note that, by the divergence theorem,

$$n \int_\Omega \sigma' \mathrm{div}(u\nabla u) = -n \int_\Omega \sigma'' u\, \partial_r u. \tag{2.20}$$

Moreover,

$$n \int_\Omega \sigma' \mathrm{div}(u\nabla u) = n \int_\Omega \sigma' |\nabla u|^2 - n \int_\Omega \sigma' u.$$

So

$$n \int_\Omega \sigma' |\nabla u|^2 = n \int_\Omega \sigma' u - n \int_\Omega \sigma'' u\, \partial_r u. \tag{2.21}$$

Substituting (2.19) and (2.21) in (2.18) we obtain

$$\begin{aligned} nc^2 \int_\Omega \sigma' > &- n \int_\Omega \sigma'' u\, \partial_r u + n \int_\Omega \sigma' u + nc^2 \int_\Omega \sigma' \\ &+ \frac{(n-2)}{2} \int_\Omega \frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}} u^2 - n \int_\Omega u\sigma'. \end{aligned}$$

Lastly, we use the identity (2.13) to deduce

$$\frac{n}{2} \int_\Omega \frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}} u^2 + \frac{(n-2)}{2} \int_\Omega \frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}} u^2 < 0,$$

i.e.

$$-(n-1)\int_\Omega \frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}}u^2 > 0; \tag{2.22}$$

and this contradicts the "compatibility" condition (2.5).
Therefore (2.17) holds true and $|\nabla u|^2 + \frac{2}{n}u$ must be constant in Ω. Since its Laplacian then vanishes, we conclude from (2.14) that equality must hold in Cauchy-Schwarz inequality, i.e. we have proved that u is a solution of (recall that $\Delta u = -1$ in Ω)

$$\nabla^2(u) = -\frac{1}{n} g_{\mathbb{M}^n_\sigma} \text{ in } \Omega. \tag{2.23}$$

Now, let $\rho := dist(o, \partial\Omega)$ and take $B_\rho(o) \subset \Omega$. Since $\partial\Omega$ is compact, there exists $p \in \partial\Omega$ such that $p \in \partial\Omega \cap \partial B_\rho(o)$. In particular, since $u = 0$ on $\partial\Omega$, we have that

$$u(p) = 0.$$

If we prove that u is a radial function in $B_\rho(o)$ then

$$u = 0 \text{ on } \partial B_\rho(o).$$

On the other hand, by the Strong Maximum Principle,

$$u > 0 \text{ in } \Omega.$$

Therefore we can conclude that $\partial B_\rho(o) \cap \Omega = \emptyset$ and, hence, $\Omega = B_\rho(o)$.
So the key point is to prove that $u : B_\rho(o) \to \mathbb{R}$, solution of (2.23), is a radial function in $B_\rho(o)$. To this end, take $x \in B_\rho(o)$. Since $\mathbb{M}^n_\sigma$ is geodesically complete there exist a minimizing and normalized geodesic $\gamma \subset B_\rho(o)$ from o to x. Let $y(t) := u \circ \gamma(t)$ and note that, along γ, equation (2.23) implies

$$\begin{aligned}
y''(t) &= \frac{d^2}{dt^2}(u \circ \gamma)(t) \\
&= \frac{d}{dt} g_{\mathbb{M}^n_\sigma}(\nabla u(\gamma(t)), \dot\gamma(t)) \\
&= g_{\mathbb{M}^n_\sigma}(D_{\dot\gamma}\nabla u(\gamma(t)), \dot\gamma(t)) + g_{\mathbb{M}^n_\sigma}(\nabla u(\gamma(t)), D_{\dot\gamma}\dot\gamma(t)) \\
&= g_{\mathbb{M}^n_\sigma}((D_{\dot\gamma(t)}\nabla u)(\gamma(t)), \dot\gamma(t)) \\
&= \nabla^2(u) \mid_{\gamma(t)} (\dot\gamma(t), \dot\gamma(t)) \\
&= -\frac{1}{n}.
\end{aligned}$$

The solutions of $y''(t) = -\dfrac{1}{n}$ are given by

$$y(t) = -\frac{1}{2n}t^2 + \alpha t + \beta$$

where $\alpha, \beta \in \mathbb{R}$. Now taking $t = r(x)$ we get

$$u(x) = u \circ \gamma(r(x)) = y(r(x)) = -\frac{1}{2n}r(x)^2 + \alpha r(x) + \beta \tag{2.24}$$

which is radial. To determine the two constant in (2.24) we recall that u satisfies the following

$$\begin{cases} u(\rho) = 0 \\ u(r) > 0 \quad \text{for } 0 < r < \rho \end{cases}$$

i.e., using the explicit formula of u we obtain

$$\begin{cases} -\dfrac{1}{2n}\rho^2 + \alpha\rho + \beta = 0 \\ \\ -\dfrac{1}{2n}\left(\dfrac{\rho}{2}\right)^2 + \alpha\dfrac{\rho}{2} + \beta > 0 \quad \text{for } r = \dfrac{\rho}{2} \end{cases}$$

substituting the expression $\beta = \dfrac{1}{2n}\rho^2 - \alpha\rho$ in the second equation we get

$$\alpha < \frac{3}{4n}\rho.$$

But, since u must be a C^2-function we have that $\alpha = 0$; indeed, if we consider the Euclidean case where $r(x) = d(x, 0) = |x|$ the gradient of u becomes

$$\nabla u(x) = -\frac{1}{n}x + \alpha\frac{x}{|x|} \tag{2.25}$$

which is not a C^1 function in the origin (i.e. the pole of the Euclidean space) unless $\alpha = 0$. In a generic model the expression (2.25) holds in a system of normal coordinates in the pole. So the same conclusion holds and $\beta = \dfrac{1}{2n}\rho^2$; with this constants the function u becomes

$$u(r) = -\frac{1}{2n}r^2 + \frac{1}{2n}\rho^2$$

which is exactly the expression (2.6); observe that, since u is radial, $\partial_\nu u = u'(r)$ and the condition $\partial_\nu u = \mathit{constant}$ in $\partial\Omega = \partial B_\rho(o)$ is automatically satisfied.
Moreover we recall that if $f : \mathbb{M}^n_\sigma \to \mathbb{R}$ is a smooth radial function, then its Hessian takes the following expression

$$\nabla^2(f) = f'' dr \otimes dr + f'\sigma\sigma' g_{\mathbb{S}^{n-1}}. \tag{2.26}$$

Using this expression with the function u we get

$$\nabla^2(u) = -\frac{1}{n}dr \otimes dr - \frac{1}{n}r\sigma\sigma' g_{\mathbb{S}^{n-1}}, \tag{2.27}$$

and using this latter in (2.23) we obtain

$$-\frac{1}{n}dr \otimes dr - \frac{1}{n}r\sigma\sigma' g_{\mathbb{S}^{n-1}} = -\frac{1}{n}(dr \otimes dr + \sigma^2 g_{\mathbb{S}^{n-1}})$$

i.e.

$$-\frac{1}{n}r\sigma\sigma' g_{\mathbb{S}^{n-1}} = -\frac{1}{n}\sigma^2 g_{\mathbb{S}^{n-1}}.$$

It follows that $\sigma(r) = r$, so in the ball $B_\rho(o)$ not only the solution of (2.23) is radial but also the metric $g_{\mathbb{M}^m_\sigma}$ is the Euclidean metric. This implies that the ball $B_\rho(o)$ is a Euclidean ball and the claim follows.

□

Remark 2.3 (An alternative end of the proof). From the equality sign in the Bochner inequality (2.14) we get

$$\mathrm{Ric}_{\mathbb{M}^m_\sigma}(\nabla u, \nabla u) = 0. \tag{2.28}$$

From the explicit expression of u (formula (2.6)) we see that the only critical point is in $r = 0$, i.e. in the pole o of the model. So the condition on the Ricci curvature becomes

$$\mathrm{Ric}_{\mathbb{M}^n_\sigma}(\nabla r, \nabla r) = 0 \text{ in } B_\rho(o) \setminus \{o\}. \tag{2.29}$$

From the explicit expression of $\mathrm{Ric}_{\mathbb{M}^n_\sigma}(\nabla r, \nabla r)$ we get $\sigma'' = 0$ in $(0, \rho)$ and we conclude that $\sigma(r) = r$, i.e. $B_\rho(o)$ is an Euclidean ball.

Remark 2.4. In [197] by Ros we can find a similar spirit where, using the Reilly's formula, he obtained a generalization of Alexandrov theorem for compact hypersurfaces with constant higher order mean curvatures; in this article equation (2.23) is used to prove a Euclidean symmetry result on a generic compact Riemannian manifold of non-negative Ricci curvature with smooth boundary with mean curvature positive everywhere.

Remark 2.5. In this remark we provide an example that shows that if the "compatibility" condition (2.5) is not satisfied then we can not have Euclidean symmetry. According to the result of Kumaresan and Prajapat result [150] we know that if we take a domain $\Omega \subset \mathbb{S}^n$ such that $\bar{\Omega} \subset \mathbb{S}^n_+$ and there exist a solution u to the Serrin's problem (2.3)-(2.4) then Ω must be a geodesic ball and u must be radially symmetric. We know that the hemisphere is isometric to the model $\mathbb{M}^n_\sigma$ with $\sigma(r) = \sin(r) \mid_{[0,\pi/2]}$; so in this example conditions (a) and (b) of Theorem 2.1 are clearly satisfied and the "compatibility" condition (2.5) becomes:

$$\int_\Omega \frac{(\sigma''\sigma^{n-1})'}{\sigma^{n-1}} u^2 = \int_\Omega -n\cos(r)u^2(r),$$

which is negative due to the monotonicity of the integral and to the fact that the function $r \mapsto \cos(r)u^2(r)$ is positive in Ω.
In conclusion the "compatibility" condition is not satisfied and the symmetry result is not Euclidean since the ball Ω is a geodesic ball, i.e. the metric in this ball is the metric of the sphere.

2.2 Symmetry results in convex cones

Before stating the results that we are going to prove, we brefly recall some notations that we already introduced in the introduction to Part I. Let Σ be an open cone in $\mathbb{R}^n$ with vertex at the origin O, i.e.

$$\Sigma = \{tx \,:\, x \in \omega,\, t \in (0,+\infty)\}$$

for some open domain $\omega \subset \mathbb{S}^{n-1}$. Given an open cone Σ such that $\partial\Sigma \setminus \{O\}$ is smooth, we consider a bounded domain $\Omega \subset \Sigma$ and denote by Γ_0 its relative boundary, i.e.

$$\Gamma_0 = \partial\Omega \cap \Sigma\,,$$

and we set

$$\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}_0\,.$$

We assume that $\mathcal{H}_{n-1}(\Gamma_1) > 0$, $\mathcal{H}_{n-1}(\Gamma_0) > 0$ and that Γ_0 is a smooth $(n-1)$-dimensional manifold, while $\partial\Gamma_0 = \partial\Gamma_1 \subset \partial\Omega \setminus \{O\}$ is a smooth $(n-2)$-dimensional manifold. Following [180], such a domain Ω is called a *sector-like domain*. In the following, we shall write $\nu = \nu_x$ to denote the exterior unit normal to $\partial\Omega$ wherever is defined (that is for $x \in \Gamma_0 \cup \Gamma_1 \setminus \{O\}$).

As already mentioned, under the assumption that Σ is a convex cone, in [180] it is proved that if Ω is a sector-like domain and there exists a classical solution $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\})$ to

$$\begin{cases} \Delta u = -1 & \text{in } \Omega\,, \\ u = 0 \text{ and } \partial_\nu u = -c & \text{on } \Gamma_0\,, \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}\,, \end{cases} \tag{2.30}$$

and such that

$$u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)\,,$$

then

$$\Omega = B_R(x_0) \cap \Sigma$$

for some $x_0 \in \mathbb{R}^n$ and u is given by (I.11). Differently from the original paper of Serrin [204], the method of moving planes is not helpful (at least when applied in a standard way) and the rigidity result in [180] is proved by using two alternative approaches. One is based on integral identities and it is inspired by [40], the other one uses a P-function approach as in [215].

In this Section, we generalize the rigidity result for Serrin's problem in [180] in two directions. The former is by considering more general operators than the Laplacian in the Euclidean space, where the operators may be of degenerate type. Here, the generalization is not trivial due to the lack of regularity of the solution (the operator may be degenerate) as well as to other technical details which are not present in the linear case.

The latter is by considering an analogous problem in space forms, i.e. the hyperbolic space and the (hemi)sphere. The operator that we consider is linear and it is interesting since it has been shown that it is a helpful generalization of the torsion problem to space forms (see [68, 189, 190]).

Overdetermined problems for quasilinear and possible degenerate operators have attracted a lot of interest in the last decades, see for instance [92, 103, 106, 107, 192, 193]. As Fosdick and Serrin noticed in [204] and [100], Serrin's overdetermined problem for quasilinear elliptic operators is also interesting for possible applications to the study of steady rectilinear motion of viscous incompressible fluids and incompressible non-Newtonian fluids (see also [106]), and in the theory of torsion of a solid straight bar. Roughly speaking, a rigidity result as the one given by Serrin proves that *the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section* or that *when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section.* There are other possible applications for Serrin's type rigidity results, and we refer to [103, Introduction] for connections to capillarity theory, torsional creep, Born-Infeld theory and other applications to quantum-physics.

As explained in [180], the study of Serrin's overdetermined problem in convex cones is related to relative isoperimetric inequality and Alexandrov soap bubble theorem. In this manuscript we extend this study to non-Euclidean manifolds, in particular to space forms. The study of isoperimetric inequality and Alexandrov theorem in non-Euclidean manifolds has recently attracted a lot of interest in the geometric analysis community (see [41, 156, 189] and references therein). We believe that, by taking inspiration from our results and the ones in [156, 189], one can study Alexandrov theorem and relative isoperimetric inequalities for sector-like domains in more general Riemannian settings.

The study of rigidity problems in convex cones appears also in the context of critical points for Sobolev inequality (which in turns can be related to Yamabe problem), see [57, 160].

We also mention that the approach used in this paper originated from [40], which in turns has been later used for proving quantitative estimates for Serrin's overdetermined problem in [39]. As for the symmetry result, this approach is also useful when considering quantitative versions of Alexandrov soap bubble theorem, in particular to describe the appearance of bubbling [58].

More general operators in the Euclidean space. Let Ω be a sector like domain in $\mathbb{R}^n$ and let $f : [0, +\infty) \to [0, +\infty)$ be such that

$$f \in C^1([0,\infty)) \cap C^3((0,\infty)) \text{ with } f(0) = f'(0) = 0,\ f''(s) > 0 \text{ for } s > 0 \\ \text{and } \lim_{s\to+\infty} \frac{f(s)}{s} = +\infty\,. \tag{2.31}$$

We consider the following mixed boundary value problem

$$\begin{cases} L_f u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}, \end{cases} \tag{2.32}$$

where the operator L_f is given by

$$L_f u = \operatorname{div}\left(f'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right), \tag{2.33}$$

and the equation $L_f u = -1$ is understood in the sense of distributions

$$\int_\Omega \frac{f'(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, dx$$

for any
$$\varphi \in T(\Omega) := \{\varphi \in C^1(\Omega) \,:\, \varphi \equiv 0 \;\; on \; \Gamma_0\}.$$
Notice that the operator L_f may be of degenerate type.

We notice that the solution to $L_f u = -1$ in $B_R(x_0)$ (a ball of radius R centered at x_0) such that $u = 0$ on $\partial B_R(x_0)$ is radial and it is given by

$$u(x) = \int_{|x-x_0|}^{R} g'\left(\frac{s}{n}\right) ds\,, \tag{2.34}$$

where g denotes the Fenchel conjugate of f (see for instance [73] or [103]), i.e.

$$g = \sup\{st - f(s) \,:\, s \geq 0\}$$

(hence for us g' is the inverse function of f'). Our first main result is the following.

Theorem 2.2. *Let f satisfy* (2.31)*. Let Σ be a convex cone such that $\Sigma \setminus \{O\}$ is smooth and let Ω be a sector-like domain in Σ. If there exists a solution $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega)$ to* (2.32) *such that*

$$\partial_\nu u = -c \;\; on \;\; \Gamma_0 \tag{2.35}$$

for some constant c, and satisfying

$$\frac{f'(|\nabla u|)}{|\nabla u|} \nabla u \in W^{1,2}(\Omega, \mathbb{R}^n)\,, \tag{2.36}$$

then there exists $x_0 \in \mathbb{R}^n$ such that $\Omega = \Sigma \cap B_R(x_0)$ with $c = g'(|\Omega|/|\Gamma_0|)$, $R = n|\Omega|/|\Gamma_0|$. Moreover u is given by (2.34)*, where x_0 is the origin or, if $\partial\Sigma$ contains flat regions, it is a point on $\partial\Sigma$.*

When $L_f = \Delta$ (i.e. $f(t) = t^2/2$), Theorem 2.2 is essentially Theorem 1.1 in [180]. Condition (2.36) holds locally in Ω for a large class of elliptic operators, such as the mean curvature operator ($f(t) = \sqrt{1+t^2}$), and for the $p-$Laplace operator ($f(t) = t^p/p$, $p > 1$), see [14, Theorem 4.1] and [54, Theorem 2.1]. We stress that the validity of (2.36) up to the boundary is more subtle, since it depends strongly on how Γ_0 and Γ_1 intersect. Some global results may be obtained by following the approach in [54], where (2.36) is proved for Dirichlet or Neumann boundary value problems of p-Laplace type in domains which are convex or satisfying minimal regularity assumptions on the boundary.

We observe that the overdetermined problem (2.32) with the condition (2.35) can be seen as a partially overdetermined problem (see for instance [94] and [95]), since we impose both Dirichlet and Neumann conditions only on a part of the boundary, namely Γ_0, while a sole homogeneous Neumann boundary condition is assigned on Γ_1 (where, however, there is the strong assumption that it is contained in the boundary of a cone).

We notice that the proof of Theorem 2.2 still works when $\Gamma_1 = \emptyset$ (hence $\partial\Omega = \Gamma_0$). In this case we obtain the celebrated result of Serrin [204] for the operator L_f (see also [29, 40, 59, 92, 103, 106, 196, 215]). Moreover, the proof is also suitable to be adapted to the anisotropic counterpart of the overdetermined problem (2.32) and (2.35) by following our approach and in [29] (see also [55] and [213]). We also mention that rigidity theorems in cones are related to the study of relative isoperimetric and Sobolev inequalities in cones, and we refer to [180] for a more detailed discussion (see also [21, 46, 97, 122, 160, 159]).

Serrin's problem in cones in space forms. A space form is a complete simply-connected Riemannian manifold (M, g) with constant sectional curvature K. Up to homotheties we may assume $K = 0,1,-1$: the case $K = 0$ corresponds to the Euclidean space $\mathbb{R}^n$, $K = -1$ is the hyperbolic space $\mathbb{H}^n$ and $K = 1$ is the unitary sphere with the round metric $\mathbb{S}^n$. More precisely, in the case $K = 1$ we consider the hemisphere $\mathbb{S}^n_+$. These three models can be described as warped product spaces $M = I \times \mathbb{S}^{n-1}$ equipped with the rotationally symmetric metric

$$g = dr^2 + h(r)^2 \, g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the round metric on the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ and

- $h(r) = r$ in the Euclidean case ($K = 0$), with $I = [0, \infty)$;
- $h(r) = \sinh(r)$ in the hyperbolic case ($K = -1$), with $I = [0, \infty)$;
- $h(r) = \sin(r)$ in the spherical case ($K = 1$), with $I = [0, \pi/2)$ for $\mathbb{S}^n_+$.

By using the warping structure of the manifold, we denote by O the pole of the model and it is natural to define a *cone* Σ *with vertex at* $\{O\}$ as the set

$$\Sigma = \{tx \, : \, x \in \omega, \, t \in I\}$$

for some open domain $\omega \subset \mathbb{S}^{n-1}$. Moreover, we say that Σ is a *convex cone* if the second fundamental form II is nonnegative defined at every $p \in \partial\Sigma$.

Serrin's overdetermined problem for semilinear equations $\Delta u + f(u) = 0$ in space forms has been studied in [150] and [171] by using the method of moving planes. If one considers the corresponding problem for sector-like domains in space forms, the method of moving planes can not be used and one has to look for alternative approaches. As already mentioned, in the Euclidean space these approaches typically use integral identities and P-functions (see [40, 215]) and have the common feature that at a crucial step of the proof they use the fact that the radial solution attains the equality sign in a Cauchy-Schwartz inequality, which implies that the Hessian matrix $\nabla^2 u$ is proportional to the identity. Since the equivalent crucial step in space forms is to prove that the Hessian matrix of the solution is proportional to the metric, then the equation $\Delta u = -1$ is no more suitable (one can easily verify that in the radial case the Hessian matrix of the solution is not proportional to the metric) and a suitable equation to be considered is

$$\Delta u + nKu = -1 \tag{2.37}$$

as done in [68] and [189], [190]. It is clear that for $K = 0$, i.e. in the Euclidean case, the equation reduces to $\Delta u = -1$. For this reason, we believe that, in this setting, (2.37) is the natural generalization of the Euclidean $\Delta u = -1$ to space forms.

A Serrin's type rigidity result for (2.37) can be proved following Weinberger's approach by using a suitable P-function associated to (2.37) (see [68] and [190]). This approach is helpful for proving the following Serrin's type rigidity result for convex cones in space forms, which is the second main result of this section.

Theorem 2.3. *Let (M, g) be the Euclidean space, hyperbolic space or the hemisphere. Let $\Sigma \subset M$ be a convex cone such that $\Sigma \setminus \{O\}$ is smooth and let Ω be a sector-like domain in Σ. Assume that there exists a solution $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ to*

$$\begin{cases} \Delta u + nKu = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}, \end{cases} \tag{2.38}$$

such that

$$\partial_\nu u = -c \ \text{ on } \ \Gamma_0 \tag{2.39}$$

for some constant c. Then $\Omega = \Sigma \cap B_R(x_0)$ where $B_R(x_0)$ is a geodesic ball of radius R centered at x_0 and u is given by

$$u(x) = \frac{H(R) - H(d(x, x_0))}{n\dot{h}(R)},$$

with

$$H(r) = \int_0^r h(s)ds$$

and where $d(x, x_0)$ denotes the distance between x and x_0.

The Section is organized as follows: in Subection 2.2.1 we introduce some notation, we recall some basic facts about elementary symmetric function of a matrix and prove some preliminary result needed to prove Theorem 2.2. Theorems 2.2 and 2.3 are proved in Subsections 2.2.2 and 2.2.3, respectively.

2.2.1 Preliminary results for Theorem 2.2

In this section we collect some preliminary results which are needed in the proof of Theorem 2.2. Let f satisfy (2.31) and consider problem (2.32)

$$\begin{cases} L_f u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0 \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\}, \end{cases}$$

where the operator L_f is given by

$$L_f u = \operatorname{div}\left(f'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right).$$

Definition 2.2. $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\})$ is a solution to Problem (2.32) if

$$\int_\Omega \frac{f'(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, dx \tag{2.40}$$

for any

$$\varphi \in T(\Omega) := \{\varphi \in C^1(\Omega) \,:\, \varphi \equiv 0 \ \text{ on } \Gamma_0\}. \tag{2.41}$$

We observe some facts that will be useful in the following. Since the outward normal ν to Γ_0 is given by

$$\nu = -\frac{\nabla u}{|\nabla u|}|_{\Gamma_0}\,, \tag{2.42}$$

then (2.35) implies that

$$|\nabla u| = c \quad \text{on} \quad \Gamma_0. \tag{2.43}$$

Moreover we observe that the constant c in the statement is given by

$$c = g'\left(\frac{|\Omega|}{|\Gamma_0|}\right), \tag{2.44}$$

as it follows by integrating the equation $L_f u = -1$, by using the divergence theorem, formula (2.43) and the fact that $\partial_\nu u = 0$ on $\Gamma_1 \setminus \{O\}$. We also notice that

$$x \cdot \nu = 0 \quad \text{on} \quad \Gamma_1. \tag{2.45}$$

It will be useful to write the operator L_f as the trace of a matrix. Let $V : \mathbb{R}^n \to \mathbb{R}$ be given by

$$V(\xi) = f(|\xi|) \quad \text{for } \xi \in \mathbb{R}^n, \tag{2.46}$$

and notice that

$$\partial_{\xi_i} V(\xi) := V_{\xi_i}(\xi) = f'(|\xi|)\frac{\xi_i}{|\xi|} \quad \text{and}$$
$$\partial^2_{\xi_i\xi_j} V(\xi) := V_{\xi_i\xi_j}(\xi) = f''(|\xi|)\frac{\xi_i\xi_j}{|\xi|^2} - f'(|\xi|)\left(\frac{\xi_i\xi_j}{|\xi|^3} - \frac{\delta_{ij}}{|\xi|}\right). \tag{2.47}$$

Hence, by setting

$$W = (w_{ij})_{i,j=1,\ldots,N}$$

where

$$w_{ij}(x) = \partial_j V_{\xi_i}(\nabla u(x))\,, \tag{2.48}$$

we have

$$L_f(u) = \mathrm{Tr}\,(W). \tag{2.49}$$

Notice that at regular points, where $\nabla u \neq 0$, it holds that

$$W = \nabla^2_\xi V(\nabla u)\nabla^2 u\,. \tag{2.50}$$

Our approach to prove Theorem 2.2 is to write several integral identities and just one pointwise inequality, involving the matrix W. Writing the operator L_f as trace of W has the advantage that we can use the generalization of the so-called Newton's inequalities, as explained in the following subsection.

Elementary symmetric functions of a matrix

Given a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, for any $k = 1, \ldots, n$ we denote by $S_k(A)$ the sum of all the principal minors of A of order k. In particular, $S_1(A) = \mathrm{Tr}\,(A)$ is the trace of A, and $S_n(A) = \det(A)$ is the determinant of A. We consider the case $k = 2$. By setting

$$S^2_{ij}(A) = -a_{ji} + \delta_{ij}\mathrm{Tr}\,(A), \tag{2.51}$$

we can write

$$S_2(A) = \frac{1}{2}\sum_{i,j} S^2_{ij}(A)a_{ij} = \frac{1}{2}((\mathrm{Tr}\,(A)^2 - \mathrm{Tr}\,(A^2))\,. \tag{2.52}$$

The elementary symmetric functions of a symmetric matrix A satisfy the so called Newton's inequalities. In particular, S_1 and S_2 are related by (cfr. (I.13) in the Introduction to Part I)

$$S_2(A) \leq \frac{n-1}{2n}(S_1(A))^2\,. \tag{2.53}$$

When the matrix $A = W$, with W given by (2.50), we have

$$S^2_{ij}(W) = -V_{\xi_j\xi_k}(\nabla u)\partial^2_{ki}u + \delta_{ij}L_f u\,, \tag{2.54}$$

and $S^2_{ij}(W)$ is divergence free in the following (weak) sense

$$\frac{\partial}{\partial x_j}S^2_{ij}(W) = 0\,. \tag{2.55}$$

If V and u are sufficiently smooth, (2.55) was proved in [55, Equation (4.14)]. In Lemma 2.7 below we will prove (2.55) under weaker regularity assumptions on V and u by approximation (notice that (2.55) is implicitly written in (2.70), as follows from (2.52)).

We will need a generalization of (2.53) to not necessarily symmetric matrices, which is given by the following lemma.

Lemma 2.3 ([55], Lemma 3.2)**.** *Let B and C be symmetric matrices in $\mathbb{R}^{n\times n}$, and let B be positive semidefinite. Set $A = BC$. Then the following inequality holds:*

$$S_2(A) \leq \frac{n-1}{2n}\mathrm{Tr}\,(A)^2. \tag{2.56}$$

Moreover, if $\mathrm{Tr}\,(A) \neq 0$ *and equality holds in* (2.56)*, then*

$$A = \frac{\mathrm{Tr}\,(A)}{n}\mathrm{Id}\,,$$

and B is, in fact, positive definite.

Some properties of solutions to (2.32)

In this subsection we collect some properties of the solutions to (2.32). We assume that the solution is of class $C^1(\Omega)\cap W^{1,\infty}(\Omega)$. From standard regularity elliptic estimates one has that u is of class $C^{1,\alpha}(\Omega)$ and $C^{2,\alpha}$ where $\nabla u \neq 0$.

In the following two lemmas we show that $u > 0$ in $\Omega \cup \Gamma_1 \setminus \{O\}$ and we prove a Pohozaev-type identity.

Lemma 2.4. *Let f satisfy* (2.31) *and let u be a solution of* (2.32)*. Then*

$$u > 0 \quad in \quad \Omega \cup \Gamma_1 \setminus \{O\}. \tag{2.57}$$

Proof. We write $u = u^+ - u^-$ and use $\varphi = u^-$ as test function in (2.40):

$$0 \geq -\int_{\Omega \cap \{u<0\}} \frac{f'(|\nabla u|)}{|\nabla u|} |\nabla u^-|^2 \, dx = \int_{\Omega \cap \{u<0\}} u^- \, dx \geq 0 \, ,$$

which implies that $u \geq 0$ in Ω. Moreover, if one assumes that $u(x_0) = 0$ at some point $x_0 \in \Omega \cup \Gamma_1 \setminus \{O\}$, then $\nabla u(x_0) = 0$. Since $x_0 \in \Omega \cup \Gamma_1 \setminus \{O\}$ and $\Gamma_1 \setminus \{O\}$ is smooth, there exists a ball $B_r \subset \Omega$ such that $x_0 \in \partial B_r$. Let v be the solution of

$$\begin{cases} L_f v = -1 & \text{in } B_r, \\ v = 0 & \text{on } \partial B_r \, . \end{cases}$$

By comparison principle we have that $v \leq u$ in $\overline{B}_r$; from $\nabla u(x_0) = 0$ and since $\nabla v(x_0) \neq 0$ we get a contradiction. □

The following Pohozaev-type identity is a typical tool to prove symmetry results. In a similar setting as the one in this paper, a Pohozaev identity was proved in [106].

Lemma 2.5 (Pohozaev-type identity). *Let Ω be a sector-like domain and assume that f satisfies* (2.31). *Let $u \in C^1(\Omega \cup \Gamma_0 \cup \Gamma_1 \setminus \{O\}) \cap W^{1,\infty}(\Omega)$ be a solution to* (2.32). *Then the following integral identity*

$$\int_\Omega [(n+1)u - nf(|\nabla u|)] \, dx = \int_{\Gamma_0} [f'(|\nabla u|)|\nabla u| - f(|\nabla u|)] x \cdot \nu \, d\sigma \qquad (2.58)$$

holds.

Proof. We argue by approximation. We first approximate f with functions f_ε such that

$$f_\varepsilon \in C^\infty([0,\infty)) \text{ with } f_\varepsilon(0) = f'_\varepsilon(0) = 0, \; f''_\varepsilon(s) > 0 \text{ for } s \geq 0 \, , \qquad (2.59)$$

and

$$f_\varepsilon \to f \quad \text{and} \quad f'_\varepsilon \to f' \quad \text{uniformly on compact sets of } [0, +\infty). \qquad (2.60)$$

We notice that such an approximation exists as shown in [106, Section 3].

We recall that $V(\xi) = f(|\xi|)$ (see (2.46)) for $\xi \in \mathbb{R}^n$, and we define $V^\varepsilon : \mathbb{R}^n \to \mathbb{R}$ as

$$V^\varepsilon(\xi) := f_\varepsilon(|\xi|).$$

We notice that $\nabla_\xi V^\varepsilon$ and $\nabla_\xi V$ can be continuously extended to 0 at $\xi = 0$.

We approximate Ω by domains Ω_δ obtained by chopping off a δ-tubular neighborhood of $\partial\Gamma_0$ and a δ-neighborhood of O. For $j \in \mathbb{N}$, we consider $u^j_\delta \in C^\infty(\Omega_\delta) \cap C^1(\bar{\Omega}_\delta)$ such that

$$u^j_\delta \to u \text{ in } C^1(\overline{\Omega}_\delta),$$

as j goes to infinity (see for instance [43, Section 2.6]).
Since

$$\operatorname{div}\left(x \cdot \nabla u^j_\delta \nabla_\xi V^\varepsilon(\nabla u^j_\delta)\right) = x \cdot \nabla u^j_\delta \operatorname{div}\left(\nabla_\xi V^\varepsilon(\nabla u^j_\delta)\right) + \nabla(x \cdot \nabla u^j_\delta) \cdot \nabla_\xi V^\varepsilon(\nabla u^j_\delta)$$

and from

$$\begin{aligned}\nabla(x \cdot \nabla u_\delta^j) \cdot \nabla_\xi V^\varepsilon(\nabla u_\delta^j) =& \nabla u_\delta^j \cdot \nabla_\xi V^\varepsilon(\nabla u_\delta^j) + x\nabla^2(u_\delta^j) \cdot \nabla_\xi V^\varepsilon(\nabla u_\delta^j) \\ =& \mathrm{div}\left(u_\delta^j \nabla_\xi V^\varepsilon(\nabla u_\delta^j)\right) - u_\delta^j \mathrm{div}\left(\nabla_\xi V^\varepsilon(\nabla u_\delta^j)\right) \\ &+ \mathrm{div}(xV^\varepsilon(\nabla u_\delta^j)) - nV^\varepsilon(\nabla u_\delta^j)\,,\end{aligned}$$

we obtain

$$\mathrm{div}\left(\varphi_j \nabla_\xi V^\varepsilon(\nabla u_\delta^j) - xV^\varepsilon(\nabla u_\delta^j)\right) = \varphi_j \mathrm{div}\left(\nabla_\xi V^\varepsilon(\nabla u_\delta^j)\right) - nV^\varepsilon(\nabla u_\delta^j)\,, \tag{2.61}$$

where

$$\varphi_j(x) = x \cdot \nabla u_\delta^j(x) - u_\delta^j(x)\,.$$

Moreover, from the divergence theorem we have

$$\begin{aligned}\int_{\Omega_\delta} \nabla_\xi V^\varepsilon(\nabla u_\delta^j) \cdot \nabla\varphi_j \, dx =& -\int_{\Omega_\delta} \varphi_j \mathrm{div}\left(\nabla_\xi V^\varepsilon(\nabla u_\delta^j)\right) dx \\ &+ \int_{\partial\Omega_\delta} \varphi_j \nabla_\xi V^\varepsilon(\nabla u_\delta^j) \cdot \nu \, d\sigma\,.\end{aligned} \tag{2.62}$$

We are going to apply the divergence theorem in Ω_δ; to this end we set

$$\Gamma_{0,\delta} = \Gamma_0 \cap \partial\Omega_\delta\,, \quad \Gamma_{1,\delta} = \Gamma_1 \cap \partial\Omega_\delta \quad \text{and} \quad \Gamma_\delta = \partial\Omega_\delta \setminus (\Gamma_{0,\delta} \cup \Gamma_{1,\delta})\,.$$

From (2.62) and by integrating (2.61) in Ω_δ we obtain

$$\begin{aligned}\int_{\Omega_\delta} \nabla_\xi V^\varepsilon(\nabla u_\delta^j) \cdot \nabla\varphi_j \, dx =& -n\int_{\Omega_\delta} V^\varepsilon(\nabla u_\delta^j)\, dx \\ &- \int_{\Omega_\delta} \mathrm{div}\left(\varphi_j \nabla_\xi V^\varepsilon(\nabla u_\delta^j)\right) dx \\ &+ \int_{\Omega_\delta} \mathrm{div}\left(xV^\varepsilon(\nabla u_\delta^j)\right) dx\,,\end{aligned}$$

and from $x \cdot \nu = 0$ on $\Gamma_{1,\delta}$, we find

$$\begin{aligned}\int_{\Omega_\delta} \nabla_\xi V^\varepsilon(\nabla u_\delta^j) \cdot \nabla\varphi_j \, dx =& -n\int_{\Omega_\delta} V^\varepsilon(\nabla u_\delta^j)\, dx \\ &- \int_{\Gamma_{0,\delta} \cup \Gamma_{1,\delta}} \varphi_j \nabla_\xi V^\varepsilon(\nabla u_\delta^j) \cdot \nu \, d\sigma \\ &+ \int_{\Gamma_{0,\delta}} V^\varepsilon(\nabla u_\delta^j) x \cdot \nu \, d\sigma \\ &- \int_{\Gamma_\delta} [\varphi_j \nabla_\xi V^\varepsilon(\nabla u_\delta^j) - xV^\varepsilon(\nabla u_\delta^j)] \cdot \nu \, d\sigma\,.\end{aligned}$$

By taking the limit as $\varepsilon \to 0$ and then as $j \to \infty$, using that $\nabla u \cdot \nu = 0$ on $\Gamma_{1,\delta}$

(since $\partial_\nu u = 0$ on Γ_1), we obtain

$$\begin{aligned}\int_{\Omega_\delta} \nabla_\xi V(\nabla u)\cdot \nabla\varphi\, dx = &- n\int_{\Omega_\delta} V(\nabla u)\, dx \\ &- \int_{\Gamma_{0,\delta}} \varphi \nabla_\xi V(\nabla u)\cdot \nu\, d\sigma \\ &+ \int_{\Gamma_{0,\delta}} V(\nabla u) x\cdot \nu\, d\sigma \\ &- \int_{\Gamma_\delta} [\varphi\nabla_\xi V(\nabla u) - xV(\nabla u)]\cdot \nu\, d\sigma\end{aligned} \tag{2.63}$$

where we let

$$\varphi(x) = x\cdot \nabla u(x) - u(x)\,. \tag{2.64}$$

Now, we take the limit as $\delta \to 0$. Since $u \in W^{1,\infty}(\Omega)$ and $\mathcal{H}_{n-1}(\Gamma_\delta)$ goes to 0 as $\delta \to 0$, we have that the last term in (2.63) vanishes and we obtain

$$\begin{aligned}\int_{\Omega} \nabla_\xi V(\nabla u)\cdot \nabla\varphi\, dx = &-n\int_{\Omega} V(\nabla u)\, dx \\ &- \int_{\Gamma_0} \varphi\nabla_\xi V(\nabla u)\cdot\nu\, d\sigma + \int_{\Gamma_0} V(\nabla u)x\cdot\nu\, d\sigma\,,\end{aligned}$$

i.e. (in terms of f)

$$\begin{aligned}\int_\Omega \frac{f'(|\nabla u|)}{|\nabla u|}\nabla u\cdot\nabla\varphi\, dx = &-n\int_\Omega f(|\nabla u|)\, dx \\ &- \int_{\Gamma_0}\varphi\frac{f'(|\nabla u|)}{|\nabla u|}\partial_\nu u\, d\sigma + \int_{\Gamma_0} f(|\nabla u|)x\cdot\nu\, d\sigma.\end{aligned}$$

Since u satisfies (2.40), we get

$$\begin{aligned}\int_\Omega \varphi\, dx = &-n\int_\Omega f(|\nabla u|)\, dx \\ &- \int_{\Gamma_0}\varphi\frac{f'(|\nabla u|)}{|\nabla u|}\partial_\nu u\, d\sigma + \int_{\Gamma_0} f(|\nabla u|)x\cdot\nu\, d\sigma.\end{aligned} \tag{2.65}$$

From (2.64) and since $u = 0$ on Γ_0 and $\partial_\nu u = 0$ on Γ_1, we have

$$\int_\Omega \varphi\, dx = -(n+1)\int_\Omega u\, dx$$

and

$$\int_{\Gamma_0}\varphi\frac{f'(|\nabla u|)}{|\nabla u|}\partial_\nu u\, d\sigma = \int_{\Gamma_0} f'(|\nabla u|)|\nabla u| x\cdot\nu\, d\sigma\,, \tag{2.66}$$

where we used the expresion of the unit exterior normal on Γ_0 given by (2.42). From (2.66) and (2.65) we obtain

$$\begin{aligned}-(n+1)\int_\Omega u\, dx + n\int_\Omega f(|\nabla u|)\, dx = &-\int_{\Gamma_0} f'(|\nabla u|)|\nabla u| x\cdot\nu\, d\sigma \\ &+ \int_{\Gamma_0} f(|\nabla u|)x\cdot\nu\, d\sigma.\end{aligned}$$

which is (2.58), and the proof is complete. □

We conclude this subsection by exploiting the boundary condition $\partial_\nu u = 0$ on Γ_1. Before doing this, we need to recall some notation from differential geometry (see also [88, Appendix A]). We denote by D the standard Levi-Civita connection. Recall that, given an $(n-1)$-dimensional smooth orientable submanifold M of $\mathbb{R}^n$ we define the *tangential gradient* of a smooth function $f : M \to \mathbb{R}$ with respect to M as

$$\nabla^T f(x) = \nabla f(x) - \nu \cdot \nabla f(x) \nu$$

for $x \in M$, where ∇f denotes the usual gradient of f in $\mathbb{R}^n$ and ν is the outward unit normal at x to M. Moreover, we recall that the *second fundamental form* of M is the bilinear and symmetric form defined on $TM \times TM$ as

$$\mathrm{II}(v, w) = D\nu(v) w \cdot \nu\,;$$

a submanifold is called *convex* if the second fundamental form is non-negative definite.

