Tesi di Dottorato

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Numerical treatment of nearly singular and bisingular integral equations by means of suitable quadrature/cubature formulas

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Tesi di Dottorato

NUMERICAL TREATMENT OF NEARLY SINGULAR AND BISINGULAR INTEGRAL EQUATIONS BY MEANS OF SUITABLE QUADRATURE/CUBATURE FORMULAS

Settore Scientifico Disciplinare MAT/08 Analisi Numerica

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DOTTORANDA

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A Marmy,
alla forza con cui è stata capace di lottare.
Alla determinazione contagiosa con cui ha creduto in questa possibilità
e mi ha spinta a realizzarla, probabilmente per entrambe.
Alle facce buffe che faceva tutte le volte che le inventavo un nuovo nome.
Al $[06\,luglio\,1988, 29\,settembre\,2014]$*: intervallo "denso" di vita.*

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Abstract

English version.

In the first part of this thesis we propose quadrature/cubature rules of "product"and "dilation"type determining conditions under which the rules are stable and convergent in suitable weighted spaces. We diffusely treat the numerical approximation of integrals with weakly singular, nearly singular and/or highly oscillating kernel functions, both, in the degenerate and not-degenerate cases. These kernels are of interest since in these cases standard rules like the Gaussian are inefficient or can fail. Some numerical examples, which confirm the theoretical estimates, are also proposed.

In the second part of the thesis, we consider numerical methods for integral equations, both, in one and two dimensions. First of all, we consider the generalized of univariate and bivariate Love's integral equation. After studying the mapping properties of the involved integral operators, in order to approximate the solution of Love's equation, we propose a Nyström method based on a revisitation of the quadrature/cubature rules previously shown. We prove the stability and the convergence, in suitable weighted spaces, of the described numerical procedures and we show the efficiency of the two methods through some numerical tests.

At last, we investigate the numerical solution of Cauchy bisingular integral equations of the first kind on the square. In particular, we propose two different methods based on a global polynomial approximation of the unknown solution. The first one is a discrete collocation method applied to the original equation and hence defined as a "direct"method. The second one is an "indirect"procedure of discrete collocation-type since we act on the so-called regularized Fredholm equation. In both cases, the convergence and the stability of the methods are proved in suitable weighted spaces of functions, and the well conditioning of the involved linear systems is shown. Also for this topic, we propose some numerical tests which confirm the efficiency of the proposed procedures.

Italian version.

Nella prima parte di questa tesi sono state proposte formule di quadratura/cubatura di tipo “prodotto”e “dilation”per le quali sono state studiate la stabilità e la convergenza in opportuni spazi pesati. È stata trattata in maniera diffusa l’approssimazione numerica di integrali con funzioni nucleo debolmente singolari, quasi singolari e/o fortemente oscillanti, sia separabili che non separabili. Questi nuclei sono di interesse poichè le formule di tipo gaussiano, in tali casi, risultano poco efficienti o addirittura falliscono. A conferma delle stime teoriche dimostrate sono stati riportati alcuni esempi numerici.

Nella seconda parte della tesi sono stati considerati metodi numerici per equazioni integrali in una e due dimensioni. Innanzitutto, è stata considerata una generalizzazione dell’equazione integrale di Love, sia nel caso univariato che bivariato. Dopo aver studiato le proprietà di mappa degli operatori integrali coinvolti, per approssimare la soluzione dell’equazione integrale di Love, è stato proposto un metodo di Nyström basato su una rivisitazione delle formule di quadratura/cubatura introdotte in precedenza. È stata provata la stabilità e la convergenza delle procedure numeriche descritte in opportuni spazi pesati ed è stata confermata l’efficienza dei metodi proposti mediante alcuni test numerici.

Infine, si è investigato sulla soluzione numerica dell’equazione integrale bisingolare di Cauchy di prima specie sul quadrato. In particolare, sono stati proposti due diversi metodi basati sull’approssimazione polinomiale globale della soluzione incognita. Il primo metodo è un metodo di tipo collocazione discreta, applicato direttamente sull’equazione originale e per questo chiamato metodo “diretto”. Il secondo metodo, sempre di tipo collocazione discreta, è una procedura “indiretta”dal momento che agisce sull’equazione regolarizzata di Fredholm. In entrambi i casi sono state provate la stabilità e la convergenza dei metodi in opportuni spazi di funzione pesati ed è stato dimostrato il buon condizionamento dei sistemi lineari coinvolti. Anche per questo topic, sono stati riportati alcuni esempi numerici che hanno confermato l’efficienza delle procedure proposte.

Introduction

English version

The proposed research line has been strongly motivated by the growing and recent attention paid to the *singular problems*, i.e. those problems that lead to equations (integral, differential, integral-differential) whose solution has an "abnormal" behavior in terms, for example, of loss of regularity, unboundedness, etc. As demonstrated by the scientific community, which dedicates several national and international conferences to the topic, the proposed research plan is very topical and concerns the numerical approximation of integrals before, and then of integral equations, showing in the known functions (kernel and/or right-hand side), and therefore in the solutions, singularities on the domain.

In particular, this thesis proposes quadrature and cubature rules in order to compute integrals with *weakly singular*, *nearly singular* and/or *highly oscillating* kernel functions [O1, S1] and their applications to Fredholm integral equations of the second kind [F2] and Cauchy bisingular integral equations of the first kind [F1].

The reader interested in the description of the applications from which the above mentioned numerical problems arise, can consult the introductions of Chapters 2, 3, 4.

The methods for the Fredholm linear integral equations of the second kind are well known in the literature. In particular, the numerical treatment of Fredholm equations is detailed in Atkinson's book [2], which illustretes the main numerical methods (projection, Galerkin, Nyström). In the last twenty years, the research group of Numerical Analysis of the University of Basilicata, has considered these equations, in the univariate case, in non-classical spaces of functions, to which the functions belong. Indeed the involved functions may have singularities, at one or both the endpoints of the definition interval (which can also be unlimited), and inside the interval itself. From a functional point of view, this means considering spaces in which the functions, multiplied by a weight, are then continuous or measurable. The literature on the subject is very wide. A survey, from which it is possible to deduce many other references, is [16]. Recently, the two–dimensional case was focused. In particular, Fredholm equations of the second kind defined on plan domains, both in the limited and unlimited cases ([66, 48, 67, 68]), were considered.

The extension to the two–dimensions requires a further effort because the multivariate polynomial approximation tools are very few and not immediately applicable. Since, as mentioned, in this thesis the study of Fredholm and Cauchy equations in the multivariate case has been discussed, it has been necessary to face functional problems, in order to understand which are the "correct" functional spaces in which to consider the solution of the equations, and also to solve problems related to the multivariate approximation and to the suitable cubature formulas.

The thesis has been structured as follows:

- Chapter 1: Notation and Preliminary Results;
- Chapter 2: Product and Dilation Quadrature/Cubature Rules for some kinds of kernels;
- Chapter 3: Numerical Treatment of the Generalized Univariate and Bivariate Love Integral Equation;
- Chapter 4: Numerical Methods for Cauchy Bisingular Integral Equations of the first kind on the Square.

More precisely: in Chapter 1 tools, notation and preliminary results, used along all the thesis, are collected. For the univariate and bivariate case, Chapter 1 follows the same structure in the presentation of contents. In particular, for the univariate case some well-known theoretical results have been recalled, while, for the bivariate case, given the poorness of the results present in the literature, it has been necessary to prove some new preliminary theoretical results. In fact, the behavior of the bivariate Lagrange and Fourier operators in suitable weighted spaces has also been proved and it has been estimated the Gaussian cubature formula for analytical functions. These auxiliary results are completely new and can be used elsewhere even in contexts different from those taken into consideration in this work.

In Chapter 2 new and efficient quadrature/cubature rules are introduced for the computation of univariate integrals of the type

$$I(f,y) = \int_{-1}^{1} f(x)k(x,y)w(x)dx, \quad y \in [-1,1]$$

and bivariate of the type

$$\boldsymbol{I}(\boldsymbol{f},\mathbf{y}) = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{f}(\mathbf{x})\boldsymbol{k}(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x})d\mathbf{x}, \quad \boldsymbol{y} \equiv (y_1,y_2) \in [-1,1]^2,$$

where k is the bivariate kernel function in the variables x and y, $\boldsymbol{k}$ is the kernel function of four variables since $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$, w is a Jacobi weight and $\mathbf{w}$ is the product of two Jacobi weights. Moreover, the kernels k and $\boldsymbol{k}$ depend on a parameter that can make pathological the behavior of the kernels themselves. Infact, our attention is devoted to kernel functions k and $\boldsymbol{k}$ *weakly singular*, *nearly singular* and/or *highly oscillating*. We also considered the combination of two aspects, i.e. integrals with nearly singular and oscillating kernels. All the results concerning the bivariate case are new and have recently been published in [O1, S1]. For completeness, the univariate case, recently presented in [F2] in a revisited form, has also been studied.