Lemma 2.6. *Let u be the solution to* (2.32). *Then*

$$\nabla_\xi V(\nabla u) \cdot \nu = 0 \quad \text{on} \quad \Gamma_1\,, \tag{2.67}$$

and

$$\nabla(\nabla_\xi V(\nabla u) \cdot \nu) \cdot \nabla u = 0 \quad \text{on} \quad \Gamma_1\,.^1 \tag{2.68}$$

Proof. Since $\partial_\nu u = 0$ on Γ_1, we immediately find (2.67). By taking the tangential derivative in (2.67) we get

$$0 = \nabla^T(\nabla_\xi V(\nabla u) \cdot \nu) = \nabla(\nabla_\xi V(\nabla u) \cdot \nu) - \nu \cdot \nabla(\nabla_\xi V(\nabla u) \cdot \nu)\nu \quad \text{on} \quad \Gamma_1\,.$$

By taking the scalar product with ∇u we obtain

$$0 = \nabla(\nabla_\xi V(\nabla u) \cdot \nu) \cdot \nabla u - \nu \cdot \nabla(\nabla_\xi V(\nabla u) \cdot \nu)\partial_\nu u\,,$$

and since $\partial_\nu u = 0$ on Γ_1, we find (2.68). □

Integral Identities for S_2

In this Subsection we prove some integral inequalities involving $S_2(W)$ and the solution to problem (2.32).

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a sector-like domain and assume that f satisfies* (2.31). *Let $u \in W^{1,\infty}(\Omega)$ be a solution of* (2.32) *such that* (2.36) *holds. Then the following inequality*

$$2\int_\Omega S_2(W) u\, dx \geq -\int_\Omega S^2_{ij}(W) V_{\xi_i}(\nabla u)\partial_j u\, dx \tag{2.69}$$

holds. Moreover the equality sign holds in (2.69) *if and only if* $\mathrm{II}(\nabla^T u, \nabla^T u) = 0$ *on Γ_1.*

[1] We remark that (2.68) is understood to be zero at points where $\nabla u = 0$.

Proof. We split the proof in two steps.

Step 1: the following identity

$$2\int_\Omega S_2(W)\phi\,dx = -\int_\Omega S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j\phi\,dx \tag{2.70}$$

holds for every $\phi \in C^1_0(\Omega)$.

For $t>0$ we set $\Omega_t = \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>t\}$. Let $\phi\in C^1_0(\Omega)$ be a test function and let $\varepsilon_0>0$ be such that $\Omega_{\varepsilon_0}\subset\Omega$ and $supp(\phi)\subset\Omega_{\varepsilon_0}$. For $\varepsilon<\varepsilon_0$ sufficiently small, we set

$$a^i(x) = V_{\xi_i}(\nabla u(x)) \quad \text{for every } \ i=1,\dots,n,\ x\in\Omega\,.$$

From (2.36) we have that $a^i\in W^{1,2}(\Omega)$, $i=1,\dots,n$. With this notation, the elements $w_{ij}=\partial_j V_{\xi_i}(\nabla u)$ of the matrix W are given by

$$w_{ij} = \partial_j a^i\,.$$

Let $\{\rho_\varepsilon\}$ be a family of mollifiers and define $a^i_\varepsilon = a^i * \rho_\varepsilon$. Let $W^\varepsilon = (w^\varepsilon_{ij})_{i,j=1,\dots,n}$ where $w^\varepsilon_{ij}=\partial_j a^i_\varepsilon$, and notice that

$$a^i_\varepsilon \to a^i \quad \text{in } W^{1,2}(\Omega_{\varepsilon_0}) \quad \text{and} \quad W^\varepsilon\to W \quad \text{in } L^2(\Omega_{\varepsilon_0})\,,$$

as $\varepsilon\to 0$. Moreover, since

$$\mathrm{Tr}\ W^\varepsilon(x) = \int_{\mathbb{R}^n}\rho_\varepsilon(y)\mathrm{Tr}\ W(x-y)\,dy$$

and $\mathrm{Tr}\ W=-1$, we have that

$$\mathrm{Tr}\ W^\varepsilon(x) = -1 \tag{2.71}$$

for every $x\in\Omega_\varepsilon$.

Let $i,j=1,\dots,n$ be fixed. We have

$$\begin{aligned} w^\varepsilon_{ji}w^\varepsilon_{ij} &= \partial_j(a^i_\varepsilon\partial_i a^j_\varepsilon) - a^i_\varepsilon\partial_j\partial_i a^j_\varepsilon \\ &= \partial_j(a^i_\varepsilon\partial_i a^j_\varepsilon) - a^i_\varepsilon\partial_i\partial_j a^j_\varepsilon \\ &= \partial_j(a^i_\varepsilon\partial_i a^j_\varepsilon) - a^i_\varepsilon\partial_i w^\varepsilon_{jj}\,, \end{aligned}$$

for every $x\in\Omega_\varepsilon$, and by summing up over $j=1,\dots,n$, using (2.71) (hence $\partial_i\sum_j w^\varepsilon_{jj}=0$), we obtain

$$\begin{aligned} \sum_j w^\varepsilon_{ji}w^\varepsilon_{ij} &= \sum_j \partial_j(a^i_\varepsilon\partial_i a^j_\varepsilon) \\ &= w^\varepsilon_{ii}\mathrm{Tr}\,W^\varepsilon - \sum_j\partial_j(S^2_{ij}(W^\varepsilon)a^i_\varepsilon), \quad x\in\Omega_\varepsilon. \end{aligned}$$

By summing over $i=1,\dots,n$, from (2.52) and (2.55) we have

$$2S_2(W^\varepsilon) = \sum_{i,j}\partial_j(S^2_{ij}(W^\varepsilon)a^i_\varepsilon)\,, \quad x\in\Omega_\varepsilon. \tag{2.72}$$

Since

$$\int_{\Omega_{\varepsilon_0}} \partial_j (S^2_{ij}(W^\varepsilon) a^i_\varepsilon)\phi \, dx + \int_{\Omega_{\varepsilon_0}} S^2_{ij}(W^\varepsilon) a^i_\varepsilon \partial_j \phi \, dx$$
$$= \int_{\partial\Omega_{\varepsilon_0}} S^2_{ij}(W^\varepsilon) a^i_\varepsilon \nu_j \phi \, d\sigma = 0 \,,$$

from (2.72) and by letting ε to zero, we obtain (2.70).

Step 2. Let $\delta > 0$ and consider a cut-off funtion $\eta^\delta \in C^\infty_0(\Omega)$ such that $\eta^\delta = 1$ in Ω_δ and $|\nabla \eta^\delta| \leq \frac{C}{\delta}$ in $\Omega \setminus \Omega_\delta$ for some constant C not depending on δ. By taking $\phi(x) = u(x)\eta^\delta(x)$ for $x \in \Omega$ in (2.70) we obtain

$$2\int_\Omega S_2(W) u \eta^\delta \, dx = -\int_\Omega S^2_{ij}(W) V_{\xi_i}(\nabla u) \partial_j u \eta^\delta \, dx$$
$$- \int_\Omega S^2_{ij}(W) V_{\xi_i}(\nabla u) u \partial_j(\eta^\delta) \, dx \,. \quad (2.73)$$

From (I.34) we have that $W \in L^2(\Omega)$ and the dominated convergence theorem yields

$$2\int_\Omega S_2(W) u \eta^\delta \, dx \to 2 \int_\Omega S_2(W) u \, dx \quad (2.74)$$

as $\delta \to 0$. Analogously,

$$\int_\Omega S^2_{ij}(W) V_{\xi_i}(\nabla u) \partial_j u \eta^\delta \, dx \to \int_\Omega S^2_{ij}(W) V_{\xi_i}(\nabla u) \partial_j u \, dx \quad (2.75)$$

as $\delta \to 0$.

Now, we consider the last term in (2.73). We write Ω in the following way:

$$\Omega = A^\delta_0 \cup A^\delta_1 \,, \quad (2.76)$$

where

$$A^\delta_0 = \{x \in \Omega \, : \, \mathrm{dist}\,(x, \Gamma_0) \leq \delta\} \quad \text{and} \quad A^\delta_1 = \Omega \setminus A^\delta_0.$$

Since $u = 0$ on Γ_0, we get that

$$u(x) \leq ||u||_{W^{1,\infty}(\Omega)} \, \mathrm{dist}\,(x, \Gamma_0) \leq ||u||_{W^{1,\infty}(\Omega)} \, \delta$$

for every $x \in A^\delta_0$ and we obtain

$$\left| \int_{A^\delta_0} S^2_{ij}(W) V_{\xi_i}(\nabla u) u \partial_j(\eta^\delta) \, dx \right| \leq C_1 |A^\delta_0| \,,$$

where C_1 is a constant depending on $||u||_{W^{1,\infty}(\Omega)}$ and $\|W\|_{L^2(\Omega)}$, which implies that

$$\lim_{\delta \to 0} \int_{A^\delta_0} S^2_{ij}(W) V_{\xi_i}(\nabla u) u \partial_j(\eta^\delta) \, dx = 0 \,. \quad (2.77)$$

Now we show that

$$\lim_{\delta \to 0} \int_{A^\delta_1} S^2_{ij}(W(x)) V_{\xi_i}(\nabla u(x)) u(x) \partial_j(\eta^\delta)(x) \, dx \geq 0 \,. \quad (2.78)$$

By choosing δ small enough, a point $x \in A_1^\delta$ can be written in the following way: $x = \bar{x} + t\nu(\bar{x})$ where $\bar{x} = \bar{x}(x) \in \Gamma_1$ and $t = |x - \bar{x}|$ with $0 < t < \delta$. Moreover, by using a standard approximation argument, η^δ can be chosen in such a way that $\eta^\delta(x) = \frac{1}{\delta}\mathrm{dist}\,(x, \Gamma_1)$ for any $x \in A_1^\delta$, so that

$$\nabla\eta^\delta(x) = -\frac{1}{\delta}\nu(\bar{x})\,, \tag{2.79}$$

for every $x \in A_1^\delta \setminus \Omega_\delta$. For simplicity of notation we set $F = (F_1, \dots, F_n)$, where

$$F_j(x) = u(x)S^2_{ij}(W(x))V_{\xi_i}(\nabla u(x)) \tag{2.80}$$

for $j = 1, \dots, n$, and hence

$$\int_{A_1^\delta} S^2_{ij}(W)V_{\xi_i}(\nabla u)u\partial_j(\eta^\delta)\,dx = \int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx\,. \tag{2.81}$$

Since $\nabla\eta^\delta = 0$ in Ω_δ and $\nabla\eta^\delta(x) = -\frac{1}{\delta}\nu(\bar{x})$, for every $x \in A_1^\delta \setminus \Omega_\delta$, we have

$$\begin{aligned}\int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx &= -\frac{1}{\delta}\int_{A_1^\delta\setminus\Omega_\delta} F(x)\cdot\nu(\bar{x})\,dx\\ &= -\frac{1}{\delta}\int_0^\delta dt\int_{\{x\in A_1^\delta\,:\,\mathrm{dist}\,(x,\Gamma_1)=t\}} F(x)\cdot\nu(\bar{x})\,d\sigma\end{aligned}$$

where we used coarea formula. Since we are in a *small* δ-tubular neighborhood of (part of) Γ_1, we can parametrize $A_1^\delta \setminus \Omega_\delta$ over (part of) Γ_1 as from [112, Formula 14.98] we obtain that

$$\int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx = -\frac{1}{\delta}\int_0^\delta dt\int_{\Gamma_1} F(\bar{x}+t\nu(\bar{x}))\cdot\nu(\bar{x})|\det(Dg)|\,d\sigma\,. \tag{2.82}$$

We notice that, by using this notation, proving (2.78) is equivalent to prove

$$\lim_{\delta\to 0}\int_{A_1^\delta} F(x)\cdot\nabla\eta^\delta(x)\,dx \geq 0\,, \tag{2.83}$$

for $\delta > 0$ sufficiently small.

From (2.79), (2.80) and the definition of S^2_{ij} (2.51), we have

$$\begin{aligned}F(x)\cdot\nu(\bar{x}) &= -\,\delta_{ij}V_{\xi_i}(\nabla u(x))u(x)\nu_j(\bar{x}) - w_{ji}(x)V_{\xi_i}(\nabla u(x))u(x)\nu_j(\bar{x})\\ &= -\,\delta_{ij}V_{\xi_i}(\nabla u(x))u(x)\nu_j(\bar{x})\\ &\quad - u(x)\frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}w_{ji}(x)\partial_i u(x)\nu_j(\bar{x})\end{aligned}$$

for almost every $x = \bar{x} + t\nu(\bar{x}) \in A_1^\delta \setminus \Omega_\delta$, with $0 \leq t \leq \delta$. Since

$$w_{ij}\nu_i\partial_j u = \partial_j(V_{\xi_i}(\nabla u)\nu_i)\partial_j u - V_{\xi_i}(\nabla u)\partial_j\nu_i\partial_j u\,,$$

we have

$$\begin{aligned}&F(x)\cdot\nu(\bar{x}) = -u(x)\nabla_\xi V(\nabla u(x))\cdot\nu(\bar{x}) - u(x)\frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}\times\\ &\left\{\nabla(\nabla_\xi V(\nabla u(x))\cdot\nu(\bar{x}))\cdot\nabla u(x) - \frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}\partial_j\nu_i(\bar{x})\partial_j u(x)\partial_i u(x)\right\}\end{aligned} \tag{2.84}$$

for almost every $x = \bar{x} + t\nu(\bar{x}) \in A_1^\delta \setminus \Omega_\delta$, with $0 \le t \le \delta$. Let

$$\Gamma_1^{\delta,t} = \{x \in A_1^\delta \,:\, \mathrm{dist}\,(x, \Gamma_1) = t\}\,.$$

We notice that if $x \in \Gamma_1^{\delta,t}$ then $\nu(\bar{x}) = \nu^t(x)$ where $\nu^t(x)$ is the outward normal to $\Gamma_1^{\delta,t}$ at x. Hence

$$\partial_j \nu_i(\bar{x}) \partial_j u(x) \partial_i u(x) = \mathrm{II}_x^{\delta,t}(\nabla^T u(x), \nabla^T u(x)) \tag{2.85}$$

where $\mathrm{II}_x^{\delta,t}$ is the second fundamental form of $\Gamma_1^{\delta,t}$ at x. Since Σ is a convex cone then the second fundamental form of $\Gamma_1 \setminus \{O\}$ is non-negative definite. This implies that the second fundamental form of $\Gamma_1^{\delta,t}$ is non-negative definite for t sufficiently small (see e.g.[112, Appendix 14.6]) and hence

$$\partial_j \nu_i(\bar{x}) \partial_j u(x) \partial_i u(x) \ge 0\,. \tag{2.86}$$

From (2.86) and (2.84) we obtain

$$\begin{aligned} F(x) \cdot \nu(\bar{x}) \ge & -u(x) \nabla_\xi V(\nabla u(x)) \cdot \nu(\bar{x}) \\ & - u(x) \frac{f'(|\nabla u(x)|)}{|\nabla u(x)|} \nabla(\nabla_\xi V(\nabla u(x)) \cdot \nu(\bar{x})) \cdot \nabla u(x) \end{aligned} \tag{2.87}$$

for almost every $x = \bar{x} + t\nu(\bar{x}) \in A_1^\delta \setminus \Omega_\delta$, with $0 \le t \le \delta$. We use (2.87) in the right-hand side of (2.82) and, by taking the limit as $\delta \to 0$, we obtain

$$\begin{aligned} \lim_{\delta \to 0} \int_{A_1^\delta} & F(x) \cdot \nabla \eta^\delta(x)\, dx \\ & \ge - \int_{\Gamma_1} u \left(\nabla_\xi V(\nabla u) \cdot \nu + \frac{f'(|\nabla u|)}{|\nabla u|} \nabla(\nabla_\xi V(\nabla u) \cdot \nu) \cdot \nabla u \right) d\sigma\,. \end{aligned}$$

From (2.67) and (2.68) we find (2.83), and hence (2.78). From (2.73), (2.74), (2.75), (2.76), (2.77) and (2.78), we obtain (2.69).

□

2.2.2 Proof of Theorem 2.2

Proof of Theorem 2.2. We divide the proof in two steps. We first show that

$$W = -\frac{1}{n} \mathrm{Id} \qquad \text{a.e. in } \Omega. \tag{2.88}$$

and

$$\mathrm{II}(\nabla^T u, \nabla^T u) = 0 \quad \text{ on } \Gamma_1\,, \tag{2.89}$$

and then we exploit (2.88) in order to prove that u is indeed radial.

Step 1. Let g be the Fenchel conjugate of f (in our case $g' = (f')^{-1}$), using (2.47) we get that

$$\begin{aligned} \mathrm{div}\,(g(|\nabla_\xi V(\nabla u)|) \nabla_\xi V(\nabla u)) = & g'(|\nabla_\xi V(\nabla u)|) \nabla |\nabla_\xi V(\nabla u)| V_\xi(\nabla u) \\ & + g(|\nabla_\xi V(\nabla u)|) \mathrm{Tr}\,(W) \\ = & g'(f'(|\nabla u|)) \frac{V_{\xi_i}(\nabla u)}{|\nabla_\xi V(\nabla u)|} \partial_j (V_{\xi_i}(\nabla u)) V_{\xi_j}(\nabla u) \\ & + g(f'(|\nabla u|)) \mathrm{Tr}\,(W)\,, \end{aligned}$$

a.e. in Ω, where we used (2.47). Since $\partial_j V_{\xi_i}(\nabla u) = w_{ij}$ and $g' = (f')^{-1}$, we obtain

$$\mathrm{div}\,(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)) = \partial_i u w_{ij} V_{\xi_j}(\nabla u) + g(f'(|\nabla u|))\mathrm{Tr}\,(W)$$

a.e. in Ω, and using again (2.47) we find

$$\mathrm{div}\,(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)) = \frac{f'(|\nabla u|)}{|\nabla u|}\partial_i u w_{ij}\partial_j u + g(f'(|\nabla u|))\mathrm{Tr}\,(W)$$

a.e. in Ω. Since

$$g(f'(t)) = tf'(t) - f(t) \tag{2.90}$$

and $\mathrm{Tr}\,(W) = -1$, we obtain

$$\begin{aligned}\mathrm{div}\,(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)) = \frac{f'(|\nabla u|)}{|\nabla u|}&\partial_i u w_{ij}\partial_j u \\ &+ f(|\nabla u|) - |\nabla u| f'(|\nabla u|) \end{aligned} \tag{2.91}$$

a.e. in Ω.

Since (2.54), (2.47) and (2.49) yield

$$-S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j u = \frac{f'(|\nabla u|)}{|\nabla u|} w_{ji}\partial_i u \partial_j u + f'(|\nabla u|)|\nabla u|\,,$$

a.e. in Ω, from (2.91) we obtain

$$\begin{aligned}- S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j u = \mathrm{div}\,&(g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)) \\ &+ 2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\,,\end{aligned} \tag{2.92}$$

a.e. in Ω.

From Lemma 2.7 and (2.92), we obtain

$$\begin{aligned} 2\int_\Omega S_2(W)u\,dx &\geq -\int_\Omega S^2_{ij}(W)V_{\xi_i}(\nabla u)\partial_j u\,dx \\ &= \int_{\partial\Omega} g(|\nabla_\xi V(\nabla u)|)\nabla_\xi V(\nabla u)\cdot\nu\,d\sigma \\ &\quad + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx\,. \end{aligned}$$

From (2.47) and (2.67) we find

$$\begin{aligned} 2\int_\Omega S_2(W)u\,dx &\geq \int_{\Gamma_0} g(|\nabla_\xi V(\nabla u)|)\frac{f'(|\nabla u|)}{|\nabla u|}\partial_\nu u\,d\sigma \\ &\quad + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx\,. \end{aligned}$$

From (2.47) and (2.35) we have

$$2\int_\Omega S_2(W)u\,dx \geq -g(f'(c))f'(c)|\Gamma_0| + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx$$

and from (2.90) we obtain

$$
\begin{aligned}
2\int_\Omega S_2(W)u\,dx \geq & -[cf'(c)-f(c)]f'(c)|\Gamma_0| \\
& + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx\,.
\end{aligned} \tag{2.93}
$$

From the Pohozaev identity (2.58) and (2.43) we get

$$
(n+1)\int_\Omega u\,dx - n\int_\Omega f(|\nabla u|)\,dx = (f'(c)c - f(c))n|\Omega|\,;
$$

which we use in (2.93) to obtain

$$
\begin{aligned}
2\int_\Omega S_2(W)u\,dx \geq & -\frac{f'(c)|\Gamma_0|}{n|\Omega|}\int_\Omega [(n+1)u - nf(|\nabla u|)]\,dx \\
& + \int_\Omega \left[2f'(|\nabla u|)|\nabla u| - f(|\nabla u|)\right]\,dx\,.
\end{aligned} \tag{2.94}
$$

We notice that from (2.44) we have

$$
|\Omega| = f'(c)|\Gamma_0|,
$$

and from (2.94) we obtain

$$
2\int_\Omega S_2(W)u\,dx \geq -\frac{n+1}{n}\int_\Omega u\,dx + 2\int_\Omega f'(|\nabla u|)|\nabla u|\,dx\,. \tag{2.95}
$$

By using u as a test function in (2.40) we have that

$$
\int_\Omega u\,dx = \int_\Omega f'(|\nabla u|)|\nabla u|\,dx\,,
$$

and from (2.95) we find

$$
2\int_\Omega S_2(W)u\,dx \geq \frac{n-1}{n}\int_\Omega u\,dx\,. \tag{2.96}
$$

From (2.56) and using the fact that $\mathrm{Tr}\,(W) = L_f u = -1$, we get that also the reverse inequality

$$
\frac{n-1}{n}\int_\Omega u\,dx \geq \int_\Omega 2S_2(W)u\,dx \tag{2.97}
$$

holds. From (2.96) and (2.97), we conclude that the equality sign must hold in (2.96) and (2.97). From Lemma 2.3 we have that

$$
W = \frac{\mathrm{Tr}\,(W)}{n}\mathrm{Id}
$$

a.e. in Ω, and since $\mathrm{Tr}\,(W) = -1$ we obtain (2.88). Moreover, Lemma 2.7 yields (2.89).

Step 2: u is a radial function. From (2.88) we have that

$$
-\frac{1}{n}\delta_{ij} = \partial_j V_{\xi_i}(\nabla u(x))\,,
$$

for every $i, j = 1, \dots, n$, which implies that there exists $x_0 \in \mathbb{R}^n$ such that

$$\nabla_\xi V(\nabla u(x)) = -\frac{1}{n}(x - x_0),$$

i.e. according to (2.47)

$$\frac{f'(|\nabla u(x)|)}{|\nabla u(x)|}\nabla u(x) = -\frac{1}{n}(x - x_0)\,.$$

Hence

$$\nabla u(x) = -g'\left(\frac{|x - x_0|}{n}\right)\frac{x - x_0}{|x - x_0|} \quad \text{in } \Omega\,.$$

Since $u = 0$ on Γ_0, we obtain (2.34) and in particular u is radial with respect to x_0. Moreover, from (2.89) we find that x_0 must be the origin or, if $\partial\Sigma$ contains flat regions, a point on $\partial\Sigma$. □

2.2.3 Cones in space forms: proof of Theorem 2.3

The goal of this section is to give an easily readable proof of Theorem 2.3. More precisely we assume more regularity on the solution than the one actually assumed in Theorem 2.3 in order to give a concise and clear idea of the proof in this setting, and we omit the technical details which are, in fact, needed. A rigorous treatment of the argument described below can be done by adapting the (technical) details in Section 2.2.2 and in [180].

Before starting the proof we declare some notations we use in the statement of Theorem 2.3 and we are going to adopt in the following. Given a n-dimensional Riemannian manifold (M, g), we denote by D the Levi-Civita connection of g. Moreover given a C^2-map $u : M \to \mathbb{R}$, we denote by ∇u the gradient of u, i.e. the dual field of the differential of u with respect to g, and by $\nabla^2 u = Ddu$ the Hessian of u. We denote by Δ the Laplace-Betrami operator induced by g; Δu can be defined as the trace of $\nabla^2 u$ with respect to g. Given a vector field X on an oriented Riemannian manifold (M, g), we denote by $\mathrm{div}X$ the divergence of X with respect to g. If $\{e_k\}$ is a local orthonormal frame on (M, g), then

$$\mathrm{div}X = \sum_{k=1}^{n} g(D_{e_k}X, e_k)\,;$$

notice that, if u is a C^1-map and if X is a C^1 vector field on M, we have the following *integration by parts* formula

$$\int_\Omega g(\nabla u, \nu)\,dx = -\int_\Omega u\mathrm{div}X\,dx + \int_{\partial\Omega} ug(X, \nu)\,d\sigma\,,$$

where ν is the outward normal to $\partial\Omega$ and Ω is a bounded domain which is regular enough. Here and in the following, dx and $d\sigma$ denote the volume form of g and the induced $(n-1)$-dimensional Hausdorff measure, respectively.

Proof of Theorem 2.3. We divide the proof in four steps.

Step 1: the P-function. Let u be the solution to problem (2.38) and, as in [68], we consider the P-function defined by

$$P = |\nabla u|^2 + \frac{2}{n}u + Ku^2\,.$$

Following [68, Lemma 2.1], P is a subharmonic function and, since $u = 0$ on Γ_0 and from (2.43), we have that $P = c^2$ on Γ_0. Moreover,

$$\nabla P = 2\nabla^2 u \nabla u + \frac{2}{n}\nabla u + 2Ku\nabla u\,. \tag{2.98}$$

From the convexity assumption of the cone Σ, we have that

$$g(\nabla^2 u \nabla u, \nu) \leq 0\,. \tag{2.99}$$

Indeed, since $\partial_\nu u = 0$ on Γ_1 and by arguing as done for (2.68), we obtain that

$$0 = g(\nabla(\partial_\nu u), \nabla u) = g(\nabla^2 u\nabla u, \nu) + \mathrm{II}(\nabla u, \nabla u) \geq g(\nabla^2 u \nabla u, \nu) \quad \text{on} \quad \Gamma_1\,,$$

which is (2.99). From (2.98) and (2.99) we obtain

$$\partial_\nu P = 2g(\nabla^2 u\nabla u, \nu) + \frac{2}{n}\partial_\nu u + 2Ku\partial_\nu u \leq 0 \quad \text{ in } \Gamma_1 \setminus \{O\}\,.$$

Hence, the function P satisfies:

$$\begin{cases} \Delta P \geq 0 & \text{in } \Omega, \\ P = c^2 & \text{on } \Gamma_0 \\ \partial_\nu P \leq 0 & \text{on } \Gamma_1 \setminus \{O\}\,. \end{cases}$$

Moreover, again from [68, Lemma 2.1], we have that

$$\Delta P = 0 \quad \text{if and only if} \quad \nabla^2 u = \left(-\frac{1}{n} - Ku\right) g\,. \tag{2.100}$$

Step 2: we have

$$P \leq c^2 \quad \textit{in } \Omega. \tag{2.101}$$

Indeed, we multiply $\Delta P \geq 0$ by $(P - c^2)^+$ and by integrating by parts we obtain

$$0 \geq \int_{\Omega\cap\{P>c^2\}} |\nabla P|^2\, dx - \int_{\partial\Omega} (P - c^2)^+ \partial_\nu P\, d\sigma\,.$$

Since $P = c^2$ on Γ_0 and $\partial_\nu P \leq 0$ on Γ_1 we obtain that

$$0 \geq \int_{\Omega\cap\{P>c^2\}} |\nabla P|^2\, dx \geq 0$$

and hence $P \leq c^2$.

Step 3: $P = c^2$. By contradiction, we assume that $P < c^2$ in Ω. Since $\dot h > 0$, we have

$$c^2 \int_\Omega \dot h\, dx > \int_\Omega \dot h |\nabla u|^2\, dx + \frac{2}{n}\int_\Omega \dot h u\, dx + K\int_\Omega \dot h u^2\, dx\,.$$

Since

$$\operatorname{div}(\dot h u \nabla u) = \dot h|\nabla u|^2 + \dot h u \Delta u + \ddot h u \partial_r u$$

and

$$\ddot h = -Kh\,,$$

and from $u = 0$ on Γ_0 and $\partial_\nu u = 0$ on $\Gamma_1 \setminus \{O\}$, we have that

$$\begin{aligned} c^2 \int_\Omega \dot h \, dx &> -\int_\Omega \dot h u \Delta u \, dx - \int_\Omega \ddot h u \partial_r u \, dx + \frac{2}{n} \int_\Omega \dot h u \, dx + K \int_\Omega \dot h u^2 \, dx \\ &= (n+1) K \int_\Omega \dot h u^2 \, dx + \left(1 + \frac{2}{n}\right) \int_\Omega \dot h u \, dx + K \int_\Omega h u \partial_r u \, dx \, . \end{aligned}$$

From $\mathrm{div}(h\partial_r) = n\dot h$ we have

$$\mathrm{div}(u^2 h \partial_r) = n \dot h u^2 + 2 h u \partial_r u \, ,$$

and from $u = 0$ on Γ_0 and $\partial_\nu u = 0$ on $\Gamma_1 \setminus \{O\}$ we obtain

$$c^2 \int_\Omega \dot h \, dx > \left(1 + \frac{2}{n}\right) \left(\int_\Omega \dot h u \, dx - K \int_\Omega h u \partial_r u \, dx \right) . \tag{2.102}$$

Now we show that if u is a solution of (2.38) satisfying (2.39) then the equality sign holds in (2.102). Indeed, let $X = h\partial_r$ be the radial vector field and, by integrating formula (2.8) in [66], we get

$$\begin{aligned} -\frac{c^2}{n} \int_{\partial\Omega} g(X,\nu) \, d\sigma + \frac{n+2}{n} \int_\Omega \dot h u \, dx - (n-2) K \int_\Omega \dot h u^2 \, dx & \\ + \left(\frac{2}{n} - 3\right) K \int_\Omega u g(X, \nabla u) \, dx = 0 \, . & \end{aligned} \tag{2.103}$$

Since $\mathrm{div} X = n \dot h$ we obtain

$$\begin{aligned} c^2 \int_\Omega \dot h \, dx = \frac{n+2}{n} \int_\Omega \dot h u \, dx - (n-2) K \int_\Omega \dot h u^2 \, dx & \\ + \left(\frac{2}{n} - 3\right) K \int_\Omega u g(X, \nabla u) \, dx \, , & \end{aligned}$$

i.e.

$$c^2 \int_\Omega \dot h \, dx = \left(1 + \frac{2}{n}\right) \left(\int_\Omega \dot h u \, dx - K \int_\Omega h u \partial_r u \, dx \right) ,$$

where we used that $u = 0$ on Γ_0, $\partial_\nu u = 0$ on $\Gamma_1 \setminus \{O\}$ and $g(X,\nu) = 0$ on Γ_1. From (2.102) we find a contradiction and hence $P \equiv c^2$ in Ω.

Step 4: u is radial. Since P is constant, then $\Delta P = 0$ and from (2.100) we find that u satisfies the following Obata-type problem

$$\begin{cases} \nabla^2 u = \left(-\frac{1}{n} - K u\right) g & \text{in } \Omega \, , \\ u = 0 & \text{on } \Gamma_0 \, , \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \setminus \{O\} \, . \end{cases} \tag{2.104}$$

We notice that the maximum and the minimum of u can not be both achieved on Γ_0 since otherwise we would have that $u \equiv 0$. Hence, at least one between the maximum and the minimum of u is achieved at a point $p \in \Omega \cup \Gamma_1$. Let $\gamma : I \to M$ be a unit speed maximal geodesic satisfying $\gamma(0) = p$ and let $f(s) = u(\gamma(s))$. From the first equation of (2.104) it follows

$$f''(s) = -\frac{1}{n} - K f(s) \, .$$

Moreover, the definition of f and the fact that $\nabla u(p) = 0$ yield

$$f'(0) = 0 \quad \text{and} \quad f(0) = u(p),$$

and therefore

$$f(s) = \left(u(p) - \frac{1}{n}\right) H(s) - \frac{1}{n}.$$

This implies that u has the same expression along any geodesic strating from p, and hence u depends only on the distance from p. This means that $\Omega = \Sigma \cap B_R$ where B_R is a geodesic ball and u depends only on the distance from the center of B_R. □

Part II

Symmetry and validity results related to functional inequalities

Introduction to Part II

Part II is about symmetry results for critical anisotropic p-Laplace equations in convex cones (in Chapter 3) and functional inequalities on a particular class of Riemannian manifolds (in Chapter 4). This part collects the results obtained in the following papers: [57, 174].

In this Introduction we present the results that we are going to prove in the Chapters 3 and 4.

Chapter 3. In the last years, Sobolev inequality (8) has been studied for more general norms as well as in convex cones of $\mathbb{R}^n$ (see [15, 46, 97, 98, 159, 160]), where it takes the following form

$$\left(\int_\Sigma u^{p^*}dx\right)^{p/p^*} \leq S_{\Sigma,H}\int_\Sigma H(\nabla u)^p dx\,, \tag{II.1}$$

where H is a norm (i.e. $H:\mathbb{R}^n\to\mathbb{R}$ is convex, positively one-homogeneous and positive) and Σ as in (I.26) is an open convex cone in $\mathbb{R}^n$, i.e. $\Sigma=\mathbb{R}^k\times\mathcal{C}$ where $k\in\{0,\dots,n\}$ and $\mathcal{C}$ is a convex cone in $\mathbb{R}^{n-k}$ with only one vertex $\{\mathcal{O}\}$.

As for the usual Sobolev inequality in $\mathbb{R}^n$, the natural functional associated to (II.1) is the following:

$$\mathcal{J}^H(u)=\frac{\int_\Sigma H(\nabla u)^p dx}{\left(\int_\Sigma |u|^{p^*}dx\right)^{\frac{p}{p^*}}}\,. \tag{II.2}$$

We observe that (II.2) is the natural generalization of (11).

We are interested in critical points of the functional $\mathcal{J}$. Let u be a (positive) critical point of (II.2) and we compute the first variation of the functional. Let $\varepsilon>0$ and $\varphi\in C_c^\infty(\mathbb{R}^n)$ be a test function, then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathcal{J}^H(u+\varepsilon\varphi)=0\,.$$

This condition leads to the following boundary value problem:

$$\begin{cases}\operatorname{div}\left(a(\nabla u)\right)+u^{p^*-1}=0 & \text{in } \Sigma\\ u>0 & \text{in } \Sigma\\ a(\nabla u)\cdot\nu=0 & \text{on } \partial\Sigma\,,\end{cases} \tag{II.3}$$

where ν is the outward normal to $\partial\Sigma$,

$$a(\xi)=H^{p-1}(\xi)\nabla H(\xi)\qquad\forall\,\xi\in\mathbb{R}^n\,. \tag{II.4}$$

We will sometimes write
$$\Delta_p^H u = \mathrm{div}\,(a(\nabla u))\,,$$
where Δ_p^H is called the *Finsler p-Laplace* (or *anisotropic p-Laplace*) operator. It is clear that when we consider the case $\Sigma = \mathbb{R}^n$ no boundary conditions are given. In [57] we study problem (II.3) which is called *critical anisotropic p-Laplace equations in convex cones.* In [57] we prove the sharp version of (II.1) by suitably adapting the optimal transportation proof of the Sobolev inequality [72] to the case of cones (see Appendix B). It is interesting to observe that our proof applies also to the case of weighted Sobolev inequalities for the class of weights considered in [46], thus generalizing [46, Theorem 1.3] to the full range of exponents $p \in (1, n)$.

Hence, as shown in Appendix B, the extremals of (II.1) are of the form
$$\mathcal{U}^H_{\lambda,x_0}(x) := \left(\frac{\lambda^{\frac{1}{p-1}} n^{\frac{1}{p}} \left(\frac{n-p}{p-1}\right)^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + \tilde{H}_0(x-x_0)^{\frac{p}{p-1}}} \right)^{\frac{n-p}{p}} \tag{II.5}$$
for some $\lambda > 0$ (see also [12, 72, 160, 217] and the references therein), where
$$\tilde{H}_0(\zeta) := H_0(-\zeta) \qquad \forall\, \zeta \in \mathbb{R}^n\,, \tag{II.6}$$
and H_0 denotes the dual norm associated to H, namely
$$H_0(\zeta) := \sup_{H(\xi)=1} \zeta \cdot \xi \qquad \forall\, \zeta \in \mathbb{R}^n.$$
Moreover, if $\Sigma = \mathbb{R}^n$ then x_0 may be any point of $\mathbb{R}^n$; if $\Sigma = \mathbb{R}^k \times \mathcal{C}$ with $k \in \{1, \dots, n-1\}$ and $\mathcal{C}$ does not contain a line, then $x_0 \in \mathbb{R}^k \times \{\mathcal{O}\}$; otherwise, $x_0 = \mathcal{O}$.

The main result of Chapter 3 is to provide a complete classification result for critical anisotropic p-Laplace equations in convex cones with the hypothesis that the solution belongs to the space $\mathcal{D}^{1,p}(\Sigma)$ (as in Theorem C in the Introduction). More precisely, we consider the problem
$$\begin{cases} \mathrm{div}\,(a(\nabla u)) + u^{p^*-1} = 0 & \text{in } \Sigma \\ u > 0 & \text{in } \Sigma \\ a(\nabla u) \cdot \nu = 0 & \text{on } \partial\Sigma \\ u \in \mathcal{D}^{1,p}(\Sigma)\,, \end{cases} \tag{II.7}$$
the space $\mathcal{D}^{1,p}(\Sigma)$ is defined as in (9) (with $\mathbb{R}^n$ replaced by Σ). The main goal of this Chapter 3 is to classify the critical points for (II.2), i.e. the classification of the solutions to (II.3). In [57] we prove the following result (which is the main result of Chapter 3)

Theorem II.A: *let $n \geq 2$, $1 < p < n$, and let $\Sigma = \mathbb{R}^k \times \mathcal{C}$ be a convex cone, where $\mathcal{C}$ does not contain a line. Let H be a norm of $\mathbb{R}^n$ such that H^2 is of class $C^2(\mathbb{R}^n \setminus \{\mathcal{O}\})$ and it is uniformly convex and $C^{1,1}$ in $\mathbb{R}^n$, namely there exist constants $0 < \lambda \leq \Lambda$ such that*
$$\lambda \mathrm{Id} \leq H(\xi)\, D^2 H(\xi) + \nabla H(\xi) \otimes \nabla H(\xi) \leq \Lambda\, \mathrm{Id}\,, \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{\mathcal{O}\}\,. \tag{II.8}$$
Let u be a solution to (II.3)*. Then $u(x) = \mathcal{U}^H_{\lambda,x_0}(x)$ for some $\lambda > 0$ and $x_0 \in \overline{\Sigma}$, where $\mathcal{U}^H_{\lambda,x_0}$ is given by* (II.5)*. Moreover,*

(*i*) *if* $k = n$ *then* $\Sigma = \mathbb{R}^n$ *and* x_0 *may be a generic point in* $\mathbb{R}^n$*;*

(*ii*) *if* $k \in \{1, \dots, n-1\}$ *then* $x_0 \in \mathbb{R}^k \times \{\mathcal{O}\}$*;*

(*iii*) *if* $k = 0$ *then* $x_0 = \mathcal{O}$*.*

As already mentioned, case (i) in the previous Theorem has been already proved in [48, 76, 203, 224] when $\Sigma = \mathbb{R}^n$ and H is the Euclidean norm. In that case, thanks to the symmetry of the problem, the authors can apply the method of moving planes. In the Euclidean case and for $p = 2$, the classification of solutions in convex cones was proved in [160, Theorem 2.4] by using the Kelvin transform and inspired by [108] and [178]. Unfortunately, the Kelvin transform and the method of moving planes are not helpful neither for anisotropic problems nor inside cones for a general $p \in (1, n)$. In this case non-existence results generalizing the one in [108] to $p \in (1, n)$ are given in [206].

In Chapter 3 we provide a new approach to the characterization of solutions to critical $p-$Laplacian equations, which is based on integral identities rather than moving planes. This approach takes inspiration from [206] and also from [29, 30, 40, 215] where classical overdetermined problems for PDEs are considered (see also [63, 180] for analogous problems in convex cones).

The strategy of the proof can be explained as follows. First, using that $u \in \mathcal{D}^{1,p}(\Sigma)$ we show that u is bounded and satisfies certain decay estimates at infinity (in particular it behaves as the fundamental solution both from above and below), so that one has optimal upper bounds on $H(\nabla u)$ in terms of the fundamental solution. We notice that, differently from [203], we do not need asymptotic lower bounds on ∇u; instead, we use a Caccioppoli-type inequality to prove some asymptotic estimates on certain integrals involving higher order derivatives. Then we consider the auxiliary function $v = u^{-\frac{p}{n-p}}$. We find the elliptic equation satisfied by v and then, thanks to the asymptotic estimates on u, we show that v and ∇v satisfy explicit growth conditions at infinity. By using integral identities, the convexity of Σ, and some suitable inequalities, we are able to prove that $\nabla(a(\nabla v))$ is a multiple of the identity matrix, from which the symmetry result follows.

We mention that also in this case the hypothesis that $u > 0$ is fundamental, indeed in [71] the authors construct a nonradial sign-changing solution when $p = 2$ and H is the Euclidean norm.

Chapter 4. In Chapter 4 we focus on a particular class of Riemannian manifolds: the so-called Cartan-Hadamard manifolds M of dimension $n \geq 3$, namely complete, noncompact, simply connected Riemannian manifold with everywhere nonpositive sectional curvatures. On the one hand, it is well known that the Sobolev inequality (8) with $p = 2$ holds also on any Cartan-Hadamard manifold M; explicitly the following Sobolev inequality holds

$$||u||_{L^{2^*}(M)} \leq C||\nabla u||_{L^2(M)} \tag{II.9}$$

for every function $u \in C^1_c(M)$. On the other hand, if the sectional curvatures are everywhere bounded from above by a negative constant k, then in addition to (II.9) also the following Poincarè inequality

$$||u||_{L^2(M)} \leq C||\nabla u||_{L^2(M)} \tag{II.10}$$

holds for every function $u \in C_c^1(M)$. We mention that (II.10) is equivalent to the fact that the infimum of the spectrum of (minus) the Laplace-Beltrami operator on M is bounded from below by $k(N-1)^2/4$, i.e. Δ has a spectral gap (see the celebrated paper by McKeans [167]). For the sake of completness we mention that, in contrast with McKean's Theorem, in [162] (see also references therein) it was shown that, on any complete non-compact Riemannian manifold, the essential spectrum of the Laplace-Beltrami operator starts from zero as soon as the Ricci curvature vanishes at infinity. This generalizes [214], where the same thesis was established upon assuming an at-least-quadratic decrease of the negative curvatures.

So, the situation is the following: if the sectional curvatures are bounded from above by $k \equiv 0$ then M supports (II.9), and as soon as such bound becomes strictly negative M also supports (II.10). In other words, there is a "jump" of the L^p exponent in the left-hand side of the inequality $||f||_{L^p(M)} \leq C||\nabla u||_{L^2(M)}$ which depends on the curvatures. So the natural question is the following: what happens in between? That is, suppose that the sectional curvatures of M satisfy the following decay estimate

$$\mathrm{Sect}(x) \leq -\frac{K}{r(x)^\beta} \quad \text{for all } x \in M \setminus B_{R_0}, \tag{II.11}$$

for some $\beta \in (0,2]$ and $K, R_0 > 0$ where $r(x)$ denotes the geodesic distance from x to a fixed point $o \in M$ (the pole of the manifold). Then, what kind of inequalities does a Cartan-Hadamard manifold M with sectional curvature satisfying (II.11) support? In [174] we find nontrivial answers and we will present them in Chapter 4. First of all one has to distinguish between two cases: $\beta \in (0,2)$ and $\beta = 2$, the first case is called the sub-hyperbolic range and the second is called the quasi-Euclidean range (this terminology was introduced in [120]). Another distinction is between radial and nonradial functions. In [174] we focus on the following Sobolev-type inequalities:

$$||u||_{L^p(M)} \leq C_p||\nabla u||_{L^2(M)} \quad \text{for some } p \in (2, 2^*]. \tag{II.12}$$

In the case $\beta \in (0,2)$ we show that (II.12) holds in the radial setting for all $p \in (2, 2^*]$, for a positive constant $C_p = C(n,p,K,R_0,\beta)$; moreover the result is optimal with respect to the dependence on p. In case $\beta = 2$ we show that (II.12) starts to holds in the radial setting from a certain exponent $\tilde{2} \in (2, 2^*)$ that depends on n and K, which tends to 2^* as $K \to 0$ and to 2 as $K \to 1$; also in this case the result is optimal with respect to p. Finally, we prove that out of the radial setting all of the above results fail: it is enough to assume that (II.11) is satisfied with the reverse inequality to be able to construct a sequence of nonradial functions that make the constant C_p in (II.12) blow up for every $p < 2^*$. We mention that the case $\beta > 2$ is not interesting because it is essentially Euclidean, i.e. the sole inequality of the type of (II.12) that holds, even if restricted to radial functions, is the standard Sobolev one: this is an immediate consequence of our results. The techniques of proof that we exploit take advantage of two main ingredients: *one-dimensional weighted* functional inequalities and *Laplacian-comparison* theorems; the idea is to first study the radial inequalities on the already mentioned *model manifolds*.

The main motivation to study inequalities like (II.12) in Cartan-Hadamard manifolds is related to the asymptotic behavior of nonnegative solutions of diffusion equation. The link between the Sobolev inequality (or Gagliardo-Nirenberg and Nash inequalities in low dimensions) and a sharp decay estimate for the heat kernel is a well-studied topic, which goes back to some pioneering works between the 50s and the 80s: see the monograph [77]. In this regard, let us also mention the recent paper [116], where the behaviour of Faber-Krahn inequalities (which are strictly related to the Sobolev ones) on Riemannian manifolds under removal of compact subsets or gluing of noncompact manifolds is investigated, with applications to heat-kernel bounds.

In the last decades, several results that connect the validity of functional inequalities of Sobolev, log-Sobolev or Poincaré type with smoothing effects for *nonlinear* diffusion equations (mostly modeled on the porous medium equation) have been established: see [119] for weighted porous medium equations in the presence of weights and Poincaré inequalities, [101, 117] for optimal short and long-time smoothing estimates for porous medium equations (or the more general *filtration equation*) on Euclidean domains in the case of homogeneous Neumann problems, and the above-mentioned paper [118] for similar analyses focused on the consequences of the validity of families of Sobolev-type inequalities of the type of (II.12). Previous results in this direction, having already in mind the manifold case, can be found in [34]. As a general reference on smoothing effects, we also quote [222]. It is worth pointing out that in order to prove some of the main theorems of these papers, the authors often exploit a very powerful equivalence tool between *families* of Gagliardo-Nirenberg-Sobolev inequalities and one *single* inequality, which is due to [16].

Among recent works that take advantage of connections between functional inequalities and *fast diffusion* flows we refer to [86, 87], where the latter have been thoroughly exploited in order to prove or disprove the achievement of optimality by radial functions in Caffarelli-Kohn-Nirenberg inequalities. For a functional-analytic investigation of fast diffusions on Cartan-Hadamard manifolds, see also [35]. Finally, the monograph [17] is devoted to a wide-scope discussion on the interplay between analytic, geometric and probabilistic features of Markov diffusion semigroups, which involves functional inequalities related to those treated here and curvature-dimension conditions in more general metric frameworks. More explicitly, it is well-known that in $\mathbb{R}^n$ there is a close connection between the Sobolev inequality (II.9) and the heat equation:

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^n \times \mathbb{R}_+ \\ u = u_0 & \text{in } \mathbb{R}^n \times \{0\}. \end{cases} \tag{II.13}$$

Indeed, by classical results (mainly due to Nash, Varopoulos, Fabes and Stroock between 1958 and 1986), the Sobolev inequality is equivalent to the following smoothing estimate for solutions of (II.13):

$$||u(t)||_\infty \le C t^{-\frac{n}{2}} ||u_0||_{L^1(\mathbb{R}^n)} . \tag{II.14}$$

Similar results have recently been extended to suitable nonlinear diffusions like the porous medium equation:

$$\begin{cases} \partial_t u = \Delta(u^m) & \text{in } M \times \mathbb{R}_+ \\ u = u_0 & \text{in } M \times \{0\}. \end{cases} \tag{II.15}$$

for $m > 1$ and $u_0 \in L^1(\mathbb{R}^n)$. That is, if $M = \mathbb{R}^n$ then the Sobolev inequality is equivalent to the smoothing estimate (see Bonforte, Grillo, Muratori between

2006 and 2013)

$$||u(t)||_\infty \le Ct^{-\frac{n}{2+n(m-1)}} ||u_0||_{L^1(\mathbb{R}^n)}^{\frac{2}{2+n(m-1)}} . \tag{II.16}$$

When $M = \mathbb{H}^n$, the porous medium equation (II.15) had been also studied (see [223]). In particular in [118], as a sole consequence of the Sobolev inequality (II.9) and Poincaré inequality (II.10) which hold in $\mathbb{H}^n$, the following smoothing estimate is proved

$$||u(t)||_\infty \le C \left(\frac{\log(2 + t||u_0||_{L^1(\mathbb{R}^n)}^{m-1})}{t} \right)^{\frac{1}{m-1}} . \tag{II.17}$$

Moreover, the porous medium equation (II.15) had been studied also when M is a Cartan-Hadamard manifold. In particular, since the Sobolev inequality (II.9) holds on any Cartan-Hadamrd manifold then one can prove an estimate like (II.16). In addition, if the sectional curvatures are negatively bounded away from zero, then (II.10) holds true and (II.16) can be improved for large times by an estimate like (II.17). A natural question is: what happens in between, i.e. for vanishing curvatures? In [118] they prove that if the sectional curvature of M satisfies (II.11) then, an estimate (II.17) does hold provided M supports (II.12), i.e. a family of Sobolev-type inequalities. That's why we study these type of inequality on Cartan-Hadamard manifolds and in [174] we have a partial result.

Chapter 3

Symmetry results for critical anisotropic p-Laplacian equations in convex cones

As mentioned in the introduction to Part II, in this chapter we consider the following problem

$$\begin{cases} \operatorname{div}\left(a(\nabla u)\right) + u^{p^*-1} = 0 & \text{in } \Sigma \\ u > 0 & \text{in } \Sigma \\ a(\nabla u) \cdot \nu = 0 & \text{on } \partial\Sigma \\ u \in \mathcal{D}^{1,p}(\Sigma) \,, \end{cases} \tag{3.1}$$

where: H is a norm[1], Σ is a convex open cone in $\mathbb{R}^n$ given by

$$\Sigma = \{tx \,:\, x \in \omega,\, t \in (0, +\infty)\} \,, \tag{3.2}$$

for some open domain $\omega \subseteq \mathbb{S}^{n-1}$, ν is the outward normal to $\partial\Sigma$,

$$a(\xi) := H^{p-1}(\xi)\nabla H(\xi) \qquad \forall\, \xi \in \mathbb{R}^n, \tag{3.3}$$

and the space $\mathcal{D}^{1,p}(\Sigma)$ is defined as follows

$$\mathcal{D}^{1,p}(\Sigma) := \left\{ u \in L^{p^*}(\Sigma) \,:\, \int_\Sigma |\nabla u|^p \, dx < \infty \right\} \,, \tag{3.4}$$

where p^* is the usual critical Sobolev exponent, i.e.

$$p^* = \frac{np}{n-p} \,. \tag{3.5}$$

Since it will be useful, we mention that, in general (up to a change of coordinates) every convex cone Σ of $\mathbb{R}^n$ is of the form

$$\Sigma = \mathbb{R}^k \times \mathcal{C} \,,$$

[1]By abuse of notation, we say that $H : \mathbb{R}^n \to \mathbb{R}$ is a norm if H is convex, positively one-homogeneous (namely, $H(\ell\xi) = \ell H(\xi)$ for all $\ell > 0$), and $H(\xi) > 0$ for all $\xi \in \mathbb{S}^{n-1}$. Note that we do not require H to be symmetric, so it may happen that $H(\xi) \neq H(-\xi)$.

where $k \in \{0, \dots, n\}$ and $\mathcal{C} \subset \mathbb{R}^{n-k}$ is a convex cone containing no lines. Moreover we will sometimes write

$$\Delta_p^H u = \operatorname{div}\left(a(\nabla u)\right),$$

where Δ_p^H is called the *Finsler p-Laplacian* (or *anisotropic p-Laplacian*) operator.

As already observed if $u \in \mathcal{D}^{1,p}(\Sigma)$ is a positive critical point for the Sobolev functional

$$\mathcal{J}^H(u) = \frac{\int_\Sigma H(\nabla u)^p dx}{\left(\int_\Sigma |u|^{p^*} dx\right)^{\frac{p}{p^*}}}, \tag{3.6}$$

then u satisfies (3.1). The main goal of this chapter is to classify the critical points for (3.6), i.e. the classification of the solutions to (3.1). The functional (3.6) is related to the sharp anisotropic Sobolev inequality that we prove in Appendix B where we also show that the analogue of the Aubin-Talenti bubbles are the following functions:

$$\mathcal{U}^H_{\lambda,x_0}(x) := \left(\frac{\lambda^{\frac{1}{p-1}}\left(n^{\frac{1}{p}}\left(\frac{n-p}{p-1}\right)^{\frac{p-1}{p}}\right)}{\lambda^{\frac{p}{p-1}} + \tilde{H}_0(x_0 - x)^{\frac{p}{p-1}}}\right)^{\frac{n-p}{p}} \tag{3.7}$$

for some $\lambda > 0$, where

$$\tilde{H}_0(\zeta) := H_0(-\zeta) \qquad \forall\, \zeta \in \mathbb{R}^n, \tag{3.8}$$

and H_0 denotes the dual norm associated to H, namely

$$H_0(\zeta) := \sup_{H(\xi)=1} \zeta \cdot \xi \qquad \forall\, \zeta \in \mathbb{R}^n.$$

Moreover, if $\Sigma = \mathbb{R}^n$ then x_0 may be any point of $\mathbb{R}^n$; if $\Sigma = \mathbb{R}^k \times \mathcal{C}$ with $k \in \{1, \dots, n-1\}$ and $\mathcal{C}$ does not contain a line, then $x_0 \in \mathbb{R}^k \times \{\mathcal{O}\}$; otherwise, $x_0 = \mathcal{O}$ (from now on, $\mathcal{O}$ denotes the origin).

The main result that we prove in this chapter is the following Liouville-type Theorem (Liouville-type Theorem II.A in the introduction to part II)

Theorem 3.1. *Let $n \geq 2$, $1 < p < n$, and let $\Sigma = \mathbb{R}^k \times \mathcal{C}$ be a convex cone, where $\mathcal{C}$ does not contain a line. Let H be a norm of $\mathbb{R}^n$ such that H^2 is of class $C^2(\mathbb{R}^n \setminus \{\mathcal{O}\})$ and it is uniformly convex and $C^{1,1}$ in $\mathbb{R}^n$, namely there exist constants $0 < \lambda \leq \Lambda$ such that*

$$\lambda \mathrm{Id} \leq H(\xi)\, D^2 H(\xi) + \nabla H(\xi) \otimes \nabla H(\xi) \leq \Lambda\, \mathrm{Id} \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{\mathcal{O}\} \tag{3.9}$$

(note that $D^2(H^2) = 2H\, D^2 H + 2\nabla H \otimes \nabla H$).

Let u be a solution to (3.1). Then $u(x) = \mathcal{U}^H_{\lambda,x_0}(x)$ for some $\lambda > 0$ and $x_0 \in \overline{\Sigma}$, where $\mathcal{U}^H_{\lambda,x_0}$ is given by (3.7). Moreover,

(i) if $k = n$ then $\Sigma = \mathbb{R}^n$ and x_0 may be a generic point in $\mathbb{R}^n$;

(ii) if $k \in \{1, \dots, n-1\}$ then $x_0 \in \mathbb{R}^k \times \{\mathcal{O}\}$;

(iii) if $k = 0$ *then* $x_0 = \mathcal{O}$.

The strategy of the proof can be explained as follows. First, using that $u \in \mathcal{D}^{1,p}(\Sigma)$ we show that u is bounded (see Subsection 3.1.1). Then, in Subsection 3.1.2 we prove that u satisfies certain decay estimates at infinity (in particular it behaves as the fundamental solution both from above and below), so that one has optimal upper bounds on $H(\nabla u)$ in terms of the fundamental solution. We notice that, differently from [203], we do not need asymptotic lower bounds on ∇u; instead, we use a Caccioppoli-type inequality to prove some asymptotic estimates on certain integrals involving higher order derivatives (see Subsection 3.1.3).

Then, in Section 3.2 we consider the auxiliary function $v = u^{-\frac{p}{n-p}}$. We find the elliptic equation satisfied by v and then, thanks to the asymptotic estimates on u, we show that v and ∇v satisfy explicit growth conditions at infinity. By using integral identities, the convexity of Σ, and some suitable inequalities, we are able to prove that $\nabla a(\nabla v)$, recall that a is given by (3.3), is a multiple of the identity matrix; from this fact the symmetry result follows.

In Appendix B we prove, via the optimal transport approach introduced by [72], the sharp version of the following anisotropic Sobolev inequality in convex cones

$$\left(\int_\Sigma u^{p^*}dx\right)^{p/p^*} \leq S_{\Sigma,H}\int_\Sigma H(\nabla u)^p dx\,, \tag{3.10}$$

for general norms and cones, and even in a weighted setting (for the general inequality, not the sharp one we refer to [46]).

Most of the chapter will focus on the case in which Σ is a convex cone with nonempty boundary. Indeed our approach perfectly works also when $\Sigma = \mathbb{R}^n$. However, since the whole space case is simpler to be proven, we prefer to focus the exposition to the case when Σ has boundary (the case when $\Sigma = \mathbb{R}^n$ can be found in Appendix C).

3.1 Preliminary results

In this section we collect some results that are well established when $\Sigma = \mathbb{R}^n$ and H is the Euclidean norm. Since we are dealing with problem (3.1) and some modifications are needed, we report here their counterpart when Σ is a convex cone and H a general norm, and provide a sketch of the proofs emphasizing the main differences.

In the whole chapter we denote by $B_r(x)$ the usual Euclidean ball, and by B_r the ball $B_r(\mathcal{O})$ centered at the origin.

3.1.1 Boundeness of solutions

In the following lemma we prove that solutions to (3.1) are bounded. The result holds for more general Neumann problems, in particular for problems with a differential operator modelled on the p-Laplace operator.

Lemma 3.1. *Let $\Sigma \subseteq \mathbb{R}^n$ be a convex cone as in (3.2) and let $u \in \mathcal{D}^{1,p}(\Sigma)$ be a solution to*

$$\begin{cases} \operatorname{div}(a(\nabla u)) + u^{p^*-1} = 0 & \text{in } \Sigma \\ u > 0 & \text{in } \Sigma \\ a(\nabla u) \cdot \nu = 0 & \text{on } \partial\Sigma\,, \end{cases} \tag{3.11}$$

where $a : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field such that the following holds: there exist $\alpha > 0$ and $0 \leq s \leq 1/2$ such that

$$|a(\xi)| \leq \alpha(|\xi|^2 + s^2)^{\frac{p-1}{2}} \quad \text{and} \quad \xi \cdot a(\xi) \geq \frac{1}{\alpha}\int_0^1 \left(t^2|\xi|^2 + s^2\right)^{\frac{p-2}{2}} |\xi|^2 \, dt\,, \tag{3.12}$$

for every $\xi \in \mathbb{R}^n$. Then there exists $\delta > 0$ with the following property: let $\rho > 0$ be such that

$$||u||_{L^{p^*}(B_\rho(x_0))} \leq \delta \qquad \forall\, x_0 \in \mathbb{R}^n.$$

Then

$$||u||_{L^\infty(\Sigma \cap B_{R/2}(x_0))} \leq CR^{-\frac{n}{p}}||u||_{L^p(\Sigma\cap B_R(x_0))} \qquad \forall\, R \leq \rho,$$

where C depends only on n, α, p and the Sobolev constant of Σ.

Proof. We closely follow [183, Theorem E.0.20] and [205, Theorem 1] and we only give a sketch of the proof. We first prove that $u \in L^{qp^*}_{\text{loc}}(\overline{\Sigma})$ for any $q < p^*/p$. Given $l > 0$ and $1 < q < \frac{p^*}{p}$, we define

$$F(u) = \begin{cases} u^q & \text{if } u \leq l \\ ql^{q-1}(u-l) + l^q & \text{if } u > l\,, \end{cases} \tag{3.13}$$

and

$$G(u) = \begin{cases} u^{(q-1)p+1} & \text{if } u \leq l \\ ((q-1)p+1)l^{(q-1)p}(u-l) + l^{(q-1)p+1} & \text{if } u > l\,. \end{cases}$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ and use

$$\xi = \eta^p G(u)$$

as a test-function in (3.11); then an integration by parts gives

$$\int_\Sigma a(\nabla u)\cdot\nabla(\eta^p G(u))\,dx = \int_\Sigma u^{p^*-1}\eta^p G(u)\,dx\,. \tag{3.14}$$

We aim at proving that

$$\begin{aligned} c\int_\Sigma \eta^p G'(u)|\nabla u|^p\,dx \leq& \int_\Sigma \eta^{p-1}G(u)|a(\nabla u)\cdot\nabla\eta|\,dx \\ &+ \int_\Sigma u^{p^*-1}\eta^p G(u)\,dx \\ &+ s^p\int_\Sigma \eta^p G'(u)\,dx \end{aligned} \tag{3.15}$$

holds for $0 \leq s \leq 1/2$. We distinguish between the cases $1 < p < 2$ and $2 \leq p < n$.

If $p \geq 2$, then (3.12) implies

$$\xi \cdot a(\xi) \geq \frac{1}{\alpha}|\xi|^p\,,$$

and from (3.14) we get

$$\frac{1}{\alpha}\int_\Sigma \eta^p G'(u)|\nabla u|^p\,dx \leq p\int_\Sigma \eta^{p-1}G(u)|a(\nabla u)\cdot\nabla\eta|\,dx + \int_\Sigma u^{p^*-1}\eta^p G(u)\,dx\,,$$

which implies (3.15).

If $1 < p < 2$ then (3.15) is obtained by using a more careful argument. We claim that

$$\int_0^1 \left(t^2|\xi|^2 + s^2\right)^{\frac{p-2}{2}} |\xi|^2\,dt \geq \frac{1}{2}\left(|\xi|^p - s^p\right). \tag{3.16}$$

To prove this we consider two cases. If $s > |\xi|$ then the left-hand side of (3.16) is negative, and so the result is clearly true. Otherwise, if $s \leq |\xi|$ then

$$t^2|\xi|^2 + s^2 \leq 2|\xi|^2 \qquad \text{for } t \in [0,1],$$

and therefore

$$\int_0^1 \left(t^2|\xi|^2 + s^2\right)^{\frac{p-2}{2}} |\xi|^2\,dt \geq \int_0^1 \left(2|\xi|^2\right)^{\frac{p-2}{2}} |\xi|^2\,dt = 2^{\frac{p-2}{2}}|\xi|^p \geq \frac{1}{2}|\xi|^p,$$

that again implies (3.16).

Thanks to (3.14), (3.12), and (3.16), we obtain

$$\begin{aligned}\frac{1}{2\alpha}\int_\Sigma \eta^p G'(u)|\nabla u|^p\,dx \leq{}& p\int_\Sigma \eta^{p-1}G(u)|a(\nabla u)\cdot\nabla\eta|\,dx\\ &+ \int_\Sigma u^{p^*-1}\eta^p G(u)\,dx + \frac{s^p}{2}\int_\Sigma \eta^p G'(u)\,dx\,,\end{aligned}$$

and the proof of (3.15) is complete.

Note now that, by Young's inequality and (3.12), for any $\epsilon \in (0,1)$ we have

$$\begin{aligned}\eta^{p-1}|a(\nabla u)\cdot\nabla\eta| &\leq \epsilon^{\frac{p}{p-1}}u^{-1}|a(\nabla u)|^{\frac{p}{p-1}}\eta^p + \epsilon^{-p}u^{p-1}|\nabla\eta|^p\\ &\leq C_0\epsilon^{\frac{p}{p-1}}u^{-1}(|\nabla u|^p + s^p)\eta^p + \epsilon^{-p}u^{p-1}|\nabla\eta|^p,\end{aligned}$$

where C_0 depends only on α and p. Thanks to this inequality and recalling (3.15), since $G(u) \leq uG'(u)$ (note that G is convex and $G(0) = 0$), for any $\epsilon \in (0,1)$ we obtain

$$\begin{aligned}c\int_\Sigma \eta^p G'(u)|\nabla u|^p\,dx \leq{}& C_0\epsilon^{\frac{p}{p-1}}\int_\Sigma \eta^p G'(u)|\nabla u|^p\,dx + (C_0+1)s^p\int_\Sigma \eta^p G'(u)\,dx\\ &+ \epsilon^{-p}\int_\Sigma G(u)u^{p-1}|\nabla\eta|^p\,dx + \int_\Sigma u^{p^*-1}\eta^p G(u)\,dx\,.\end{aligned}$$

Hence, choosing ϵ small enough so that $C_0\epsilon^{\frac{p}{p-1}} = c/2$, we deduce that

$$\begin{aligned}c'\int_\Sigma \eta^p G'(u)|\nabla u|^p\,dx \leq{}& s^p\int_\Sigma \eta^p G'(u)\,dx + \int_\Sigma G(u)u^{p-1}|\nabla\eta|^p\,dx\\ &+ \int_\Sigma u^{p^*-1}\eta^p G(u)\,dx\,,\end{aligned} \tag{3.17}$$

where $c' > 0$ depends only on n, α, and p. Using now that $G'(u) \geq c[F']^p$ and that $u^{p-1}G(u) \leq C[F(u)]^p$, we obtain

$$\hat{c}\int_\Sigma |\nabla(\eta F(u))|^p\,dx \leq s^p\int_\Sigma \eta^p G'(u)\,dx + \int_\Sigma |\nabla\eta|^p F^p(u)\,dx \\ + \int_\Sigma \eta^p u^{p^*-p}F^p(u)\,dx\,.$$

Hence, thanks to the Sobolev inequality (3.10) we get

$$\bar{c}\left(\int_\Sigma F^{p^*}(u)\eta^{p^*}\,dx\right)^{\frac{p}{p^*}} \leq s^p\int_\Sigma \eta^p G'(u)\,dx + \int_\Sigma |\nabla\eta|^p F^p(u)\,dx \\ + \int_\Sigma \eta^p u^{p^*-p}F^p(u)\,dx\,, \quad (3.18)$$

where $\bar{c} > 0$ depends only on n, α, p and the Sobolev constant for Σ.

Now, choose $\delta = (\bar{c}/2)^{1/(p^*-p)}$, so that for any $R \leq \rho$ it holds

$$||u||^{p^*-p}_{L^{p^*}(B_R(x_0))} \leq \frac{\bar{c}}{2} \qquad \forall\, x_0 \in \mathbb{R}^n.$$

Then, if we choose η such that $\mathrm{supp}(\eta) \subset B_R(x_0)$, it follows from Holder's inequality that we can reabsorb the last term in (3.18), and we get

$$\frac{\bar{c}}{2}\left(\int_\Sigma F^{p^*}(u)\eta^{p^*}\,dx\right)^{\frac{p}{p^*}} \leq s^p\int_{\Sigma\cap B_R(x_0)} \eta^p G'(u)\,dx \\ + \int_{\Sigma\cap B_R(x_0)} |\nabla\eta|^p F^p(u)\,dx\,.$$

Hence, taking the limit as $l \to \infty$ in the definition of F and G, by monotone convergence we conclude

$$\frac{\bar{c}}{2}\left(\int_{\Sigma\cap B_R(x_0)} \eta^{p^*}u^{qp^*}\,dx\right)^{\frac{p}{p^*}} \leq s^p\int_{\Sigma\cap B_R(x_0)} u^{(q-1)p}\,dx \\ + ||\nabla\eta||^p_\infty\int_{\Sigma\cap B_R(x_0)} u^{qp}\,dx\,.$$

Since $qp < p^*$ it follows that the right hand side is finite, hence by the inequality above and the arbitrariness of x_0 we conclude that $u \in L^{qp^*}_{\mathrm{loc}}(\overline{\Sigma})$.

Thanks to this information, we can rewrite the equation satisfied by u as follows:

$$-\mathrm{div}\,(a(\nabla u)) = f(x)u^{p-1} + g(x)$$

where

$$f(x) = \begin{cases} 0 & \text{if } u < 1 \\ u^{p^*-p} & \text{if } u \geq 1\,, \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if } u > 1 \\ u^{p^*-1} & \text{if } u \leq 1\,. \end{cases}$$

Since $u \in L^{qp^*}_{\text{loc}}$ we get that $f \in L^r$ with $r > \frac{n}{p}$ and $g \in L^\infty$. Hence, as in the proof of [205, Theorem 1], a classical Moser iteration argument yields the result. □

Remark 3.1. As observed in the proof of [183, Theorem E.0.20], the Moser iteration argument can also be used to show that u is uniformly $C^{0,\theta}$ up to the boundary.

3.1.2 Asymptotic bounds on u and ∇u

The main goal of this subsection is to prove Proposition 3.1 below. Proposition 3.1 is a generalization of [224, Theorem 1.1] to the conical-anisotropic setting. The proof of Proposition 3.1 follows the one given in [224], although the lack of smoothness of Σ creates some nontrivial extra difficulties.

Proposition 3.1. *Let $1 < p < n$ and let u be a solution to* (3.1)*. Then there exist two positive constants C_0 and C_1 such that*

$$\frac{C_0}{1+|x|^{\frac{n-p}{p-1}}} \leq u(x) \leq \frac{C_1}{1+|x|^{\frac{n-p}{p-1}}} \quad \text{and} \quad |\nabla u(x)| \leq \frac{C_1}{1+|x|^{\frac{n-1}{p-1}}}, \tag{3.19}$$

for all $x \in \Sigma$.

Before giving the proof of Proposition 3.1, we first introduce a useful definition.

Definition 3.1. Given $L > 0$, we say that a convex cone $\mathcal{C}$ is L-Lipschitz if for any point $x \in \partial\mathcal{C}$ there exist $r_x > 0$ and a unit vector ν_x such that

$$B_{r_x}(x + Lr_x\nu_x) \subset \mathcal{C}.$$

Note that, by convexity of $\mathcal{C}$, also the convex hull of $B_{r_x}(x + L\nu_x) \cup \{x\}$ is contained in $\mathcal{C}$.

In the spirit of [224, Lemma 2.3], we now prove a general lower bound on the L^{p^*} norms of solutions to our equation in convex cones, with a bound depending only on the Lipschitz constant.

Lemma 3.2 (Lower bound on the mass)**.** *Let u be a nontrivial solution to*

$$\begin{cases} \operatorname{div}(a(\nabla u)) + u^{p^*-1} = 0 & \text{in } \mathcal{C} \\ u > 0 & \text{in } \mathcal{C} \\ a(\nabla u)\cdot\nu = 0 & \text{on } \partial\mathcal{C} \\ u \in \mathcal{D}^{1,p}(\mathcal{C}), \end{cases} \tag{3.20}$$

where $\mathcal{C}$ is a L-Lipschitz convex cone and $a(\xi)$ is as in (3.3)*. Then there exists a constant $k_0 > 0$, depending only on n, p, L, and* $\min_{\mathbb{S}^{n-1}} H$*, such that*

$$\|u\|_{L^{p^*}(\mathcal{C})} \geq k_0.$$

Proof. As in [224, Lemma 2.3], the proof is based on the Sobolev inequality in $\mathcal{C}$, and on the integral identity that one obtains by multiplying (3.20) by u

and integrating in $\mathcal{C}$. However in this case a bit more carefulness is needed, especially to quantify the dependencies.

First of all, up to a translation, we can assume that $\mathcal{C}$ has vertex at O. Then, since $\mathcal{C}$ is L-Lipschitz, there exist $r_0 > 0$ and a unit vector ν_0 such that $B_{r_0}(Lr_0\nu_0) \subset \mathcal{C}$. Therefore, since $\mathcal{C}$ is a convex cone, this implies that the cone

$$\hat{\mathcal{C}}_L := \bigcup_{r>0} B_r(Lr\nu_0)$$

is contained inside $\mathcal{C}$.

We now want to estimate the Sobolev constant of $\mathcal{C}$. To this aim we define the constant $\mathcal{S}_L$ as

$$\inf\left\{ \frac{\left(\int_\Omega |\nabla\varphi|^p dx\right)^{1/p}}{\left(\int_\Omega |\varphi|^{p^*} dx\right)^{1/p^*}} : \Omega \text{ is convex, } B_1 \cap \hat{\mathcal{C}}_L \subset \Omega \subset B_1,\ \varphi \in C^1(\Omega),\ \varphi|_{\partial B_1 \cap \hat{\mathcal{C}}_L} = 0 \right\}.$$

Since the set of convex domains $\Omega \subset B_1$ containing $B_1 \cap \hat{\mathcal{C}}_L$ are uniformly Lipschitz, standard arguments in the calculus of variations show that $\mathcal{S}_L$ is positive.

We now notice that, given any function $\psi \in C^1_c(\overline{\mathcal{C}})$, there exists $\lambda > 0$ large such that $\psi_\lambda(x) := \psi(\lambda x)$ satisfies $\psi_\lambda \in C^1(\mathcal{C})$ and $\psi_\lambda|_{\partial B_1 \cap \hat{\mathcal{C}}_L} = 0$ (since $\partial B_1 \cap \hat{\mathcal{C}}_L \subset \partial B_1 \cap \mathcal{C}$). Hence, we can bound

$$\frac{\left(\int_\mathcal{C} |\nabla\psi|^p dx\right)^{1/p}}{\left(\int_\mathcal{C} |\psi|^{p^*} dx\right)^{1/p^*}} = \frac{\left(\int_\mathcal{C} |\nabla\psi_\lambda|^p dx\right)^{1/p}}{\left(\int_\mathcal{C} |\psi_\lambda|^{p^*} dx\right)^{1/p^*}} \geq \mathcal{S}_L.$$

Since $\psi \in C^1_c(\overline{\mathcal{C}})$ is arbitrary, it follows by approximation that

$$\left(\int_\mathcal{C} |\nabla\psi|^p dx\right)^{1/p} \geq \mathcal{S}_L \left(\int_\mathcal{C} |\psi|^{p^*} dx\right)^{1/p^*} \qquad \forall\, \psi \in \mathcal{D}^{1,p}(\mathcal{C}).$$

Applying this inequality to u and defining $c_H := \min_{|\xi|=1} H(\xi)$, we get

$$\int_\mathcal{C} H(\nabla u)^p dx \geq c_H^p \int_\mathcal{C} |\nabla u|^p dx \geq (c_H \mathcal{S}_L)^p \left(\int_\mathcal{C} u^{p^*} dx\right)^{p/p^*}.$$

On the other hand, multiplying (3.20) by u and integrating in $\mathcal{C}$, we get

$$\int_\mathcal{C} H(\nabla u)^p dx = \int_\mathcal{C} u^{p^*} dx.$$

Combining the last two equations yield the desired lower bound. □

Remark 3.2. An alternative proof of Lemma 3.2 can be obtained by computing the optimal Sobolev constant of $\mathcal{C}$ (using Appendix B) and noticing that this constant is bounded below in terms only of n, p, H_0, and the volume of $\mathcal{C} \cap B_1$. In particular, whenever $\mathcal{C}$ is L-Lipschitz then $\hat{\mathcal{C}}_L \subseteq \mathcal{C}$ and $|\mathcal{C} \cap B_1| \geq |\hat{\mathcal{C}}_L \cap B_1|$, and one concludes that the Sobolev constant of $\mathcal{C}$ is controlled by (actually, it is larger or equal than) the one of $\hat{\mathcal{C}}_L$.

We shall also need a doubling-type property on u which is proved in [187, Lemma 5.1] (see also [224, Lemma 3.1]). Below we state a version of this doubling property which is suitable for our setting.

Note that, by convexity, there exists a constant $L_\Sigma > 0$ such that Σ is L_Σ-Lipschitz. Then we let $k_0 > 0$ be the constant provided by Lemma 3.2 with $L = L_\Sigma$.

Lemma 3.3 (Doubling property [187]). *Let u be a solution to* (3.20)*, let L_Σ be the Lipschitz constant of Σ, and let $k_0 > 0$ be the constant provided by Lemma 3.2 with $L = L_\Sigma$.*

Let $k \in (0, k_0)$, $r > 0$, and $r' \in (0, r)$ be fixed, and set

$$r'' = \frac{r + r'}{2}.$$

Then for any $x \in \overline{\Sigma} \setminus B_{r''}$ and $\alpha > 0$ such that the distance d between x and $\Sigma \cap B_{r''}$ satisfies

$$d(x, \Sigma \cap B_{r''}) u(x)^{\frac{p}{n-p}} > 2\alpha, \tag{3.21}$$

there exists a point $y_0 \in \overline{\Sigma} \setminus B_{r''}$ such that

$$d(y_0, \Sigma \cap B_{r''}) u(x)^{\frac{p}{n-p}} > 2\alpha, \quad u(x_0) \le u(y_0), \tag{3.22}$$

and

$$u(y) \le 2^{\frac{n-p}{p}} u(y_0) \quad \textit{for all } y \in \Sigma \cap B_{\bar r}(y_0), \tag{3.23}$$

where $\bar r = \alpha u(y_0)^{-\frac{p}{n-p}}$.

Proof of Proposition 3.1. We divide the proof of Proposition 3.1 in three steps. In Step 1 we give a preliminary decay estimate on u (which is not sharp). In Step 2 we prove that $u \in L^{\hat p - 1, \infty}(\Sigma)$ for a suitable $\hat p$. Finally, in Step 3 we prove (3.19).

- *Step 1: Let u be a solution of* (3.1)*, and for $k \in (0, k_0)$ define*

$$r_k(u) := \inf\{r > 0 \,:\, ||u||_{L^{p^*}(\Sigma \setminus B_r)} < k\}. \tag{3.24}$$

Then, for any fixed $k \in (0, k_0)$ and $r > r_k(u)$, there exists a constant K_0 such that

$$|u(x)| \le K_0 H_0(x)^{\frac{p-n}{p}} \quad \textit{for all } x \in \overline{\Sigma} \setminus B_r. \tag{3.25}$$

In order to prove the assertion, it suffices to show the existence of a constant K_1 such that

$$d(x, \Sigma \cap B_{r''}) u(x)^{\frac{p}{n-p}} \le K_1 \quad \text{for all } x \in \overline{\Sigma} \setminus B_r, \tag{3.26}$$

where $r'' = (r + r')/2$ and $r' \in (0, r)$ is fixed. We prove (3.26) by contradiction.

Suppose there exists a sequence of points $\{x_\alpha\}_{\alpha \in \mathbb{N}} \subset \overline{\Sigma} \setminus B_r$ such that

$$d(x_\alpha, \Sigma \cap B_{r''}) u(x_\alpha)^{\frac{p}{n-p}} > 2\alpha. \tag{3.27}$$

Since $B_{r''} \subset B_r$, it follows from (3.27) and Lemma 3.3 that there exists a sequence of points $\{y_\alpha\}_{\alpha \in \mathbb{N}} \subset \overline{\Sigma} \setminus B_{r''}$ such that

$$d(y_\alpha, \Sigma \cap B_{r''}) u(y_\alpha)^{\frac{p}{n-p}} > 2\alpha, \quad u(x_\alpha) \le u(y_\alpha), \tag{3.28}$$

and

$$u(y) \leq 2^{\frac{n-p}{p}} u(y_\alpha) \quad \text{for all } \; y \in \Sigma \cap B_{\bar{r}}(y_\alpha)\,. \tag{3.29}$$

We observe that, since u is bounded, the sequences $\{x_\alpha\}_{\alpha\in\mathbb{N}}$ and $\{y_\alpha\}_{\alpha\in\mathbb{N}}$ are both divergent as $\alpha \to \infty$.

For any $\alpha \in \mathbb{N}$ and $y \in \overline{\Sigma}$, we define

$$\tilde{u}_\alpha(y) := u(y_\alpha)^{-1} u(m_\alpha^{-1} y + y_\alpha) \tag{3.30}$$

where $m_\alpha := u(y_\alpha)^{\frac{-p}{n-p}}$. From (3.1) we obtain

$$\begin{cases} -\Delta_p^H \tilde{u}_\alpha = \tilde{u}_\alpha^{p^*-1} & \text{in } \Sigma_\alpha \\ \tilde{u}_\alpha(\mathcal{O}) = 1, & \\ a(\nabla \tilde{u}_\alpha)\cdot \nu = 0 & \text{on } \partial\Sigma_\alpha\,, \end{cases} \tag{3.31}$$

where

$$\Sigma_\alpha := m_\alpha(\Sigma - y_\alpha) = \{y \in \mathbb{R}^n \,:\, m_\alpha^{-1} y + y_\alpha \in \Sigma\}$$

is a convex cone.

It is immediate to check that the cones Σ_α are L_Σ-Lipschitz. Furthermore, if we set $\mu_\alpha := u(y_\alpha)^{-1}$, (3.29) and (3.30) yield that

$$\tilde{u}_\alpha(-y_\alpha m_\alpha) = \mu_\alpha u(\mathcal{O}) \neq 0 \quad \text{and} \quad \tilde{u}_\alpha(y) \leq 2^{\frac{n-p}{p}} \quad \text{for all } \; y \in \Sigma_\alpha \cap B_\alpha\,. \tag{3.32}$$

At this point we consider the ratio

$$q_\alpha := \frac{m_\alpha}{|y_\alpha|}\,.$$

Observe that (by (3.28)) $q_\alpha \to 0$ as $\alpha \to \infty$.

Since $|y_\alpha| \to +\infty$, the ratio between $-y_\alpha m_\alpha$ and the scaling factor m_α goes to infinity. Hence, one of the following two cases may occur as $\alpha \to \infty$:

(i) the sequence of cones $\{\Sigma_\alpha\}_{\alpha\in\mathbb{N}}$ converges to $\mathbb{R}^n$ (this happens if the distance between $m_\alpha y_\alpha$ and $\partial\Sigma_\alpha$ goes to infinity);

(ii) the sequence of cones $\{\Sigma_\alpha\}_{\alpha\in\mathbb{N}}$ converges to a L_Σ-Lipschitz convex cone $\mathcal{C}$, not necessarily centered at the origin (this happens if the distance between $m_\alpha y_\alpha$ and $\partial\Sigma_\alpha$ remains bounded).