In this Chapter we propose a *product quadratute/cubature formula* in order to approximate the numerical solution of the above mentioned integrals. These formulas are obtained by replacing the function f, respectively $\boldsymbol{f}$, by a univariate/bivariate Lagrange polynomial based on a set of knots (made by zeros of univariate Jacobi ortogonal polynomials) chosen such that the stability and the convergence of the rule is assured. For such quadrature/cubature formulas, stability and convergence in suitable weighted spaces are shown. Although the proposed approach, for these formulas, may seem to be very "simple", the computation of their coefficients is not yet an easy task. In the analogous univariate case, in order to compute the corresponding coefficients, it is necessary to determine modified moments by means of recurrence relations, and to examine the stability of such relations (see for instance [26, 73, 41, 81, 53]). This approach, however, does not appear feasible for not degenerate kernels, mostly in the multivariate case. In this thesis we present a unique approach for computing the coefficients of the aforesaid quadrature/cubature rule when k (respectively $\boldsymbol{k}$) belongs to the types described above. Such method, that we called *Dilation Method*, is based on a preliminary "dilation"of the domain of definition and, by suitable transformations, on the successive reduction of the initial integral to the sum of ones again defined on $[-1, 1]$ ($[-1, 1]^2$ respectively). These manipulations produce a "relaxation"in some sense, of the "too fast"behavior of k (respectively $\boldsymbol{k}$) when the parameter appearing in the definition of the kernel grows. After that, the problem is reduced to compute the integrals appearing in the sum by a suitable Gaussian rules. The proposed method is a generalization of the one–dimensional Dilation quadrature method proposed in [71] for nearly singular kernels and in [17] for highly oscillatory functions, both of them considered in the unweighted case (the integrand does not contain weight functions and all the estimates are proved in standard spaces of functions). The possibility of generalizing everything into weighted functional spaces is totally new, as well as new is the possibility to treat kernels that are a combination of the two previous kernels (i.e. *nearly singular and highly oscillating*). For a correct use of the 1D/2D *Dilation Method*, which could be also applied directly to compute integrals with kernels of the kind described above, we stated conditions under which the rule is stable and convergent.

In Chapter 3 numerical methods solving Love's integral equation [F2] were developed, both in the univariate:

$$f(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{\omega^{-1}}{(x-y)^2+\omega^{-2}} f(y)w(y)dy = g(x), \quad |x| < 1,$$

and bivariate cases:

$$\boldsymbol{f}(\mathbf{x}) - \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{\omega^{-1}}{|\mathbf{x}-\mathbf{y}|^2+\omega^{-2}} \boldsymbol{f}(\mathbf{y})\mathbf{w}(\mathbf{y})d\mathbf{y} = \boldsymbol{g}(\mathbf{x}), \ \mathbf{x} = (x_1, x_2) \in [-1,1]^2,$$

where $0 < \omega \in \mathbb{R}$ is a parameter, w is a Jacobi weight, $\mathbf{w}$ is the product of two Jacobi weights, g and $\mathbf{g}$ are the right-hand sides, f and $\boldsymbol{f}$ are the unknow functions. These equations present *nearly singular* behaviors in a fixed point of the domain.

The idea is to apply a revisitation of the quadrature/cubature formulas introduced in Chapter 2 [O1], in order to approximate the integral operator of Love's integral equation. The extension to the bivariate case was not immediate and it has required further studies. The needed tools are not trivial at all and required the use of "ad hoc" results of the functional analysis and of the approximation theory. In fact, the equation has to be studied in a proper functional space and this goal involves the study of mapping properties of the involved integral operators. All the results presented in Chapter 3 are original and can be found in [F2].

Chapter 4 has been devoted to the development of numerical methods solving Cauchy bisingular integral equations, defined on plane domains. The bisingular equation of the first kind was therefore considered

$$(D+K)\boldsymbol{f} = \boldsymbol{g}$$

where $\boldsymbol{f}$ is the bivariate unknown function and $\boldsymbol{g}$ is a given functions defined on the unit square $S = [-1,1] \times [-1,1]$, D is the dominant operator defined as

$$D\boldsymbol{f}(t,s) = \frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)} \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}}\, dx\, dy$$

where here and in the sequel the symbol $\oint$ means that the integral has to be interpreted in the Cauchy Principal Value sense and K is the perturbation operator

$$K\boldsymbol{f}(t,s) = \int_S \boldsymbol{k}(x,y,t,s)\boldsymbol{f}(x,y) \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-y}{1+y}}\, dx\, dy,$$

where $\boldsymbol{k}$ is the kernel function defined on S^2. For the numerical treatment of the equation, we propose two different approaches, both based on a global polynomial approximation of the unknown bivariate function f. The first one is a *direct* method since we act directly on the equation, while the second one is an *indirect* procedure, since it solves an equivalent regularized Fredholm equation. As a preliminary study, we show the mapping properties of the operator D and K in suitable weighted space equipped with the 2-norm, and we give a complete analysis of the proposed methods in suitable weighted L^2 subspaces. In details, we examine the stability, show the related convergence results and error estimates, and discuss the condition numbers of the involved linear systems. All the results presented in Chapter 4 are original and have been recently published in [F1].

For more details, the reader can consult the prefaces of the individual Chapters. Moreover, some numerical examples, which confirm the theoretical estimates, are also proposed at the end of Chapters 2, 3 and 4.

Italian version

Il filone di ricerca proposto è stato fortemente motivato dalla crescente e recente attenzione ai *problemi singolari*, ossia a quei problemi che si concretizzano in equazioni (integrali, differenziali, integro-differenziali) la cui soluzione ha un comportamento "anomalo"in termini, ad esempio, di perdita di regolarità, illimitatezza etc. Come dimostrato dalla comunità scientifica, che a questo argomento dedica diversi convegni in ambito nazionale e internazionale, il piano di ricerca affrontato nel presente lavoro di tesi, è molto attuale e riguarda la risoluzione numerica di integrali prima, e di equazioni integrali poi, che presentano nelle funzioni note (nucleo e/o termine noto), e quindi nelle soluzioni, delle singolarità sul dominio di definizione.

In particolare, sono state trattate formule di quadratura e cubatura per integrali con nuclei *debolmente singolari*, *quasi singolari* e/o *fortemente oscillanti* [O1, S1] e loro applicazioni alle equazioni integrali di Fredholm di seconda specie [F2] ed equazioni integrali bisingolari di Cauchy di prima specie [F1].

Per i dettagli relativi alle applicazioni dalle quali scaturiscono i suddetti problemi numerici, si rimanda alle introduzioni dei Capitoli 2, 3, 4.

I metodi per le equazioni integrali di Fredholm di seconda specie, di tipo lineare, sono ben collaudati e appartengono oramai alla letteratura classica sull'argomento. In particolare, la trattazione numerica delle equazioni di Fredholm è dettagliatamente esposta nel libro di Atkinson [2], in cui sono raccolti i principali metodi risolutivi (di proiezione, di Galerkin, di Nyström). Negli ultimi venti anni, il gruppo di ricerca di Analisi Numerica dell'Università degli Studi della Basilicata, ha considerato tali equazioni, nel caso unidimensionale, in spazi di funzioni non classici, ai quali appartengono funzioni che possono avere singolarità, sia agli estremi dell'intervallo di definizione (che può anche essere non limitato), sia interne all'intervallo stesso. Dal punto di vista funzionale, questo significa considerare spazi nei quali le funzioni, moltiplicate per un peso, risultino poi continue o misurabili. La letteratura sull'argomento è davvero molto ampia. Un lavoro di survey dal quale possono essere desunte molte altre indicazioni bibliografiche è [16]. Recentemente l'attenzione si è spostata sul caso bidimensionale. Sono state in particolare considerate equazioni di Fredholm di seconda specie definite su domini piani, sia limitati che non limitati ([66, 48, 67, 68]).

Il passaggio alle due dimensioni richiede un ulteriore sforzo in quanto gli strumenti di approssimazione polinomiale in più variabili sono pochi e non immediatamente applicabili. Poichè, come detto, in questo lavoro è stato affrontato lo studio di equazioni lineari di Fredholm e singolari di Cauchy nel caso multivariato, è stato necessario affrontare problemi sia di tipo funzionale, per capire quali fossero gli spazi di funzione "corretti"nei quali considerare le funzioni soluzione delle equazioni, sia problemi legati invece all'approssimazione in più variabili e a formule di cubatura opportune.

Il lavoro di tesi è stato strutturato nel seguente modo:

- Capitolo 1: Notazione e Risultati Preliminari;
- Capitolo 2: Formule di Quadratura e Cubatura di Tipo Prodotto e Dilation per Alcuni Tipi di Nuclei;
- Capitolo 3: Trattamento Numerico dell'Equazione Integrale Generalizzata di Love nel caso Univariato e Bivariato;
- Capitolo 4: Metodi Numerici per le Equazioni Integrali Bisingolari di Cauchy di Prima Specie sul Quadrato.

Più precisamente: nel Capitolo 1 sono stati raccolti strumenti, notazioni e risultati preliminari, usati poi nel resto della tesi. La trattazione del Capitolo 1 segue la stessa struttura, per il caso univariato e bivariato, per ciò che concerne la presentazione dei contenuti. In particolare, per il caso univariato sono stati riportati alcuni risultati teorici ben noti in letteratura, mentre, per il caso bivariato, data la scarsità dei risultati presenti in letteratura, è stato necessario dimostrarne anche di nuovi. È stato infatti provato il comportamento degli operatori bivariati di Lagrange e di Fourier in opportuni spazi pesati ed è stata fornita una stima per la formula di cubatura gaussiana nel caso di funzioni analitiche. Questi risultati ausiliari sono del tutto nuovi e possono essere usati anche in contesti diversi da quello preso in considerazione.