We now look in both cases at the behaviour of the functions $\{u_\alpha\}_{\alpha\in\mathbb{N}}$. We consider the two cases separately.

- Case (i): fix a ball B_R. Then there exists $\overline{\alpha} \in \mathbb{N}$ such that $\Sigma_\alpha \cap B_R = B_R$ for every $\alpha \geq \overline{\alpha}$; moreover $\tilde{u}_\alpha$ (for every $\alpha \geq \overline{\alpha}$) is a solution of (3.31) in B_R. From (3.9), (3.32), and [85], there exist a constant $C > 0$ and a real number $\theta \in (0,1)$ such that

$$||\tilde{u}_\alpha||_{C^{1,\theta}(B_{R/2})} \leq C \tag{3.33}$$

for any $\alpha \geq \overline{\alpha}$. Since $R > 0$ is arbitrary, Ascoli-Arzelà Theorem and a diagonal argument imply that $\{\tilde{u}_\alpha\}_{\alpha\in\mathbb{N}}$ converges (up to subsequence) in $C^1_{\text{loc}}(\mathbb{R}^n)$ to some function $\tilde{u}_\infty$. By construction we have that $\tilde{u}_\infty \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, $\tilde{u}_\infty(\mathcal{O}) = 1$, and $\tilde{u}_\infty$ is a weak solution of

$$-\,\Delta_p^H \tilde{u}_\infty = \tilde{u}_\infty^{p^*-1} \quad \text{in } \mathbb{R}^n\,. \tag{3.34}$$

- Case (ii): consider a ball B_R. Then for every compact set $K \subset\subset B_R \cap \mathcal{C}$ there exists $\overline{\alpha} \in \mathbb{N}$ such that $K \subset \Sigma_\alpha \cap B_R$ for every $\alpha \geq \overline{\alpha}$. As in Case ($i$), for every $\alpha \geq \overline{\alpha}$ the function $\tilde{u}_\alpha$ is a solution of (3.31) in K, and there exist a constant $C > 0$ and a real number $\theta \in (0,1)$ such that

$$||\tilde{u}_\alpha||_{C^{1,\theta}(K')} \leq C \tag{3.35}$$

for any $\alpha \geq \overline{\alpha}$ and $K' \subset\subset K$. In addition, it follows by Remark 3.1 that the functions $\tilde{u}_\alpha$ are uniformly $C^{0,\theta}$ inside $B_R \cap \overline{\mathcal{C}}$ for any $R > 0$. Hence, again Ascoli-Arzelà Theorem and a diagonal argument imply that $\{\tilde{u}_\alpha\}_{\alpha\in\mathbb{N}}$ converges (up to subsequence) in $C^0(B_R \cap \overline{\mathcal{C}}) \cap C^1_{\mathrm{loc}}(B_R \cap \mathcal{C})$ to some function $\tilde{u}_\infty$, for any $R > 0$. Taking the limit in the weak formulation of the equation, we obtain that $\tilde{u}_\infty \in \mathcal{D}^{1,p}(\mathcal{C})$, $\tilde{u}_\infty(\mathcal{O}) = 1$, and $\tilde{u}_\infty$ is a weak solution of

$$\begin{cases} -\Delta_p^H \tilde{u}_\infty = \tilde{u}_\infty^{p^*-1} & \text{in } \mathcal{C} \\ a(\nabla \tilde{u}_\infty) \cdot \nu = 0 & \text{on } \partial\mathcal{C}\,. \end{cases} \tag{3.36}$$

We now notice that, in both cases, for any $\rho > 0$ we have

$$||\tilde{u}_\alpha||_{L^{p^*}(\Sigma_\alpha \cap B_\rho)} = ||u||_{L^{p^*}(\Sigma \cap B_{\rho m_\alpha}(y_\alpha))}\,. \tag{3.37}$$

Also, by (3.28), since $r_k(u) < r''$ we get

$$B_{\rho m_\alpha}(y_\alpha) \cap B_{r_k(u)} = \emptyset \tag{3.38}$$

for α large. Thus, from (3.37), (3.38), and by definition of $r_k(u)$, we obtain

$$||\tilde{u}_\alpha||_{L^{p^*}(\Sigma_\alpha \cap B_\rho)} \leq k \tag{3.39}$$

for α large. Thus, taking the limit in (3.39) as $\alpha \to \infty$ and then as $\rho \to \infty$, yields

$$||\tilde{u}_\infty||_{L^{p^*}(\mathbb{R}^n)} \leq k \quad \text{or} \quad ||\tilde{u}_\infty||_{L^{p^*}(\mathcal{C})} \leq k\,, \tag{3.40}$$

in Case (i) or Case (ii), respectively. Since $k < k_0$ with $k_0 > 0$ as in Lemma 3.2, it follows by (3.34) (resp. (3.36)) and (3.40) that $\tilde{u}_\infty \equiv 0$ in Case (i) (resp. Case (ii)), a contradiction to the fact that $\tilde{u}_\infty(\mathcal{O}) = 1$. This completes the proof of the assertion of Step 1.

• *Step 2: Let u be a solution of* (3.20). *Then $u \in L^{\hat{p}-1,\infty}(\Sigma)$ for $\hat{p} := \frac{p(n-1)}{n-p}$.*

Recall that, given a set Ω and $r \geq 1$, one defines the space $L^{r,\infty}(\Omega)$ as the set of all measurable functions $v : \Omega \to \mathbb{R}$ such that

$$||v||_{L^{r,\infty}(\Omega)} := \sup_{h>0} \left\{ h \,\mathrm{meas}\,(\{|u| > h\})^{1/r} \right\} < \infty\,. \tag{3.41}$$

Using the Sobolev inequality in cones, the proof of this step can be easily adapted from the case of $\mathbb{R}^n$ (see [224, Lemma 2.2]) and for this reason is omitted.

• *Step 3: Proof of* (3.19).

The proof of this step closely follows the proof of [224, Theorem 1.1], which in turn uses [220, Theorem 1.3] and [205, Theorem 5]. Even if [220, Theorem 1.3] and [205, Theorem 5] are stated in a local setting, thanks to the homogeneous Neumann boundary condition they can be easily extended to our setting. For this reason we only give a sketch of the proof, following the argument of [224, Theorem 1.1].

Let k and r be as in Step 1. For any $R > 0$ and $y \in \Sigma$, we define

$$u_R(y) := R^{\frac{n-p}{p-1}} u(Ry)\,. \tag{3.42}$$

From (3.1) we obtain

$$-\Delta_p^H u_R = R^{-\frac{p}{p-1}} u_R^{p^*-1} \qquad \text{in}\,\Sigma\,. \tag{3.43}$$

Also, writing $u_R^{p^*-1} = u_R^{p^*-p} u_R^{p-1}$ and using (3.25), we have

$$R^{-\frac{p}{p-1}} u_R^{p^*-1} \le K_0^{p^*-p} u_R^{p-1} \quad \text{in}\,\overline{\Sigma} \setminus B_1\,, \tag{3.44}$$

provided that $R \ge r$. Thus, it follows from (3.43), (3.44), and [220, Theorem 1.3], that for any $\varepsilon > 0$ it holds

$$||u_R||_{L^\infty(\Sigma \cap (B_4 \setminus B_2))} \le C_\varepsilon ||u_R||_{L^{p-1+\varepsilon}(\Sigma \cap (B_5 \setminus B_1))} \tag{3.45}$$

for some constant $C_\varepsilon > 0$. We fix $\varepsilon_0 = \varepsilon_0(n,p)$ such that $0 < \varepsilon_0 < \hat{p} - p$, where $\hat{p}$ is as in Step 2. Since

$$||u_R||_{L^{p-1+\varepsilon_0}(\Sigma \cap (B_5 \setminus B_1))} \le C_0 ||u_R||_{L^{\hat{p}-1,\infty}(\Sigma \cap (B_5 \setminus B_1))}\,,$$

for $C_0 = C_0(n,p)$, recalling Step 2 we obtain that

$$||u_R||_{L^\infty(\Sigma \cap (B_4 \setminus B_2))} \le C_1 \tag{3.46}$$

for some constant C_1. Hence, by (3.43), (3.46), and elliptic regularity theory for p-Laplacian type equations [85, 218], we get

$$||\nabla u_R||_{L^\infty(\Sigma \cap (B_{7/2} \setminus B_{5/2}))} \le C_2 \tag{3.47}$$

for some constant C_2. Here we notice that, even if (3.47) is proved in [85, Section 3] in a local setting (see also [53], where the authors prove global Lipschitz regularity in convex domains for the case when H coincides with the Euclidean norm), the argument easily extends to our setting by an approximation argument. Indeed, as in the proof of Proposition 3.2 below, one can work in regularized domains and, because of the presence of the boundary, with respect to [85, Section 3] it appears an extra boundary term. However, this can be dropped since the second fundamental form of $\partial\Sigma$ is nonnegative definite (compare with (3.57)-(3.60) below, or with [53, Proof of Theorem 1.2, Step 1]).

Finally, for any $x \in \mathbb{R}^n \setminus B_{3r}$, applying (3.46) and (3.47) with $R = |x|/3$ we obtain

$$u(x) \le C_3 |x|^{\frac{p-n}{p-1}} \quad \text{and} \quad |\nabla u(x)| \le C_3 |x|^{\frac{1-n}{p-1}} \tag{3.48}$$

for some constant C_3. Since u and ∇u are uniformly bounded in B_{3r}, (3.19) follows. Finally, to prove the lower bound in (3.19) one argues as in [224, pages 159-160]. □

3.1.3 Asymptotic estimates on higher order derivatives

By using a Caccioppoli-type inequality, in this subsection we prove Proposition 3.2 below which will be useful in the proof of Theorem 3.1. In particular it will avoid the use of an asymptotic lower bound on $|\nabla u|$, which is crucial in [203].

Proposition 3.2. *Let Σ be a convex cone, and let u be a solution to* (3.1) *with $a(\cdot)$ given by* (3.3)*, where H satisfies the assumptions of Theorem 3.1. Then $a(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\overline{\Sigma})$, and for any $\gamma \in \mathbb{R}$ the following asymptotic estimate holds:*

$$\int_{B_r \cap \Sigma} |\nabla(a(\nabla u))|^2 u^\gamma \, dx \leq C\Big(1 + r^{-n-\gamma\frac{n-p}{p-1}}\Big) \qquad \forall\, r \geq 1, \tag{3.49}$$

where C is a positive constant independent of r.

Proof. The estimate (3.49) is obtained by using a Caccioppoli-type inequality. We argue by approximation, following the approach in [14, 54].

We approximate Σ by a sequence of convex cones $\{\Sigma_k\}$ such that $\Sigma_k \subseteq \Sigma$ and $\partial\Sigma_k \setminus \{\mathcal{O}\}$ is smooth. Also, we fix a point $\bar{x} \in \cap_k \Sigma_k$, and for k fixed we let u_k be the solution of[2]

$$\begin{cases} \operatorname{div}(a(\nabla u_k)) + u^{p^*-1} = 0 & \text{in } \Sigma_k \\ u_k(\bar{x}) = u(\bar{x}) & \\ a(\nabla u_k)\cdot \nu = 0 & \text{on } \partial\Sigma_k\,. \end{cases} \tag{3.50}$$

Set

$$a^\ell(z) := (a * \phi_\ell)(z) \qquad \text{for } z \in \mathbb{R}^n\,, \tag{3.51}$$

where $\{\phi_\ell\}$ is a family of radially symmetric smooth mollifiers. Standard properties of convolution and the fact $a(\cdot)$ is continuous imply $a^\ell \to a$ uniformly on compact subset of $\mathbb{R}^n$. From [99, Lemma 2.4] we have that a^ℓ satisfies the first condition in (3.12) with s replaced by s_ℓ, where $s_\ell \to 0$ as $\ell \to \infty$. In addition, since

$$\frac{1}{\tilde{\alpha}}(|z|^2 + s_\ell^2)^{\frac{p-2}{2}}|\xi|^2 \leq \nabla a^\ell(z)\xi\cdot\xi\,, \qquad \text{for every } \xi, z \in \mathbb{R}^n,$$

for some $\tilde{\alpha} > 0$, we obtain that a^ℓ satisfies also the second condition in (3.12).

Let $u_{k,\ell}$ be a solution of

$$\begin{cases} \operatorname{div}(a^\ell(\nabla u_{k,\ell})) + u^{p^*-1} = 0 & \text{in } \Sigma_k \\ a^\ell(\nabla u_{k,\ell})\cdot \nu = 0 & \text{on } \partial\Sigma_k \end{cases} \tag{3.52}$$

(this solution can be constructed analogously to u_k).

We notice that $u_{k,\ell}$ is unique up to an additive constant. Also, because u is locally bounded, the functions $u_{k,\ell}$ are $C^{1,\theta}_{\mathrm{loc}}(\overline{\Sigma}_k \setminus \{\mathcal{O}\}) \cap C^{0,\theta}_{\mathrm{loc}}(\overline{\Sigma}_k)$, uniformly in ℓ. In particular, assuming without loss of generality that $u_{k,\ell}(\bar{x}) = u(\bar{x})$ for

[2] The function u_k can be found by considering first the minimizer $v_{k,R}$ of the minimization problem

$$\min_v \left\{ \int_{\Sigma_k \cap B_R} \left[\frac{1}{p} H(\nabla v)^p - u^{p^*-1} v\right] dx \,:\, v = 0 \text{ on } \Sigma_k \cap \partial B_R \right\},$$

then setting $u_{k,R}(x) := v_{k,R}(x) + u(\bar{x}) - v_{k,R}(\bar{x})$, and finally taking the limit of $u_{k,R}$ as $R \to \infty$ (note that the functions $\tilde{u}_{k,R}$ are uniformly $C^{1,\theta}$ in every compact subset of Σ, and uniformly Hölder continuous up to the boundary).

some fixed point $\bar{x} \in \Sigma_k$, as $\ell \to \infty$ one sees that $u_{k,\ell}$ converges in C^1_{loc} to the unique solution $\bar{u}_k$ of

$$\begin{cases} \operatorname{div}(a(\nabla \bar{u}_k)) + u^{p^*-1} = 0 & \text{in } \Sigma_k \\ \bar{u}_k(\bar{x}) = u(\bar{x}) & \\ a(\nabla \bar{u}_k) \cdot \nu = 0 & \text{on } \partial\Sigma_k \,. \end{cases} \tag{3.53}$$

Since u_k is also a solution of the problem above, it follows by uniqueness that $\bar{u}_k = u_k$ and therefore $u_{k,\ell}$ converges to u_k as $\ell \to \infty$. Analogously, $u_k \to u$ as $k \to \infty$.

Given $R > 1$ large, we define

$$\Omega_k := \Sigma_k \cap B_R \,, \quad \Gamma_{k,0} := \Sigma_k \cap \partial B_R \,, \quad \Gamma_{k,1} := \partial\Sigma_k \cap B_R \,.$$

Note that, since u is uniformly positive inside Σ (see Proposition 3.1), for k large enough (depending on R) also u_k is uniformly positive inside Ω_k, and hence for ℓ large enough we have that $u_{k,\ell}$ is also uniformly positive inside Ω_k. In the sequel we shall always assume that k and ℓ are sufficiently large so that this positivity property holds. We now fix k and deal with the functions $u_{k,\ell}$. To simplify the notation, we shall drop the dependency on k and we write $u_\ell, \Sigma, \Omega, \Gamma_0, \Gamma_1$ instead of $u_{k,\ell}, \Sigma_k, \Omega_k, \Gamma_{k,0}, \Gamma_{k,1}$, respectively.

The idea is to prove a Caccioppoli-type inequality for u_ℓ and then let $\ell \to \infty$. Since u_ℓ solves a non-degenerate equation, we have that $u_\ell \in C^1 \cap W^{2,2}_{\mathrm{loc}}(\overline{\Sigma})$ and furthermore we have $a^\ell(\nabla u_\ell) \in W^{1,2}_{\mathrm{loc}}(\overline{\Sigma})$. In addition, since Σ is smooth outside the origin, u_ℓ is of class C^2 in $\overline{\Omega}$ away from $\Gamma_1 \cup \{\mathcal{O}\}$.

Multiply (3.52) by $\psi \in C^\infty_c(B_R \setminus B_{1/R})$ and integrate over Ω to get

$$\int_\Omega \operatorname{div}(a^\ell(\nabla u_\ell))\psi \, dx = -\int_\Omega u^{p^*-1}\psi \, dx,$$

that together with the divergence theorem gives

$$-\int_\Omega a^\ell(\nabla u_\ell) \cdot \nabla\psi \, dx + \int_{\partial\Omega} \psi a^\ell(\nabla u_\ell) \cdot \nu \, d\sigma = -\int_\Omega u^{p^*-1}\psi \, dx \,. \tag{3.54}$$

Since

$$\int_{\partial\Omega} \psi a^\ell(\nabla u_\ell) \cdot \nu \, d\sigma = \int_{\Gamma_1} \psi a^\ell(\nabla u_\ell) \cdot \nu \, d\sigma + \int_{\Gamma_0} \psi a^\ell(\nabla u_\ell) \cdot \nu \, d\sigma \,,$$

from the fact that $\psi \in C^\infty_c(B_R \setminus B_{1/R})$ and from the boundary condition in (3.52), we obtain that the second term in (3.54) vanishes; hence (3.54) becomes

$$-\int_\Omega a^\ell(\nabla u_\ell) \cdot \nabla\psi \, dx = -\int_\Omega u^{p^*-1}\psi \, dx \,. \tag{3.55}$$

Let $\varphi \in C^\infty_c(B_R \setminus B_{1/R})$, and for $\delta > 0$ small define the set

$$\Omega_\delta := \{x \in \Omega \, : \, \operatorname{dist}(x, \partial\Omega) > \delta\} \,.$$

Since $\Omega \cap \operatorname{supp}(\varphi)$ is smooth, for δ small enough we see that $\Omega_\delta \setminus \Omega_{2\delta}$ is of class C^∞ inside the support of φ. In particular, every point $x \in (\Omega_\delta \setminus \Omega_{2\delta}) \cap \operatorname{supp}(\varphi)$ can be written as

$$x = y - |x - y|\nu(y)$$

where $y = y(x) \in \partial\Omega_\delta$ is the projection of x on $\partial\Omega_\delta$ and $\nu(y)$ is the outward normal to $\partial\Omega_\delta$ at y. Moreover the set $(\Omega_\delta \setminus \Omega_{2\delta}) \cap \operatorname{supp}(\varphi)$ can be parametrized on $\partial\Omega_\delta$ by a C^1 function g (see [112, Formula 14.98]).

Let $\zeta_\delta : \Omega \to [0,1]$ be a cut-off function such that $\zeta_\delta = 1$ in $\Omega_{2\delta}$, $\zeta_\delta = 0$ in $\Omega \setminus \Omega_\delta$, and

$$\nabla\zeta_\delta(x) = -\frac{1}{\delta}\nu(y(x)) \qquad \text{inside } \Omega_\delta \setminus \Omega_{2\delta}\,.$$

Using $\psi = \partial_m(\varphi\zeta_\delta)$ in (3.55) with $m \in \{1,\dots,n\}$ and integrating by parts, we get

$$\sum_{i=1}^{n}\left(\int_\Omega \partial_m a_i^\ell(\nabla u_\ell)\zeta_\delta\partial_i\varphi\,dx + \int_\Omega \partial_m a_i^\ell(\nabla u_\ell)\varphi\partial_i\zeta_\delta\,dx\right) = \int_\Omega \partial_m(u^{p^*-1})\varphi\zeta_\delta\,dx\,,$$

where we use the notation $a^\ell = (a_1^\ell,\dots,a_n^\ell)$ to denote the components of the vector field a^ℓ.

Observe that, from the definition of ζ_δ, we have

$$\lim_{\delta\to0}\int_\Omega \partial_m a_i^\ell(\nabla u_\ell)\zeta_\delta\partial_i\varphi\,dx = \int_\Omega \partial_m a_i^\ell(\nabla u_\ell)\partial_i\varphi\,dx\,.$$

Also, if we set

$$f(x) = \partial_m a_i^\ell(\nabla u_\ell(x))\varphi(x)\,,$$

by the coarea formula we have

$$\begin{aligned}\int_{\Omega_\delta\setminus\Omega_{2\delta}} f\partial_i\zeta_\delta\,dx &= -\frac{1}{\delta}\int_{\Omega_\delta\setminus\Omega_{2\delta}} \nu_i(y(x))f\,dx\\ &= -\frac{1}{\delta}\int_\delta^{2\delta} dt\int_{\partial\Omega_\delta} \nu_i(y(x))f(y-t\nu(y))|\det(Dg)|d\sigma(y)\\ &= -\int_1^2 ds\int_{\partial\Omega_{s\delta}} f(y-s\delta\nu(y))\nu_i(y)|\det(Dg)|d\sigma(y)\,.\end{aligned}$$

Since $f \in C^0$, we can pass to the limit and obtain

$$\lim_{\delta\to0}\int_\Omega \partial_m a_i^\ell(\nabla u_\ell)\varphi\partial_i\zeta_\delta\,dx = -\int_{\partial\Omega} \partial_m a_i^\ell(\nabla u_\ell)\varphi\nu_i d\sigma\,.$$

Hence, we proved that

$$\sum_{i=1}^{n}\left(\int_\Omega \partial_m a_i^\ell(\nabla u_\ell)\partial_i\varphi\,dx - \int_{\partial\Omega} \partial_m a_i^\ell(\nabla u_\ell)\varphi\nu_i d\sigma\right) = \int_\Omega \partial_m(u^{p^*-1})\varphi\,dx\,. \tag{3.56}$$

Now, let

$$\Omega_\delta^t := \{x \in \Omega_\delta \,:\, \operatorname{dist}(x,\partial\Omega_\delta) > t\}\,.$$

We notice that, if $x \in (\Omega_\delta \setminus \Omega_{2\delta}) \cap \operatorname{supp}(\varphi)$ with $x = y - t\nu(y)$, then $x \in \partial\Omega_\delta^t$ and the outward normal to $\partial\Omega_\delta^t$ at x coincides with the outward normal to $\partial\Omega_\delta$ at y. Hence, by writing $\nu(x)$ in place of $\nu(y)$, we have

$$\begin{aligned}\partial_m a_i^\ell(\nabla u_\ell(x))\varphi(x)\nu_i(x) =&\, \varphi(x)\partial_m(a^\ell(\nabla u_\ell(x))\cdot\nu(x))\\ &- \varphi(x)a_i^\ell(\nabla u_\ell(x))\partial_m\nu_i(x)\,.\end{aligned} \tag{3.57}$$

Now, we take a cut-off function $\eta \in C_c^\infty(B_R \setminus B_{1/R})$, and for $m \in \{1, \dots, n\}$ we set $\varphi = a^\ell_m(\nabla u_\ell) u_\ell^\gamma \eta^2$ where $\gamma \in \mathbb{R}$, and in (3.57) we obtain

$$\begin{aligned}\partial_m a_i^\ell(\nabla u_\ell(x))\varphi(x)\nu_i(x) = & a^\ell_m(\nabla u_\ell(x)) u_\ell^\gamma(x)\eta^2(x)\partial_m\big(a^\ell(\nabla u_\ell(x))\cdot\nu(x)\big)\\ & - a^\ell_m(\nabla u_\ell(x)) u_\ell^\gamma(x)\eta^2(x) a^\ell_i(\nabla u_\ell(x))\partial_m \nu_i(x)\,.\end{aligned} \tag{3.58}$$

We notice that $\partial_m \nu_i(x)$ is the second fundamental form II^t_x of $\partial\Omega^t_\delta$ at x:

$$\sum_{i,m=1}^n \partial_m\nu_i(x) a^\ell_i(\nabla u_\ell(x)) a^\ell_m(\nabla u_\ell(x)) = \mathrm{II}^t_x(a^\ell(\nabla u_\ell(x)), a^\ell(\nabla u_\ell(x)))\,.$$

Since the cone Σ is convex then II^t_x is non-negative definite, which implies that

$$\sum_{i,m=1}^n \partial_m\nu_i(x) a^\ell_i(\nabla u_\ell(x)) a^\ell_m(\nabla u_\ell(x)) \geq 0\,. \tag{3.59}$$

Hence (3.58) becomes

$$\begin{aligned}&\sum_{i,m=1}^n \partial_m a_i^\ell(\nabla u_\ell(x))\varphi(x)\nu_i(x)\\ &\qquad\leq \sum_{i,m=1}^n a^\ell_m(\nabla u_n(x)) u_\ell^\gamma(x)\eta^2(x)\partial_m\big(a^\ell(\nabla u_\ell(x))\cdot\nu(x)\big)\,,\end{aligned} \tag{3.60}$$

and so, with the choice $\varphi = a^\ell_m(\nabla u_\ell) u_\ell^\gamma\eta^2$, we obtain

$$\begin{aligned}\sum_{i,m=1}^n \int_{\partial\Omega} \partial_m a^\ell_i(\nabla u_\ell)\varphi\nu_i d\sigma &\leq \sum_{i,m=1}^n \int_{\partial\Omega} u_\ell^\gamma\eta^2 a^\ell_m(\nabla u_\ell)\partial_m\big(a^\ell(\nabla u_\ell)\cdot\nu\big)\,dx\\ &= \sum_{i=1}^n \int_{\partial\Omega} u_\ell^\gamma\eta^2\, a^\ell(\nabla u_\ell)\cdot\nabla\big(a^\ell(\nabla u_\ell)\cdot\nu\big)\,dx = 0\,,\end{aligned}$$

where the last equality follows from the condition $a^\ell(\nabla u_\ell)\cdot\nu = 0$ on $\partial\Sigma$. Indeed, this condition implies that $a^\ell(\nabla u_\ell)$ is a tangent vector-field and that the tangential derivative of $a^\ell(\nabla u_\ell)\cdot\nu$ vanishes on $\partial\Sigma$.

Hence, recalling (3.56), we proved that

$$\sum_{i,m=1}^n \int_\Omega \partial_m a^\ell_i(\nabla u_\ell)\partial_i\left(a^\ell_m(\nabla u_\ell)u_\ell^\gamma\eta^2\right)\,dx \leq n\int_\Omega |\nabla(u^{p^*-1})||a^\ell(\nabla u_\ell)|u_\ell^\gamma\eta^2\,dx\,. \tag{3.61}$$

Inequality (3.61) can be used in place of Equation (4.11) in [14, Proof of Theorem 4.1], and by arguing as in [14] we obtain

$$\begin{aligned}&\int_\Omega |\nabla(a^\ell(\nabla u_\ell))|^2\eta^2 u_\ell^\gamma\,dx \leq\\ &C\int_\Omega |\nabla(a^\ell(\nabla u_\ell))||a^\ell(\nabla u_\ell)|\eta u_\ell^{\frac{\gamma}{2}}|\nabla(\eta u_\ell^{\frac{\gamma}{2}})|\,dx + C\int_\Omega |\nabla(u^{p^*-1})||a^\ell(\nabla u_\ell)|u_\ell^\gamma\eta^2\,dx\,.\end{aligned}$$

From Hölder and Young inequalities, for any $\epsilon \in (0,1)$ we can bound

$$C \int_\Omega |\nabla(a^\ell(\nabla u_\ell))||a^\ell(\nabla u_\ell)|\eta u_\ell^{\frac{\gamma}{2}} |\nabla(\eta u_\ell^{\frac{\gamma}{2}})|\, dx \\ \le C\epsilon \int_\Omega |\nabla(a^\ell(\nabla u_\ell))|^2 \eta^2 u_\ell^\gamma\, dx + \frac{C}{\epsilon}\int_\Omega |a^\ell(\nabla u_\ell)|^2 |\nabla(\eta u_\ell^{\frac{\gamma}{2}})|^2\, dx,$$

so choosing ϵ small enough such that $C\epsilon = 1/2$, we obtain

$$\int_\Omega |\nabla(a^\ell(\nabla u_\ell))|^2 \eta^2 u_\ell^\gamma\, dx \le C\int_\Omega |a^\ell(\nabla u_\ell)|^2 |\nabla(\eta u_\ell^{\frac{\gamma}{2}})|^2\, dx \\ + C\int_\Omega |\nabla(u^{p^*-1})||a^\ell(\nabla u_\ell)| u_\ell^\gamma \eta^2\, dx.$$

Recall that here $\eta \in C_c^\infty(B_R \setminus B_{1/R})$. However, by approximation the same property holds for any $\eta \in C_c^\infty(\mathbb{R}^n)$.

Now, we recall that we were writing u_ℓ in place of $u_{k,\ell}$. Then, since $u_{k,\ell} \to u_k$ in C^1_{loc} and $a^\ell \to a$ locally uniformly, we can let $\ell \to \infty$ to deduce that

$$\int_{\Omega_k} |\nabla(a(\nabla u_k))|^2 \eta^2 u_k^\gamma\, dx \\ \le C\int_{\Omega_k} |a(\nabla u_k)|^2 |\nabla(\eta u_k^{\frac{\gamma}{2}})|^2\, dx + C\int_{\Omega_k} |\nabla(u^{p^*-1})||a(\nabla u_k)| u_k^\gamma \eta^2\, dx. \tag{3.62}$$

In particular, taking $\gamma = 0$, (3.62) proves that $a(\nabla u_k) \in W^{1,2}_{\mathrm{loc}}(\overline{\Sigma}_k)$, and $\{a(\nabla u_k)\}_{k\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}_{\mathrm{loc}}$. Hence, letting $k \to \infty$ in (3.62) we obtain

$$\int_\Omega |\nabla(a(\nabla u))|^2 \eta^2 u^\gamma\, dx \\ \le C\int_\Omega |a(\nabla u)|^2 |\nabla(\eta u^{\frac{\gamma}{2}})|^2\, dx + C\int_\Omega |\nabla(u^{p^*-1})||a(\nabla u)| u^\gamma \eta^2\, dx.$$

Finally, the asymptotic estimate (3.49) follows from (3.19). □

3.2 Proof of Theorem 3.1

As already mentioned in the introduction, we consider the auxiliary function

$$v = u^{-\frac{p}{n-p}} \tag{3.63}$$

where u is a solution of (3.1). A straightforward computation shows that $v > 0$ satisfies the following problem

$$\begin{cases} \Delta_p^{\tilde H} v = f(v, \nabla v) & \text{in } \Sigma \\ \tilde a(\nabla v)\cdot \nu = 0 & \text{on } \partial\Sigma\,, \end{cases} \tag{3.64}$$

where $\Delta_p^{\tilde H} v = \mathrm{div}\,(\tilde a(\nabla v))$ with

$$\tilde a(\xi) = \tilde H^{p-1}(\xi)\nabla \tilde H(\xi) \qquad \forall \xi \in \mathbb{R}^n\,, \tag{3.65}$$

and where we set

$$f(v,\nabla v)=\left(\frac{p}{n-p}\right)^{p-1}\frac{1}{v}+\frac{n(p-1)}{p}\frac{\tilde{H}^p(\nabla v)}{v}\,, \tag{3.66}$$

with

$$\tilde{H}(\xi)=H(-\xi)\qquad \forall\xi\in\mathbb{R}^n\,. \tag{3.67}$$

It is clear that v inherits some properties from u. In particular $v\in C^{1,\theta}_{\rm loc}$, and it follows from Proposition 3.1 that there exist constants $C_0, C_1>0$ such that

$$C_0|x|^{-\frac{p}{p-1}}\le v(x)\le C_1|x|^{-\frac{p}{p-1}} \tag{3.68}$$

and

$$|\nabla v(x)|\le C_1|x|^{-\frac{1}{p-1}} \tag{3.69}$$

for $|x|$ sufficiently large. Higher regularity results for v are summarized in the following lemma.

Lemma 3.4. *Let v be given by* (3.63)*. Then, for every $\sigma\in\mathbb{R}$, the asymptotic estimate*

$$\int_{B_r\cap\Sigma}|\nabla(\tilde{a}(\nabla v))|^2v^\sigma\,dx\le C\Big(1+r^{n+\frac{\sigma p}{p-1}}\Big)\qquad\forall\, r\ge 1 \tag{3.70}$$

holds.

Proof. We notice that

$$\tilde{a}(\nabla v)=-\left(\frac{p}{n-p}\right)^{p-1}u^{-\frac{n(p-1)}{n-p}}a(\nabla u)$$

and

$$\nabla(\tilde{a}(\nabla v))=-\left(\frac{p}{n-p}\right)^{p-1}\times \left[u^{-\frac{n(p-1)}{n-p}}\nabla(a(\nabla u))-\frac{n(p-1)}{n-p}u^{\frac{p(1-n)}{n-p}}\nabla u\otimes a(\nabla u)\right]\,,$$

so it follows from Proposition 3.2 that

$$\tilde{a}(\nabla v)\in W^{1,2}_{\rm loc}(\overline{\Sigma})\,. \tag{3.71}$$

Finally, the asymptotic estimate (3.70) follows from (3.49) and (3.19). □

3.2.1 An integral inequality

In this subsection, by using the convexity of the cone, we show that v satisfies an integral inequality.

We recall that the second symmetric function $S^2(M)$ of a $n\times n$ matrix $M=(m_{ij})$ is the sum of all the principal minors of A of order two, and we have

$$S^2(M)=\frac{1}{2}\sum_{i,j}S^2_{ij}(M)m_{ij}\,, \tag{3.72}$$

where

$$S^2_{ij}(M) = -m_{ji} + \delta_{ij}\mathrm{Tr}\,(M)\,.$$

As proved in [55, Lemma 3.2] (see also Lemma 2.3), given two symmetric matrices $B, C \in \mathbb{R}^{n\times n}$ with B positive semidefinite, and by setting $M = BC$, we have the following Newton's type inequality:

$$S^2(M) \leq \frac{n-1}{2n}\mathrm{Tr}\,(M)^2\,. \tag{3.73}$$

Moreover, if $\mathrm{Tr}\,(M) \neq 0$ and equality holds in (3.73), then

$$M = \frac{\mathrm{Tr}\,(M)}{n}\,\mathrm{Id}\,,$$

and B is positive definite. As we will describe later, we will apply (3.73) to the matrix $M = \nabla[\tilde{a}(\nabla v)]$.

We start from the following differential identity (see [29]). We use the Einstein convention of summation over repeated indices.

Lemma 3.5. *Let v be a positive function of class C^3 and let $V : \mathbb{R}^n \to \mathbb{R}^+$ be of class $C^3(\mathbb{R}^n)$ and such that $V(\nabla v)\mathrm{div}\,(\nabla V(\nabla v))$ can be continuously extended to zero at $\nabla v = 0$. Let*

$$W = \nabla[\nabla_\xi V(\nabla v)] = V_{\xi_i\xi_j}(\nabla v)\partial^2_{ij}v\,. \tag{3.74}$$

Then, for any $\gamma \in \mathbb{R}$ we have

$$2v^\gamma S^2(W) = \mathrm{div}\,(v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v)) - \gamma v^{\gamma-1}S^2_{ij}(W)V_{\xi_i}(\nabla v)\partial_j v \tag{3.75}$$

and

$$\begin{aligned}
&\mathrm{div}\,\big(v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v) + \gamma(p-1)v^{\gamma-1}V(\nabla v)V_{\xi_j}(\nabla v)\big)\\
&= 2v^\gamma S^2(W) + \gamma(\gamma-1)(p-1)v^{\gamma-2}V(\nabla v)V_{\xi_i}(\nabla v)\partial_i v\\
&\quad + \gamma v^{\gamma-1}\left((p-1)V(\nabla v) + V_{\xi_i}(\nabla v)\partial_i v\right)\mathrm{Tr}\,(W)\\
&\quad + \gamma v^{\gamma-1}\big((p-1)V_{\xi_i}(\nabla v)V_{\xi_j}(\nabla v)\partial^2_{ij}v + V_{\xi_j\xi_l}(\nabla v)\partial^2_{li}vV_{\xi_i}(\nabla v)\partial_j v\big)\,.
\end{aligned} \tag{3.76}$$

In particular, if

$$V(\xi) = \frac{\tilde{H}^p(\xi)}{p} \qquad \textit{for } p>1 \textit{ and } \xi \in \mathbb{R}^n\,, \tag{3.77}$$

where $\tilde{H}$ is a norm, then

$$\begin{aligned}
2v^\gamma S^2(W) = &\,\mathrm{div}\,\big(v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v) + \gamma(p-1)v^{\gamma-1}V(\nabla v)\nabla_\xi V(\nabla v)\big)\\
&- \gamma(\gamma-1)p(p-1)v^{\gamma-2}V^2(\nabla v)\\
&- \gamma(2p-1)v^{\gamma-1}V(\nabla v)\Delta^{\tilde{H}}_p v\,,
\end{aligned} \tag{3.78}$$

where $\Delta^{\tilde{H}}_p v = \mathrm{div}\,(\tilde{a}(\nabla v))$ and $\tilde{a}(\cdot)$ is given by (3.65). Observe that, in this particular case,

$$W(x) := \nabla[\tilde{a}(\nabla v(x))]\,.$$

Proof. See [29, Lemma 4.1]. □

The idea is to apply the above lemma to the function v solving (3.64) and integrate the identity above on Σ. Due to the lack of regularity of v, Lemma 3.5 cannot be applied directly but we can still prove its integral counterpart.

Lemma 3.6. *Let v be given by* (3.63)*, let V be as in* (3.77)*, and W as in* (3.74)*. Then, for any $\varphi \in C_c^\infty(\Sigma)$, we have*

$$\int_\Sigma \Big(2v^\gamma S^2(W) + \gamma(\gamma-1)p(p-1)v^{\gamma-2}V^2(\nabla v) + \gamma(2p-1)v^{\gamma-1}V(\nabla v)\Delta_p^{\tilde{H}} v\Big)\,\varphi\,dx$$
$$= -\int_\Sigma \partial_j\varphi\big(v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v) + \gamma(p-1)v^{\gamma-1}V(\nabla v)V_{\xi_j}(\nabla v)\big)\,dx\,. \tag{3.79}$$

Proof. We argue by approximation. So, first we extend v as 0 outside Σ, and then for $\varepsilon > 0$ we define $v^\varepsilon = v * \rho^\varepsilon$ and $V^\varepsilon = V * \rho^\varepsilon$, where ρ^ε is a standard mollifier. Also, we set $\tilde{a}^\varepsilon = \nabla V^\varepsilon$ and $W^\varepsilon = (w^\varepsilon_{ij})_{i,j=1,\dots,n}$ where $w^\varepsilon_{ij} = \partial_j(\tilde{a}^\varepsilon_i(\nabla v^\varepsilon))$.

Since $V \in C^1(\mathbb{R}^n)$ then $\tilde{a}^\varepsilon_i = \tilde{a}_i * \rho^\varepsilon$ for $i = 1,\dots,n$, where $\tilde{a}$ is given by (3.65). Also, since $\tilde{a}(\nabla v) \in W^{1,2}_{\mathrm{loc}}(\Sigma)$, then $\tilde{a}^\varepsilon_i(\nabla v^\varepsilon) \to \tilde{a}_i(\nabla v)$ and $w^\varepsilon_{ij} \to w_{ij}$ in $L^2_{\mathrm{loc}}(\Sigma)$. Moreover, since

$$\tilde{H}_0(\nabla \tilde{H}(\xi)) = 1 \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}\,,$$

we have that

$$\tilde{H}_0(\tilde{a}(\xi)) = \tilde{H}^{p-1}(\xi) \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}\,,$$

which implies that

$$pV(\xi) = \tilde{H}_0^{\frac{p}{p-1}}(\tilde{a}(\xi)) \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}\,.$$

Since $\tilde{H}_0^{\frac{p}{p-1}}$ is locally Lipschitz and $\tilde{a}(\nabla v) \in W^{1,2}_{\mathrm{loc}}(\Sigma)$ then $V(\nabla v) \in W^{1,2}_{\mathrm{loc}}(\Sigma)$ and we have that

$$\partial_{x_j}(V^\varepsilon(\nabla v^\varepsilon)) \to \partial_{x_j}(V(\nabla v)) \quad \text{in } L^2_{\mathrm{loc}}(\Sigma).$$

Now we write (3.76) for the approximating functions v^ε, V^ε and W^ε, we multiply by $\varphi \in C_c^\infty(\Sigma)$ and integrate over Σ. Since φ has compact support inside Σ, it follows from the divergence theorem that

$$\int_\Sigma \big(2(v^\varepsilon)^\gamma S^2(W^\varepsilon) + \gamma(\gamma-1)(p-1)(v^\varepsilon)^{\gamma-2}V^\varepsilon(\nabla v^\varepsilon)V^\varepsilon_{\xi_i}(\nabla v^\varepsilon)\partial_i v^\varepsilon\big)\,\varphi\,dx$$
$$+\int_\Sigma \gamma(v^\varepsilon)^{\gamma-1}\big((p-1)V^\varepsilon(\nabla v^\varepsilon) + V^\varepsilon_{\xi_i}(\nabla v^\varepsilon)\partial_i v^\varepsilon\big)\,\mathrm{Tr}\,(W^\varepsilon)\varphi\,dx$$
$$+\int_\Sigma \gamma(v^\varepsilon)^{\gamma-1}\Big((p-1)V^\varepsilon_{\xi_i}(\nabla v^\varepsilon)V^\varepsilon_{\xi_j}(\nabla v^\varepsilon)\partial^2_{ij}v^\varepsilon + V^\varepsilon_{\xi_j\xi_l}(\nabla v^\varepsilon)\partial^2_{li}v^\varepsilon V^\varepsilon_{\xi_i}(\nabla v^\varepsilon)\partial_j v^\varepsilon\Big)\,\varphi\,dx$$
$$= -\int_\Sigma \partial_j\varphi\big((v^\varepsilon)^\gamma S^2_{ij}(W^\varepsilon)V^\varepsilon_{\xi_i}(\nabla v^\varepsilon) + \gamma(p-1)(v^\varepsilon)^{\gamma-1}V^\varepsilon(\nabla v^\varepsilon)V^\varepsilon_{\xi_j}(\nabla v^\varepsilon)\big)\,dx\,. \tag{3.80}$$

Since $V^\varepsilon_{\xi_i}(\nabla v^\varepsilon)\partial^2_{ij}v^\varepsilon = \partial_{x_j}(V^\varepsilon(\nabla v^\varepsilon))$, recalling (3.78) we conclude easily by letting $\varepsilon \to 0$. □

Now we extend Lemma 3.6 to cut-off functions defined on a ball centered at the origin. Here, the convexity of Σ plays a crucial role.