Nel Capitolo 2 sono state introdotte nuove ed efficienti formule di quadratura/cubatura per la computazione di integrali univariati del tipo:

$$I(f,y) = \int_{-1}^{1} f(x)k(x,y)w(x)dx, \quad y \in [-1,1]$$

e bivariati del tipo:

$$\boldsymbol{I}(\boldsymbol{f},\mathbf{y}) = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{f}(\mathbf{x})\boldsymbol{k}(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x})d\mathbf{x}, \quad \boldsymbol{y} \equiv (y_1,y_2) \in [-1,1]^2,$$

dove k è una funzione bivariata nelle variabili x e y, $\boldsymbol{k}$ è una funzione di quattro variabili dal momento che $\boldsymbol{x} = (x_1, x_2)$ e $\boldsymbol{y} = (y_1, y_2)$, w è un peso di Jacobi e $\mathbf{w}$ è il prodotto di due pesi di Jacobi. Inoltre, i nuclei k e $\boldsymbol{k}$ dipendono da un parametro che può rendere patologico il comportamento del nucleo stesso. Infatti, la nostra attenzione è stata focalizzata su funzioni nucleo k e $\boldsymbol{k}$ *debolmente singolari*, *quasi singolari* o *fortemente oscillanti*. Abbiamo inoltre considerato la combinazione di due aspetti, cioè integrali con nuclei che presentano ambedue le patologie: *quasi singolari e fortemente oscillanti*. Tutti i risultati inerenti il caso bivariato sono originali e sono stati recentemente pubblicati in [O1, S1]. Per completezza di trattazione, è stato anche studiato il caso univariato, recentemente presentato, in forma rivisitata, in [F2].

Quello che proponiamo in questo Capitolo è una formula di quadratura/cubatura di tipo prodotto in grado di fornire una soluzione approssimata dell'integrale iniziale. Tale formula, è stata ottenuta rimpiazzando la funzione f, analogamente per $\boldsymbol{f}$, con un polinomio univariato/bivariato di Lagrange basato su un set di nodi (zeri di polinomi ortogonali univariati di Jacobi) scelti in modo da assicurare la stabilità e la convergenza della formula. Per tali formule è stata provata la stabilità e la convergenza in opportuni spazi funzionali pesati. Nonostante l'approccio proposto possa apparire molto "semplice", il calcolo dei coefficienti, della formula di quadratura/cubatura proposta, risulta complicato. Per analoghe problematiche nel caso univariato, sono state determinate relazioni di ricorrenza per i momenti modificati [26, 73, 41, 81, 53]. Tali tecniche risultano però inapplicabili in presenza di nuclei non degeneri, soprattutto per il caso multivariato. Quello che proponiamo, dunque, è la possibilità di computare i coefficienti della formula di quadratura/cubatura con un unico approccio quando il nucleo k (analogamente per $\boldsymbol{k}$) presenta, una o più delle patologie sopra descritte. Tale metodo, che abbiamo chiamato *Dilation Method*, è basato su una preliminare dilatazione del dominio, in grado di ridurre le problematiche della funzione integranda. Successivamente, l'integrale iniziale viene splittato nella somma di un certo numero di integrali e, ciascuno di questi, viene poi approssimato con un'opportuna formula di quadratura/cubatura gaussiana opportunamente pesata. Queste manipolazioni hanno lo scopo di "rilassare", in un certo senso, il comportamento rapidamente oscillante e/o con picchi elevati intorno alla singolarità x_0 (analogamente per $\mathbf{x}_0 = (s_0, t_0)$), del nucleo k (analogamente per $\boldsymbol{k}$) al crescere del parametro presente nella definizione del nucleo. Il metodo proposto risulta essere una generalizzazione del *Dilation Method* unidimensionale proposto in [71] per nuclei *quasi singolari* e in [17] per nuclei *fortemente oscillanti*. In entrambi i lavori però, gli autori si sono occupati soltanto del caso *non pesato* (la funzione integranda non contiene funzioni peso e tutte le stime sono state fatte in spazi di funzioni non pesati). La possibilità di generalizzare il tutto in spazi di funzioni pesati è totalmente nuova, come nuova è la possibilità di trattare, con un unico approccio, nuclei che siano una combinazione dei due nuclei precedenti (i.e. *quasi singolari e fortemente oscillanti*). D'altra parte, il *Dilation Method* in 1D/2D, può essere applicato direttamente per il calcolo approssimato di integrali del tipo descritto sopra e, in ogni caso, per un corretto utilizzo di tale approssimazione nel calcolo dei coefficienti della formula prodotto, è stato necessario provare la stabilità e la convergenza anche per la formula di quadratura/cubatura di tipo *dilation*.

Nel Capitolo 3 sono stati sviluppati metodi numerici per la risoluzione dell'equazione integrale di Love generalizzata [F2], sia nel caso univariato:

$$f(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{\omega^{-1}}{(x-y)^2+\omega^{-2}} f(y)w(y)dy = g(x), \quad |x| < 1,$$

che bivariato:

$$\boldsymbol{f}(\mathbf{x}) - \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{\omega^{-1}}{|\mathbf{x}-\mathbf{y}|^2+\omega^{-2}} \boldsymbol{f}(\mathbf{y})\mathbf{w}(\mathbf{y})d\mathbf{y} = \boldsymbol{g}(\mathbf{x}),\ \mathbf{x} = (x_1, x_2) \in [-1,1]^2,$$

dove $0 < \omega \in \mathbb{R}$ è un parametro, w è un peso di Jacobi, $\mathbf{w}$ è il prodotto di due pesi di Jacobi, g e $\boldsymbol{g}$ sono le funzioni termine noto, f e $\boldsymbol{f}$ sono le funzioni incognite. Tale equazione presenta comportamenti *quasi singolari* in un punto fisso del dominio di definizione.

L'idea è quella di applicare una rivisitazione delle formule di quadratura/cubatura introdotte nel Capitolo 2 [O1], per l'approssimazione dell'operatore integrale dell'equazione di Love. Il passaggio al caso bivariato, non è stato immediato e ha richiesto ulteriori approfondimenti. Infatti, dopo una preliminare familiarizzazione con gli operatori integrali in due dimensioni, è stato necessario studiare le proprietà di mappa degli operatori integrali bivariati coinvolti, in opportuni spazi di funzioni pesati. Gli strumenti necessari per questa impostazione sono stati non banali e hanno richiesto l'utilizzo di risultati "ad hoc"sia di analisi funzionale che di teoria dell'approssimazione. Tutti i risultati del Capitolo 3 sono originali e sono contenuti in [F2].

Il Capitolo 4 è stato dedicato allo sviluppo di metodi numerici per la risoluzione di equazioni integrali bisingolari con nucleo di Cauchy, definite su domini del piano. È stata considerata quindi l'equazione bisingolare di prima specie

$$(D+K)\boldsymbol{f} = \boldsymbol{g}$$

dove $\boldsymbol{f}$ è la funzione bivariata da determinare, $\boldsymbol{g}$ è il termine noto definito sul quadrato unitario $S = [-1,1] \times [-1,1]$, D è l'*operatore dominante*:

$$D\boldsymbol{f}(t,s) = \frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-y}{1+y}}\, dx\, dy$$

dove il simbolo $\oint$ significa che l'integrale doppio è da intendere come composizione di due integrali nel senso del valor principale di Cauchy e K è l'*operatore di perturbazione* definito come segue

$$K\boldsymbol{f}(t,s) = \int_S \boldsymbol{k}(x,y,t,s)\boldsymbol{f}(x,y) \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-y}{1+y}}\, dx\, dy,$$

con $\boldsymbol{k}$ funzione nucleo definita su S^2.

Per la risoluzione dell'equazione iniziale sono stati sviluppati due metodi: il primo, di tipo collocazione discreta, agisce sull'equazione originale e pertanto è detto *diretto*; il secondo, sempre di tipo collocazione discreta, è *indiretto* e risolve un'equivalente equazione regolarizzata di Fredholm. Entrambi i metodi sono di tipo globale e si basano sull'approssimazione polinomiale della soluzione incognita. Nel corso del capitolo sono state dimostrate proprietà di mappa degli operatori D e K in opportuni spazi L^2 pesati, risultati di convergenza, stabilità e buon condizionamento del sistema lineare equivalente all'equazione finito dimensionale che risolve il problema. Tutti i risultati del Capitolo 4 sono originali e sono stati recentemente pubblicati in [F1].

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Chapter 1

Notation and Preliminary Results

In this Chapter we collect the main notation and tools we will use from now on. For the univariate case some well known results have been recalled, while, for the bivariate case, given the poorness of the results present in the literature, it was necessary to prove some new preliminary theoretical results with respect to the behaviour of the bivariate Lagrange and Fourier operators in suitable weighted spaces. Moreover, the Gaussian cubature formula for analytical functions was estimated. These auxiliary results are completely new and can be used elsewhere even in contexts different from those taken into consideration in this work.

Along all the thesis the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. Moreover from now on we will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \ldots$, and $\mathcal{C} = \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. Moreover, if $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a constant $0 < \mathcal{C} \neq \mathcal{C}(A, B)$ such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

1.1 The univariate case

In this Section we collect the main notation and tools we will use in the univariate case. From now on $\mathbb{P}_m$ denotes the space of the univariate algebraic polynomials of degree at most m. In what follows we use the notation $w := v^{\alpha,\beta}$ for a Jacobi weight of parameters $\alpha, \beta > -1$, i.e.

$$w(x) := v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta},\ x \in (-1, 1). \tag{1.1.1}$$

Moreover we will denote by N_1^m the set $\{1, 2, \ldots, m\}$.