Lemma 3.7. *Let v be given by* (3.63)*, let V be as in* (3.77)*, and W as in* (3.74)*. Consider a non-negative cut-off function $\eta \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\int_\Sigma \Big(2v^\gamma S^2(W) + \gamma(\gamma-1)p(p-1)v^{\gamma-2}V^2(\nabla v) + \gamma(2p-1)v^{\gamma-1}V(\nabla v)\Delta_p^{\tilde{H}} v\Big)\,\eta\,dx$$
$$\geq -\int_\Sigma \partial_j\eta\big(v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v) + \gamma(p-1)v^{\gamma-1}V(\nabla v)V_{\xi_j}(\nabla v)\big)\,dx\,. \tag{3.81}$$

Proof. As in the proof of Proposition 3.2, this proof requires a regularization argument considering the solutions of the approximating problems

$$\begin{cases} \operatorname{div}(\tilde{a}^\ell(\nabla v_{k,\ell})) = f(v,\nabla v) & \text{in } \Sigma_k \\ \tilde{a}^\ell(\nabla v_{k,\ell})\cdot\nu = 0 & \text{on } \partial\Sigma_k\,, \end{cases}$$

where $\tilde{a}^\ell$ are defined as in (3.51) with a replaced by $\tilde{a}$ given by (3.65) and $f(v,\nabla v)$ is given by (3.66). Note that, since $v \in C^{1,\theta}_{\mathrm{loc}}(\Sigma\setminus\{\mathcal{O}\})$, the functions $v_{k,\ell}$ are of class $C^{2,\theta}_{\mathrm{loc}}$ in $\overline{\Sigma}_k\setminus\{\mathcal{O}\}$, and this allows one to perform all the desired computations on the functions $v_{k,\ell}$, and then let ℓ and k to infinity. Since this approximation argument is very similar to the one in the proof of Proposition 3.2, to simplify the notation and emphasize the main ideas we shall work directly with v, assuming that v is of class $C^{2,\theta}_{\mathrm{loc}}$ in $\overline{\Sigma}\setminus\{\mathcal{O}\}$ in order to justify all the computations.

Set

$$F = 2v^\gamma S^2(W) + \gamma(\gamma-1)p(p-1)v^{\gamma-2}V^2(\nabla v) + \gamma(2p-1)v^{\gamma-1}V(\nabla v)\Delta_p^{\tilde{H}} v \tag{3.82}$$

and $L = (L_1,\dots,L_n)$ with

$$L_j = v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v) + \gamma(p-1)v^{\gamma-1}V(\nabla v)V_{\xi_j}(\nabla v)$$

for $j=1,\dots,n$. Then we apply Lemma 3.6 with $\varphi = \eta\zeta_\delta$, where $\eta\in C_c^\infty(\mathbb{R}^n)$ is a cut-off function as in the statement, and $\zeta_\delta \in C_c^\infty(\Sigma)$ is a cut-off function of the distance from $\partial\Sigma$ that converges to 1 inside Σ as $\delta\to 0$. In this way, as in the proof of (3.56), letting $\delta\to0$ the term involving $\nabla\zeta_\delta$ gives rise to a boundary term: more precisely, we obtain

$$\int_\Sigma F\eta\,dx = -\int_\Sigma \nabla\eta\cdot L\,dx + \int_{\partial\Sigma}\eta L\cdot\nu d\sigma\,. \tag{3.83}$$

Now, to conclude the proof, we need to show that the last integral in (3.83) is non-negative; indeed, for $x\in\partial\Sigma\setminus\{\mathcal{O}\}$, by using the explicit expression of L and of $S^2_{ij}(W)$ we get

$$\begin{aligned} &L(x)\cdot\nu(x) = \\ &v^\gamma(x)\tilde{a}(\nabla v(x))\cdot\nu(x)\left[\operatorname{Tr}(W)(x) + \gamma(p-1)v^{-1}(x)V(\nabla v(x))\right] \\ &- v^\gamma(x)\partial_i(\tilde{a}_j(\nabla v(x)))\tilde{a}_i(\nabla v(x))\nu_\ell(x)\,, \end{aligned} \tag{3.84}$$

where we used that $w_{ji}(x) = \partial_i\tilde{a}_j(\nabla v(x))$ and $V_{\xi_i} = \tilde{a}_i$.

We notice now that $\partial_i \nu_\ell(x)$ is the second fundamental form of $\partial\Sigma$ at x, which is non-negative definite by the convexity of Σ. Hence

$$\partial_i \nu_\ell(x) \tilde{a}_j(\nabla v(x)) \tilde{a}_i(\nabla v(x)) \geq 0. \tag{3.85}$$

From (3.84) and (3.85) we get

$$\begin{aligned} L(x) \cdot \nu(x) \geq & v^\gamma(x) \tilde{a}(\nabla v(x)) \cdot \nu(y) \left[\mathrm{Tr}\,(W)(x) + \gamma(p-1) v^{-1}(x) V(\nabla v(x)) \right] \\ & - v^\gamma(x) \nabla(\tilde{a}(\nabla v(x)) \cdot \nu(y)) \cdot \tilde{a}(\nabla v(x)). \end{aligned}$$

Now, since $\tilde{a}(\nabla v) \cdot \nu = 0$ on $\partial\Sigma$, the first term on the right-hand side vanishes. Moreover, since the tangential derivative of $\tilde{a}(\nabla v) \cdot \nu$ vanishes on $\partial\Sigma$ and $\tilde{a}(\nabla v)$ is a tangential vector-field, also the second term vanishes. This proves that $L \cdot \nu \geq 0$ on $\partial\Sigma \setminus \{\mathcal{O}\}$, that together with (3.83) (recall that $\eta \geq 0$) concludes the proof. □

Proposition 3.3. *Let v be given by* (3.63)*, let V be as in* (3.77)*, and W as in* (3.74)*. Then*

$$2 \int_\Sigma v^\gamma S^2(W)\, dx + \gamma(\gamma-1)p(p-1) \int_\Sigma v^{\gamma-2} V^2(\nabla v)\, dx + \gamma(2p-1) \int_\Sigma v^{\gamma-1} V(\nabla v) \Delta_p^{\tilde{H}} v\, dx \geq 0\,, \tag{3.86}$$

for any $\gamma < -\frac{n(p-1)}{p}$.

Proof. From (3.64), (3.68), and (3.69) we know that $|\Delta_p^{\tilde{H}} v| \leq C$ in Σ, and from Newton's inequality (3.73) we also have $|S^2(W)| \leq C$ (recall that $\mathrm{Tr}\,(W) = \Delta_p^{\tilde{H}} v$).

Now, let η be a non-negative radial cut-off function such that $\eta = 1$ in B_R, $\eta = 0$ outside B_{2R}, and $|\nabla\eta| \leq \frac{2}{R}$. Thanks to (3.68) and (3.69), we can take the limit as $R \to \infty$ in the left-hand side of (3.81) to obtain the left-hand side of (3.86). Hence, in order to prove (3.86) it is enough to show that

$$\lim_{R\to\infty} \int_{E_R} \partial_j \eta \big(v^\gamma S^2_{ij}(W) V_{\xi_i}(\nabla v) + \gamma(p-1) v^{\gamma-1} V(\nabla v) V_{\xi_j}(\nabla v) \big)\, dx = 0\,, \tag{3.87}$$

where we set for simplicity

$$E_R := \Sigma \cap (B_{2R} \setminus B_R)\,.$$

Since $|S^2_{ij}(W)| \leq |W|$, using Holder's inequality we get

$$\left| \int_{E_R} \partial_j \eta v^\gamma S^2_{ij}(W) V_{\xi_i}(\nabla v)\, dx \right| \leq \frac{c(n)}{R} \|W\|_{L^2(E_R)} \times \left(\int_{E_R} v^{2\gamma} |\nabla V(\nabla v)|^2\, dx \right)^{\frac{1}{2}}.$$

Observe that (3.70) yields

$$\|W\|^2_{L^2(E_R)} \leq C R^n\,.$$

Also, from (3.68) and (3.69) we have

$$\int_{E_R} v^{2\gamma}|\nabla V(\nabla v)|^2\,dx \le CR^{\frac{2\gamma p}{p-1}+n+2}\,.$$

Hence, since by assumption $\gamma < -\frac{n(p-1)}{p}$, this proves that

$$\lim_{R\to\infty}\int_{E_R} \partial_j\eta v^{\gamma} S^2_{ij}(W)V_{\xi_i}(\nabla v)\,dx = 0\,.$$

Analogously, using (3.68) and (3.69), the second term in (3.87) can be bounded as

$$\left|\int_{E_R} \partial_j\eta v^{\gamma-1}V(\nabla v)V_{\xi_j}(\nabla v)\,dx\right| \le CR^{\frac{p\gamma}{p-1}+n}\,, \tag{3.88}$$

which also goes to zero as $R\to\infty$ since $\gamma < -\frac{n(p-1)}{p}$. This proves (3.87) and hence (3.86). □

3.2.2 Conclusion

We multiply (3.64) by v^{-n} and integrate over Σ. By using the divergence theorem, the boundary condition in (3.64), and the decay estimates (3.68) and (3.69), we get

$$\left(\frac{p}{n-p}\right)^{p-1}\int_\Sigma v^{-n-1}\,dx - \frac{n}{p}\int_\Sigma v^{-n-1}\tilde{H}^p(\nabla v)\,dx = 0\,. \tag{3.89}$$

Now we use Newton's inequality applied to W in (3.86). More precisely, since $\mathrm{Tr}\,(W) = \Delta_p^{\tilde{H}} v$, we have

$$2S^2(W) \le \frac{n-1}{n}(\Delta_p^{\tilde{H}} v)^2\,, \tag{3.90}$$

and from (3.86) we obtain

$$\frac{n-1}{n}\int_\Sigma v^{\gamma}(\Delta_p^{\tilde{H}} v)^2\,dx + \gamma(\gamma-1)p(p-1)\int_\Sigma v^{\gamma-2}V^2(\nabla v)\,dx \\ + \gamma(2p-1)\int_\Sigma v^{\gamma-1}V(\nabla v)\Delta_p^{\tilde{H}} v\,dx \ge 0\,, \tag{3.91}$$

for any $\gamma < -\frac{n(p-1)}{p}$. Since $p<n$ we can choose $\gamma = 1-n$ in (3.91), and using (3.64), (3.66), and (3.77), we obtain

$$\left(\frac{p}{n-p}\right)^{p-1}\int_\Sigma v^{-n-1}\,dx - \frac{n}{p}\int_\Sigma v^{-n-1}\tilde{H}^p(\nabla v)\,dx \ge 0\,. \tag{3.92}$$

Recalling (3.89), this implies that the equality case must hold in (3.92). Hence the equality case must hold in (3.90) a.e., which implies that

$$W(x) = \lambda(x)\mathrm{Id} \quad \text{for a.e. } x\in\Sigma\,, \tag{3.93}$$

for some function $\lambda : \Sigma \to \mathbb{R}$, where Id is the identity matrix.

Now we show that the function λ is constant. Since

$$\lambda(x) = \frac{1}{n}\mathrm{Tr}\,(W) = \frac{1}{n}\Delta_p^{\tilde{H}} v(x) = \frac{1}{n} f(v, \nabla v)$$

(see (3.64)), and since $v \in C^{1,\theta}_{\mathrm{loc}}(\Sigma)$, we get that $\lambda \in C^{0,\theta}_{\mathrm{loc}}(\Sigma)$. Moreover, elliptic regularity theory yields that $v \in C^{2,\theta}_{\mathrm{loc}}(\Sigma \cap \{\nabla v \neq 0\})$, which implies that $\lambda \in C^{1,\theta}_{\mathrm{loc}}(\Sigma \cap \{\nabla v \neq 0\})$. From (3.93) we have that

$$\partial_i(\tilde{a}_j(\nabla v(x))) = \lambda(x)\delta_{ij} \tag{3.94}$$

for $i, j \in \{1, \ldots, n\}$, which implies that $\tilde{a}(\nabla v) \in C^{2,\theta}_{\mathrm{loc}}(\Sigma \cap \{\nabla v \neq 0\})$.

Then, given $i \in \{1, \ldots, n\}$, choosing $j \neq i$ and using (3.94) we obtain

$$\partial_i \lambda(x) = \partial_i\big(\partial_j(\tilde{a}_j(\nabla v(x)))\big) = \partial_j\big(\partial_i(\tilde{a}_j(\nabla v(x)))\big) = 0$$

for any $x \in \Sigma \cap \{\nabla v \neq 0\}$, which implies that λ is constant on each connected component of $\Sigma \cap \{\nabla v \neq 0\}$. Since λ is continuous in Σ and $\{\nabla v = 0\}$ has no interior points (this follows easily from (3.64)), we deduce that λ is constant. In particular, recalling (3.93), we get

$$\nabla[\tilde{a}(\nabla v(x))] = W(x) = \lambda\,\mathrm{Id} \qquad \text{in } \Sigma\,.$$

Hence $\tilde{a}(\nabla v(x)) = \lambda(x - x_0)$ for some $x_0 \in \overline{\Sigma}$, and from the boundary condition in (3.64) we obtain that $x_0 \in \partial\Sigma$. This implies that $v(x) = c_1 + c_2 \tilde{H}_0(x - x_0)^{\frac{p}{p-1}}$, or equivalently (recalling (3.63)) $u(x) = \mathcal{U}^H_{\mu,x_0}(x)$ for some $\mu > 0$. Finally, it is clear that:

- if $\Sigma = \mathbb{R}^n$ and x_0 may be a generic point in $\mathbb{R}^n$;
- if $k \in \{1, \ldots, n-1\}$ then $x_0 \in \mathbb{R}^k \times \{\mathcal{O}\}$;
- if $k = 0$ then $x_0 = \mathcal{O}$.

This completes the proof of Theorem 3.1.

Chapter 4

Sobolev-type inequalities on Cartan-Hadamard manifolds

4.1 Statements of the main results

As already mentioned in the Introduction, in this Chapter we consider Sobolev-type inequalities on *Cartan-Hadamard* manifold M of dimension $n \geq 3$, namely on complete, noncompact, simply connected Riemannian manifold with everywhere nonpositive sectional curvatures. The starting remark which motivates our study is the following: from one hand it is well known that on any *Cartan-Hadamard* manifold M the Euclidean *Sobolev inequality*

$$\|f\|_{L^{2^*}(M)} \leq C_S \|\nabla f\|_{L^2(M)} \quad \forall f \in C_c^1(M), \qquad 2^* := \frac{2n}{n-2} \tag{4.1}$$

holds. On the other hand, if the sectional curvatures are everywhere bounded from above by a negative constant $-k$, then in addition to (4.1) also the *Poincaré inequality*

$$\|f\|_{L^2(M)} \leq \frac{2}{\sqrt{k}\,(n-1)} \|\nabla f\|_{L^2(M)} \qquad \forall f \in C_c^1(M) \tag{4.2}$$

holds, or equivalently the infimum of the spectrum of (minus) the Laplace-Beltrami operator on M is bounded from below by $k(n-1)^2/4$, i.e. Δ has a *spectral gap*. This is a celebrated result due to H.P. McKean [167], which we discuss extensively in Section 4.4 (Theorem 4.4). Note that the spectral bound is optimal since it is attained by the *hyperbolic space* $\mathbb{H}^n$ of curvature $-k$.

As already mentioned in the Introduction, in this Chapter, we suppose that the sectional curvatures of the Cartan-Hadamard manifold satisfy a bound of the following type

$$\mathrm{Sect}(x) \leq -K\, r^{-\beta} \qquad \forall x \in M \setminus B_{R_0}, \tag{4.3}$$

for some $\beta \in (0, 2]$ and $K, R_0 > 0$, where $r = r(x) := \mathrm{dist}(x, o)$ denotes the geodesic distance from x to a fixed point o (the *pole* of the manifold). Then the question is:

what kind of inequalities does M support?

The answers we show in the Chapter are nontrivial. First of all, one has to distinguish between the so-called *sub-hyperbolic* range ($\beta \in (0,2)$) and the *quasi-Euclidean* range ($\beta = 2$), following a terminology originally introduced in [120]. Another crucial difference is between *radial* and *nonradial* functions. We shall focus on the following *Sobolev-type inequalities*

$$\|f\|_{L^p(M)} \le C_p \, \|\nabla f\|_{L^2(M)} \, , \tag{4.4}$$

where p is a suitable exponent belonging to the interval $(2, 2^*]$.

Let us briefly describe the results we prove, which are stated precisely later. In the case $\beta \in (0,2)$ we show that (4.4) holds in the *radial* setting for all $p \in (2, 2^*]$, for a positive constant C_p of the form

$$C_p \equiv \frac{C \, p^{\frac{2+\beta}{2(2-\beta)}}}{(p-2)^{\frac{\beta}{2-\beta}}} \, , \tag{4.5}$$

C being another positive constant that depends only on the constants n, β, K, R_0 appearing in (4.3). The result is optimal with respect to the dependence on p (see Theorem 4.1). In the case $\beta = 2$, namely negative curvatures that can decay quadratically at infinity, inequalities (4.4) (still in the radial setting) start to hold from a certain exponent $\tilde{2} \in (2, 2^*)$ that depends on n and K, which tends to 2^* as $K \to 0$ and to 2 as $K \to \infty$. Hence, one is no more allowed to let $p \downarrow 2$. This result is also optimal with respect to p, see Theorem 4.2 for the details. Finally, we prove in Theorem 4.3 that out of the radial setting all of the above results *fail*: namely, it is enough to assume that (4.3) is satisfied with reverse inequality (in fact it suffices to require the same bound on the *Ricci* curvature) to be able to construct a sequence of nonradial functions that make the constant C_p in (4.4) blow up for every $p < 2^*$. We point out that the case $\beta > 2$ is not interesting because it is essentially Euclidean, i.e. the sole inequality of the type of (4.4) that holds, even if restricted to radial functions, is the standard Sobolev one: this is an immediate consequence of our results, see Remark 4.2.

The techniques of proof that we exploit take advantage of two main ingredients: *one-dimensional weighted* functional inequalities and *Laplacian-comparison* theorems, which are recalled in Subsections 4.2.4 and 4.2.2, respectively. The idea is to first study the radial inequalities on *model manifolds*, namely spherically-symmetric Riemannian manifolds whose metric g can be written as

$$g \equiv dr^2 + \psi(r)^2 \, d\theta^2_{\mathbb{S}^{n-1}}$$

for some regular "model" function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ (see Definition 4.7), where $d\theta^2_{\mathbb{S}^{n-1}}$ is the usual metric on the $(n-1)$-dimensional unit sphere. In this context, (4.4) becomes a family of one-dimensional weighted inequalities, where the associated weight is just $\psi(r)^{n-1}$. Then, by resorting again to Laplacian comparison, as well as to *surface-measure comparison* (see also Subsection 4.2.1), one can extend the results to general Cartan-Hadamard manifolds.

As we mentioned in the Introduction an important motivation to study the validity of (4.4) on Cartan-Hadamard manifolds under curvature bounds like (4.3) came from the recent work [120], where the authors investigate the asymptotic behavior of nonnegative solutions to the *porous medium equation* (see the monograph [223])

$$\begin{cases} u_t = \Delta(u^m) & \text{in } M \times \mathbb{R}^+ \,, \\ u = u_0 & \text{on } M \times \{0\} \,, \end{cases} \tag{4.6}$$

where $m > 1$. They prove (in particular) that if $\mathrm{Sect}(x) \sim r^{-\beta}$ for some $\beta \in (0, 2)$ then solutions to (4.6) starting from nontrivial compactly-supported data satisfy

$$u(x,t) \sim t^{-\frac{1}{m-1}} \left[\gamma \left(\log t \right)^{\frac{2+\beta}{2-\beta}} - r^{\frac{2+\beta}{2}} \right]_+^{\frac{1}{m-1}} \qquad \text{as } t \to \infty \,,$$

for a suitable positive constant γ. Such bounds are compatible with the L^1-L^∞ *smoothing effects* proved in [118] under the sole assumption that inequalities (4.4) are satisfied with a constant C_p as in (4.5). It was by combining these results that we conjectured that, at least in the radial framework, the above inequalities might hold indeed. Completely analogous connections can be established in the quasi-Euclidean case. For more details on this discussion, we refer to Section 4.6.

Concerning the organization, the Chapter is organized as follows: in Section 4.1 we state our main results, after a brief introduction to notations. Section 4.2 recalls some preliminary tools in Riemannian geometry and functional inequalities, which are key concepts in order to carry out our methods of proof. In Section 4.3 we provide the proofs of the radial inequalities (Theorems 4.1 and 4.2), while in Section 4.4 we focus on the Poincaré case $p = 2$ and give an alternative proof of McKean's Theorem, under somewhat weaker assumptions (see Theorems 4.5 and 4.6). Section 4.5 deals with the nonradial case, namely the disproof of the analogues of Theorems 4.1 and 4.2 for nonradial functions (see Theorem 4.3 and Remark 4.2). The last section shows how the functional inequalities established here yield optimal smoothing estimates for the (radial) porous medium equation flow on the manifold at hand (Theorems 4.7 and 4.8).

4.1.1 Basic notations

Given an n-dimensional ($n \geq 2$) Riemannian manifold (M, g) and $x \in M$, we shall denote by $\mathrm{Sect}(x)$ the sectional curvature w.r.t. *any* 2-dimensional tangent subspace at $x \in M$ and by $\mathrm{Ric}(x)$ the Ricci curvature at $x \in M$, as a quadratic form on T_xM, the latter being the tangent space at x.

We shall denote by $C^1_{c:\mathrm{rad}}(M)$ the space of all C^1 functions on M, with compact support, which are radial with respect to some (fixed) point $o \in M$, i.e.

$$C^1_{c:\mathrm{rad}}(M) := \left\{ f \in C^1_c(M) : \; f(x) \equiv f(\mathrm{d}(x, o)) \quad \forall x \in M \right\},$$

where $r = r(x) := \mathrm{d}(x, o)$ is the geodesic distance from o, which is called *pole*, to x. Furthermore, for all $1 \leq p < \infty$

$$\|f\|_{L^p(M)} := \left(\int_M |f|^p \right)^{\frac{1}{p}} ,$$

where the integral is computed with respect to $d\nu$: the Riemannian measure of M; while $\|f\|_{L^\infty(M)}$ stands for the essential supremum of $|f|$. The Sobolev critical exponent is, as usual,

$$2^* := \begin{cases} \frac{2n}{n-2} & \text{if } n \geq 3\,, \\ \infty & \text{if } n = 2\,. \end{cases}$$

In many parts of the Chapter we shall need to deal with spherically symmetric, complete and noncompact manifolds M for which the Riemannian metric has the following special structure:

$$g \equiv dr^2 + \psi(r)^2\, d\theta^2_{\mathbb{S}^{n-1}}\,,$$

where $d\theta^2_{\mathbb{S}^{n-1}}$ is the standard metric on $\mathbb{S}^{n-1}$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function belonging to the class $\mathcal{A}$, which is defined as

$$\mathcal{A} := \left\{ \psi \in C^\infty((0,\infty)) \cap C^1([0,\infty)) : \ \psi(0) = 0\,,\ \psi(r) > 0 \ \forall r > 0\,,\ \psi'(0) = 1 \right\}. \tag{4.7}$$

Such Riemannian manifolds are referred to as *model manifolds*, and will be denoted by $\mathbb{M}^n_\psi$. For example, the Euclidean space corresponds to $\psi(r) = r$, while the hyperbolic space corresponds to $\psi(r) = \sinh r$.

By an n-dimensional *Cartan-Hadamard* manifold $M \equiv \mathbb{M}^n$ we mean a complete, noncompact, simply connected Riemannian manifold with everywhere nonpositive sectional curvatures. First of all, we observe that on Cartan-Hadamard manifolds the cut locus of any point o is empty; hence, for every $x \in M \setminus \{o\}$ one can define polar coordinates with pole at o, namely $r(x) = \mathrm{d}(x,o)$ and $\theta \in \mathbb{S}^{N-1}$. We then denote by B_r the Riemannian ball of radius r centered at o and $S_r := \partial B_r$. In the case of Cartan-Hadamard manifolds, we also denote by $\mathrm{Sect}_\omega(x)$ the sectional curvature w.r.t. to any 2-dimensional tangent subspace ω at x containing a *radial* direction, and by $\mathrm{Ric}_o(x)$ the Ricci curvature at x evaluated in the radial direction corresponding to the pole o.

4.1.2 The sub-hyperbolic radial case

In the first result of this paper, we prove that if the (radial) sectional curvatures of a Cartan-Hadamard manifold do not decay too fast at infinity, namely slower than an inverse-quadratic rate with respect to r, then a suitable family of Sobolev-type inequalities holds in the *radial* framework. The terminology *sub-hyperbolic* is borrowed from [120].

Theorem 4.1. *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold such that*

$$\mathrm{Sect}_\omega(x) \leq -C_0\, r^{-\beta} \qquad \forall x \in \mathbb{M}^n \setminus B_{R_0}\,, \tag{4.8}$$

for some $\beta \in (0,2)$ and $C_0, R_0 > 0$. Then there exists a positive constant C, depending only on n, β, C_0, R_0, such that for every $p \in (2, 2^*] \cap (2,\infty)$ the following radial Sobolev-type inequalities

$$\|f\|_{L^p(\mathbb{M}^n)} \leq \frac{C\, p^{\frac{2+\beta}{2(2-\beta)}}}{(p-2)^{\frac{\beta}{2-\beta}}}\, \|\nabla f\|_{L^2(\mathbb{M}^n)} \qquad \forall f \in C^1_{c;\mathrm{rad}}(\mathbb{M}^n) \tag{4.9}$$

hold. Moreover, the dependence on p of the multiplying constant in (4.9) *is optimal, in the sense that for each $\beta \in (0,2)$ there exists a model manifold $\mathbb{M}^n_\psi$,* *complying with* (4.8)*, such that*

$$\inf_{f\in C^1_{c:\mathrm{rad}}(\mathbb{M}^n_\psi),\, f\not\equiv 0} \frac{\|\nabla f\|_{L^2(\mathbb{M}^n_\psi)}}{\|f\|_{L^p(\mathbb{M}^n_\psi)}} \le \frac{(p-2)^{\frac{\beta}{2-\beta}}}{C\,p^{\frac{2+\beta}{2(2-\beta)}}} \qquad \forall p\in(2,2^*]\cap(2,\infty) \tag{4.10}$$

for another positive constant C depending on n,β,C_0,R_0.

4.1.3 The quasi-Euclidean radial case

In the case of curvatures that decay with a rate which is *at most* quadratic, we still have radial Sobolev-type inequalities: however, in this case, they start to hold from a certain exponent which is *strictly* larger than 2. Again, the terminology *quasi-Euclidean* is in agreement with [120].

Theorem 4.2. *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold such that*

$$\mathrm{Sect}_\omega(x) \le -C_1\, r^{-2} \qquad \forall x\in\mathbb{M}^n\setminus B_{R_0} \tag{4.11}$$

for some C_1 and $R_0>0$. Then there exists a positive constant C, depending only on n,C_1,R_0, such that for every $p\in\left[\tilde{2},2^*\right]\cap\left[\tilde{2},\infty\right)$ the following radial Sobolev-type inequalities

$$\|f\|_{L^p(\mathbb{M}^n)} \le C\,\sqrt{p}\,\|\nabla f\|_{L^2(\mathbb{M}^n)} \qquad \forall f\in C^1_{c:\mathrm{rad}}(\mathbb{M}^n) \tag{4.12}$$

hold, where

$$\tilde{2} := \frac{2\tilde{n}}{\tilde{n}-2}\,, \qquad \tilde{N} := \frac{n+1+\sqrt{1+4C_1}\,(n-1)}{2}\,. \tag{4.13}$$

Moreover, the result is optimal w.r.t. to the exponent p, in the sense that for each $C_1>0$ there exists a model manifold $\mathbb{M}^n_\psi$, complying with (4.11)*, such that* (4.12) *fails for all $p\in[2,\tilde{2})$ and, in the case $n=2$, the optimal constant in* (4.12) *does behave like $\sqrt{p}$ (up to multiplicative constants) as $p\to\infty$.*

Remark 4.1 (Ricci bounds from above). It is worth pointing out that actually the thesis of Theorem 4.1 still holds if one replaces $\mathrm{Sect}_\omega(x)$ with $\mathrm{Ric}_o(x)$ in assumption (4.8): this is due to the fact that Laplacian-comparison results with model manifolds, which we exploit extensively throughout the paper, can also be established under such a weaker hypothesis. The argument applies to Theorem 4.2 as well, except that in this case the analogue of exponent $\tilde{2}$ in (4.13) is no more optimal, just because one has (4.24) in place of (4.20). Same observations can be made with regards to Theorems 4.5 (no optimal constant however) and 4.6, both dealing with the Poincaré inequality. The key result on which these extensions rely can be found in the monograph [228]: see Subsection 4.2.2.

4.1.4 Failure of the nonradial inequalities

Surprisingly enough, all of the above inequalities (except the *Euclidean* Sobolev inequality) *fail* in the nonradial framework under reverse curvature bounds. Indeed, we have the following.

Theorem 4.3. *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold such that*

$$\mathrm{Ric}(x) \geq -C_2\, r^{-\beta} \qquad \forall x \in \mathbb{M}^n \setminus B_{R_0}\,, \tag{4.14}$$

for some $\beta \in (0,2)$ and $R_0, C_2 > 0$. Let $p \in [2, 2^*)$. Then there exists no positive constant C for which the following Sobolev-type inequality

$$\|f\|_{L^p(\mathbb{M}^n)} \leq C\, \|\nabla f\|_{L^2(\mathbb{M}^n)} \qquad \forall f \in C^1_c(\mathbb{M}^n) \tag{4.15}$$

holds.

Remark 4.2 (Optimality of the curvature bounds). It is plain that Theorem 4.3 implies that the conclusions of Theorems 4.1 and 4.2 *cannot* hold, in general, in the nonradial framework: indeed, for each given $\beta \in (0,2]$, it is enough to consider a Cartan-Hadamard manifold (e.g. a model) satisfying

$$\mathrm{Sect}(x) \sim r^{-\beta} \qquad \text{as } r \to \infty\,.$$

Any such a manifold clearly complies with both the assumptions of Theorem 4.1 (or Theorem 4.2 if $\beta = 2$) and the ones of Theorem 4.3, so that the thesis of the latter prevents the validity of (4.9) (or (4.12)) extended to *nonradial* functions.

For analogous reasons, we do not treat the case $\beta > 2$: by following a strategy similar to proof of optimality of Theorem 4.2, it is not difficult to check that for any $\beta > 2$ one can construct a model manifold such that $\mathrm{Sect}(x) \sim r^{-\beta}$ as $r \to \infty$, for which any of the radial inequalities (4.9) fails as long as $p < 2^*$ (we omit details and refer to [120, Section 2.3, Type IV]). Hence, in general no inequality of the type of (4.9) different from the classical Sobolev one can hold.

Remark 4.3 (Validity of the inequalities for more general functions). For simplicity, we have stated the above results of Theorems 4.1–4.2 (as well as those of Theorems 4.4, 4.5 and 4.6 below) for functions in $C^1_c(\mathbb{M}^n)$. Nevertheless, by means of standard density arguments, it is apparent that they also hold for compactly supported *Lipschitz* functions or, more in general, for all functions belonging to the closure of $C^1_c(\mathbb{M}^n)$ w.r.t. the L^2 norm of the gradient.

4.2 Geometric and functional preliminaries

In the following, we recall some basic facts in Riemannian geometry concerning volume, surface and Laplacian comparison (Subsections 4.2.1–4.2.3), along with a key result related to weighted one-dimensional Sobolev-type inequalities (Subsection 4.2.4).

4.2.1 Notations from Riemannian geometry

We shall adopt the same notations as in Section 4.1.1. If $\mathbb{M}^n$ is an n-dimensional Cartan-Hadamard manifold, then one the surface measure of spheres reads

$$\mathrm{meas}\,(S_r) = \int_{\mathbb{S}^{n-1}} A(r,\theta)\, d\theta\,, \qquad \text{where } d\theta := d\theta_1 \ldots d\theta_{n-1} \tag{4.16}$$

and $A(r,\theta)$ is the weight associated with the volume measure of $\mathbb{M}^n$ w.r.t. polar coordinates, which turns out to be the square root of the determinant of the metric matrix written in such coordinates (see e.g. [115, Section 3]). In particular, the latter is a nonnegative $C^\infty(\mathbb{R}^+ \times \mathbb{S}^{n-1})$ function. Hence, if f is a radial function as in (4.9) or in (4.12), then by Fubini's Theorem

$$\begin{aligned}\|f\|_{L^p(M)} = \left(\int_M |f|^p\right)^{\frac{1}{p}} &= \left(\int_0^\infty \int_{\mathbb{S}^{n-1}} |f(r)|^p \, A(r,\theta)\, d\theta\, dr\right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty |f(r)|^p \operatorname{meas}(S_r)\, dr\right)^{\frac{1}{p}},\end{aligned} \tag{4.17}$$

so that radial Sobolev-type inequalities can be rewritten as one-dimensional weighted inequalities (see Subsection 4.2.4).

It is direct to see that the Laplace-Beltrami operator on $\mathbb{M}^n$ in polar coordinates has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \mathsf{m}(r,\theta)\, \frac{\partial}{\partial r} + \Delta_{S_r}\,,$$

where Δ_{S_r} is the Laplace-Beltrami operator on the submanifold S_r and

$$\mathsf{m}(r,\theta) = \frac{\partial}{\partial r}\left(\log A(r,\theta)\right) \qquad \forall x \equiv (r,\theta) \in \mathbb{R}^+ \times \left(\mathbb{S}^{n-1} \setminus \mathcal{P}\right), \tag{4.18}$$

where $\mathcal{P}$ is the (finite) set of poles on $\mathbb{S}^{n-1}$, namely all angles $\theta \in \mathbb{S}^{n-1}$ at which $A(r,\theta)$ vanishes identically. Elsewhere, $A(r,\theta)$ is always strictly positive. Note that $\mathsf{m}(r,\theta)$ is just the Laplacian of the distance function $x \equiv (r,\theta) \mapsto r$. So, by integrating (4.18) from a fixed $r_0 > 0$ to $r > r_0$ we deduce that

$$\int_{r_0}^r \mathsf{m}(s,\theta)\, ds = \log A(r,\theta) - \log A(r_0,\theta)\,,$$

i.e.

$$A(r,\theta) = e^{\int_{r_0}^r \mathsf{m}(s,\theta)\, ds + c_\theta}$$

with $c_\theta := \log A(r_0,\theta)$. As a result, recalling (4.16), we get that

$$\operatorname{meas}(S_r) = \int_{\mathbb{S}^{n-1}} e^{\int_{r_0}^r \mathsf{m}(s,\theta)\, ds + c_\theta}\, d\theta\,.$$

4.2.2 Laplacian comparison

We recall here some classical results which allow one to compare the Laplacian (as well as the Hessian in some cases) of the distance function of a Cartan-Hadamard manifold with the Laplacian of the distance function of a suitable model manifold corresponding to the curvature bounds (as a reference see e.g. [114, Section 2] or [115, Section 15]). More precisely, if

$$\operatorname{Sect}_\omega(x) \le -\frac{\psi''(r)}{\psi(r)} \qquad \forall x \equiv (r,\theta) \in M \setminus \{o\} \tag{4.19}$$

for some function $\psi \in \mathcal{A}$, then

$$\mathsf{m}(r,\theta) \ge (n-1)\, \frac{\psi'(r)}{\psi(r)} \qquad \forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}\,. \tag{4.20}$$

Similarly, if

$$\mathrm{Ric}_o(x) \geq -(n-1)\,\frac{\psi''(r)}{\psi(r)} \qquad \forall x \equiv (r,\theta) \in M \setminus \{o\}$$

for another function $\psi \in \mathcal{A}$, then

$$\mathsf{m}(r,\theta) \leq (n-1)\,\frac{\psi'(r)}{\psi(r)} \qquad \forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}\,.$$

Though we shall mostly use them in the Cartan-Hadamard setting, let us point out that the above inequalities are true in more general Riemannian manifolds (at least manifolds with a pole).

As a simple consequence of Laplacian comparison with the Euclidean space (just use (4.19)–(4.20) with $\psi(r) = r$), on any Cartan-Hadamard manifold there holds

$$\mathsf{m}(r,\theta) \geq \frac{n-1}{r} \qquad \forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}\,. \tag{4.21}$$

In particular, thanks to (4.18) and the fact that $\mathbb{M}^n$ is locally Euclidean, we immediately deduce that

$$\text{the function } r \mapsto A(r,\theta) \text{ is nondecreasing for all } \theta \in \mathbb{S}^{n-1} \setminus \mathcal{P} \tag{4.22}$$

and

$$\frac{\partial A}{\partial r}(0,\theta) > 0 \qquad \forall \theta \in \mathbb{S}^{n-1} \setminus \mathcal{P}\,. \tag{4.23}$$

Actually, in the special case of Cartan-Hadamard manifolds, a comparison result similar to the first one can be deduced by replacing the (radial) sectional curvatures $\mathrm{Sect}_\omega(x)$ with the Ricci curvature evaluated in the radial direction w.r.t. the pole o, which we denote $\mathrm{Ric}_o(x)$. Namely, if

$$\mathrm{Ric}_o(x) \leq -(n-1)\,\frac{\psi''(r)}{\psi(r)} \qquad \forall x \equiv (r,\theta) \in \mathbb{M}^n \setminus \{o\}$$

for some function $\psi \in \mathcal{A}$, then

$$\mathsf{m}(r,\theta) \geq \sqrt{n-1}\,\frac{\psi'\big(\sqrt{n-1}\,r\big)}{\psi\big(\sqrt{n-1}\,r\big)} \qquad \forall (r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}\,. \tag{4.24}$$

This is basically due to the fact that the *Hessian* of the distance function on $\mathbb{M}^n$ has nonnegative eigenvalues: we refer to [228, Theorem 2.15].

By exploiting Laplacian comparison with carefully chosen model functions $\psi \in \mathcal{A}$, one can easily prove the following (for the details see e.g. [120, Lemma 4.1]).

Lemma 4.1. *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold satisfying* (4.8) *for some $\beta \in [0,2)$ and $C_0, R_0 > 0$. Then there exist $r_0 = r_0(\beta, C_0, R_0) > 0$ and $c = c(n,\beta,C_0,R_0) > 0$ such that*

$$\mathsf{m}(r,\theta) \geq c\,r^{-\frac{\beta}{2}} \qquad \forall (r,\theta) \in [r_0,\infty) \times \mathbb{S}^{n-1}\,.$$

4.2.3 Geometric interpretation of the scalar curvature

Given a generic n-dimensional Riemannian manifold (M, g), it is well known (see for instance [28, Introduction] or [49, p. 133]) that *the scalar curvature* S_g is linked to the volume of balls: in particular, when S_g is positive the volume of the balls in M is smaller than the volume of the balls of the same radius in the Euclidean space. On the other hand, when S_g is negative we have the opposite relation. This facts can be made quantitative, at least for small balls. More precisely, the ratio between the volume of a ball $B_\varepsilon(p) \subset M$ of radius $\varepsilon > 0$ centered at $p \in M$ and the Euclidean volume of the corresponding ball $\mathbb{B}_\varepsilon \subset \mathbb{R}^n$ centered at the origin is given by

$$\frac{\mathrm{Vol}(B_\varepsilon(p))}{|\mathbb{B}_\varepsilon|} = 1 - \frac{S_g}{6(n+2)}\,\varepsilon^2 + O(\varepsilon^4)\,,$$

where $|\cdot|$ stands for the Euclidean volume. Moreover, the boundaries of these balls, i.e. $\partial B_\varepsilon(p) = S_\varepsilon(p)$ and $\partial \mathbb{B}_\varepsilon = \mathbb{S}_\varepsilon$, are $(n-1)$-dimensional geodesic spheres of radius $\varepsilon > 0$ whose ratio of the corresponding surface (or Hausdorff) measures satisfies

$$\frac{\mathrm{meas}\,(S_\varepsilon(p))}{\mathrm{meas}\,(\mathbb{S}_\varepsilon)} = 1 - \frac{S_g}{6n}\,\varepsilon^2 + O(\varepsilon^4)\,. \tag{4.25}$$

4.2.4 One-dimensional weighted inequalities

In the following, by a *weight* in $\mathbb{R}^+$ we simply mean any positive $L^1_{\mathrm{loc}}([0,\infty))$ function, even though in the rest of the paper we shall in fact deal with more regular functions. The techniques we exploit in Sections 4.3 and 4.4 take advantage of some results for *one-dimensional* weighted Sobolev-type inequalities (or Hardy-type inequalities according to the terminology of [149]), which have been known for a long time. In this regard, we shall mainly refer to the monograph [149] by A. Kufner and P. Opic, which collects several results from this perspective (not only in the one-dimensional framework by the way).