1.1.1 Functional spaces

Let

$$\sigma(x) := v^{\gamma,\delta}(x) = (1-x)^\gamma(1+x)^\delta, \tag{1.1.2}$$

with $\gamma, \delta \geq 0$. Define

$$C_\sigma = \left\{ f \in C((-1,1)) : \lim_{x\to\pm1^\mp} (f\sigma)(x) = 0 \right\},$$

equipped with the weighted uniform norm

$$\|f\|_{C_\sigma} = \|f\sigma\|_\infty = \max_{x\in[-1,1]} |(f\sigma)(x)|,$$

and in the sequel we denote by $\|f\sigma\|_{\infty,A} = \max_{x\in A} |(f\sigma)(x)|$ the uniform norm on the set $A \subset [-1,1]$. The limit conditions in the definition of C_σ guarantee that C_σ is a Banach space. Whenever one or more of the parameters γ, δ, are greater than 0, C_σ includes functions that can be singular on one or both the endpoints of $[-1,1]$. In the case $\gamma = \delta = 0$, $C_\sigma = C^0([-1,1])$. i.e. C_σ reduces to the space of continuous functions.

For smoother functions, i.e. for functions having some derivatives which can be continuous on $(-1,1)$, for $r \in \mathbb{N}$ we introduce the following Sobolev–type space

$$\mathcal{W}^r_{\sigma,\infty} = \left\{ f \in C_\sigma : \left\| f^{(r)}\varphi^r\sigma \right\|_\infty < \infty \right\},$$

where the superscrit (r) denotes the rth derivative of the function f and $\varphi(x) = \sqrt{1-x^2}$. We equip $\mathcal{W}^r_{\sigma,\infty}$ with the norm

$$\|f\|_{\mathcal{W}^r_{\sigma,\infty}} = \|f\sigma\|_\infty + \left\| f^{(r)}\varphi^r\sigma \right\|_\infty.$$

Furthermore, for any $p \in \mathbb{N}_0 \cup \{\infty\}$, we denote by $C^p([-1,1])$ the set of all continuous functions having p continuous derivatives.

Denoted by $E_m(f)_{\sigma,\infty}$ the error of best polynomial approximation in C_σ, i.e.

$$E_m(f)_{\sigma,\infty} = \inf_{P\in\mathbb{P}_m} \|(f-P)\sigma\|_\infty,$$

the following iterated Favard's inequality holds true [47] for each function $f \in \mathcal{W}^r_{\sigma,\infty}$

$$E_m(f)_{\sigma,\infty} \leq \frac{\mathcal{C}}{m^r} \left\| f^{(r)}\varphi^r\sigma \right\|_\infty, \quad 0 < \mathcal{C} \neq \mathcal{C}(m,f). \tag{1.1.3}$$

Moreover for any $f, g \in C_\sigma$, the following inequality can be easily proved

$$E_m(fg)_{\sigma,\infty} \leq \|g\sigma\|_\infty E_M(f)_{\sigma,\infty} + \|f\sigma\|_\infty E_M(g)_{\sigma,\infty}, \tag{1.1.4}$$

where $M = \left\lfloor \frac{m}{2} \right\rfloor$.

From now on, let us denote by $L^2_w \equiv L^2_w([-1,1])$ the weighted Hilbert space, with inner product

$$\langle f, g\rangle_w = \int_{-1}^{1} f(x)g(x)w(x)\,dx,$$

equipped with the norm

$$\|f\|_{L^2_w} = \|f\sqrt{w}\|_2 = \sqrt{\langle f, f\rangle_w}. \tag{1.1.5}$$

In L^2_w, for a integer $r \geq 1$, we introduce the following Sobolev-type subspace

$$\mathcal{W}^r_{w,2} = \{f \in L^2_w : f^{(r-1)} \in AC((-1,1)), \|f\|_{\mathcal{W}^r_{w,2}} = \|f\|_{L^2_w} + \|f^{(r)}\varphi^r\|_{L^2_w} < \infty\} \tag{1.1.6}$$

where $AC((-1,1))$ denotes the set of all continuous functions f which are absolutely continuous on every closed subinterval of $(-1,\ 1)$.

1.1.2 Fourier and Lagrange operators

Let us consider the weight function $w = v^{\alpha,\beta}$ defined in (1.1.1) and let $\{p_m(w)\}_{m=0}^{\infty}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e.

$$p_m(w,x) = \gamma_m(w)x^m + \text{ terms of lower degree}, \quad \gamma_m(w) > 0, \tag{1.1.7}$$

and let $\{\xi^w_i\}_{i=1}^m$ be the zeros of $p_m(w,x)$.

Fourier sums

For a given function $f \in L^2_w$, where L^2_w is defined in (1.1.5), let

$$S_m(f,w,x) = \sum_{i=0}^{m-1} c_i(f,w)p_i(w,x) \tag{1.1.8}$$

be the mth Fourier sum, where

$$c_i(f,w) = \int_{-1}^{1} f(x)p_i(w,x)w(x)dx = \langle f, p_i\rangle_w, \quad i = 0, ...m-1,$$

are the Fourier coefficients.

Next two propositions (see [10, 15, 47]) show the behaviour of $S_m(f,w)$ in the case $f \in L^2_w$ or $f \in \mathcal{W}^r_{w,2}$, where $\mathcal{W}^r_{w,2}$ is defined in (1.1.6).

Proposition 1.1.1. *Let $f \in \mathcal{W}^r_{w,2}$ and let $S_m(f,w)$ be the mth Fourier sum with respect to the weight w defined in* (1.1.8). *Let r_1 and r be two positive integers such that $r_1 \leq r$. Then there exists a positive constant $\mathcal{C} \neq \mathcal{C}(m,f)$ such that the following estimates hold true*

$$\|f - \mathcal{S}_m(f,w)\|_{\mathcal{W}^{r_1}_{w,2}} \leq \frac{\mathcal{C}}{m^{r-r_1}} \|f\|_{\mathcal{W}^r_{w,2}}, \tag{1.1.9}$$

$$\|S_m(f,w)\|_{\mathcal{W}^r_{w,2}} \leq \mathcal{C} \, \|f\|_{\mathcal{W}^r_{w,2}}, \tag{1.1.10}$$

where in all the inequalities $\mathcal{C} \neq \mathcal{C}(m,f)$.

Let us define now the error of best polynomial approximation in L^2_w as

$$E_m(f)_{w,2} = \inf_{P \in \mathbb{P}_m} \|f - P\|_{L^2_w}.$$

The following proposition shows the connection between $S_m(f,w)$ and $E_m(f)_{w,2}$.

Proposition 1.1.2. *Let $f \in L^2_w$. Then*

$$E^2_m(f)_{w,2} = \|f - S_m(f,w)\|^2_{L^2_w} = \|f\|^2_{L^2_w} - \sum_{i=0}^{m-1} c^2_i(f,w). \tag{1.1.11}$$

Thus, according to the previous result (see [51] and the reference therein), $\mathcal{S}_m(f,w)$ is the best polynomial approximation of $f \in L^2_w$ and, if the Weierstrass Theorem holds true, by (1.1.11) we get the Parseval identity

$$\|f\|_{L^2_w} = \sqrt{\sum_{i=0}^{\infty} c^2_i(f,w)}$$

and, consequently

$$E_m(f)_{w,2} = \sqrt{\sum_{i \geq m} c^2_i(f,w)}.$$

Lagrange interpolating polynomials in $[-1,1]$

For a given function $f \in C((-1,1))$, let $\mathcal{L}_m(f,w,x)$ be the Lagrange polynomial interpolating f at the zeros $\{\xi^w_i, \; i \in N^m_1\}$ of the mth Jacobi polynomial $p_m(w,x)$, i.e.

$$\mathcal{L}_m(f,w,\xi^w_i) = f(\xi^w_i), \quad i \in N^m_1.$$

The polynomial $\mathcal{L}_m(f,w) \in \mathbb{P}_{m-1}$ and $\mathcal{L}_m(P,w) = P$, for any $P \in \mathbb{P}_{m-1}$. An expression of $\mathcal{L}_m(f,w)$ is given by

$$\mathcal{L}_m(w,f,x) = \sum_{i=1}^{m} \ell_i^w(x) f(\xi_i^w), \tag{1.1.12}$$

where

$$\ell_i^w(x) = \frac{p_m(w,x)}{p'_m(w,\xi_i^w)(x-\xi_i^w)} \tag{1.1.13}$$

is the ith *fundamental Lagrange polynomial*. It is known that ℓ_i^w can be expressed in equivalent form as [47]

$$\ell_i^w(x) = \lambda_i^w D_m(w,x,\xi_i^w), \tag{1.1.14}$$

where D_m is the *Darboux kernel* defined as

$$\begin{aligned} D_m(w,x,t) &:= \frac{\gamma_{m-1}}{\gamma_m} \frac{p_m(w,x)p_{m-1}(w,t) - p_m(w,t)p_{m-1}(w,x)}{x-t} \\ &= \sum_{k=0}^{m-1} p_k(w,x)p_k(w,t). \end{aligned}$$

About $\mathcal{L}_m(w,f,x)$, the following estimates hold true [10, 15, 47].

Proposition 1.1.3. *Let $f \in \mathcal{W}^r_{w,2}$ and let $\mathcal{L}_m(f,w)$ be the Lagrange polynomial defined in* (1.1.12)*. Then, the following estimates hold true:*

$$\|f - \mathcal{L}_m(f,w)\|_{L^2_w} \le \frac{\mathcal{C}}{m^r} \|f\|_{\mathcal{W}^r_{w,2}}, \tag{1.1.15}$$

$$\|\mathcal{L}_m(f,w)\|_{L^2_w} \le \|f\|_{L^2_w} + \frac{\mathcal{C}}{m^r} \|f\|_{\mathcal{W}^r_{w,2}}, \tag{1.1.16}$$

$$\|[f - \mathcal{L}_m(f,w)]^{(r)}\varphi^r\|_{L^2_w} \le \mathcal{C}\left(\|f^{(r)}\varphi^r\|_{L^2_w} + m^r\|f - \mathcal{L}_m(f,w)\|_{L^2_w}\right), \tag{1.1.17}$$

where in all the inequalities $\mathcal{C} \neq \mathcal{C}(m,f)$.

1.1.3 Gauss-Jacobi quadrature rules

Let $w = v^{\alpha,\beta}$ be defined as in (1.1.1). The Gauss-Jacobi quadrature rule, which will be essential for our aims, takes the following expression

$$\begin{aligned} \int_{-1}^{1} f(x)w(x)\,dx &= \sum_{i=1}^{m} \lambda_i^w f(\xi_i^w) + \mathcal{R}_m^{\mathcal{G}}(f) \\ &:= \mathcal{G}_m^w(f) + \mathcal{R}_m^{\mathcal{G}}(f), \end{aligned} \tag{1.1.18}$$

where $\{\xi_i^w\}_{i=1}^m$ are the zeros of the Jacobi polynomial $p_m(w,x)$ defined in (1.1.7), λ_i^w, $i = 1, \dots, m$, denote the Christoffel numbers with respect to w, i.e.

$$\lambda_i^w = \int_{-1}^1 \ell_i^w(x)w(x)dx = \int_{-1}^1 \left(\ell_i^w(x)\right)^2 w(x)dx,$$

and the remainder term $\mathcal{R}_m^{\mathcal{G}}(P) = 0$ for any $P \in \mathbb{P}_{2m-1}$. About the error estimate, the following result is well known (see, for instance, [47])

Proposition 1.1.4. *Let $f \in C_\sigma$. Under the assumption*

$$\int_{-1}^1 \frac{w(x)}{\sigma(x)} dx < +\infty,$$

we have

$$\left|\mathcal{R}_m^{\mathcal{G}}(f)\right| \leq \mathcal{C} E_{2m-1}(f)_\sigma, \tag{1.1.19}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Moreover if $f \in C^{2m}([-1,1])$ then

$$\left|\mathcal{R}_m^{\mathcal{G}}(f)\right| \leq \frac{\left\|f^{(2m)}\right\|_\infty}{(2m)!\ \gamma_m^2(w)}, \tag{1.1.20}$$

where $\gamma_m(w)$ is the positive leading coefficient of $p_m(w,x)$ introduced in (1.1.7).

1.2 The bivariate case

In this Section we collect the main notation and tools we will use in the bivariate case. From now on $\mathbb{P}_{m,m}$ denotes the space of the two–dimensional algebraic polynomials of degree at most m in each variable and $S := [-1,1]^2$, $\dot{S} = S \backslash \partial S = (-1,1)^2$, where ∂S denotes the boundary of S, $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in S$. Moreover, we use the notation

$$\mathbf{w}(\mathbf{z}) := w_1(z_1)w_2(z_2) = (1-z_1)^{\alpha_1}(1+z_1)^{\beta_1}(1-z_2)^{\alpha_2}(1+z_2)^{\beta_2}, \tag{1.2.1}$$

where $\mathbf{z} = (z_1, z_2) \in S$, for denoting a product of two Jacobi weights of parameters α_1, α_2, β_1, $\beta_2 > -1$.

Finally we will denote by $N_1^m \times N_1^m$ the set $\{1, 2, \dots, m\} \times \{1, 2, \dots, m\}$.

1.2.1 Functional spaces

Let

$$\boldsymbol{\sigma}(\mathbf{x}) := \sigma_1(x_1)\sigma_2(x_2) = v^{\gamma_1,\delta_1}(x_1)v^{\gamma_2,\delta_2}(x_2), \tag{1.2.2}$$

with $\gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2 \geq 0$.Define

$$C_{\boldsymbol{\sigma}} = \left\{ \boldsymbol{f} \in C(\dot{S}) : \lim_{\mathbf{x}\to\partial S} (\boldsymbol{f}\boldsymbol{\sigma})(\mathbf{x}) = 0 \right\},$$

equipped with the weighted uniform norm

$$\|\boldsymbol{f}\|_{C_{\boldsymbol{\sigma}}} = \|\boldsymbol{f}\boldsymbol{\sigma}\|_\infty = \max_{\mathbf{x}\in S} |(\boldsymbol{f}\boldsymbol{\sigma})(\mathbf{x})|,$$

and in the sequel we denote by $\|\boldsymbol{f}\boldsymbol{\sigma}\|_{\infty,B} := \max_{\mathbf{x}\in B} |(\boldsymbol{f}\boldsymbol{\sigma})(\mathbf{x})|$, the uniform norm on the set $B \subset S$. Whenever one or more of the parameters $\gamma_1, \delta_1, \gamma_2, \delta_2$ are greater than 0, functions in $C_{\boldsymbol{\sigma}}$ can be singular on one or more sides of S. In the case $\gamma_1 = \delta_1 = \gamma_2 = \delta_2 = 0$ we set $C_{\boldsymbol{\sigma}} = C^0(S)$. For smoother functions, i.e. for functions having some partial derivatives which can be discontinuous on ∂S, for $r \in \mathbb{N}$ we introduce the following Sobolev–type space

$$W^r_{\boldsymbol{\sigma},\infty} = \left\{ \boldsymbol{f} \in C_{\boldsymbol{\sigma}} : \|\boldsymbol{f}\|_{W^r_{\boldsymbol{\sigma},\infty}} = \|\boldsymbol{f}\boldsymbol{\sigma}\|_\infty + \mathcal{M}_r(\boldsymbol{f}, \boldsymbol{\sigma}) < \infty \right\} \tag{1.2.3}$$

where

$$\mathcal{M}_r(\boldsymbol{f}, \boldsymbol{\sigma}) := \max \left\{ \left\| \frac{\partial^r \boldsymbol{f}}{\partial x_1^r} \varphi_1^r \boldsymbol{\sigma} \right\|_\infty, \left\| \frac{\partial^r \boldsymbol{f}}{\partial x_2^r} \varphi_2^r \boldsymbol{\sigma} \right\|_\infty \right\}$$

and $\varphi_i(z_i) = \sqrt{1 - z_i^2}$, $i \in \{1, 2\}$.

Furthermore, for any $p \in \mathbb{N}_0 \cup \{\infty\}$, we denote by $C^p(S)$ the set of all bivariate continuous functions having p continuous partial derivatives.

Denoted by $E_{m,m}(\boldsymbol{f})_{\boldsymbol{\sigma}}$ the error of best polynomial approximation in $C_{\boldsymbol{\sigma}}$, i.e.

$$E_{m,m}(\boldsymbol{f})_{\boldsymbol{\sigma},\infty} = \inf_{\boldsymbol{P}\in\mathbb{P}_{m,m}} \|(\boldsymbol{f} - \boldsymbol{P})\boldsymbol{\sigma}\|_\infty,$$

in [66] (see also [48]) it was proved that for any $\boldsymbol{f} \in W^r_{\boldsymbol{\sigma},\infty}$

$$E_{m,m}(\boldsymbol{f})_{\boldsymbol{\sigma},\infty} \leq \mathcal{C} \frac{\mathcal{M}_r(\boldsymbol{f}, \boldsymbol{\sigma})}{m^r}, \tag{1.2.4}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, \boldsymbol{f})$ and $\mathcal{M}_r(\boldsymbol{f}, \boldsymbol{\sigma})$ is defined in (1.2.3).

For $\boldsymbol{f}, \boldsymbol{g} \in C_{\boldsymbol{\sigma}}$, the following inequality can be easily proved

$$E_{m,m}(\boldsymbol{f}\boldsymbol{g})_{\boldsymbol{\sigma},\infty} \leq \|\boldsymbol{g}\boldsymbol{\sigma}\|_\infty E_{M,M}(\boldsymbol{f})_{\boldsymbol{\sigma},\infty} + \|\boldsymbol{f}\boldsymbol{\sigma}\|_\infty E_{M,M}(\boldsymbol{g})_{\boldsymbol{\sigma},\infty}, \tag{1.2.5}$$

where $M = \left\lfloor \frac{m}{2} \right\rfloor$.