Proposition 4.1 ([149, Theorem 6.2]). *Let w be a weight in $\mathbb{R}^+$. Let $p \in [2,\infty)$. Then the Sobolev-type inequality*

$$\left(\int_0^\infty |g(r)|^p\, w(r)\, dr\right)^{\frac{1}{p}} \le C \left(\int_0^\infty |g'(r)|^2\, w(r)\, dr\right)^{\frac{1}{2}} \qquad \forall g \in C^1_c([0,\infty)) \tag{4.26}$$

holds for some $C > 0$ if and only if

$$\mathcal{B}(w,p) := \sup_{r\in(0,\infty)} \left(\int_0^r w(s)\, ds\right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{w(s)}\, ds\right)^{\frac{1}{2}} < \infty\,, \tag{4.27}$$

and the optimal constant C appearing in (4.26) *satisfies the two-sided bound*

$$\mathcal{B}(w,p) \le C \le \left(1 + \frac{p}{2}\right)^{\frac{1}{p}} \left(1 + \frac{2}{p}\right)^{\frac{1}{2}} \mathcal{B}(w,p)\,. \tag{4.28}$$

4.3 Proofs of the radial results

We devote this section to the proof of Theorems 4.1 and 4.2, so that here we shall focus only on radial functions. Nonradial issues will then be addressed in Sections 4.4 and 4.5.

4.3.1 The sub-hyperbolic case

We start this subsection by a key result showing that, under suitable assumptions on the weight $w(r) \equiv \psi(r)^{n-1}$, where $\psi \in \mathcal{A}$ (recall (4.7)) is any function corresponding to sub-hyperbolic model manifolds (according to the terminology adopted in Subsection 4.1.2), the supremum appearing in the statement of Theorem 4.1 can be bounded from above in a quantitative way.

Lemma 4.2. *Let $\psi \in \mathcal{A}$ satisfy the following assumptions:*

$$\psi(r) \geq r \quad \forall r \geq 0 \,, \qquad \psi'(r) \geq 0 \quad \forall r \geq 0 \,, \qquad \frac{\psi'(r)}{\psi(r)} \geq \frac{c}{r^\alpha} \quad \forall r \geq r_0 \,, \tag{4.29}$$

for some $\alpha \in (0,1)$ and $c, r_0 > 0$. Let $n \in \mathbb{N}$ be larger than or equal to 2. Then there exists a positive constant C, depending only on n, α, c, r_0, such that for every $p \in (2, 2^*)$ there holds

$$\sup_{r \in (0,\infty)} \left(\int_0^r \psi(s)^{n-1} \, ds \right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}} \, ds \right)^{\frac{1}{2}} \leq \frac{C \, p^{\frac{1+\alpha}{2(1-\alpha)}}}{(p-2)^{\frac{\alpha}{1-\alpha}}} \,. \tag{4.30}$$

Proof. First of all, let us establish that the l.h.s. of (4.30) is finite. To this end, set

$$Q(r) := \left(\int_0^r \psi(s)^{n-1} \, ds \right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}} \, ds \right)^{\frac{1}{2}} \qquad \forall r > 0 \,. \tag{4.31}$$

The integration of the last inequality in (4.29) from r_0 to $r > r_0$, along with the first inequality, yields the bound from below

$$\psi(r) \geq k \, e^{\frac{c}{1-\alpha} r^{1-\alpha}} \quad \forall r \geq r_0 \,, \qquad \text{where } k = k(\alpha, c, r_0) := r_0 \, e^{-\frac{c}{1-\alpha} r_0^{1-\alpha}} \,, \tag{4.32}$$

which readily ensures that $Q(r)$ is a smooth function on $(0, \infty)$. In addition, because $\psi(r) \sim r$ as $r \to 0$ and $p < 2^*$, it is immediate to check that $\lim_{r \to 0} Q(r) = 0$. In order to deal with the behaviour of $Q(r)$ at infinity, we need some more integral estimates. To this aim, upon rewriting the last inequality in (4.29) as

$$\psi(r)^{n-1} \leq \frac{r^\alpha}{c(n-1)} \frac{d}{dr} \left(\psi^{n-1}\right)(r) \qquad \forall r \geq r_0$$

and integrating (by parts) between r_0 and $r > r_0$, we obtain:

$$\begin{aligned} \int_{r_0}^r \psi(s)^{n-1} \, ds &\leq \frac{1}{c(n-1)} \int_{r_0}^r s^\alpha \frac{d}{ds} \left(\psi^{n-1}\right)(s) \, ds \\ &= \frac{1}{c(n-1)} \left[r^\alpha \, \psi(r)^{n-1} - r_0^\alpha \, \psi(r_0)^{n-1} - \alpha \int_{r_0}^r \frac{\psi(s)^{n-1}}{s^{1-\alpha}} \, ds \right] \\ &\leq \frac{1}{c(n-1)} \, r^\alpha \, \psi(r)^{n-1} \,. \end{aligned} \tag{4.33}$$

By plugging estimate (4.33) in (4.31), exploiting (4.32) and the fact that $\psi(r)$ is nondecreasing, we deduce that

$$\begin{aligned}
Q(r) &= \left(\int_0^{r_0} \psi(s)^{n-1}\,ds + \int_{r_0}^{r} \psi(s)^{n-1}\,ds\right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds\right)^{\frac{1}{2}} \\
&\le \left(\int_0^{r_0} \psi(s)^{n-1}\,ds + \frac{1}{c(n-1)}\, r^\alpha\, \psi(r)^{n-1}\right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds\right)^{\frac{1}{2}} \\
&= \left(\frac{1}{r^\alpha\, \psi(r)^{n-1}} \int_0^{r_0} \psi(s)^{n-1}\,ds + \frac{1}{c(n-1)}\right)^{\frac{1}{p}} \times \\
&\quad \left[\left(r^\alpha\, \psi(r)^{n-1}\right)^{\frac{2}{p}} \int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds\right]^{\frac{1}{2}} \\
&\le \left(\frac{1}{r^\alpha\, \psi(r)^{n-1}} \int_0^{r_0} \psi(s)^{n-1}\,ds + \frac{1}{c(n-1)}\right)^{\frac{1}{p}} \left(\int_r^\infty \frac{s^{\frac{2\alpha}{p}}}{\psi(s)^{\frac{(n-1)(p-2)}{p}}}\,ds\right)^{\frac{1}{2}} \\
&\le \left(\frac{e^{-\frac{c(n-1)}{1-\alpha} r^{1-\alpha}}}{k^{n-1}\, r^\alpha} \int_0^{r_0} \psi(s)^{n-1}\,ds + \frac{1}{c(n-1)}\right)^{\frac{1}{p}} \times \\
&\quad \left(\int_r^\infty \frac{s^{\frac{2\alpha}{p}}\, e^{-\frac{c(n-1)(p-2)}{(1-\alpha)p} s^{1-\alpha}}}{k^{\frac{(n-1)(p-2)}{p}}}\,ds\right)^{\frac{1}{2}}
\end{aligned}$$

for all $r > r_0$, whence $\lim_{r\to\infty} Q(r) = 0$. As a consequence, because $Q(r)$ is smooth and positive in $(0,\infty)$, it admits a maximum at some $\overline{r} > 0$, which is a critical point. Since

$$\begin{aligned}
Q'(r) = {} & \frac{\psi(r)^{n-1}}{p} \left(\int_0^r \psi(s)^{n-1}\,ds\right)^{\frac{1}{p}-1} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds\right)^{\frac{1}{2}} \\
& - \frac{1}{2\,\psi(r)^{n-1}} \left(\int_0^r \psi(s)^{n-1}\,ds\right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds\right)^{\frac{1}{2}-1},
\end{aligned}$$

at $r = \overline{r}$ we find the identity

$$\int_{\overline{r}}^\infty \frac{1}{\psi(s)^{n-1}}\,ds = \frac{p}{2}\,\frac{1}{\psi(\overline{r})^{2n-2}} \int_0^{\overline{r}} \psi(s)^{n-1}\,ds\,, \tag{4.34}$$

so that

$$Q(\overline{r}) = \left(\frac{p}{2}\right)^{\frac{1}{2}} \frac{1}{\psi(\overline{r})^{n-1}} \left(\int_0^{\overline{r}} \psi(s)^{n-1}\,ds\right)^{\frac{p+2}{2p}}. \tag{4.35}$$

In particular,

$$\sup_{r\in(0,\infty)} Q(r) \le \left(\frac{p}{2}\right)^{\frac{1}{2}} \sup_{r\in(0,\infty)} \frac{1}{\psi(r)^{n-1}} \left(\int_0^{r} \psi(s)^{n-1}\,ds\right)^{\frac{p+2}{2p}}. \tag{4.36}$$

Therefore, in order to establish (4.30), it is enough to prove an analogous upper bound on the r.h.s. of (4.36). To this aim, first of all note that the first two inequalities of (4.29) entail

$$\frac{1}{\psi(r)^{n-1}}\left(\int_0^r \psi(s)^{n-1}\,ds\right)^{\frac{p+2}{2p}} \le \frac{r^{\frac{p+2}{2p}}}{\psi(r)^{\frac{(n-1)(p-2)}{2p}}} \le r^{\frac{(n-2)(2^*-p)}{2p}},$$

that is, upon taking the supremum over $(0, r_0)$,

$$\sup_{r\in(0,r_0)} \frac{1}{\psi(r)^{n-1}}\left(\int_0^r \psi(s)^{n-1}\,ds\right)^{\frac{p+2}{2p}} \le r_0^{\frac{(n-2)(2^*-p)}{2p}}, \tag{4.37}$$

where in the case $n = 2$ we mean $(n-2)(2^*-p) = 4$. On the other hand, by exploiting (4.32), (4.33), (4.37) and the fact that ψ is nondecreasing, we obtain:

$$\begin{aligned}
&\sup_{r\in(r_0,\infty)} \frac{1}{\psi(r)^{n-1}}\left(\int_0^r \psi(s)^{n-1}\,ds\right)^{\frac{p+2}{2p}} \\
&\le \sup_{r\in(r_0,\infty)}\left(\frac{1}{\psi(r_0)^{\frac{2p(n-1)}{p+2}}}\int_0^{r_0}\psi(s)^{n-1}\,ds + \frac{1}{\psi(r)^{\frac{2p(n-1)}{p+2}}}\int_{r_0}^r \psi(s)^{n-1}\,ds\right)^{\frac{p+2}{2p}} \\
&\le \sup_{r\in(r_0,\infty)}\left(r_0^{\frac{(n-2)(2^*-p)}{p+2}} + \frac{r^\alpha}{c(n-1)\,\psi(r)^{\frac{(n-1)(p-2)}{p+2}}}\right)^{\frac{p+2}{2p}} \\
&\le \sup_{r\in(r_0,\infty)}\left(r_0^{\frac{(n-2)(2^*-p)}{p+2}} + \frac{r^\alpha}{c(n-1)\,k^{\frac{(n-1)(p-2)}{p+2}}\,e^{\frac{c(n-1)(p-2)}{(p+2)(1-\alpha)}r^{1-\alpha}}}\right)^{\frac{p+2}{2p}} \\
&\le \left(r_0^{\frac{(n-2)(2^*-p)}{p+2}} + \frac{[\alpha(p+2)]^{\frac{\alpha}{1-\alpha}}}{[c(n-1)]^{\frac{1}{1-\alpha}}\,(p-2)^{\frac{\alpha}{1-\alpha}}}\,r_0^{-\frac{(n-1)(p-2)}{p+2}}\,e^{\frac{c(n-1)(p-2)}{(p+2)(1-\alpha)}r_0^{1-\alpha}-\frac{\alpha}{1-\alpha}}\right)^{\frac{p+2}{2p}},
\end{aligned} \tag{4.38}$$

where we have computed explicitly the last supremum in (4.38) (over the whole $\mathbb{R}^+$ actually) recalling the definition of k given in (4.32). Hence, by combining (4.36), (4.37) and (4.38), we end up with

$$\sup_{r\in(0,\infty)}\left(\int_0^r \psi(s)^{n-1}\,ds\right)^{\frac{1}{p}}\left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds\right)^{\frac{1}{2}} \le \left(\frac{p}{2}\right)^{\frac{1}{2}} \times$$

$$\left(r_0^{\frac{(n-2)(2^*-p)}{p+2}} + \frac{[\alpha(p+2)]^{\frac{\alpha}{1-\alpha}}}{[c(n-1)]^{\frac{1}{1-\alpha}}\,(p-2)^{\frac{\alpha}{1-\alpha}}}\,r_0^{-\frac{(n-1)(p-2)}{p+2}}\,e^{\frac{c(n-1)(p-2)}{(1-\alpha)(p+2)}r_0^{1-\alpha}-\frac{\alpha}{1-\alpha}}\right)^{\frac{p+2}{2p}},$$

from which (4.30) easily follows just by letting $p \downarrow 2$ and (in the case $n = 2$ only) $p \to \infty$. □

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. We consider only the case $p < 2^*$ since, as recalled in the Introduction, it is well known that the Euclidean Sobolev inequality holds on any Cartan-Hadamard manifold.

Let us first establish the validity of (4.9) and then show optimality according to (4.10). To our purposes, we introduce the following function:

$$\psi_\star(r) := \left(\frac{\operatorname{meas}(S_r)}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \qquad \forall r \ge 0, \tag{4.39}$$

where ω_{n-1} is the Hausdorff measure of the Euclidean unit sphere $\mathbb{S}^{n-1}$ and S_r is the sphere of radius r in $\mathbb{M}^n$ centered at the pole o. It is an elementary fact that $\psi_\star \in \mathcal{A}$. Indeed, recalling (4.16), it is apparent that $\psi_\star \in C^\infty((0,\infty))$ thanks to the regularity of $A(r,\theta)$ outside the pole (recall the corresponding discussion in Subsection 4.2.1). On the other hand, by applying (4.25) with $\varepsilon = r$, we easily deduce that $\psi_\star \in C^1([0,\infty))$, $\psi_\star(0) = 0$ and $\psi_\star'(0) = 1$. Now we aim at showing that $\psi_\star$ fulfills the hypotheses (4.29) of Lemma 4.2 for some positive $r_0 = r_0(\beta, C_0, R_0)$, $c = c(n, \beta, C_0, R_0)$ and $\alpha = \beta/2$. Indeed, thanks to (4.18), (4.21) and Lemma 4.1, the following inequalities hold:

$$\frac{\frac{\partial}{\partial r} A(r,\theta)}{A(r,\theta)} \ge \frac{n-1}{r} \qquad \forall (r,\theta) \in (0,\infty) \times \mathbb{S}^{n-1} \setminus \mathcal{P} \tag{4.40}$$

and

$$\frac{\frac{\partial}{\partial r} A(r,\theta)}{A(r,\theta)} \ge c\, r^{-\frac{\beta}{2}} \qquad \forall (r,\theta) \in [r_0,\infty) \times \mathbb{S}^{n-1} \setminus \mathcal{P}\,, \tag{4.41}$$

for suitable constants $r_0, c > 0$ as above. By integrating (4.40) from $\varepsilon > 0$ to $r > \varepsilon$, we obtain:

$$A(r,\theta) \ge \frac{A(\varepsilon,\theta)}{\varepsilon^{n-1}}\, r^{n-1} \qquad \forall (r,\theta) \in (\varepsilon,\infty) \times \mathbb{S}^{n-1} \setminus \mathcal{P}\,,$$

whence, upon integrating over $\mathbb{S}^{n-1}$,

$$\operatorname{meas}(S_r) \ge \omega_{n-1}\, r^{n-1}\, \frac{\operatorname{meas}(S_\varepsilon)}{\operatorname{meas}(\mathbb{S}_\varepsilon)} = \omega_{n-1}\, r^{n-1} \left(1 + O(\varepsilon^2)\right), \tag{4.42}$$

where we have used (4.16) and again (4.25). If we let $\varepsilon \downarrow 0$ in (4.42), we therefore end up with

$$\psi_\star(r) \ge r \qquad \forall r \ge 0\,.$$

The fact that $\psi_\star'(r) \ge 0$ everywhere is a trivial consequence of (4.40), so that we are left with establishing the last inequality of (4.29). To this aim, note that the integration of (4.41) over $\mathbb{S}^{n-1}$ yields

$$\begin{aligned} \frac{d}{dr} \operatorname{meas}(S_r) &= \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial r} A(r,\theta)\, d\theta \\ &\ge c\, r^{-\frac{\beta}{2}} \int_{\mathbb{S}^{n-1}} A(r,\theta)\, d\theta \\ &= c\, r^{-\frac{\beta}{2}} \operatorname{meas}(S_r) \qquad \forall r \ge r_0\,, \end{aligned}$$

which readily entails

$$\frac{\psi_\star'(r)}{\psi_\star(r)} \ge \frac{c}{n-1}\, r^{-\frac{\beta}{2}} \qquad \forall r \ge r_0\,,$$

namely the last inequality of (4.29) with $\alpha = \beta/2$, upon relabeling c. Hence, we have proved that the function $\psi_\star$ defined in (4.39) satisfies all of the assumptions of Lemma 4.2, with $\alpha = \beta/2$. Thus, as a consequence of (4.30), we deduce that

$$\sup_{r \in (0,\infty)} \left(\int_0^r \psi_\star(s)^{n-1}\, ds \right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi_\star(s)^{n-1}}\, ds \right)^{\frac{1}{2}} \le \frac{C\, p^{\frac{2+\beta}{2(2-\beta)}}}{(p-2)^{\frac{\beta}{2-\beta}}} \tag{4.43}$$

for a suitable $C = C(n, \beta, C_0, R_0) > 0$. Thanks to (4.43), we can apply Proposition 4.1 with $w(r) = \psi_\star(r)^{n-1}$, which ensures the validity of the Sobolev-type inequalities

$$\begin{gathered} \left(\int_0^\infty |g(r)|^p \, \psi_\star(r)^{N-1} \, dr \right)^{\frac{1}{p}} \le \\ \left(1+\frac{p}{2}\right)^{\frac{1}{p}} \left(1+\frac{2}{p}\right)^{\frac{1}{2}} \frac{C\, p^{\frac{2+\beta}{2(2-\beta)}}}{(p-2)^{\frac{\beta}{2-\beta}}} \left(\int_0^\infty \left|g'(r)\right|^2 \, \psi_\star(r)^{N-1} \, dr \right)^{\frac{1}{2}} \\ \forall g \in C^1_c([0,\infty))\,, \qquad \forall p \in (2, 2^*)\,. \end{gathered} \tag{4.44}$$

Finally, we need to show how to pass from (4.44) to (4.9). To this purpose, it is enough to observe that $f \in C^1_{c:\mathrm{rad}}(\mathbb{M}^n)$ implies $r \mapsto f(r) \in C^1_c([0,\infty))$, which, along with Fubini's Theorem, allows us to apply (4.44) in the following way:

$$\begin{aligned} \|f\|_{L^p(\mathbb{M}^n)} =& \left(\int_0^\infty \int_{\mathbb{S}^{n-1}} |f(r)|^p \, A(r,\theta) \, d\theta \, dr \right)^{\frac{1}{p}} \\ =& \left(\omega_{n-1} \int_0^\infty |f(r)|^p \, \psi_\star(r)^{n-1} \, dr \right)^{\frac{1}{p}} \\ \le& \, \omega_{n-1}^{\frac{1}{p}} \left(1+\frac{p}{2}\right)^{\frac{1}{p}} \left(1+\frac{2}{p}\right)^{\frac{1}{2}} \frac{C\, p^{\frac{2+\beta}{2(2-\beta)}}}{(p-2)^{\frac{\beta}{2-\beta}}} \left(\int_0^\infty \left|f'(r)\right|^2 \, \psi_\star(r)^{n-1} \, dr \right)^{\frac{1}{2}} \\ =& \, \omega_{n-1}^{\frac{1}{p}-\frac{1}{2}} \left(1+\frac{p}{2}\right)^{\frac{1}{p}} \left(1+\frac{2}{p}\right)^{\frac{1}{2}} \frac{C\, p^{\frac{2+\beta}{2(2-\beta)}}}{(p-2)^{\frac{\beta}{2-\beta}}} \times \\ & \left(\int_0^\infty \int_{\mathbb{S}^{n-1}} \left|f'(r)\right|^2 A(r,\theta) \, d\theta \, dr \right)^{\frac{1}{2}} \\ =& \, \omega_{n-1}^{\frac{1}{p}-\frac{1}{2}} \left(1+\frac{p}{2}\right)^{\frac{1}{p}} \left(1+\frac{2}{p}\right)^{\frac{1}{2}} \frac{C\, p^{\frac{2+\beta}{2(2-\beta)}}}{(p-2)^{\frac{\beta}{2-\beta}}} \, \|\nabla f\|_{L^2(\mathbb{M}^n)}\,, \end{aligned} \tag{4.45}$$

namely (4.9) upon relabelling C.

Let us now deal with optimality. It is enough to consider any function $\psi \in \mathcal{A}$ such that

$$\psi''(r) \ge 0 \quad \forall r > 0\,, \qquad \frac{\psi''(r)}{\psi(r)} = C_0 \, r^{-\beta} \quad \forall r \ge R_0\,, \tag{4.46}$$

which ensures that the associated model manifold $\mathbb{M}^M_\psi$ is Cartan-Hadamard and complies with (4.8). Indeed, following e.g. [120, Lemma 4.1], it is not difficult to prove that (4.46) implies

$$\frac{\psi'(r)}{\psi(r)} \sim \sqrt{C_0}\, r^{-\frac{\beta}{2}} \qquad \text{as } r \to \infty\,, \tag{4.47}$$

where by $a(r) \sim b(r)$ we mean that the ratio $a(r)/b(r)$ tends to 1. In particular, (4.47) entails

$$\frac{\sqrt{C_0}}{2}\, r^{-\frac{\beta}{2}} \le \frac{\psi'(r)}{\psi(r)} \le 2\sqrt{C_0}\, r^{-\frac{\beta}{2}} \quad \forall r \ge r_0 \quad \Longrightarrow \quad \psi(r) \ge c_1 \, e^{c_2 \, r^{\frac{2-\beta}{2}}} \quad \forall r \ge r_0\,, \tag{4.48}$$

$r_0, c_1, c_2 > 0$ being suitable constants that depend on ψ but not on $p \in (2, 2^*] \cap (2, \infty)$, whose exact values are not relevant to our purposes. Hence, (4.48) plus a simple integration by parts in the same spirit as above, yield

$$\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \geq \frac{r^{\frac{\beta}{2}}}{2\sqrt{C_0}(n-1)\,\psi(r)^{n-1}} \qquad \forall r \geq r_0\,. \tag{4.49}$$

Similarly,

$$\int_{r_0}^r \psi(s)^{n-1}\, ds \geq$$
$$\frac{1}{2\sqrt{C_0}(n-1)}\left[r^{\frac{\beta}{2}}\,\psi(r)^{n-1} - r_0^{\frac{\beta}{2}}\,\psi(r_0)^{n-1} - \frac{\beta}{2}\int_{r_0}^r \frac{\psi(s)^{n-1}}{s^{\frac{2-\beta}{2}}}\, ds\right] \qquad \forall r \geq r_0\,,$$

which implies, upon picking r_0 so large that

$$\frac{\beta}{4\sqrt{C_0}(n-1)\, r_0^{\frac{2-\beta}{2}}} \leq 1\,,$$

the validity of the estimate

$$\int_{r_0}^r \psi(s)^{n-1}\, ds \geq \frac{1}{4\sqrt{C_0}(n-1)}\left[r^{\frac{\beta}{2}}\,\psi(r)^{n-1} - r_0^{\frac{\beta}{2}}\,\psi(r_0)^{n-1}\right] \qquad \forall r \geq r_0\,. \tag{4.50}$$

From (4.50), it is then apparent that one can select another $\widehat{r}_0 > r_0$ in such a way that

$$\int_{r_0}^r \psi(s)^{n-1}\, ds \geq \frac{r^{\frac{\beta}{2}}\,\psi(r)^{n-1}}{8\sqrt{C_0}(n-1)} \qquad \forall r \geq \widehat{r}_0\,. \tag{4.51}$$

Hence, by combining (4.48), (4.49) and (4.51), we deduce that

$$\begin{aligned}\left(\int_0^r \psi(s)^{n-1}\, ds\right)^{\frac{1}{p}}\left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds\right)^{\frac{1}{2}} &\geq C\,\frac{r^{\frac{\beta(p+2)}{4p}}}{\psi(r)^{\frac{(n-1)(p-2)}{2p}}} \\ &\geq C\,\frac{r^{\frac{\beta(p+2)}{4p}}}{e^{c_2\,\frac{(n-1)(p-2)}{2p}\,r^{\frac{2-\beta}{2}}}} \qquad \forall r \geq \widehat{r}_0\,,\end{aligned} \tag{4.52}$$

where from here on C denotes a general positive constant that can be taken independent of $p \in (2, 2^*] \cap (2, \infty)$, that we shall not relabel. A straightforward calculation shows that the maximum over $r \in (0, \infty)$ of the last term in (4.52) is attained at

$$\overline{r} = \left[\frac{\beta(p+2)}{c_2(n-1)(2-\beta)(p-2)}\right]^{\frac{2}{2-\beta}},$$

which ensures that

$$\sup_{r\in(0,\infty)}\left(\int_0^r \psi(s)^{n-1}\, ds\right)^{\frac{1}{p}}\left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds\right)^{\frac{1}{2}} \geq C\left(\frac{p+2}{p-2}\right)^{\frac{\beta}{2-\beta}} \tag{4.53}$$

provided

$$\left[\frac{\beta(p+2)}{c_2(n-1)(2-\beta)(p-2)}\right]^{\frac{2}{2-\beta}} \geq \widehat{r}_0\,. \tag{4.54}$$

It is therefore clear that (4.53)–(4.54), together with (4.28), yield (4.10) at least in the case $n \geq 3$, where $2^* < \infty$. What is left in the case $n = 2$ is just the correct estimate on the behaviour of the supremum in the l.h.s. of (4.53) as $p \to \infty$. To this aim, it is enough to observe that, as a simple consequence of the fact that $\psi(r) \sim r$ as $r \to 0$, there holds

$$\sup_{r\in(0,\infty)} \left(\int_0^r \psi(s)\, ds \right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)}\, ds \right)^{\frac{1}{2}} \geq C \sup_{r\in\left(0,\frac{1}{\sqrt{e}}\right)} r^{\frac{2}{p}} \left(-\log r\right)^{\frac{1}{2}} \geq C\, p^{\frac{1}{2}} ,$$

which, upon exploiting again (4.28), ensures the validity of (4.10) also as $p \to \infty$. □

4.3.2 The quasi-Euclidean case

Similarly to Subsection 4.3.1, we start by a crucial result showing that, for appropriate weights $w(r) \equiv \psi(r)^{n-1}$ ($\psi \in \mathcal{A}$) associated with quasi-Euclidean model manifolds (still according to the terminology of Subsection 4.1.2), one can bound quantitatively the supremum appearing in the statement of Theorem 4.1.

Lemma 4.3. *Let $\psi \in \mathcal{A}$ satisfy the following assumptions:*

$$\psi(r) \geq r \quad \forall r > 0 , \qquad \psi'(r) \geq 0 \quad \forall r > 0 , \qquad \frac{\psi'(r)}{\psi(r)} \geq \frac{c}{r} - \frac{c'}{r^q} \quad \forall r \geq r_0 , \qquad (4.55)$$

for some $c > 1$, $c' > 0$, $q > 1$ and $r_0 > 0$. Let $n \in \mathbb{N}$ be larger than or equal to 2. Then

$$\sup_{r\in(0,\infty)} \left(\int_0^r \psi(s)^{n-1}\, ds \right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \right)^{\frac{1}{2}} \leq C\sqrt{p} \qquad \forall p \in \left[\frac{2\tilde{n}}{\tilde{n}-2}, 2^* \right) , \qquad (4.56)$$

where $\tilde{n} := \tilde{n}(n,c) := c(n-1)+1$ and C is a positive constant depending only on n, c, c', q, r_0.

Proof. The strategy relies on arguments close to the ones used in the proof Lemma 4.2. Indeed, let $Q(r)$ be defined by (4.31). By integrating the last differential inequality in (4.55) from r_0 to $r > r_0$ (and taking advantage of the first one as well), we obtain:

$$\psi(r) \geq k\, r^c \quad \forall r \geq r_0 , \qquad \text{where } k = k(c', q, r_0) > 0 , \qquad (4.57)$$

which ensures that $Q(r)$ is a smooth function of $r > 0$, given the finiteness of the integrals involved. Moreover, because $\psi(r) \sim r$ as $r \to 0$ and $p < 2^*$, it is immediate to check that $\lim_{r\to 0} Q(r) = 0$. In order to deal with the limit at infinity, we need again some integral bounds. Clearly, the last inequality of (4.55) yields

$$\frac{\psi'(r)}{\psi(r)} \geq \frac{c+1}{2r} \quad \forall r \geq \tilde{r}_0 , \qquad \text{for some } \tilde{r}_0 = \tilde{r}_0(c', q, r_0) > r_0 ; \qquad (4.58)$$

note that inequality (4.58) can be rewritten as

$$\psi(r)^{n-1} \leq \frac{2r}{(c+1)(n-1)} \frac{d}{dr} \left(\psi^{n-1}\right)(r) \qquad \forall r \geq \tilde{r}_0 . \qquad (4.59)$$

Hence, if we integrate (by parts) (4.59) between $\tilde{r}_0$ and $r > \tilde{r}_0$, we deduce that

$$\begin{aligned}\int_{\tilde{r}_0}^{r} \psi(s)^{n-1}\, ds &\leq \frac{2}{(c+1)(n-1)} \int_{\tilde{r}_0}^{r} s\, \frac{d}{ds}\left(\psi^{n-1}\right)(s)\, ds \\ &= \frac{2}{(c+1)(n-1)} \left[r\, \psi(r)^{n-1} - \tilde{r}_0\, \psi(\tilde{r}_0)^{n-1} - \int_{\tilde{r}_0}^{r} \psi(s)^{n-1}\, ds \right] \\ &\leq \frac{2r\, \psi(r)^{n-1}}{(c+1)(n-1)}\,. \end{aligned} \tag{4.60}$$

Let us now rewrite (4.58) as

$$\frac{1}{\psi(r)^{n-1}} \leq -\frac{2r}{(c+1)(n-1)} \frac{d}{dr}\left(\frac{1}{\psi(r)^{n-1}} \right) \qquad \forall r \geq \tilde{r}_0\,. \tag{4.61}$$

The integration (by parts) of (4.61) between $r > \tilde{r}_0$ and ∞ (along with (4.57)) ensures that

$$\begin{aligned}\int_{r}^{\infty} \frac{1}{\psi(s)^{n-1}}\, ds \leq - &\int_{r}^{\infty} \frac{2s}{(c+1)(n-1)} \frac{d}{ds}\left(\frac{1}{\psi(s)^{n-1}} \right) ds \\ = &\frac{2}{(c+1)(n-1)} \frac{r}{\psi(r)^{n-1}} \\ &+ \frac{2}{(c+1)(n-1)} \int_{r}^{\infty} \frac{1}{\psi(s)^{n-1}}\, ds\,, \end{aligned}$$

whence

$$\int_{r}^{\infty} \frac{1}{\psi(s)^{n-1}}\, ds \leq \frac{2}{(c-1)(n-1)+2(n-2)} \frac{r}{\psi(r)^{n-1}} \qquad \forall r \geq \tilde{r}_0\,. \tag{4.62}$$

By plugging estimate (4.60) into (4.31), exploiting (4.57), the fact that $\psi(r)$ is nondecreasing and (4.62), we obtain:

$$\begin{aligned} Q(r) &= \left(\int_0^{\tilde{r}_0} \psi(s)^{n-1}\, ds + \int_{\tilde{r}_0}^{r} \psi(s)^{n-1}\, ds \right)^{\frac{1}{p}} \left(\int_r^{\infty} \frac{1}{\psi(s)^{n-1}}\, ds \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^{\tilde{r}_0} \psi(s)^{n-1}\, ds + \frac{2r\, \psi(r)^{n-1}}{(c+1)(n-1)} \right)^{\frac{1}{p}} \left(\int_r^{\infty} \frac{1}{\psi(s)^{n-1}}\, ds \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{r\, \psi(r)^{n-1}} \int_0^{\tilde{r}_0} \psi(s)^{n-1}\, ds + \frac{2}{(c+1)(n-1)} \right)^{\frac{1}{p}} \times \\ &\quad \left[\left(r\, \psi(r)^{n-1} \right)^{\frac{2}{p}} \int_r^{\infty} \frac{1}{\psi(s)^{n-1}}\, ds \right]^{\frac{1}{2}} \\ &\leq \left(\frac{1}{r\, \psi(r)^{n-1}} \int_0^{\tilde{r}_0} \psi(s)^{n-1}\, ds + \frac{2}{(c+1)(n-1)} \right)^{\frac{1}{p}} \left(\frac{2\, r^{\frac{p+2}{p}}\, \psi(r)^{-\frac{(n-1)(p-2)}{p}}}{(c-1)(n-1)+2(n-2)} \right)^{\frac{1}{2}} \\ &\leq C\, \frac{r^{\frac{p+2}{2p}}}{\psi(r)^{\frac{(n-1)(p-2)}{2p}}} \\ &\leq C\, r^{\frac{p+2}{2p} - \frac{c(n-1)(p-2)}{2p}}\,, \end{aligned}$$

for all $r > \tilde{r}_0$, where from here on C stands for a general suitable positive constant depending only on n, c, c', q, r_0 (that we shall not relabel). In particular,

$$\limsup_{r\to\infty} Q(r) \le C$$

since

$$p \ge \frac{2\tilde{n}}{\tilde{n}-2} \qquad \Longrightarrow \qquad \frac{p+2}{2p} - \frac{c(n-1)(p-2)}{2p} \le 0 \,. \tag{4.63}$$

There are two possibilities: either $Q(r)$ does not admit an internal maximum, in which case

$$Q(r) < C \qquad \forall r > 0 \,, \tag{4.64}$$

or $Q(r)$ does admit an internal maximum at some $\overline{r} > 0$, which is a critical point. Note that, in this case, by carrying out the same computations as in the proof of Lemma 4.2, equations (4.34), (4.36) and (4.37) (with r_0 replaced by $\tilde{r}_0$) are still true. On the other hand, by exploiting (4.37) (with r_0 replaced by $\tilde{r}_0$), (4.57), (4.60) and the fact that $\psi(r)$ is nondecreasing, we end up with

$$\begin{aligned}
&\sup_{r\in(\tilde{r}_0,\infty)} \frac{1}{\psi(r)^{n-1}} \left(\int_0^r \psi(s)^{n-1}\, ds \right)^{\frac{p+2}{2p}} \\
\le& \sup_{r\in(\tilde{r}_0,\infty)} \left(\frac{1}{\psi(\tilde{r}_0)^{\frac{2p(n-1)}{p+2}}} \int_0^{\tilde{r}_0} \psi(s)^{n-1}\, ds + \frac{1}{\psi(r)^{\frac{2p(n-1)}{p+2}}} \int_{\tilde{r}_0}^r \psi(s)^{n-1}\, ds \right)^{\frac{p+2}{2p}} \\
\le& \sup_{r\in(\tilde{r}_0,\infty)} \left(\tilde{r}_0^{\frac{(n-2)(2^*-p)}{p+2}} + \frac{1}{\psi(r)^{\frac{2p(n-1)}{p+2}}} \frac{2r\,\psi(r)^{n-1}}{(c+1)(n-1)} \right)^{\frac{p+2}{2p}} \\
\le& \sup_{r\in(\tilde{r}_0,\infty)} \left(\tilde{r}_0^{\frac{(n-2)(2^*-p)}{p+2}} + \frac{2}{(c+1)(n-1)} \frac{r}{\psi(r)^{\frac{(n-1)(p-2)}{p+2}}} \right)^{\frac{p+2}{2p}} \\
\le& \sup_{r\in(\tilde{r}_0,\infty)} \left(\tilde{r}_0^{\frac{(n-2)(2^*-p)}{p+2}} + C\, r^{1-\frac{c(n-1)(p-2)}{p+2}} \right)^{\frac{p+2}{2p}} \\
=& \left(\tilde{r}_0^{\frac{(n-2)(2^*-p)}{p+2}} + C\, \tilde{r}_0^{1-\frac{c(n-1)(p-2)}{p+2}} \right)^{\frac{p+2}{2p}} ,
\end{aligned}$$

where in the last line we have taken advantage of (4.63). Hence, by collecting (4.36), (4.37) (with r_0 replaced by $\tilde{r}_0$) and (4.64), we finally obtain

$$\begin{aligned}
&\sup_{r\in(0,\infty)} \left(\int_0^r \psi(s)^{n-1}\, ds \right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \right)^{\frac{1}{2}} \\
&\le \left(\frac{p}{2}\right)^{\frac{1}{2}} \left(\tilde{r}_0^{\frac{(n-2)(2^*-p)}{p+2}} + C\, \tilde{r}_0^{1-\frac{c(n-1)(p-2)}{p+2}} \right)^{\frac{p+2}{2p}} \vee C \,,
\end{aligned}$$

which establishes (4.56) up to relabelling C. □

We can now prove Theorem 4.2.

Proof of Theorem 4.2. We argue similarly to the proof of Theorem 4.1: our aim is to show that the function $\psi_\star$ defined by (4.39) satisfies the hypotheses of

Lemma 4.3. In order to establish the validity of the first two inequalities of (4.55), as well as the fact that $\psi_\star \in \mathcal{A}$, one can reason exactly in the same way. As concerns the third one, some adaptations have to be performed: we shall mainly refer to [120, Subsection 8.1]. First of all, note that the general solution of the differential equation

$$\phi''(r) = C_1 \, r^{-2} \, \phi(r) \qquad \forall r \in \mathbb{R}^+$$

is explicit, i.e.

$$\phi(r) = a_1 \, r^{q_1} + a_2 \, r^{q_2} \qquad \forall r \in \mathbb{R}^+$$

for arbitrary real constants a_1 and a_2, where $q_{1,2} = (1 \pm \sqrt{1+4C_1})/2$. It is not difficult to show (just by following the same ideas as in [120, Subsection 8.1]) that one can construct a function $\psi \in \mathcal{A}$ such that

$$\psi''(r) = C_1 \, r^{-2} \, \psi(r) \quad \forall r \geq 2R_0 \, , \qquad \psi''(r) \leq C_1 \, r^{-2} \, \psi(r) \quad \forall r \in (R_0, 2R_0) \, ,$$
$$\psi''(r) = 0 \quad \forall r \in [0, R_0] \, ,$$

which therefore complies with (4.3.2) for every $r \geq 2R_0 =: r_0$ and constants $a_1 > 0$, $a_2 \in \mathbb{R}$ depending only on C_1, R_0. In view of (4.11), we are in position to apply the Laplacian-comparison results of Subsection 4.2.2 (specifically (4.20)), guaranteeing that

$$\mathsf{m}(r,\theta) \geq (n-1) \, \frac{\psi'(r)}{\psi(r)} \geq (n-1) \left(\frac{q_1}{r} - \frac{h}{r^{1+\sqrt{1+4C_1}}} \right) \tag{4.65}$$

for all $r \geq r_0$, where h is a suitable positive constant depending on a_1, a_2, q_1, q_2, r_0. Thanks to (4.65), we can now proceed as in the proof of Theorem 4.1, observing that

$$\begin{aligned}
\psi_\star'(r) =& \frac{1}{\omega_{n-1}(n-1)} \left(\frac{\operatorname{meas}(S_r)}{\omega_{n-1}} \right)^{\frac{1}{n-1}-1} \int_{\mathbb{S}^{n-1}} \mathsf{m}(r,\theta) \, A(r,\theta) \, d\theta \\
\geq & \frac{1}{\omega_{n-1}(n-1)} \left(\frac{\operatorname{meas}(S_r)}{\omega_{n-1}} \right)^{\frac{1}{n-1}-1} \times \\
& (n-1) \left(\frac{q_1}{r} - \frac{h}{r^{1+\sqrt{1+4C_1}}} \right) \int_{\mathbb{S}^{n-1}} A(r,\theta) \, d\theta \\
= & \left(\frac{q_1}{r} - \frac{h}{r^{1+\sqrt{1+4C_1}}} \right) \psi_\star(r)
\end{aligned}$$

for all $r \geq r_0$. Hence, the function $\psi_\star$ satisfies the hypotheses of Lemma 4.3 with $c = q_1$, $c' = h$ and $q = 1 + \sqrt{1+4C_1}$. As a consequence, we deduce that

$$\sup_{r \in (0,\infty)} \left(\int_0^r \psi_\star(s)^{n-1} \, ds \right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi_\star(s)^{n-1}} \, ds \right)^{\frac{1}{2}} \leq C \sqrt{p} \qquad p \in \left[\tilde{2}, 2^*\right) , \tag{4.66}$$

where $\tilde{2}$ is defined in (4.13) and C is a positive constant as in the statement. Once (4.66) has been established, the conclusion follows as in the proof of Theorem 4.1, i.e. by applying Proposition 4.1 and carrying out the same computations that led to (4.45).