Let us define also the weighted Hilbert space $L^2_{\mathbf{w}} \equiv L^2_{\mathbf{w}}(S)$ as the set of all weighted square integrable functions $\boldsymbol{f} : S \to \mathbb{R}$ equipped with the inner product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_{\mathbf{w}} = \int_S \boldsymbol{f}(\mathbf{x})\boldsymbol{g}(\mathbf{x})\mathbf{w}(\mathbf{x})\, d\mathbf{x}$$

and define the norm

$$\|\boldsymbol{f}\|_{L^2_{\mathbf{w}}} = \|\boldsymbol{f}\sqrt{\mathbf{w}}\|_2 = \sqrt{\langle \boldsymbol{f}, \boldsymbol{f}\rangle_{\mathbf{w}}}. \tag{1.2.6}$$

In $L^2_{\mathbf{w}}$, for a integer $r \geq 1$, we introduce the following Sobolev-type subspace

$$W^r_{\mathbf{w},2} = \{\boldsymbol{f} \in L^2_{\mathbf{w}} : \boldsymbol{f}^{(r-1)} \in AC(\dot{S}), \|\boldsymbol{f}\|_{W^r_{\mathbf{w}}} = \|\boldsymbol{f}\|_{L^2_{\mathbf{w}}} + \mathcal{M}_r(\boldsymbol{f}, \mathbf{w}) < \infty\} \tag{1.2.7}$$

where $AC(\dot{S})$ denotes the set of all continuous functions $\boldsymbol{f}$ which are absolutely continuous on every closed subdomain of $\dot{S}$ and

$$\mathcal{M}_r(\boldsymbol{f}, \mathbf{w}) = \max\left\{ \left(\int_S \left| \frac{\partial^r \boldsymbol{f}(\mathbf{x})}{\partial x_1^r} \varphi_1^r(x_1) \right|^2 \mathbf{w}(\mathbf{x})\, d\mathbf{x} \right)^{1/2}, \left(\int_S \left| \frac{\partial^r \boldsymbol{f}(\mathbf{x})}{\partial x_2^r} \varphi_2^r(x_2) \right|^2 \mathbf{w}(\mathbf{x})\, d\mathbf{x} \right)^{1/2} \right\} \tag{1.2.8}$$

with $\varphi_i(z) = \sqrt{1 - z_i^2}$, $i \in \{1, 2\}$.

1.2.2 Fourier and Lagrange operators

Let us consider the weight functions defined in (1.2.1) and let $\{p_m(w_i)\}_{m=0}^{\infty}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients defined as in (1.1.7), with respect to the weights w_i, for $i \in \{1, 2\}$, respectively.

Fourier sums

For a function $\boldsymbol{f} \in L^2_{\mathbf{w}}$, where $L^2_{\mathbf{w}}$ is defined in (1.2.6), we define the bivariate Fourier sum as

$$S_{m,m}(\boldsymbol{f}, \mathbf{w}, \mathbf{x}) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{ij}(\boldsymbol{f}, \mathbf{w}) p_i(w_1, x_1) p_j(w_2, x_2) \tag{1.2.9}$$

where

$$c_{ij}(\boldsymbol{f}, \mathbf{w}) = \int_S \boldsymbol{f}(\mathbf{x}) p_i(w_1, x_1) p_j(w_2, x_2) \mathbf{w}(\mathbf{x}) d\mathbf{x}, \quad i, j = 0, ..., m-1, \tag{1.2.10}$$

are the Fourier coefficients.

The next two propositions show the behaviour of $S_{m,m}$ in the case when $\boldsymbol{f} \in L^2_{\mathbf{w}}$ or $\boldsymbol{f} \in W^r_{\mathbf{w},2}$, where $W^r_{\mathbf{w},2}$ is defined in (1.2.7). We underline that these auxiliary results are new and can also be used elsewhere. In order to state the results, let us define the error of best polynomial approximation in $L^2_{\mathbf{w}}$ as

$$E_{m,m}(\boldsymbol{f})_{\mathbf{w},2} = \inf_{\boldsymbol{P} \in \mathbb{P}_{m,m}} \|\boldsymbol{f} - \boldsymbol{P}\|_{L^2_{\mathbf{w}}}.$$

Proposition 1.2.1. *Let* $\boldsymbol{f} \in L^2_{\mathbf{w}}$. *Then*

$$E^2_{m,m}(\boldsymbol{f})_{\mathbf{w},2} = \|\boldsymbol{f} - \mathcal{S}_{m,m}(\boldsymbol{f}, \mathbf{w})\|^2_{L^2_{\mathbf{w}}} = \|\boldsymbol{f}\|^2_{L^2_{\mathbf{w}}} - \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c^2_{ij}(\boldsymbol{f}, \mathbf{w}). \quad (1.2.11)$$

Proof. We only give the main idea of the proof since the thesis can be proved, mutatis mutandis, in the same way of the univariate case (see [51] and the references therein).

Let $Q_{m-1,m-1}$ be an arbitrary polynomial of degree $m-1$ in each variable:

$$Q_{m-1,m-1}(x_1, x_2) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} b_{ij}\, p_i(w_1, x_1) p_j(w_2, x_2).$$

Then, by standard arguments, we get

$$\|\boldsymbol{f} - Q_{m-1,m-1}\|_{L^2_{\mathbf{w}}} = \|\boldsymbol{f}\|_{L^2_{\mathbf{w}}} + \|Q_{m-1,m-1}\|_{L^2_{\mathbf{w}}} - 2\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} b_{ij} c_{ij}(\boldsymbol{f}, \mathbf{w})$$

where $c_{ij}(\boldsymbol{f}, \mathbf{w})$ are the Fourier coefficients of the function $\boldsymbol{f}$ defined in (1.2.10). Then, since in virtue of the orthogonality of $\{p_m(w_1)\}_m$ and $\{p_m(w_2)\}_m$, we have

$$\|Q_{m-1,m-1}\|^2_{L^2_{\mathbf{w}}} = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} b^2_{ij},$$

we can claim that

$$\|\boldsymbol{f} - Q_{m-1,m-1}\|_{L^2_{\mathbf{w}}} = \|\boldsymbol{f}\|_{L^2_{\mathbf{w}}} + \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}(b_{ij} - c_{ij}(\boldsymbol{f}, \mathbf{w}))^2 - \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c^2_{ij}(\boldsymbol{f}, \mathbf{w}).$$

Hence, by replacing b_{ij} with $c_{ij}(\boldsymbol{f}, \mathbf{w})$ we get the thesis. □

Thus, according to the previous result, as in the univariate case (see [51] and the references therein), $\mathcal{S}_{m,m}$ turns to be the best polynomial approximation of $\boldsymbol{f} \in L^2_{\mathbf{w}}$ and, if the Weierstrass Theorem holds true, by (1.2.11) we get the Parseval identity

$$\|\boldsymbol{f}\|_{L^2_{\mathbf{w}}} = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}^2(\boldsymbol{f},\mathbf{w})} \tag{1.2.12}$$

and, consequently

$$E_{m,m}(\boldsymbol{f})_{\mathbf{w},2} = \sqrt{\sum_{i\geq m}\sum_{j\geq m} c_{ij}^2(\boldsymbol{f},\mathbf{w})}.$$

In order to prove the next proposition, let us note that the bivariate Fourier operator defined in (1.2.9) can be thought as a composition of two univariate Fourier operators, namely

$$\mathcal{S}_{m,m}(\boldsymbol{f},\mathbf{w},\mathbf{x}) = \mathcal{S}_m\left(\mathcal{S}_m(f_y,w_1,x_1),w_2,x_2\right) = \mathcal{S}_m\left(\mathcal{S}_m(f_x,w_2,x_2),w_1,x_1\right),$$

where $\mathcal{S}_m$ identifies the univariate Fourier sum defined as in (1.1.8) and f_{x_1} and f_{x_2} denote the function f as a univariate function of the only variables x_2 and x_1, respectively.

Proposition 1.2.2. *Let $\boldsymbol{f} \in W^r_{\mathbf{w},2}$ and r_1 and r be two positive integers such that $r_1 \leq r$. Then there exists a positive constant $\mathcal{C} \neq \mathcal{C}(m,\boldsymbol{f})$ such that the following estimate holds true*

$$\|\boldsymbol{f} - \mathcal{S}_{m,m}(\boldsymbol{f},\mathbf{w})\|_{W^{r_1}_{\mathbf{w},2}} \leq \frac{\mathcal{C}}{m^{r-r_1}}\|\boldsymbol{f}\|_{W^r_{\mathbf{w},2}}.$$

Proof. We write

$$\begin{aligned}
&\|\boldsymbol{f} - \mathcal{S}_{m,m}(\boldsymbol{f},\mathbf{w})\|_{W^{r_1}_{\mathbf{w}}}\\
&\leq \|\boldsymbol{f} - \mathcal{S}_m(\boldsymbol{f},w_2)\|_{W^{r_1}_{\mathbf{w}}} + \|\mathcal{S}_m(\boldsymbol{f},w_2) - \mathcal{S}_m(\mathcal{S}_m(f_y,w_1),w_2)\|_{W^{r_1}_{\mathbf{w}}}\\
&= \left(\int_{-1}^{1}\|f_x - \mathcal{S}_m(f_x,w_2)\|^2_{\mathcal{W}^{r_1}_{w_2}}\, w_1(x)\,dx\right)^{1/2}\\
&+ \quad \left(\int_{-1}^{1}\|\mathcal{S}_m(f_y - \mathcal{S}_m(f_y,w_1),w_2)\|^2_{\mathcal{W}^{r_1}_{w_2}}\, w_1(x)\,dx\right)^{1/2}.
\end{aligned}$$

Then, by applying (1.1.9) to the norm of the first term, (1.1.10) and again (1.1.9) to the norm of the second one, we get

$$\begin{aligned}
\|\boldsymbol{f} - \mathcal{S}_{m,m}(\boldsymbol{f},\mathbf{w})\|_{W^{r_1}_{\mathbf{w}}} &\leq \frac{\mathcal{C}}{m^{r-r_1}}\left(\int_{-1}^{1}\|f_x\|^2_{\mathcal{W}^{r}_{w_2}}\, w_1(x)\,dx\right)^{1/2}\\
&+ \frac{\mathcal{C}}{m^{r-r_1}}\left(\int_{-1}^{1}\|f_y\|^2_{\mathcal{W}^{r}_{w_1}}\, w_2(y)\,dy\right)^{1/2}\\
&\leq \frac{\mathcal{C}}{m^{r-r_1}}\|\boldsymbol{f}\|_{W^r_{\mathbf{w}}}.
\end{aligned}$$