Let us finally deal with optimality. To this aim, it is enough to consider any function $\psi \in \mathcal{A}$ such that

$$\psi''(r) \geq 0 \quad \forall r > 0 \qquad \text{and} \qquad \psi(r) \asymp r^{\frac{\tilde{n}-1}{n-1}} \quad \text{as } r \to \infty\,,$$

where by $a(r) \asymp b(r)$ we mean that the ratios $a(r)/b(r)$, $a'(r)/b'(r)$, $a''(r)/b''(r)$ tend to some positive numbers and $\tilde{n}$ is related to C_1 by (4.13). This ensures that the associated model manifold $\mathbb{M}^n_\psi$ is Cartan-Hadamard and complies with (4.11). Recalling that $\tilde{n} - 2 > 0$, we therefore obtain

$$\left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \right)^{\frac{1}{2}} \asymp r^{-\frac{\tilde{n}-2}{2}} \qquad \text{as } r \to \infty \tag{4.67}$$

and

$$\left(\int_0^r \psi(s)^{n-1}\, ds \right)^{\frac{1}{p}} \asymp r^{\frac{\tilde{n}}{p}} \qquad \text{as } r \to \infty\,, \tag{4.68}$$

for all $p \geq 2$. The combination of (4.67)–(4.68) then yields

$$\left(\int_0^r \psi(s)^{n-1}\, ds \right)^{\frac{1}{p}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \right)^{\frac{1}{2}} \asymp r^{\frac{2\tilde{n}-p\tilde{n}+2p}{2p}} \qquad \text{as } r \to \infty\,. \tag{4.69}$$

Clearly, the r.h.s. of (4.69) stays bounded as $r \to \infty$ if and only if $2\tilde{n} - p\tilde{n} + 2p \leq 0$, namely $p \geq \tilde{2}$. Hence, thanks to Proposition 4.1, we can conclude that in this case (4.12) fails for all $p \in [2, \tilde{2})$. As for the behaviour of the optimal constant as $p \to \infty$ (for $n = 2$), one reasons exactly as in the end of the proof of Theorem 4.1. □

Remark 4.4 (On the Cartan-Hadamard assumption). It is worth pointing out that, in Theorems 4.1 and 4.2, in general it is not possible to drop the assumption that M is a Cartan-Hadamard manifold, i.e. it is not enough to require that the sectional curvatures satisfy only (4.8) or (4.11). Indeed, consider a model manifold $\mathbb{M}^n_\psi$ with $\psi \in \mathcal{A}$ such that $\psi(r) \asymp e^{-r^\alpha}$ as $r \to \infty$, where $\alpha = \beta/2 \in (0, 1)$. It is straightforward to check that such a manifold complies with (4.8); however, it is apparent that all of the inequalities (4.9) fail, since the supremum appearing in (4.27) in Proposition 4.1 is identically ∞ because of the second integral. Similarly, with regards to Theorem 4.2, one can consider a model manifold $\mathbb{M}^N_\psi$ with $\psi(r) = r^{q_2}$ for large r, where $q_2 < 0$ is the power appearing in (4.3.2). It is plain that all such manifolds are not Cartan-Hadamard, since a local change of sign of the second derivative of ψ necessarily occurs.

Remark 4.5 (The case $p \in [1, 2)$). Throughout the whole paper we have assumed that $p \geq 2$. In fact there is a simple reason for such a restriction: it was proved in [118, Theorem 4.6] that on *any* Cartan-Hadamard manifold the inequality $\|f\|_p \leq C\,\|\nabla f\|_2$ always *fails* as soon as p is strictly smaller than 2. Moreover, since the argument used in the corresponding proof relies only on radial functions, the inequality is false even if restricted to $C^1_{c;\mathrm{rad}}(\mathbb{M}^n)$.

4.4 The Poincaré inequality: McKean's Theorem and related issues

As discussed in the Introduction, one of the main motivations for this work was a celebrated paper by H.P. McKean [167], which is fully devoted to the proof of the following result.

Theorem 4.4 (McKean 1970, original statement). *Consider a smooth, n-dimensional, simply-connected Riemannian manifold M with negative sectional curvatures* Sect *bounded away from* 0*: specifically, suppose* $\mathrm{Sect} \le -k$ *for some constant $k > 0$. Then the spectrum of the corresponding Laplace-Beltrami operator Δ acting in $L^2(M)$ is also bounded from* 0*: specifically, the top of the spectrum lies to the left of*

$$-\frac{k\,(n-1)^2}{4}\,,$$

and this bound is sharp.

We point out that Theorem 4.4 can be rephrased equivalently by asserting that on any Cartan-Hadamard manifold $\mathbb{M}^n$ with sectional curvatures bounded from above by $-k < 0$, the following *Poincaré inequality*

$$\|f\|_{L^2(\mathbb{M}^n)} \le \frac{2}{\sqrt{k}\,(n-1)}\,\|\nabla f\|_{L^2(\mathbb{M}^n)} \qquad \forall f \in C^1_c(\mathbb{M}^n)$$

holds. This is the form of the statement that we shall refer to below.

The original proof of McKean is far from trivial. He had already understood that it all amounted to establishing the inequality for *radial* functions, since the extension from radial to nonradial in the pure Poincaré case ($p = 2$) is straightforward, see the proof of Theorem 4.5 below. However, in order to prove that the weight associated with the volume measure on $\mathbb{M}^n$ (recall Subsection 4.2.1) satisfies a differential inequality of the type of (4.70) (actually of second order) w.r.t. the variable r, which is at the core of the problem, he employs several technical tools that involve the second fundamental form, Jacobi fields and the so-called index form of Morse theory. Here we shall only use elementary arguments related to weighted one-dimensional inequalities, in the spirit of Section 4.3. Of course the main nontrivial result behind our methods lies in the Laplacian-comparison Theorem recalled in Subsection 4.2.2, which allows one to pass from model manifolds to general manifolds with very little effort. Furthermore, through these techniques, we are able to slightly generalize McKean's Theorem, by requiring that only the *radial* sectional curvatures are negative away from zero.

In order to carry out our alternative proof, we need a preliminary lemma.

Lemma 4.4. *Let $\psi \in C^1([0,\infty)) \cap C^\infty((0,\infty))$ be a positive function on $(0,\infty)$ such that $\psi(0) = 0$. Let $n \in \mathbb{N}$ be larger than or equal to 2. If*

$$\frac{\psi'(r)}{\psi(r)} \ge \sqrt{k} \qquad \forall r > 0\,, \quad \textit{for some } k > 0\,, \tag{4.70}$$

then

$$\sup_{r\in(0,\infty)} \left(\int_0^r \psi(s)^{n-1}\,ds\right)^{\frac12} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds\right)^{\frac12} \le \frac{1}{\sqrt{k}\,(n-1)}\,. \tag{4.71}$$

Proof. For convenience, let us assume that $k = 1$: the general case can be obtained by a simple scaling argument, as we shall see in the end of the proof. So, given any $\varepsilon > 0$, upon integrating (4.70) from ε to r we infer that

$$\psi(r) \ge \psi(\varepsilon)\,e^{r-\varepsilon} \qquad \forall r \ge \varepsilon\,. \tag{4.72}$$

On the other hand, inequality (4.70) can be rewritten as

$$\psi(r)^{n-1} \le \frac{1}{n-1}\,\frac{d}{dr}\left(\psi^{n-1}\right)(r) \qquad \forall r>0\,,$$

so that an integration between 0 and r yields (recall that $\psi(0)=0$)

$$\int_0^r \psi(s)^{n-1}\,ds \le \frac{1}{n-1}\,\psi(r)^{n-1} \qquad \forall r>0\,. \tag{4.73}$$

Similarly, another way of rewriting (4.70) is

$$\frac{1}{\psi(r)^{n-1}} \le -\frac{1}{n-1}\,\frac{d}{dr}\left(\psi^{-n+1}\right)(r) \qquad \forall r>0\,; \tag{4.74}$$

by integrating (4.74) from r to ∞ we obtain

$$\int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds \le \frac{1}{n-1}\,\frac{1}{\psi(r)^{n-1}} \qquad \forall r>0\,, \tag{4.75}$$

where we have exploited the fact that $\lim_{r\to\infty}\psi(r)=\infty$, trivial consequence of (4.72). By combining (4.73) and (4.75), we finally deduce that

$$\int_0^r \psi(s)^{n-1}\,ds \int_r^\infty \frac{1}{\psi(s)^{n-1}}\,ds \le \frac{1}{(n-1)^2} \qquad \forall r>0\,,$$

namely (4.71) for $k=1$. In order to deal with the general case, it is enough to apply the just proved result to $r\mapsto \sqrt{k}\,\psi\big(r/\sqrt{k}\big)$. □

We are now ready to give an elementary proof of McKean's Theorem, by taking advantage of Lemma 4.4 along with the basic facts in Riemannian geometry recalled in Subsections 4.2.1–4.2.3.

Theorem 4.5 (McKean's Theorem revisited)**.** *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold satisfying*

$$\mathrm{Sect}_\omega(x) \le -k \qquad \forall x\in\mathbb{M}^n\,, \tag{4.76}$$

where $k>0$ and $\mathrm{Sect}_\omega(x)$ denotes the sectional curvature w.r.t. any 2-dimensional tangent subspace ω at x containing a radial direction. Then

$$\|f\|_{L^2(\mathbb{M}^n)} \le \frac{2}{\sqrt{k}\,(n-1)}\,\|\nabla f\|_{L^2(\mathbb{M}^n)} \qquad \forall f\in C^1_c(\mathbb{M}^n)\,. \tag{4.77}$$

Proof. Thanks to (4.76), we can apply the Laplacian-comparison result recalled in Subsection 4.2.2 with the explicit model function $\psi(r)=\sinh\big(\sqrt{k}r\big)$, which corresponds to the hyperbolic space of curvature $-k$ and trivially satisfies

$$\frac{\psi''(r)}{\psi(r)} = k \qquad \forall r>0\,.$$

Hence, upon recalling identity (4.18), we deduce that

$$\begin{aligned}
\frac{\frac{\partial}{\partial r}A(r,\theta)}{A(r,\theta)} =&\,\mathsf{m}(r,\theta)\\
\ge&(n-1)\,\frac{\psi'(r)}{\psi(r)}\\
=&(n-1)\,\sqrt{k}\,\coth\big(\sqrt{k}r\big)\\
\ge&(n-1)\,\sqrt{k} \qquad \forall r>0\,,\quad \forall\theta\in\mathbb{S}^{n-1}\setminus\mathcal{P}\,,
\end{aligned}$$

namely

$$\frac{\frac{\partial}{\partial r}\psi_A(r,\theta)}{\psi_A(r,\theta)} \geq \sqrt{k} \qquad \forall r>0\,, \quad \forall \theta \in \mathbb{S}^{n-1}\setminus \mathcal{P}\,, \tag{4.78}$$

where $\psi_A(r,\theta) := A(r,\theta)^{\frac{1}{n-1}}$.

Thanks to the basic properties of the function $(r,\theta)\mapsto A(r,\theta)$ described in Subsection 4.2.1, in view of (4.78) we can apply Lemma 4.4 to $\psi \equiv \psi_A(\cdot,\theta)$, for every fixed $\theta \in \mathbb{S}^{n-1}\setminus\mathcal{P}$, which ensures that

$$\sup_{r\in(0,\infty)} \left(\int_0^r \psi_A(s,\theta)^{n-1}\,ds\right)^{\frac{1}{2}} \left(\int_r^\infty \frac{1}{\psi_A(s,\theta)^{n-1}}\,ds\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{k}\,(n-1)} \qquad \forall\theta\in\mathbb{S}^{n-1}\setminus\mathcal{P}\,. \tag{4.79}$$

As a consequence, from Proposition 4.1 with $p=2$ and $w(r)\equiv\psi_A(r,\theta)$ we deduce that

$$\int_0^\infty g(r)^2\,A(s,\theta)\,dr \leq \frac{4}{k\,(N-1)^2}\int_0^\infty g'(r)^2\,A(r,\theta)\,dr \qquad \forall g\in C^1_c([0,\infty))\,, \quad \forall\theta\in\mathbb{S}^{n-1}\setminus\mathcal{P}\,. \tag{4.80}$$

On the other hand, if $f\in C^1_c(\mathbb{M}^n)$ then $r\mapsto f(r,\theta)\in C^1_c([0,\infty))$ for every $\theta\in\mathbb{S}^{N-1}$, so that by exploiting (4.80) with $g(r)\equiv f(r,\theta)$, integrating over $\mathbb{S}^{n-1}$ and using Fubini's Theorem, we end up with

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}\int_0^\infty f(r,\theta)^2\,A(r,\theta)\,dr\,d\theta &\leq \frac{4}{k\,(n-1)^2}\int_{\mathbb{S}^{n-1}}\int_0^\infty \left|\frac{\partial}{\partial r}f(r,\theta)\right|^2 A(r,\theta)\,dr\,d\theta \\ &\leq \frac{4}{k\,(n-1)^2}\int_{\mathbb{S}^{n-1}}\int_0^\infty |\nabla f(r,\theta)|^2\,A(r,\theta)\,dr\,d\theta\,,\end{aligned}$$

namely (4.77), recalling (4.17). □

As concerns the sharpness of the constant, which of course had already been established by McKean, note that it is easily verified e.g. by observing that the r.h.s. of (4.79) is attained on hyperbolic space, i.e. when $\psi_A(r,\theta)\equiv\sinh(\sqrt{k}r)$.

4.4.1 Negative curvatures outside a ball

In fact the previous techniques allow us to obtain a McKean-type result under the weaker assumption that curvatures are bounded above by a negative constant only in the complement of a ball. To the best of our knowledge, this result is new even if, in some sense, expectable.

We first establish the following lemma.

Lemma 4.5. *Let $\psi\in C^1([0,\infty))\cap C^\infty((0,\infty))$ be a positive function on $(0,\infty)$ such that $\psi(0)=0$ and $\psi'(0)>0$. Let $n\in\mathbb{N}$ be larger than or equal to 2. If*

$$\psi'(r)\geq 0 \quad \forall r\geq 0 \qquad \text{and} \qquad \frac{\psi'(r)}{\psi(r)}\geq c \quad \forall r\geq r_0 \tag{4.81}$$

for some $c, r_0 > 0$*, then there exists a positive constant* C*, depending only on* N, c, r_0*, such that*

$$\sup_{r\in(0,\infty)} \left(\int_0^r \psi(s)^{n-1}\, ds \right)^{\frac{1}{2}} \left(\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \right)^{\frac{1}{2}} \le C\,. \tag{4.82}$$

Proof. First of all we observe that the integration of the last inequality in (4.81) from r_0 to $r > r_0$, along with the fact that ψ is positive, yields the bound from below

$$\psi(r) \ge K\, e^{cr} \quad \forall r \ge r_0\,, \qquad \text{where } K := \psi(r_0)\, e^{-cr_0}\,. \tag{4.83}$$

We proceed similarly to the proof of Lemma 4.4. Upon rewriting the last inequality in (4.81) as

$$\psi(r)^{n-1} \le \frac{1}{c(N-1)} \frac{d}{dr} \left(\psi^{n-1}\right)(r) \qquad \forall r \ge r_0$$

and integrating between r_0 and $r > r_0$, we obtain:

$$\begin{aligned} \int_{r_0}^r \psi(s)^{n-1}\, ds &\le \frac{1}{c(n-1)} \int_{r_0}^r \frac{d}{ds} \left(\psi^{n-1}\right)(s)\, ds \\ &= \frac{1}{c(n-1)} \left[\psi(r)^{n-1} - \psi(r_0)^{n-1}\right] \\ &\le \frac{1}{c(n-1)}\, \psi(r)^{n-1}\,. \end{aligned} \tag{4.84}$$

Another way of rewriting such an inequality is

$$\frac{1}{\psi(r)^{n-1}} \le -\frac{1}{c(n-1)} \frac{d}{dr} \left(\psi^{-n+1}\right)(r) \qquad \forall r \ge r_0\,, \tag{4.85}$$

whence by integrating (4.85) from r to ∞ we infer that

$$\int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \le \frac{1}{c(n-1)} \frac{1}{\psi(r)^{n-1}} \qquad \forall r \ge r_0\,, \tag{4.86}$$

where we have used the property $\lim_{r\to\infty} \psi(r) = \infty$, consequence of (4.83). The combination of (4.84), (4.86) and the first inequality of (4.81) yields

$$\begin{aligned} \hat{Q}(r) &:= \int_0^r \psi(s)^{n-1}\, ds \int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \\ &= \left(\int_0^{r_0} \psi(s)^{n-1}\, ds + \int_{r_0}^r \psi(s)^{n-1}\, ds \right) \int_r^\infty \frac{1}{\psi(s)^{n-1}}\, ds \\ &\le \frac{1}{c(n-1)} \frac{1}{\psi(r)^{n-1}} \int_0^{r_0} \psi(s)^{n-1}\, ds + \frac{1}{c^2(n-1)^2} \\ &\le \frac{r_0}{c(n-1)} + \frac{1}{c^2(n-1)^2} \end{aligned}$$

for all $r \ge r_0$, namely

$$\sup_{r\in[r_0,\infty)} \hat{Q}(r) \le \frac{r_0}{c(n-1)} + \frac{1}{c^2(n-1)^2} =: C^2\,. \tag{4.87}$$

We are left with bounding the analogous supremum for r ranging between 0 and r_0. There are two possibilities: either $\sup_{r\in(0,r_0)} \hat{Q}(r) \le C^2$, in which case (4.82) trivially follows, or $\sup_{r\in(0,r_0)} \hat{Q}(r) > C^2$, in which case there necessarily exists $\overline{r} \in (0, r_0)$ such that $\hat{Q}(\overline{r}) = \sup_{r\in(0,\infty)} \hat{Q}(r)$ (note that $\lim_{r\to 0} \hat{Q}(r) = 0$). In particular, we can reason exactly as in the proof of Lemma 4.2 to deduce the analogue of (4.35) with $p = 2$:

$$\sqrt{\hat{Q}(\overline{r})} = \frac{1}{\psi(\overline{r})^{n-1}} \int_0^{\overline{r}} \psi(s)^{n-1}\, ds \le \overline{r} \le r_0 \,, \tag{4.88}$$

where we have taken advantage again of the elementary fact that $\psi(r)$ is non-decreasing. Hence, as a consequence of (4.87)–(4.88),

$$\sup_{r\in(0,\infty)} \sqrt{\hat{Q}(r)} \le C \vee r_0 \,,$$

namely (4.82) up to relabelling C. □

We are now ready to prove the following version of McKean's Theorem outside a ball.

Theorem 4.6 (McKean's Theorem outside a ball)**.** *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold such that*

$$\mathrm{Sect}_\omega(x) \le -k \qquad \forall x \in \mathbb{M}^n \setminus B_{R_0}$$

for some $k, R_0 > 0$, where $\mathrm{Sect}_\omega(x)$ denotes the sectional curvature w.r.t. any 2-dimensional tangent subspace ω at x containing a radial direction. Then there exists a positive constant C_P, depending only on k and R_0, such that

$$\|f\|_{L^2(\mathbb{M}^n)} \le C_P \,\|\nabla f\|_{L^2(\mathbb{M}^n)} \qquad \forall f \in C^1_c(\mathbb{M}^n)\,. \tag{4.89}$$

Proof. By applying Lemma 4.1 with $\beta = 0$, we infer that

$$\mathsf{m}(r,\theta) \ge c \qquad \forall (r,\theta) \in [r_0,\infty) \times \mathbb{S}^{n-1}\,,$$

for suitable positive constants $r_0 = r_0(k, R_0)$ and $c = c(n, k, R_0)$, that is

$$\frac{\frac{\partial}{\partial r}\psi_A(r,\theta)}{\psi_A(r,\theta)} \ge c \qquad \forall r > r_0\,, \quad \forall \theta \in \mathbb{S}^{n-1}\backslash\mathcal{P}\,, \qquad \text{where } \psi_A(r,\theta) := A(r,\theta)^{\frac{1}{n-1}}\,. \tag{4.90}$$

Thanks to (4.22) and (4.23), in view of (4.90) we can apply Lemma 4.5 to $\psi \equiv \psi_A(\cdot,\theta)$ at each fixed $\theta \in \mathbb{S}^{n-1} \setminus \mathcal{P}$, which ensures that

$$\sup_{r\in(0,\infty)} \left(\int_0^r \psi_A(s,\theta)^{n-1}\, ds\right)^{\frac12} \left(\int_r^\infty \frac{1}{\psi_A(s,\theta)^{n-1}}\, ds\right)^{\frac12} \le C \qquad \forall \theta \in \mathbb{S}^{n-1} \setminus \mathcal{P}\,,$$

where C is the same constant as in (4.82). Hence, by Proposition 4.1 with $p = 2$ and $w(r) \equiv \psi_A(r,\theta)$, we end up with

$$\int_0^\infty g(r)^2\, A(s,\theta)\, dr \le 4C^2 \int_0^\infty g'(r)^2\, A(r,\theta)\, dr \qquad \forall g \in C^1_c([0,\infty))\,, \quad \forall \theta \in \mathbb{S}^{n-1} \setminus \mathcal{P}\,. \tag{4.91}$$

Once (4.91) has been established, inequality (4.89) follows with $C_P = 2C$ just by reasoning as in the final part of the proof of Theorem 4.5. □

4.5 Failure of the inequalities in the nonradial framework

In this section we prove Theorem 4.3 by constructing an explicit sequence of nonradial functions that make the Rayleigh quotient associated with inequality (4.15) blow up.

Proof of Theorem 4.3. For later convenience, we set $\lambda := (2-\beta)/2 \in (0,1)$. We want to prove that the Sobolev-type inequality

$$\left(\int_{\mathbb{M}^n} |f|^p\right)^{\frac{1}{p}} \le C\left(\int_{\mathbb{M}^n} |\nabla f|^2\right)^{\frac{1}{2}}, \tag{4.92}$$

supposed to be valid for all $f \in C_c^1(\mathbb{M}^n)$, actually *fails* as soon as $p < 2^*$ under the running assumptions on $\mathbb{M}^n$. To this purpose, we provide the following family of functions that we denote by f_R, for which the Rayleigh quotient of (4.92) blows up as $R \to +\infty$:

$$f_R(x) := \left(1 - \frac{\mathrm{d}(x, o_R)}{R^{1-\lambda}}\right)_+ \qquad \forall x \in \mathbb{M}^n, \tag{4.93}$$

where $o_R \in S_R$, namely $R = \mathrm{d}(o_R, o)$. Note that each f_R is in fact only Lipschitz regular, but this is not an issue (one can always regularize it in order to obtain a C_c^1 function close enough to f_R, see also Remark 4.3). In view of (4.93), we have:

$$|\nabla f_R(x)| = \frac{1}{R^{1-\lambda}}\, \chi_{B_{R^{1-\lambda}}(o_R)}(x) \qquad \forall x \in \mathbb{M}^n, \tag{4.94}$$

$$|f_R(x)| \ge \frac{1}{2}\, \chi_{B_{\frac{R^{1-\lambda}}{2}}(o_R)}(x) \qquad \forall x \in \mathbb{M}^n, \tag{4.95}$$

$B_r(o_R)$ being the Riemannian ball of radius $r > 0$ centered at o_R. Thanks to (4.95), the L^p norm of f_R is readily estimated from below:

$$\int_{\mathbb{M}^n} |f_R|^p \ge \frac{1}{2^p}\, \nu\Big(B_{\frac{R^{1-\lambda}}{2}}(o_R)\Big) \ge \frac{\omega_{n-1}}{2^{p+n}\, n}\, R^{(1-\lambda)n}, \tag{4.96}$$

where in the last inequality we have used the simple fact that, because $\mathbb{M}^n$ is Cartan-Hadamard, the volume of balls (w.r.t. any pole) grows at least with Euclidean rate (recall that ω_{n-1} is the Hausdorff measure of the Euclidean unit sphere of dimension $n-1$). This is just a consequence of Laplacian comparison, see Section 4.2 (in particular Subsections 4.2.1–4.2.3). Let us now deal with the L^2 norm of the gradient. By (4.94), we have:

$$\int_{\mathbb{M}^N} |\nabla f_R|^2\, d\nu = \frac{1}{R^{2(1-\lambda)}}\, \nu(B_{R^{1-\lambda}}(o_R)); \tag{4.97}$$

in order to estimate the volume in the r.h.s. of (4.97), we need to exploit (4.14).

First of all, note that in $B_{R^{1-\lambda}}(o_R)$ condition (4.14) can be rewritten as follows:

$$\begin{aligned} \operatorname{Ric}(x) &\geq -\frac{C_2}{\mathrm{d}(x,o)^{2-2\lambda}} \\ &\geq -\frac{C_2}{[\mathrm{d}(o,o_R)-\mathrm{d}(x,o_R)]^{2-2\lambda}} \\ &\geq -\frac{C_2}{(R-R^{1-\lambda})^{2-2\lambda}} \\ &\geq -\frac{2C_2}{R^{2-2\lambda}} \qquad \forall x \in B_{R^{1-\lambda}}(o_R)\,, \end{aligned} \tag{4.98}$$

provided R is so large that $(1-R^{-\lambda})^{2-2\lambda} \geq 1/2$ and $R - R^{1-\lambda} \geq R_0$. Thanks to (4.98), we can then apply comparison with the surface measure of the balls in the hyperbolic space with sectional curvature

$$-\frac{2C_2}{(N-1)R^{2-2\lambda}} := -\frac{\hat{C}^2}{R^{2-2\lambda}}\,,$$

which corresponds to the model function

$$\psi(r) = \frac{R^{1-\lambda}}{\hat{C}} \sinh\!\left(\frac{\hat{C}}{R^{1-\lambda}}\, r\right)$$

(recall again Subsections 4.2.1–4.2.3, here the reference "pole" is o_R). To this aim we take advantage, in particular, of the validity of (4.98) along radial directions emanating from o_R, which yields

$$\begin{aligned} \nu(B_{R^{1-\lambda}}(o_R)) &\leq \omega_{n-1}\, \frac{R^{(1-\lambda)(n-1)}}{\hat{C}^{n-1}} \int_0^{R^{1-\lambda}} \left[\sinh\!\left(\tfrac{\hat{C}}{R^{1-\lambda}}\, r\right)\right]^{n-1} dr \\ &= \omega_{n-1}\, \frac{R^{(1-\lambda)n}}{\hat{C}^n} \int_0^{\hat{C}} \sinh(s)^{n-1}\, ds\,. \end{aligned} \tag{4.99}$$

Hence, by virtue of (4.97) and (4.99), we obtain:

$$\int_{\mathbb{M}^n} |\nabla f_R|^2\, d\nu \leq \frac{\omega_{n-1} \int_0^{\hat{C}} \sinh(s)^{n-1}\, ds}{\hat{C}^n}\, R^{(1-\lambda)(n-2)}\,. \tag{4.100}$$

So, if inequality (4.15) was true, in view of (4.96) and (4.100) we would end up with

$$\left(\frac{\omega_{n-1}\, R^{(1-\lambda)n}}{2^{p+n}\, n}\right)^{\frac{1}{p}} \leq C \left(\frac{\omega_{n-1} \int_0^{\hat{C}} \sinh(s)^{n-1}\, ds}{\hat{C}^n}\, R^{(1-\lambda)(N-2)}\right)^{\frac{1}{2}},$$

namely

$$R^{\frac{1}{p}-\frac{1}{2^*}} \leq \overline{C}(n,\lambda,C,p)\,,$$

and the contradiction is achieved upon letting $R \to \infty$, since $p < 2^*$. $\square$

4.6 The porous medium equation on Cartan-Hadamard manifolds

Theorem 4.1 has some interesting consequences concerning smoothing effects for the porous medium equation (4.6), at least when the initial datum belongs to $L^1(\mathbb{M}^n)$ and is radially symmetric with respect to the pole o. We shall denote by $L^1_{\rm rad}(\mathbb{M}^n)$ the space constituted by all such functions.

Theorem 4.7. *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold such that*

$$\mathrm{Sect}_\omega(x) \le -C_0\, r^{-\beta} \qquad \forall x \in \mathbb{M}^n \setminus B_{R_0}\,, \tag{4.101}$$

for some $\beta \in (0,2)$ and $C_0, R_0 > 0$. Then there exists a positive constant $K > 0$, depending only on m, n, β, C_0, R_0, such that for any initial datum $u_0 \in L^1_{\rm rad}(\mathbb{M}^n)$ the solution u of (4.6) satisfies the smoothing estimate

$$\|u(t)\|_{L^\infty(\mathbb{M}^n)} \le K \left[\log\Big(t\, \|u_0\|_{L^1(\mathbb{M}^n)}^{m-1} + e\Big)\right]^{\frac{2+\beta}{(m-1)(2-\beta)}} t^{-\frac{1}{m-1}} \qquad \forall t > 0\,. \tag{4.102}$$

Moreover, the result is optimal w.r.t. long-time dependence, in the sense that if (4.101) holds with reverse inequality and $\mathrm{Sect}_\omega(x)$ replaced by $\mathrm{Ric}_o(x)$, then there exist initial data $u_0 \in L^1_{\rm rad}(\mathbb{M}^n)$ for which the analogue of (4.102) holds with reverse inequality for large t.

The above result is a consequence of arguments that follow the lines of [118, Theorem 3.1] and [120, Theorem 3.2]. For the reader's convenience here we write down a concise proof (mostly borrowed from the proof of [118, Theorem 3.1]), which should allow one to realize how the Sobolev-type inequalities (4.9) come into play.

Proof. Let $q > 0$ and $\sigma = p/2 \in (1, 2^*/2]$, both being for the moment free parameters. We can suppose with no loss of generality that $u_0 \in L^1(\mathbb{M}^n) \cap L^\infty(\mathbb{M}^n)$. In order to make rigorous the computations we shall carry out, one needs some approximation procedures, which we skip because they are out of the scope of this section: see [118] and references therein for more details. To improve readability, throughout the proof we mean $\|\cdot\|_q \equiv \|\cdot\|_{L^q(\mathbb{M}^n)}$.

So, by multiplying the differential equation in (4.6) by u^q, integrating by parts and using (4.9), we obtain:

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{q+1}^{q+1} &= -\,\frac{4q(q+1)m}{(m+q)^2}\left\|\nabla\Big(u^{\frac{q+m}{2}}\Big)(t)\right\|_2^2 \\ &\le -\,\frac{4q(q+1)m}{(m+q)^2\, C_\sigma^2}\,\|u(t)\|_{\sigma(q+m)}^{q+m}\,, \end{aligned} \tag{4.103}$$

where

$$C_\sigma := \frac{C\, p^{\frac{2+\beta}{2(2-\beta)}}}{(p-2)^{\frac{\beta}{2-\beta}}} \tag{4.104}$$

and C is the same constant as in (4.9). Taking advantage of standard interpolation and the well-known fact that the L^1 norm does not increase along the evolution, we infer that

$$\|u(t)\|_{q+1} \le \|u(t)\|_{\sigma(q+m)}^{\frac{\sigma(q+m)q}{[\sigma(q+m)-1](q+1)}}\, \|u_0\|_1^{\frac{\sigma(q+m)-(q+1)}{[\sigma(q+m)-1](q+1)}} \qquad \forall t > 0\,. \tag{4.105}$$

For notational convenience, let us assume that $\|u_0\|_1 = 1$ (the case of a general L^1 norm can be handled by a routine time-scaling argument). As a consequence, from (4.103) and (4.105) there follows

$$\frac{d}{dt}\|u(t)\|_{q+1}^{q+1} \le -\frac{4q(q+1)m}{(m+q)^2\, C_\sigma^2}\, \|u(t)\|_{q+1}^{(q+1)\frac{\sigma(q+m)-1}{\sigma q}} . \tag{4.106}$$

The integration of (4.106) yields

$$y(t)^{\frac{\sigma m-1}{\sigma q}} \le \frac{1}{\frac{1}{y(0)^{\frac{\sigma m-1}{\sigma q}}} + \frac{4m(q+1)(\sigma m-1)}{\sigma(q+m)^2 C_\sigma^2}\, t} \qquad \forall t > 0 , \qquad y(t) := \|u(t)\|_{q+1}^{q+1} ,$$

whence

$$\|u(t)\|_{q+1} \le \left[\frac{\sigma(q+m)^2 C_\sigma^2}{4m(q+1)(\sigma m-1)}\right]^{\frac{\sigma q}{(q+1)(\sigma m-1)}} t^{-\frac{\sigma q}{(q+1)(\sigma m-1)}} \qquad \forall t > 0 . \tag{4.107}$$

By previous results (see e.g. [117, Corollary 5.6] or [34, Theorem 1.5]), the validity of (4.1) for a fixed $p/2 = \sigma = \sigma_0 \in (1, 2^*/2)$ entails the smoothing estimate

$$\|u(t)\|_\infty \le K\, t^{-\frac{\sigma_0}{(\sigma_0-1)(q+1)+\sigma_0(m-1)}}\, \|u_0\|_{q+1}^{\frac{(\sigma_0-1)(q+1)}{(\sigma_0-1)(q+1)+\sigma_0(m-1)}} \qquad \forall t > 0 , \tag{4.108}$$

where from here on by K we shall denote a general positive constant that depends only on m, n, β, C_0, R_0 (which will not be relabelled). Therefore, the combination of (4.107) (evaluated at time $t/2$) and (4.108) (with the time origin shifted from 0 to $t/2$) yields

$$\|u(t)\|_\infty \le K \left[\frac{\sigma(q+m)^2 C_\sigma^2}{4m(q+1)(\sigma m-1)}\right]^{\frac{\sigma q(\sigma_0-1)}{(\sigma m-1)[(\sigma_0-1)(q+1)+\sigma_0(m-1)]}} \times \\ t^{-\frac{\sigma_0(\sigma m-1)+\sigma q(\sigma_0-1)}{(\sigma m-1)[(\sigma_0-1)(q+1)+\sigma_0(m-1)]}} \tag{4.109}$$

for all $t > 0$. Because $q > 0$ is a free parameter and (4.109) holds at any time for any such q, we can let $q = \log(t+e)$ in (4.109) to obtain

$$\begin{aligned} \|u(t)\|_\infty \le & K \left\{\frac{\sigma[\log(t+e)+m]^2}{4m[\log(t+e)+1](\sigma m-1)}\right\}^{-\frac{\sigma(\sigma_0 m-1)}{(\sigma m-1)\{(\sigma_0-1)[\log(t+e)+1]+\sigma_0(m-1)\}}} \\ & \times\, t^{\frac{\sigma_0-\sigma}{(\sigma m-1)\{(\sigma_0-1)[\log(t+e)+1]+\sigma_0(m-1)\}}} \left\{\frac{\sigma[1+m/\log(t+e)]^2}{4m[1+1/\log(t+e)](\sigma m-1)}\right\}^{\frac{\sigma}{\sigma m-1}} \\ & \times\, C_\sigma^{-\frac{2\sigma(\sigma_0 m-1)}{(\sigma m-1)\{(\sigma_0-1)[\log(t+e)+1]+\sigma_0(m-1)\}}} \left[\log(t+e)\, C_\sigma^2\right]^{\frac{\sigma}{\sigma m-1}} t^{-\frac{\sigma}{\sigma m-1}} , \end{aligned} \tag{4.110}$$

for all $t > 0$. If $\sigma \in (1, \sigma_0)$, it is apparent that the first two factors in the r.h.s. of (4.110) can be bounded from above by another general positive constant K, so that (4.110) reads

$$\|u(t)\|_\infty \le K\, C_\sigma^{-\frac{2\sigma(\sigma_0 m-1)}{(\sigma m-1)\{(\sigma_0-1)[\log(t+e)+1]+\sigma_0(m-1)\}}} \times \\ \left[\log(t+e)\, C_\sigma^2\right]^{\frac{\sigma}{\sigma m-1}} t^{-\frac{\sigma}{\sigma m-1}} , \tag{4.111}$$

for all $t > 0$, which implies, upon recalling (4.104),

$$\|u(t)\|_\infty \le K\,(\sigma-1)^{\frac{2\beta\sigma(\sigma_0 m-1)}{(2-\beta)(\sigma m-1)\{(\sigma_0-1)[\log(t+e)+1]+\sigma_0(m-1)\}}} \times \left[\log(t+e)\,(\sigma-1)^{-\frac{2\beta}{2-\beta}}\right]^{\frac{\sigma}{\sigma m-1}} t^{-\frac{\sigma}{\sigma m-1}} \qquad (4.112)$$

for all $t > 0$. We can now set

$$\sigma = 1 + \frac{\sigma_0 - 1}{\log(t+e)}\,,$$

so that from (4.112) (using the fact that the first factor stays bounded) one deduces that

$$\|u(t)\|_\infty \le K\,[\log(t+e)]^{\frac{2+\beta}{(m-1)(2-\beta)} - \frac{(2+\beta)(\sigma_0-1)}{[(m-1)\log(t+e)+m(\sigma_0-1)](m-1)(2-\beta)}} \times t^{-\frac{1}{m-1}+\frac{\sigma_0-1}{[(m-1)\log(t+e)+m(\sigma_0-1)](m-1)}}$$

for all $t > 0$, i.e.

$$\|u(t)\|_\infty \le K\,[\log(t+e)]^{\frac{2+\beta}{(m-1)(2-\beta)}}\, t^{-\frac{1}{m-1}} \qquad \forall t > 0\,,$$

which is equivalent to (4.102) in the case $\|u_0\|_1 = 1$.

As concerns optimality, it is enough to recall that [120, Theorem 3.2] ensures, provided the curvature assumption (4.101) holds with reverse inequality and $\mathrm{Sect}_\omega(x)$ is replaced by $\mathrm{Ric}_o(x)$, that any (nontrivial) bounded, compactly supported and positive initial datum u_0 gives rise to a solution of (4.6) satisfying (in particular) the lower bound

$$\|u(t)\|_\infty^{m-1} \ge \hat{C}\,\frac{(\log t)^{\frac{2+\beta}{2-\beta}}}{t} \qquad \text{for large } t\,, \qquad (4.113)$$

where $\hat{C}$ is a suitable positive constant depending on $\mathbb{M}^n, m, u_0$. It is plain that (4.113) matches (4.102) (with respect to time behaviour) from below. □

Remark 4.6 (The case $\beta = 0$). It is worth pointing out that Theorem 4.7 actually holds for $\beta = 0$ as well: in fact in such case the result is true for *all* $L^1(\mathbb{M}^n)$ initial data, not only the radial ones. This is a direct consequence of Theorem 4.6 and [118, Theorem 2.1], whereas optimality follows from the sharp estimates of [221].

Finally, in the quasi-Euclidean case, thanks to Theorem 4.2 we can obtain the analogue of Theorem 4.7. The proof follows by combining [117, Corollary 5.6] (the fact that it is stated on Euclidean domains is inessential) and [120, Theorem 6.2], so we shall omit it since the argument is just a simplified version of the one used in the proof of Theorem 4.7.

Theorem 4.8. *Let $\mathbb{M}^n$ be a Cartan-Hadamard manifold such that*

$$\mathrm{Sect}_\omega(x) \le -C_1\, r^{-2} \qquad \forall x \in \mathbb{M}^n \setminus B_{R_0} \qquad (4.114)$$

for some C_1 and $R_0 > 0$. Then there exists a positive constant $K > 0$, depending only on N, C_1, R_0, such that for any initial datum $u_0 \in L^1_{\mathrm{rad}}(\mathbb{M}^n)$ the solution u of (4.6) *satisfies the smoothing estimate*

$$\|u(t)\|_{L^\infty(\mathbb{M}^n)} \le K\, t^{-\frac{\tilde{n}}{2+\tilde{n}(m-1)}}\, \|u_0\|_{L^1(\mathbb{M}^n)}^{\frac{2}{2+\tilde{n}(m-1)}} \qquad \forall t > 0\,, \qquad (4.115)$$

where $\tilde{n}$ *is defined in* (4.13).

Moreover, the result is optimal w.r.t. long-time dependence, in the sense that if (4.114) *holds with reverse inequality and* $\mathrm{Sect}_\omega(x)$ *replaced by* $\mathrm{Ric}_o(x)/(n-1)$, *then there exist initial data* $u_0 \in L^1_{\mathrm{rad}}(\mathbb{M}^n)$ *for which the analogue of* (4.115) *holds with reverse inequality for large* t.

Part III

Appendixes

Appendix A

A general result on Riemannian manifolds with boundary

In this appendix we prove a general result on Riemannian manifolds with boundary, in particular let (M, g) be a n-dimensional orientable compact Riemannian C^2-manifold with boundary ∂M. For $\delta \in \mathbb{R}_+$ we denote

$$M_\delta := \{p \in M \, : \, d_M(p, \partial M) > \delta\} \, ,$$

and for $r \in \mathbb{R}_+$ and $z \in M$ we denote by $\mathcal{B}_r(z)$ the geodesic ball centred at z of radius r, i.e.

$$\mathcal{B}_r(z) = \{p \in M \, : \, d_M(z, p) < r\} \, ,$$

where d_M is the geodesic distance on M induced by g. Moreover we denote by $|\cdot|_g$ the Riemannian volume with respect to g.

Proposition A.1. *Assume that there exist positive constants c and δ_0 such that*

$$|\mathcal{B}_r(z)|_g \geq c r^n \, , \tag{A.1}$$

and $B_r(z)$ belongs to the image of the exponential map, for every $z \in M_\delta$ and $0 < r \leq \delta < \delta_0$.

Fix p and q in a connected component of M_δ. Then there exists a piecewise geodesic path $\gamma : [0, 1] \to M_{\delta/2}$ connecting p and q of length bounded by δN_δ, where

$$N_\delta := \max\left(4, \frac{2^n |M|_g}{c\delta^n}\right) \, . \tag{A.2}$$

Proof. We proceed in three steps:

1. We observe that we can join p and q by a continuous path $\tilde{\gamma} : [0, 1] \to M_\delta$ such that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}(1) = q$.

2. We construct a chain of pairwise disjoint geodesic balls $\{\mathcal{B}_0, \dots, \mathcal{B}_{N+1}\}$ of radius $\delta/2$ such that:

    – $\mathcal{B}_0$ is centered at p;

– $\mathcal{B}_i$ is centered at $\tilde{\gamma}(t_i)$, where $\{t_i\}$ is an increasing sequence in $[0,1]$;
– $\mathcal{B}_{N+1}$ contains q;
– $\mathcal{B}_i$ is tangent to $\mathcal{B}_{i+1}$ for any $i = 0, \dots, N$.

In order to construct this chain we consider the increasing sequence $\{t_0, t_1 \dots, t_N\}$ in $[0,1]$ recursively defined as follows:

$$t_0 = 0\,,$$

and

$$t_{i+1} := \inf \left\{ t \in [0,1] \,:\, \mathcal{B}_{\delta/2}(\tilde{\gamma}(s)) \cap \bigcup_{j=0}^{i} \mathcal{B}_{\delta/2}(\tilde{\gamma}(t_j)) = \emptyset\,,\, \forall s \in [t,1] \right\}\,, \quad \text{(A.3)}$$

if the set in brackets is non-empty, and $t_{i+1} = t_N$ otherwise. Therefore, by construction, $\{t_0, t_1 \dots, t_N\}$ is an increasing sequence in $[0,1]$ satisfying

$$\mathcal{B}_{\delta/2}(\tilde{\gamma}(t_i)) \cap \mathcal{B}_{\delta/2}(\tilde{\gamma}(t_j)) = \emptyset \quad \text{for } i \neq j,\, i,j = 0, \dots, N\,, \qquad \text{(A.4)}$$

and

$$\mathcal{B}_{\delta/2}(\tilde{\gamma}(t_i)) \subset M_{\delta/2} \quad i = 0, \dots, N\,.$$

We complete the sequence by adding $t_{N+1} = 1$ as the last term. Since

$$\left| \bigcup_{j=0}^{N} \mathcal{B}_{\delta/2}(\tilde{\gamma}(t_j)) \right|_g \leq |M|_g\,,$$

from (A.1) and (A.4) we get

$$N + 1 \leq N_\delta\,. \qquad \text{(A.5)}$$

From (A.3) it is clear that

$$\overline{\mathcal{B}}_{\delta/2}(\tilde{\gamma}(t_i)) \cap \bigcup_{j=0}^{i-1} \overline{\mathcal{B}}_{\delta/2}(\tilde{\gamma}(t_j)) \neq \emptyset\,, \quad \text{for every } i = 1, \dots, N.$$

3. We construct the piecewise geodesic path γ. The idea is the following: for every $i = 0, \dots, N$ we choose a tangency point p_i between $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$. The piecewise geodesic path γ is constructed by connecting $\tilde{\gamma}(t_i)$ with p_i and p_i with $\tilde{\gamma}(t_{i+1})$ by using geodesic radii, for $i = 0, \dots, N-1$, and connecting $\tilde{\gamma}(t_N)$ with q by using a geodesic path contained in $\mathcal{B}_{N+1}$. Hence

$$\text{length}(\gamma) \leq N\delta \leq \delta N_\delta\,,$$

as required.

In order to construct the path γ we set

$$\sigma(i) = \max\{j > i \,:\, \overline{\mathcal{B}}_{\delta/2}(\tilde{\gamma}(t_i)) \cap \overline{\mathcal{B}}_{\delta/2}(\tilde{\gamma}(t_j)) \neq \emptyset\}\,.$$

Then we set $\sigma^2(i) = \sigma(\sigma(i))$, $\sigma^3(i) = \sigma(\sigma(\sigma(i)))$ and so on and fix $\tau \in \mathbb{N}$ such that $\sigma^\tau(0) = N$. We define γ_1 as a minimal geodesic joining p and $\tilde{\gamma}(t_{\sigma(0)})$ and such that

$$\gamma_1 \subset \overline{\mathcal{B}}_{\delta/2}(p) \cup \overline{\mathcal{B}}_{\delta/2}(\tilde{\gamma}(t_{\sigma(0)}))\,;$$

for $i = 2, \ldots, \tau$, we let γ_i be a minimal geodesic joining $\tilde{\gamma}(t_{\sigma^i(0)})$ and $\tilde{\gamma}(t_{\sigma^{i+1}(0)})$ and such that

$$\gamma_i \subset \overline{\mathcal{B}}_{\delta/2}(\tilde{\gamma}(t_{\sigma^i(0)})) \cup \overline{\mathcal{B}}_{\delta/2}(\tilde{\gamma}(t_{\sigma^{i+1}(0)}))\,.$$

Moreover, we let $\gamma_{\tau+1}$ be a minimal geodesic joining $\tilde{\gamma}(t_N)$ and q and such that

$$\gamma_{\tau+1} \subset \overline{\mathcal{B}}_{\delta/2}(\tilde{\gamma}(t_{\sigma^{\tau+1}(0)})) \cup \overline{\mathcal{B}}_{\delta/2}(q)\,.$$

Let γ be the piecewise geodesic obtained as the union of $\gamma_1, \ldots, \gamma_{\tau+1}$. It is clear that each γ_i has length δ for $i = 1, \ldots, \tau$, and $\leq \delta$ for $i = \tau + 1$. Since $\tau \leq N$, from (A.5) we obtain

$$\mathrm{length}(\gamma) \leq (\tau + 1)\delta \leq \delta N_\delta\,,$$

as required. □

The second result of this appendix is the following Proposition in which we give an upper bound of the diameter of M when $\partial M = \emptyset$. The proof is analogue to the one of Proposition A.1 and it is omitted.

Proposition A.2. *Assume $\partial M = \emptyset$ and that there exist positive constants c and δ such that*

$$|\mathcal{B}_r(z)|_g \geq c r^n\,, \tag{A.6}$$

for every $z \in M$ and $0 < r \leq \delta$. Fix p and q in M, then there exists a piecewise geodesic path $\gamma : [0,1] \to M$ connecting p to q of length bounded by δN_δ where N_δ is given by (A.2).

In particular the diameter of M is bounded by δN_δ.

Appendix B

Sharp anisotropic Sobolev inequalities with weight in convex cones

In this appendix we prove a sharp version of the anisotropic Sobolev inequality in cones by suitably adapting the optimal transportation proof of the Sobolev inequality in [72, Theorem 2]. As we shall see, the proof not only applies to the case of arbitrary norms, but it also allows us to cover a large class of weights. In particular, our result extends the weighted isoperimetric inequalities from [46, Theorem 1.3] to the full Sobolev range $p \in (1, n)$ (note that the case $p = 1$ can be recovered letting $p \to 1^+$).

Theorem B.1. *Let $p \in (1, n)$. Let Σ be a convex cone and H a norm in $\mathbb{R}^n$. Let $w \in C^0(\overline{\Sigma})$ be positive in Σ, homogeneous of degree $a \geq 0$, and such that $w^{1/a}$ is concave in case $a > 0$. Then for any $f \in \mathcal{D}^{1,p}(\Sigma)$ we have*

$$\left(\int_\Sigma |f(x)|^\beta w(x)\, dx \right)^{p/\beta} \leq C_\Sigma(n, p, a, H, w) \int_\Sigma H^p(\nabla f(x))\, w(x)\, dx \qquad \text{(B.1)}$$

where

$$\beta = \frac{p(n+a)}{n+a-p}\,. \qquad \text{(B.2)}$$

Moreover, inequality (B.1) *is sharp and the equality is attained if and only if $f = \mathcal{U}^{H,a}_{\lambda,x_0}$, where*

$$\mathcal{U}^{H,a}_{\lambda,x_0}(x) := \left(\frac{\lambda^{\frac{1}{p-1}} c(n,p,a,H,w)}{\lambda^{\frac{p}{p-1}} + \tilde{H}_0(x-x_0)^{\frac{p}{p-1}}} \right)^{\frac{n+a-p}{p}} \qquad \text{(B.3)}$$

with $\lambda > 0$ where

$$\tilde{H}_0(\zeta) := H_0(-\zeta) \qquad \forall\, \zeta \in \mathbb{R}^n\,, \qquad \text{(B.4)}$$

and H_0 denotes the dual norm associated to H, namely

$$H_0(\zeta) := \sup_{H(\xi)=1} \zeta \cdot \xi \qquad \forall\, \zeta \in \mathbb{R}^n.$$

Furthermore, writing $\Sigma = \mathbb{R}^k \times \mathcal{C}$ with $k \in \{0, \dots, n\}$ and with $\mathcal{C} \subset \mathbb{R}^{n-k}$ a convex cone that does not contain a line, then:

(i) if $k = n$ then $\Sigma = \mathbb{R}^n$ and x_0 may be a generic point in $\mathbb{R}^n$;

(ii) if $k \in \{1, \dots, n-1\}$ then $x_0 \in \mathbb{R}^k \times \{\mathcal{O}\}$;

(iii) if $k = 0$ then $x_0 = \mathcal{O}$.

Proof. We aim at proving that for any nonnegative $f, g \in L^\beta(\Sigma)$ with $\|f\|_{L^\beta(\Sigma)} = \|g\|_{L^\beta(\Sigma)}$ and such that $\nabla f \in L^p(\Sigma)$, we have that

$$\int_\Sigma g^\gamma w\, dx \le \frac{\gamma}{n+a} \left(\int_\Sigma H^p(\nabla f)\, w\, dx \right)^{1/p} \left(\int_\Sigma H_0^{p'} g^\beta w\, dx \right)^{1/p'}, \tag{B.5}$$

with equality if $f = g = \mathcal{U}^{H,a}_{\lambda,x_0}$. The value of γ will be specified later. As shown in [72], inequality (B.5) implies the Sobolev inequality (B.1).

Let F and G be probability densities on Σ and let $T : \Sigma \to \Sigma$ be the optimal transport map (see e.g. [225]).[1] It is well known that, by the transport condition $T_\# F = G$, one has

$$|\det(DT)| = \frac{F}{G \circ T}$$

(see for instance [**?**, Section 3]). Then, if we choose

$$F = f^\beta w \quad \text{and} \quad G = g^\beta w\,,$$

the Jacobian equation for T becomes

$$|\det(DT)|\, \frac{w \circ T}{w} = \frac{f^\beta}{g^\beta \circ T}\,.$$

We observe that, since

$$T_\#(f^\beta w) = g^\beta w\,,$$

then for any $0 < \gamma < \beta$ we have

$$\int_\Sigma g^\gamma w\, dx = \int_\Sigma (g^{\gamma-\beta} \circ T) f^\beta w\, dx = \int_\Sigma \left[|\det(DT)|\, \frac{w \circ T}{w} \right]^{\frac{\beta-\gamma}{\beta}} f^\gamma w\, dx\,. \tag{B.6}$$

We choose γ such that

$$\frac{\beta - \gamma}{\beta} = \frac{1}{n+a} \quad \text{i.e.} \quad \gamma = \frac{p(n+a-1)}{n+a-p}\,.$$

[1] As explained in [97] (see also [98]), the argument that follows can be made rigorous using the fine properties of BV functions (we note that T belongs to BV, being the gradient of a convex function). However, to emphasize the main ideas, we shall write the whole argument when $T : \Sigma \to \Sigma$ is a C^1 diffeomorphism, and we invite the interested reader to look at the proof of [97, Theorem 2.2] to understand how to adapt the argument using only that $T \in BV_{\mathrm{loc}}(\Sigma; \Sigma)$.

Alternatively, arguing by approximation, one can assume that w is strictly positive in $\overline{\Sigma} \setminus \{0\}$, and that f and g are both strictly positive and smooth inside $\overline{\Sigma}$. Then, if $T : \Sigma \to \Sigma$ denotes the optimal transport map from $f^\beta w$ to $g^\beta w$, [**?**, Theorem 1 and Remark 4] ensure that $T : \Sigma \to \Sigma$ is a diffeomorphism. This allows one to perform the proof of (B.5) avoiding the use of the fine properties of BV functions.

Since $T = \nabla\varphi$ for some convex function φ, then DT is symmetric and nonnegative definite. In particular $\det(DT) \geq 0$, and it follows from Young and the arithmetic-geometric inequalities that

$$\begin{aligned}\left[|\det(DT)|\,\frac{w\circ T}{w}\right]^{\frac{1}{n+a}} &\leq \frac{n}{n+a}\det(DT)^{1/n} + \frac{a}{n+a}\left(\frac{w\circ T}{w}\right)^{1/a}\\ &\leq \frac{1}{n+a}\left[\mathrm{div}\,(T) + a\left(\frac{w\circ T}{w}\right)^{1/a}\right].\end{aligned}$$

Also, from the concavity of $w^{1/a}$ we have that

$$a\left(\frac{w\circ T}{w}\right)^{1/a} \leq \frac{\nabla w\cdot T}{w}$$

(see [46, Lemma 5.1]), hence

$$\left[|\det(DT)|\,\frac{w\circ T}{w}\right]^{\frac{1}{n+a}} \leq \frac{1}{n+a}\left(\mathrm{div}\,(T) + \frac{\nabla w\cdot T}{w}\right). \tag{B.7}$$

(If $a = 0$ then w is just constant and (B.7) corresponds to the arithmetic-geometric inequality.) Noticing that

$$\mathrm{div}\,(T) + \frac{\nabla w\cdot T}{w} = \frac{1}{w}\mathrm{div}\,(Tw),$$

combining (B.6) and (B.7) we have

$$\begin{aligned}\int_\Sigma g^\gamma w\,dx &\leq \frac{1}{n+a}\int_\Sigma \mathrm{div}\,(Tw) f^\gamma\,dx\\ &= -\frac{\gamma}{n+a}\int_\Sigma w f^{\gamma-1} T\cdot\nabla f\,dx + \frac{1}{n+a}\int_{\partial\Sigma} w f^\gamma T\cdot\nu\,d\sigma.\end{aligned}$$

Here we notice that, since $T(x) \in \overline{\Sigma}$ for any $x \in \overline{\Sigma}$, the convexity of Σ implies that $T\cdot\nu \leq 0$ on $\partial\Sigma$. Thus we obtain

$$\int_\Sigma g^\gamma w\,dx \leq -\frac{\gamma}{n+a}\int_\Sigma f^{\gamma-1} T\cdot\nabla f\,w\,dx \leq \frac{\gamma}{n+a}\int_\Sigma f^{\gamma-1}\tilde{H}_0(T)H(\nabla f)\,w\,dx,$$

where the last inequality follows from the definition of the dual norm H_0 of H and from the definition of $\tilde{H}_0$ (B.4). Finally, setting $p' = \frac{p}{p-1}$, it follows by Holder's inequality that

$$\begin{aligned}\int_\Sigma f^{\gamma-1}\tilde{H}_0(T)H(\nabla f)\,w\,dx &\leq \left(\int_\Sigma f^{p(\gamma-1)-\frac{p\beta}{p'}} H^p(\nabla f)\,w\,dx\right)^{1/p} \times\\ &\quad\left(\int_\Sigma \tilde{H}_0^{p'}(T)\,f^\beta w\,dx\right)^{1/p'}\\ &= \left(\int_\Sigma H^p(\nabla f)\,w\,dx\right)^{1/p}\left(\int_\Sigma \tilde{H}_0^{p'} g^\beta w\,dx\right)^{1/p'},\end{aligned}$$

where we used the transport condition $T_\#(f^\beta w) = g^\beta w$ and the identity

$$\gamma - 1 - \frac{\beta}{p'} = 0.$$

Hence, by this chain of inequalities we get (B.5).

In order to prove the sharpness of our Sobolev inequality we choose $f = g = \mathcal{U}^{H,a}_{1,\mathcal{O}}$. In this particular case the transport map reduces to the identity map $T(x) = \nabla\varphi(x) = x$ and $\det(DT) = 1$. Also the homogeneity of w implies that $\nabla w \cdot x = a\, w$. This implies that all the inequalities in the previous computations become equalities and we obtain (B.1).

Finally, to prove the characterization of the minimizers one can argue as in [98, Appendix A] and [72, Section 4]. More precisely, choose $g = \mathcal{U}^{H,a}_{1,\mathcal{O}}$ and let f be a minimizer. As noticed in the proof of [72, Theorem 5], one can assume that $f \geq 0$.

First one shows that the support of f is indecomposable (this is a measure-theoretic notion of the concept that $\{f > 0\}$ is connected, see [98, Appendix A] for a definition and more details). Indeed, otherwise one could write $f = f_1 + f_2$ with

$$\int_\Sigma H^p(\nabla f) w(x) dx = \int_\Sigma H^p(\nabla f_1) w(x) dx + \int_\Sigma H^p(\nabla f_2) w(x) dx$$

and then by applying (B.1) and the fact that f is a minimizer, we would get

$$\left(\int_\Sigma f^\beta w(x) dx\right)^{p/\beta} \geq \left(\int_\Sigma f_1^\beta w(x) dx\right)^{p/\beta} + \left(\int_\Sigma f_2^\beta w(x) dx\right)^{p/\beta}.$$

Since

$$\int_\Sigma f^\beta w(x) dx = \int_\Sigma f_1^\beta w(x) dx + \int_\Sigma f_2^\beta w(x) dx$$

(because f_1 and f_2 have disjoint support), by concavity of the function $t \mapsto t^{p/\beta}$ we conclude that either f_1 or f_2 vanishes.

Once this is proved, one can then argue as in the proof of [72, Proposition 6] to deduce (from the fact that all the inequalities in the proof given above much be equalities) that T must be of the form $T(x) = \lambda(x - x_0)$ for some $\lambda > 0$ and $x_0 \in \Sigma$, from which the result follows easily. Finally, properties $(i) - (ii) - (iii)$ on the location of x_0 follow for instance from the fact that T has to map Σ onto Σ. □

Appendix C

Symmetry results for critical p-Laplace equation in $\mathbb{R}^n$

The goal of this section is to revisit the proof of Theorem 3.1 when the cone Σ is $\mathbb{R}^n$ and the norm $H(\cdot)$ is the Euclidean norm $|\cdot|$, which is the simplest possible case. In this case, a proof of Theorem 3.1 is already available in literature [48, 203, 224] and asymptotic estimates on u and ∇u are already known (see Lemma C.1 below). The knowledge of those asymptotic estimates allows us to give a more readable version of the proof of Theorem 3.1 and to emphasize the main ideas without entering in technical details.

The theorem that we are going to prove in this Appendix is the following:

Theorem C.1. *Let $n \geq 2$ and $1 < p < n$. Let u be a solution to*

$$\begin{cases} \Delta_p u + u^{p^*-1} = 0 & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^n) \end{cases} \tag{C.1}$$

then $u(x) = \mathcal{U}_{\lambda,x_0}(x)$, for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, where

$$\mathcal{U}_{\lambda,x_0}(x) := \left(\frac{\lambda^{\frac{1}{p-1}} \left(n^{\frac{1}{p}} \left(\frac{n-p}{p-1} \right)^{\frac{p-1}{p}} \right)}{\lambda^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}}} \right)^{\frac{n-p}{p}} . \tag{C.2}$$

We will need the following three preliminary results which are collected in the following three lemmas: in the first lemma we prove explicit growth conditions on u, ∇u and $\nabla^2 u$, in the second lemma we recall the Newton's inequality and in the third lemma we recall a general differential identity which holds in a very general setting.

Lemma C.1. *Let $n \geq 2$, $1 < p < n$ and u be a solution to* (C.1). *Then*

(*i*) *there exist two constants C_1 and C_2 depending only on n, p and u such that*

$$\frac{C_1}{1 + |x|^{\frac{n-p}{p-1}}} \leq u(x) \leq \frac{C_2}{1 + |x|^{\frac{n-p}{p-1}}} \quad \text{and} \quad |\nabla u(x)| \leq \frac{C_2}{1 + |x|^{\frac{n-1}{p-1}}} , \tag{C.3}$$

for every $x \in \mathbb{R}^n$.

(ii) *There exist a radius* R_0 *and a constant* C_3 *depending only on* n, p *and* u *sucht that*

$$|\nabla u(x)| \geq \frac{C_3}{|x|^{\frac{n-1}{p-1}}}\,, \tag{C.4}$$

for every $x \in \mathbb{R}^n \setminus B_{R_0}$.

(iii) *There exists a costant* C_4 *depending only on* n, p *and* u *sucht that such that*

$$|\nabla^2 u(x)| \leq \frac{C_4}{|x|^{\frac{n-1}{p-1}+1}}\,, \tag{C.5}$$

for every $x \in \mathbb{R}^n \setminus B_{R_0}$.

Proof. (i) and (ii) are shown in [224, Theorem 1.1] and in [203, Theorem 2.2], respectively. In order to prove (iii) we argue as in [29, Theorem 3.3]; let $\rho > 4R_0$ be fixed. For $y \in E = \overline{B_4} \setminus B_{1/4}$, we define

$$\tilde{u}(y) = \rho^{\frac{n-p}{p-1}} u(\rho y)\,. \tag{C.6}$$

We notice that, from (C.1),

$$-\Delta_p \tilde{u} = \frac{1}{\rho^{\frac{p}{p-1}}} \tilde{u}^{p^*-1} \quad \text{in} \quad E\,. \tag{C.7}$$

Hence $\tilde{u}$ satisfies an elliptic equation (thanks to (C.4)) of the form

$$\sum_{i,j=1}^{n} a_{ij} \partial^2_{ij} \tilde{u} = -\frac{1}{\rho^{\frac{p}{p-1}}} \tilde{u}^{p^*-1}\,,$$

where the coefficients a_{ij} are given by

$$a_{ij}(y) = |\nabla \tilde{u}(y)|^{p-2} \left\{ (p-2) \frac{\partial_i \tilde{u}(y) \tilde{\partial}_j u(y)}{|\nabla \tilde{u}(y)|^2} + \partial^2_{ij} \tilde{u}(y) \right\}.$$

From (C.3) and (C.4) we have that there exists γ depending only on n and p such that

$$\frac{1}{\gamma} |\xi|^2 \leq a_{ij}(y) \xi_i \xi_j \leq \gamma |\xi|^2$$

for every $y \in E$ and $\xi \in \mathbb{R}^n$. Notice that interior Schauder's estimates (see e.g. [112]) apply to $\tilde{u}(y)$. This entails $|\nabla^2 \tilde{u}| \leq C_4$ for some positive constant C_4, that is

$$|\nabla^2 u(\rho y)| \leq C_2 \rho^{\frac{p-n}{p-1}-2}\,,$$

for $y \in B_2 \setminus \overline{B_{1/2}}$ and (iii) follows. □

Lemma C.2 ([55], Lemma 3.2). *Let* B *and* C *be symmetric matrices in* $\mathbb{R}^{n\times n}$, *and let* B *be positive semidefinite. Set* $A = BC$. *Then the following inequality holds:*

$$S_2(A) \leq \frac{n-1}{2n} \mathrm{Tr}\,(A)^2. \tag{C.8}$$

Moreover, if $\mathrm{Tr}\,(A) \neq 0$ *and equality holds in* (C.8)*, then*

$$A = \frac{\mathrm{Tr}\,(A)}{n}\mathrm{Id}\,,$$

and B *is, in fact, positive definite.*

Lemma C.3 ([29], Lemma 4.1)**.** *Let* v *be a positive function of class* C^3 *and let* $V : \mathbb{R}^n \to \mathbb{R}^+$ *be of class* $C^3(\mathbb{R}^n)$ *and such that* $V(\nabla v)\mathrm{div}\,(\nabla V(\nabla v))$ *can be continuously extended to zero at* $\nabla v = 0$*. Let*

$$W = \nabla[\nabla_\xi V(\nabla v)] = V_{\xi_i \xi_j}(\nabla v)\partial^2_{ij} v\,. \tag{C.9}$$

Then, for any $\gamma \in \mathbb{R}$ *we have*

$$2v^\gamma S^2(W) = \mathrm{div}\,(v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v)) - \gamma v^{\gamma-1} S^2_{ij}(W)V_{\xi_i}(\nabla v)\partial_j v \tag{C.10}$$

and

$$\begin{aligned}
&\mathrm{div}\,\big(v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v) + \gamma(p-1)v^{\gamma-1}V(\nabla v)V_{\xi_j}(\nabla v)\big)\\
&\quad = 2v^\gamma S^2(W) + \gamma(\gamma-1)(p-1)v^{\gamma-2}V(\nabla v)V_{\xi_i}(\nabla v)\partial_i v\\
&\qquad + \gamma v^{\gamma-1}\left((p-1)V(\nabla v) + V_{\xi_i}(\nabla v)\partial_i v\right)\mathrm{Tr}\,(W)\\
&\qquad + \gamma v^{\gamma-1}\big((p-1)V_{\xi_i}(\nabla v)V_{\xi_j}(\nabla v)\partial^2_{ij}v + V_{\xi_j\xi_l}(\nabla v)\partial^2_{li}v V_{\xi_i}(\nabla v)\partial_j v\big)\,.
\end{aligned} \tag{C.11}$$

In particular, if

$$V(\xi) = \frac{|\xi|^p}{p} \qquad \textit{for } p > 1 \textit{ and } \xi \in \mathbb{R}^n\,, \tag{C.12}$$

then

$$\begin{aligned}
2v^\gamma S^2(W) = &\,\mathrm{div}\,\big(v^\gamma S^2_{ij}(W)V_{\xi_i}(\nabla v) + \gamma(p-1)v^{\gamma-1}V(\nabla v)\nabla_\xi V(\nabla v)\big)\\
&- \gamma(\gamma-1)p(p-1)v^{\gamma-2}V^2(\nabla v)\\
&- \gamma(2p-1)v^{\gamma-1}V(\nabla v)\Delta_p v\,.
\end{aligned} \tag{C.13}$$

Observe that, in this particular case,

$$W(x) := \nabla[|\nabla v(x)|^{p-2}\nabla v(x)]\,.$$

We are now ready to give the proof of Theorem C.1.

Proof of Theorem C.1. We consider the following auxiliary function:

$$v = u^{-\frac{p}{n-p}}\,. \tag{C.14}$$

An easy computation shows that v is a positive solution to

$$-\Delta_p v + \left(\frac{p}{n-p}\right)^{p-1}\frac{1}{v} + \frac{n(p-1)}{p}\frac{|\nabla v|^p}{v} = 0\,. \tag{C.15}$$

By using Newton's inequality (C.8) we get

$$2v^\gamma S_2(W) \leq \frac{n-1}{n}v^\gamma(\Delta_p v)^2\,, \tag{C.16}$$

for any $\gamma \in \mathbb{R}$, where W is as in Lemma C.3. From (C.16) and from formula (C.13) (here we can perform an approximation argument as in Subsection 3.2.1 which is simpler in this case because, thanks to Lemma C.1, we know that $u \in C^{2,\alpha}(\mathbb{R}^n \setminus B_{R_0})$) we obtain

$$
\begin{aligned}
\frac{n-1}{n} v^\gamma (\Delta_p v)^2 \geq & \operatorname{div}\left(v^\gamma S^2_{ij}(W) V_{\xi_i}(\nabla v) + \gamma(p-1) v^{\gamma-1} V(\nabla v) \nabla_\xi V(\nabla v)\right) \\
& - \gamma(\gamma-1) p(p-1) v^{\gamma-2} V^2(\nabla v) \\
& - \gamma(2p-1) v^{\gamma-1} V(\nabla v) \Delta_p v \,.
\end{aligned} \tag{C.17}
$$

Observe that choosing $\gamma = 1-n$ from Lemma C.1 we get that

$$
v^\gamma S^2_{ij}(W) V_{\xi_i}(\nabla v) + \gamma(p-1) v^{\gamma-1} V(\nabla v) \nabla_\xi V(\nabla v) = o(|x|^{n-1}) \,, \tag{C.18}
$$

as $|x| \to \infty$. Hence, by integrating (C.17) in a ball of radius R, by using the divergence Theorem and by sending R to infinity (using (C.18) and recalling that $|\nabla u|^{p-2}\nabla u \in W^{1,2}_{loc}(\mathbb{R}^n)$, see Proposition 3.2) we obtain (here we can perform an approximation argument as in Subsection 3.1.3)

$$
\begin{aligned}
\frac{n-1}{n} \int_{\mathbb{R}^N} v^\gamma (\Delta_p v)^2 \, dx \geq & - \gamma(\gamma-1)(p-1)p \int_{\mathbb{R}^N} v^{\gamma-2} V^2(\nabla v) \, dx \\
& - \gamma(2p-1) \int_{\mathbb{R}^N} v^{\gamma-1} V(\nabla v) \Delta_p v \, dx \,,
\end{aligned} \tag{C.19}
$$

and by using (C.15) (recall that $\gamma = 1-n$) we get

$$
-(n-1) \int_{\mathbb{R}^n} v^{-n-1} V(\nabla v) \, dx + \frac{n-1}{n} \left(\frac{p}{n-p} \right)^{p-1} \int_{\mathbb{R}^n} v^{-n-1} \, dx \geq 0 \,. \tag{C.20}
$$

Now we show that the equality holds in (C.20), indeed by multiplying (C.15) by v^{-n} and integrating by parts we obtain

$$
-(n-1) \int_{\mathbb{R}^n} v^{-n-1} V(\nabla v) \, dx + \frac{n-1}{n} \left(\frac{p}{n-p} \right)^{p-1} \int_{\mathbb{R}^n} v^{-n-1} \, dx = 0 \,.
$$

This implies that all the previous inequalities are equalities and then the equality holds in Newton's inequality (C.16), that is (from Lemma C.2)

$$
W(x) = \lambda(x) \mathrm{Id} \qquad \text{for a.e. } x \in \mathbb{R}^n \,, \tag{C.21}
$$

for some function $\lambda : \mathbb{R}^n \to \mathbb{R}$. To conclude the proof, we show that the function λ is constant. Since

$$
\begin{aligned}
\lambda(x) &= \frac{1}{n} \mathrm{Tr}\,(W(x)) \\
&= \frac{1}{n} \Delta_p v(x) \\
&= \frac{1}{n} \left(\frac{p}{n-p} \right)^{p-1} \frac{1}{v(x)} + \frac{p-1}{p} \frac{|\nabla v(x)|^p}{v(x)} \,,
\end{aligned}
$$

and since $v \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$, we get that $\lambda \in C^{0,\alpha}_{loc}(\mathbb{R}^n)$. Moreover, elliptic regularity theory yields that $v \in C^{2,\alpha}_{loc}(\mathbb{R}^n \setminus B_{R_0})$, which implies that $\lambda \in C^{1,\alpha}_{loc}(\mathbb{R}^n \setminus B_{R_0})$. From (C.21) we have that

$$
\partial_j (|\nabla v(x)|^{p-2} \partial_i v(x)) = \lambda(x) \delta_{ij} \qquad \text{for } i, j \in \{1, \ldots, n\}, \tag{C.22}
$$

which implies that $|\nabla v|^{p-2}\nabla v \in C^{2,\theta}_{loc}(\mathbb{R}^n \setminus B_{R_0})$. Then, given $i \in \{1, \ldots, n\}$, choosing $j \neq i$ and using (C.22) we obtain

$$\partial_i \lambda(x) = \partial_i(\partial_j(|\nabla v(x)|^{p-2}\partial_j v(x))) = \partial_j(\partial_i(|\nabla v(x)|^{p-2}\partial_j v(x))) = 0\,,$$

for any $x \in \mathbb{R}^n \setminus B_{R_0}$, which implies that λ is constant on $\mathbb{R}^n \setminus B_{R_0}$. In order to deduce that λ is constant in the whole $\mathbb{R}^n$, we can argue as in Subsection 3.2.2. For this reason we omit the details and conclude that λ is constant. In particular, recalling (C.21)

$$\nabla[|\nabla v|^{p-2}\nabla v] = W = \lambda \mathrm{Id} \quad \text{ in } \mathbb{R}^n\,.$$

Hence $|\nabla v(x)|^{p-2}\nabla v(x) = \lambda(x - x_0)$ for some $x_0 \in \mathbb{R}^n$; this implies that

$$v(x) = c_1 + c_2|x - x_0|^{\frac{p}{p-1}}\,,$$

or equivalently (recalling (C.14)) $u(x) = \mathcal{U}_{\lambda,x_0}(x)$. This completes the proof of Theorem C.1. □

Appendix D

Serrin vs Alexandrov

In this appendix we show that the celebrated theorems by Alexandrov and Serrin (Theorem A and Theorem B in the Introduction) are in some sense equivalent. In particular, one can use Serrin's Theorem to prove Alexandrov's one and viceversa. The Appendix is divided in two sections: in the first one we show how to deduce Serrin's Theorem from Alexandrov's Theorem, while in the second one we show how to deduce Alexandrov's Theorem from Serrin's Theorem.

We recall that Serrin's result is the following:

let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2. Then there exists a solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ to

$$\begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = c & \text{on } \partial\Omega\,. \end{cases}$$

if and only if Ω is a ball.

While Alexandrov's result is the following:

the sphere is the only C^2-regular, connected, closed hypersurface embedded in the Euclidean space with constan mean curvature.

D.1 Serrin implies Alexandrov

Let S be a connected, C^2-regular and closed hypersurface embedded in $\mathbb{R}^n$ with constant mean curvature. Thanks to the embededness we may assume that $S = \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. We want to prove that $\partial\Omega$ is a sphere. In order to do this, we consider the unique solution to the following Dirichlet problem

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{D.1}$$

Our goal is to assume that H, the mean curvature of S, is constant and to show that u satisfies

$$\partial_\nu u = c \quad \text{on } \partial\Omega. \tag{D.2}$$

Then according to Serrin's result, applied to $-u$, Ω must be a ball and we prove Alexandrov's Theorem.

To prove (D.2), we follow the approach in [194, Theorem 5] (see also [30, Appendix B]). Thanks to (I.15) and thanks to Newton's inequality (I.13) we get, for all $x \in \Omega$,

$$\frac{n-1}{2n} = \frac{n-1}{2n}(\Delta u)^2 \geq \frac{1}{2}\operatorname{div}(S^2_{ij}(\nabla^2 u)\partial_i u)\,. \tag{D.3}$$

By integrating over Ω, using the divergence theorem on the right hand side and using that $\nu = \nabla u/|\nabla u|$ we obtain

$$\begin{aligned}\frac{n-1}{2n}|\Omega| &\geq \frac{1}{2}\int_{\partial\Omega} S^2_{ij}(\nabla^2 u)\partial_i u \nu_j \, d\sigma \\ &= \frac{1}{2}\int_{\partial\Omega} S^2_{ij}(\nabla^2 u)\partial_i u \frac{\partial_j u}{|\nabla u|}\, d\sigma \\ &= \frac{n-1}{2} H \int_{\partial\Omega} |\nabla u|^2 \, d\sigma\,,\end{aligned}$$

where we used the following formula (see e.g. [30, Formula 61])

$$S^2_{ij}(\nabla^2 u)\partial_i u \partial_j u = (n-1)H|\nabla u|^3\,,$$

and the fact the mean curvature H of $\partial\Omega$ is constant. Hence we get

$$\int_{\partial\Omega} |\nabla u|^2 \, d\sigma \leq \frac{|\Omega|}{nH}\,. \tag{D.4}$$

On the other hand, from Holder's inequality, we have

$$\left(\int_{\partial\Omega} |\nabla u| \, d\sigma\right)^2 \leq |\partial\Omega| \int_{\partial\Omega} |\nabla u|^2 \, d\sigma\,. \tag{D.5}$$

Moreover, from the divergence theorem,

$$|\Omega| = \int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} |\nabla u| \, d\sigma\,. \tag{D.6}$$

Hence, from (D.5) and (D.6) we get

$$|\Omega|^2 \leq |\partial\Omega| \int_{\partial\Omega} |\nabla u|^2 \, d\sigma\,.$$

Recalling (D.4) we have proved that

$$|\Omega|^2 \leq |\partial\Omega| \frac{|\Omega|}{nH}\,. \tag{D.7}$$

Now, thanks to the *Minkowski's identity* (see e.g. [195])

$$\int_{\partial\Omega} Hx \cdot \nu \, d\sigma = |\partial\Omega|\,,$$

and the fact that, by assumption, the mean curvature H of $\partial\Omega$ is constant we obtain that

$$H = \frac{|\partial\Omega|}{n|\Omega|}\,.$$

and hence the equality in (D.7) holds. This entails that equality holds in both Newton and Holder inequalities and hence $|\nabla u|$ must be constant on $\partial\Omega$ and from this fact we immediately obtain that (D.2) holds true and we conclude.

D.2 Alexandrov implies Serrin

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2 and suppose that there exists a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ to

$$\begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = c & \text{on } \partial\Omega\,. \end{cases} \tag{D.8}$$

If we show that the mean curvature H of $\partial\Omega$ is constant, then according to Alexandrov's Theorem, $\partial\Omega$ must be a sphere.

To prove that the mean curvature of $\partial\Omega$ is constant, we argue as in [92, 215]. We use the already cited P-function introduced by Weinberger

$$P(x) = |\nabla u(x)|^2 + \frac{2}{n} u(x)\,. \tag{D.9}$$

We notice that P is subharmonic. Indeed

$$\Delta P = 2|\nabla^2 u|^2 + \frac{2}{n}\Delta u = 2\left(n|\nabla^2 u|^2 - \frac{1}{n}(\Delta u)^2\right)\,,$$

and this is non-negative according to Cauchy-Schwarz inequality. Moreover, P is constant on $\partial\Omega$, hence from the strong maximum principle we obtain that either

$$P \equiv c^2 \quad \text{in } \Omega. \tag{D.10}$$

or

$$P < c^2 \quad \text{in } \Omega. \tag{D.11}$$

Arguing as in Weinberger's proof of Serrin's theorem (see Introduction to Part I) one can prove that (D.10) holds true. From (D.10) and (D.9) we get

$$|\nabla u(x)| = \sqrt{c^2 - 2u(x)} := g(u(x))\,, \tag{D.12}$$

for all $x \in \Omega$. Obviously g is a function of class C^1 in $(0, \max u)$. Moreover, since the function $t \to \sqrt{t}$ is strictly monotone we have that $\nabla u \equiv 0$ only where u attains its maximum on Ω; then the vector field $\nu = -\nabla u/|\nabla u|$ is well-defined on the set $\mathcal{U} := \{x \in \Omega \,:\, u(x) \in (0, \max u)\}$.

We observe that $\partial_\nu u = -|\nabla u| = -g(u)$ and that

$$\Delta u = \partial^2_{\nu\nu} u + (n-1) H \partial_\nu u \tag{D.13}$$

where H is the mean curvature of the level set of u. From (D.13) and (D.8) we get that H depends only on u, actually on $g(u)$ and on $\partial^2_{\nu\nu} u$; moreover we observe that, on the one hand

$$\partial_\nu(|\nabla u|^2) = 2\partial_\nu u \partial^2_{\nu\nu} u\,,$$

and on the other hand, from (D.12),

$$\partial_\nu(|\nabla u|^2) = 2g(u)g'(u)\partial_\nu u\,,$$

so

$$\partial^2_{\nu\nu} u = g(u)g'(u)\,. \tag{D.14}$$

Hence, from (D.13) and (D.14), we get

$$H = \frac{1 + g(u)g'(u)}{(n-1)g(u)} \quad \text{in } \mathcal{U},$$

and this identity says that the mean curvature of each level set of u at height between 0 and $\max u$ is constant. By Alexandrov's Theorem we deduce that the connected components of each level set must be spheres. Since $\partial\Omega$ is connected this implies that Ω must be simply connected (otherwise a particular level set would contain two nested spheres of equal radius, which is a contradiction). Therefore each level set consists of exactly one sphere, and because of (D.12) these spheres are concentric and Ω is a ball.

Part IV

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