□

Lagrange interpolating polynomials in $[-1,1]^2$

For a function $\boldsymbol{f} \in C(\dot{S})$, we define the bivariate Lagrange polynomial $\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w},\mathbf{x})$ interpolating a given function $\boldsymbol{f}$ at the points

$$\{\left(\xi_i^{w_1},\xi_j^{w_2}\right),(i,j)\in N_1^m\times N_1^m\},$$

where $\{\xi_i^{w_1}\}_{i=1}^m$ and $\left\{\xi_j^{w_2}\right\}_{j=1}^m$ are the zeros of the univariate Jacobi polynomials $\{p_m\left(w_1\right)\}_{m=0}^{\infty}$ and $\{p_m\left(w_2\right)\}_{m=0}^{\infty}$, respectively, i.e.

$$\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w},\xi_{i,j}^{w_1,w_2}) = \boldsymbol{f}(\xi_{i,j}^{w_1,w_2}), \quad (i,j)\in N_1^m\times N_1^m, \tag{1.2.13}$$

where, for the sake of brevity, $\{\xi_{i,j}^{w_1,w_2}\} := \{\left(\xi_i^{w_1},\xi_j^{w_2}\right)\}$.

The polynomial $\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w})\in\mathbb{P}_{m-1,m-1}$ and $\mathcal{L}_{m,m}(\boldsymbol{P},\mathbf{w}) = \boldsymbol{P}$, for any $\boldsymbol{P}\in\mathbb{P}_{m-1,m-1}$. An expression of $\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w})$ is given by

$$\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w},\mathbf{x}) = \sum_{i=1}^m\sum_{j=1}^m \ell_{i,j}^{w_1,w_2}(\mathbf{x})\boldsymbol{f}(\xi_{i,j}^{w_1,w_2}), \tag{1.2.14}$$

where $\ell_{i,j}^{w_1,w_2}(\mathbf{x}) = \ell_i^{w_1}(x_1)\ell_j^{w_2}(x_2)$ and

$$\ell_i^{w_k}(z) = \frac{p_m(w_k,z)}{p_m'(w_k,\xi_i^{w_k})(z-\xi_i^{w_k})}, \quad k\in\{1,2\}.$$

Next proposition shows the weighted-L^2 convergence of the Lagrange interpolating polynomial for every $\boldsymbol{f}\in W_{\mathbf{w},2}^r$. We underline that this auxiliary result is new and can also be used elsewhere.

In order to prove the next proposition, let us note that as already underlined for the bivariate Fourier operator, the bivariate Lagrange operator defined in (1.2.13), can be thought as a composition of two univariate Lagrange operators, namely

$$\mathcal{L}_{m,m}\left(\boldsymbol{f},\mathbf{w},\mathbf{x}\right) = \mathcal{L}_m\left(\mathcal{L}_m(f_y,w_1,x_1),w_2,x_2\right) = \mathcal{L}_m\left(\mathcal{L}_m(f_x,w_2,x_2),w_1,x_1\right),$$

where $\mathcal{L}_m$ denotes the univariate Lagrange polynomial defined in (1.1.12).

Proposition 1.2.3. *Let $\boldsymbol{f}\in W_{\mathbf{w},2}^r$. Then there exists a positive constant $\mathcal{C}\neq\mathcal{C}(m,\boldsymbol{f})$ such that the following estimate holds true*

$$\left\|\boldsymbol{f}-\mathcal{L}_{m,m}\left(\boldsymbol{f},\mathbf{w}\right)\right\|_{L_{\mathbf{w}}^2} \le \frac{\mathcal{C}}{m^r}\left\|\boldsymbol{f}\right\|_{W_{\mathbf{w},2}^r}. \tag{1.2.15}$$

Proof. We begin by writing

$$\begin{aligned}
&\|\boldsymbol{f}-\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w})\|_{L^2_w}\\
&\le \|\boldsymbol{f}-\mathcal{L}_m(\boldsymbol{f},w_2)\|_{L^2_{\mathbf{w}}} + \|\mathcal{L}_m(\boldsymbol{f},w_2)-\mathcal{L}_m(\mathcal{L}_m(f_y,w_1),w_2)\|_{L^2_{\mathbf{w}}}\\
&= \left(\int_{-1}^{1} \|f_x-\mathcal{L}_m(f_x,w_2)\|^2_{L^2_{w_2}}\, w_1(x)\,dx\right)^{1/2}\\
&+ \quad \left(\int_{-1}^{1} \|\mathcal{L}_m(f_y-\mathcal{L}_m(f_y,w_1),w_2)\|^2_{L^2_{w_2}}\, w_1(x)\,dx\right)^{1/2}.
\end{aligned}$$

Hence by using (1.1.15) to the first term, (1.1.15), (1.1.16) and (1.1.17) to the second one, we get

$$\|\boldsymbol{f}-\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w})\|_{L^2_{\mathbf{w}}} \le \frac{\mathcal{C}}{m^r}\left(\int_{-1}^{1} \|f_x-\mathcal{L}_m(f_x,w_2)\|^2_{\mathcal{W}^r_{w_2}}\, w_1(x)\,dx\right)^{1/2}$$

from which we deduce the thesis. □

1.2.3 Gauss-Jacobi cubature rules

Let $\mathbf{w} := w_1 w_2 = v^{\alpha_1,\beta_1} v^{\alpha_2,\beta_2}$ be defined in (1.2.1) and let $\{p_m(w_i)\}_{m=0}^{\infty}$ be, for $i \in \{1,2\}$, the corresponding sequences of orthonormal polynomials with positive leading coefficients defined as in (1.1.7), with respect to the weight w_1 and w_2, respectively.

The tensor-product Gaussian rule, which will be essential for our aims, reads as [66]

$$\begin{aligned}
\int_S \boldsymbol{f}(\mathbf{x})\mathbf{w}(\mathbf{x})\, d\mathbf{x} &= \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i^{w_1}\lambda_j^{w_2} \boldsymbol{f}(\xi_{i,j}^{w_1,w_2}) + \mathcal{R}^{\mathcal{G}}_{m,m}(\boldsymbol{f})\\
&:= \mathcal{G}^{w_1,w_2}_{m,m}(\boldsymbol{f}) + \mathcal{R}^{\mathcal{G}}_{m,m}(\boldsymbol{f}), \qquad (1.2.16)
\end{aligned}$$

where $\xi_{i,j}^{w_1,w_2} := \left(\xi_i^{w_1}, \xi_j^{w_2}\right)$ with $\{\xi_i^{w_1}\}_{i=1}^{m}$ and $\left\{\xi_j^{w_2}\right\}_{j=1}^{m}$ the zeros of the Jacobi polynomials $\{p_m(w_1,x_1)\}_{m=0}^{\infty}$ and $\{p_m(w_2,x_2)\}_{m=0}^{\infty}$ respectively, $\lambda_i^{w_j}$, $i = 1,\ldots,m$, denote the ith-Christoffel numbers with respect to w_j, $j \in \{1,2\}$, and the remainder term $\mathcal{R}^{\mathcal{G}}_{m,m}(\boldsymbol{P}) = 0$ for any $\boldsymbol{P} \in \mathbb{P}_{2m-1,2m-1}$. About the error estimate, we get the following (see [66]):

Proposition 1.2.4. *Let $\boldsymbol{f} \in C_{\boldsymbol{\sigma}}$. Under the assumption*

$$\int_S \frac{\mathbf{w}(\mathbf{x})}{\boldsymbol{\sigma}(\mathbf{x})} d\mathbf{x} < +\infty,$$

we have

$$\left|\mathcal{R}^{\mathcal{G}}_{m,m}(\boldsymbol{f})\right| \leq \mathcal{C}E_{2m-1,2m-1}(\boldsymbol{f})_{\boldsymbol{\sigma}}, \tag{1.2.17}$$

where $\mathcal{C} \neq \mathcal{C}(m, \boldsymbol{f})$.

The following proposition gives a new estimate for $\mathcal{R}^{\mathcal{G}}_{m,m}(\mathbf{f})$ which is useful for analytical functions.

Proposition 1.2.5. *Let* $\mathbf{f}(x_1, x_2)$ *be a bivariate function defined on* S *having* $2m$ *continuous partial derivatives with respect to each variable. Then*

$$\left|\mathcal{R}^{\mathcal{G}}_{m,m}(\boldsymbol{f})\right| \leq \mathcal{C}\frac{\Gamma(\boldsymbol{f})}{\gamma^2_m(w_1)\gamma^2_m(w_2)(2m)!},$$

where $\gamma_m(w_1)$ *and* $\gamma_m(w_2)$ *are the leading coefficients of* $p_m(w_1, x_1)$ *and* $p_m(w_2, x_2)$, *respectively and* $\Gamma(\boldsymbol{f}) = \max\left\{\left\|\frac{\partial^{2m}\boldsymbol{f}}{\partial x_1^{2m}}\right\|_\infty, \left\|\frac{\partial^{2m}\boldsymbol{f}}{\partial x_2^{2m}}\right\|_\infty\right\}$.

Proof. In order to prove the thesis, let us denote by $\mathcal{L}_{2m,2m}$ the bivariate Lagrange polynomial of degree $2m-1$ in each variable [66] interpolating the function $\boldsymbol{f}$ at the points $(\xi_i^{w_1}, \xi_i^{w_2})$ and (t_i, s_i) for $i = 1, ..., m$ with $\{\xi_i^{w_1}\}_{i=1}^m$ and $\{\xi_i^{w_2}\}_{i=1}^m$ the zeros of $p_m(w_1, x_1)$ and $p_m(w_2, x_2)$, resepctively and $\{t_i\}_{i=1}^m$ and $\{s_i\}_{i=1}^m$ the zeros, of the monic polynomials of degree m defined as

$$r_m(x_1) = \prod_{i=1}^m (x_1 - t_i), \qquad s_m(x_2) = \prod_{i=1}^m (x_2 - s_i).$$

Taking into account that the Gaussian cubature rule is exact for algebraic polynomials of degree $2m-1$ in each variable, we have

$$\mathcal{R}^{\mathcal{G}}_{m,m}(\boldsymbol{f}) = \mathcal{R}_{m,m}(\boldsymbol{f} - \mathcal{L}_{2m,2m}) = \int_S [\boldsymbol{f}(\mathbf{x}) - \mathcal{L}_{2m,2m}(\boldsymbol{f}, \mathbf{x})]\, \mathbf{w}(\mathbf{x}) d\mathbf{x}.$$

Then, being

$$\boldsymbol{f}(\mathbf{x}) - \mathcal{L}_{2m,2m}(\boldsymbol{f}, \mathbf{x}) = \mathcal{C}\frac{\Gamma(\boldsymbol{f})}{(2m)!} p_m(w_1, x_1) r_m(x_1) p_m(w_2, x_2) s_m(x_2),$$

with $\Gamma(\boldsymbol{f}) = \max\left\{\left\|\frac{\partial^{2m}\boldsymbol{f}}{\partial x_1^{2m}}\right\|, \left\|\frac{\partial^{2m}\boldsymbol{f}}{\partial x_2^{2m}}\right\|\right\}$, we have in virtue of the orthonomality

$$|\mathcal{R}^{\mathcal{G}}_{m,m}(\boldsymbol{f})| \leq \mathcal{C}\frac{\Gamma(\boldsymbol{f})}{\gamma^2_m(w_1)\gamma^2_m(w_2)(2m)!}.$$

□

Chapter 2

Product and Dilation Quadrature/Cubature Rules for Some Kinds of Kernels

This Chapter deals with quadrature and cubature rules on $[-1,1]$ and on the square $S = [-1,1]^2$ respectively, discussing on their stability and convergence and on their numerical construction. In particular, we treat the numerical approximation of integrals of the type

$$I(f,y) = \int_{-1}^{1} f(x)k(x,y)w(x)dx, \quad y \in [-1,1]$$

and

$$\boldsymbol{I}(\boldsymbol{f},\mathbf{y}) = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{f}(\mathbf{x})\boldsymbol{k}(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x})d\mathbf{x}, \quad \boldsymbol{y} \equiv (y_1,y_2) \in [-1,1]^2,$$

where k is a bivariate function in the variables x and y and $\boldsymbol{k}$ is a function of four variables since $\boldsymbol{x} = (x_1,x_2)$ and $\boldsymbol{y} = (y_1,y_2)$. We assume that the kernel functions k and $\boldsymbol{k}$ can be *weakly singular*, *nearly singular* or *highly oscillating.* We consider also the combination of two aspects, i.e. integrals with *nearly singular and highly oscillating* kernels. We underline that, all the results in this Chapter, for the bivariate case, are new and have recently been presented in [69, 77]. For completeness, we also deduced some new results for the univariate case. In particular, this Chapter is organized as follows. Section 2.1 is completely devoted to the product quadrature rule with results on the stability and convergence for a wide class of kernels. In Subsections 2.1.1 and 2.1.2 we give some details for computing the coefficients of the product quadrature rule when the kernels are *weakly* singular, *nearly singular* and/or *highly oscillating* and we describe the 1D-dilation rule in a general form, proving results about the stability and the rate of convergence of the error. In Subsection 2.1.3 we show some cases of complexity reduction. Section 2.2 is dedicated to the product cubature rule with results on the stability and convergence for a wide class of kernels. Subsection 2.2.1 contains some details for computing the coefficients of the product cubature rule when the kernel functions are *weakly singular*. In Subsection 2.2.2 we give some details for computing the coefficients of the product cubature rule when the kernel functions are *nearly singular* and/or *highly oscillating* and we describe the 2D-dilation rule in a general form, proving results about the stability and the rate of convergence of the error. In Subsection 2.2.4 we suggest some criteria on the choice of the stretching parameter in the 2D-dilation formula and in Subsection 2.2.3 we propose a test for comparing our cubature formulae with respect to their CPU time. Finally, in Section 2.3, we present some numerical examples, where our results are compared with those achieved by other methods.

2.1 Product Quadrature Rules on $[-1,1]$

Consider

$$I(f,y) = \int_{-1}^{1} f(x)k(x,y)w(x)dx, \quad y \in [-1,1].$$

By replacing the function f with the Lagrange polynomial $\mathcal{L}_m(f,w,x)$ defined in (1.1.12), we obtain the following *product quadrature rule*

$$I(f,y) = \sum_{h=1}^{m} A_h(y) f\left(\xi_h^w\right) + \mathcal{R}_m(f,y) =: I_m(f,y) + \mathcal{R}_m(f,y) \qquad (2.1.1)$$

where

$$A_h(y) = \int_{-1}^{1} \ell_h^w(x)k(x,y)w(x)dx, \qquad (2.1.2)$$

with ℓ_h^w defined in (1.1.13), and

$$\mathcal{R}_m(f,y) =: I(f,y) - I_m(f,y)$$

is the remainder term.

We recall that the quadrature rule is exact for algebraic polynomials of degree $m-1$, i.e. $\mathcal{R}_m(P,y) = 0$, $\forall P \in \mathbb{P}_{m-1}$, $\forall y \in [-1,1]$.

In order to prove the stability and convergence of the product formula, we use the following result, which can be deduced from a theorem of Nevai (see, for instance, [65, 47, 77]) and the interested reader can find it in [47] for the case $w = 1$.

Theorem 2.1.1. *Assume $w = v^{\alpha,\beta}$, $\sigma = v^{\gamma,\delta}$, $\gamma, \delta \geq 0$, $\varphi(x) = \sqrt{1-x^2}$ and*

$$\sup_{|y|\leq 1} \int_{-1}^{1} \frac{|k(x,y)|\, w(x)}{\sigma(x)} \log\left(2 + \frac{|k(x,y)|\, w(x)}{\sigma(x)}\right) dx < +\infty. \qquad (2.1.3)$$

Then, for all functions $f \in C_\sigma$, we have

$$\sup_{|y|\leq 1} \int_{-1}^{1} |\mathcal{L}_m(f,w,x) k(x,y)|\; w(x)\, dx < \mathcal{C} \, \|f\sigma\|_\infty \,, \; \mathcal{C} \neq \mathcal{C}(m,f), \qquad (2.1.4)$$

if and only if

$$\sup_{|y|\leq 1} \int_{-1}^{1} \frac{|k(x,y)|\, \sigma(x)}{\sqrt{w(x)\varphi(x)}} dx < +\infty \quad \textit{and} \quad \int_{-1}^{1} \frac{\sqrt{w(x)\varphi(x)}}{\sigma(x)} dx < +\infty. \; (2.1.5)$$

Remark 2.1.2. *Let us remark that if the parameters of the weight w are such that $\alpha, \beta < \frac{3}{2}$, then the parameters of the weight σ could also be chosen equal to zero.*

About the stability of $I_m(f,y)$, the following theorem holds true.

Theorem 2.1.3. *Under the same assumptions of Theorem 2.1.1, for any $f \in C_\sigma$, the rule is stable, i.e.*

$$\sup_{|y|\leq 1} |I_m(f,y)| \leq \mathcal{C}\, \|f\sigma\|_\infty \,, \quad \mathcal{C} \neq \mathcal{C}(m,f).$$

Proof. By Theorem 2.1.1, for any fixed y

$$\int_{-1}^{1} |\mathcal{L}_m(f,w,x)k(x,y)|\, w(x) dx \leq \mathcal{C}\, \|f\sigma\|_\infty \,.$$

Therefore

$$\begin{aligned} |I_m(f,y)| &= \left| \int_{-1}^{1} \mathcal{L}_m(f,w,x)k(x,y)w(x) \right| dx \\ &\leq \int_{-1}^{1} |\mathcal{L}_m(f,w,x)k(x,y)|\, w(x)dx \leq \mathcal{C}\, \|f\sigma\|_\infty \end{aligned}$$

and taking the sup on $y \in [-1,1]$, the thesis follows. □

About the convergence, the following theorem holds true.

Theorem 2.1.4. *Under the same assumptions of Theorem 2.1.1, for any* $f \in C_\sigma$,

$$\sup_{|y|\le 1} |\mathcal{R}_m(f,y)| \le \mathcal{C} E_{m-1}(f)_\sigma, \quad \mathcal{C} \ne \mathcal{C}(m,f). \tag{2.1.6}$$

Proof. Consider

$$\begin{aligned}\mathcal{R}_m(f,y) &=: I(f,y) - I_m(f,y)\\ &= \int_{-1}^{1} f(x)k(x,y)w(x)dx - \sum_{h=1}^{m} A_h(y) f\left(\xi_h^w\right).\end{aligned}$$

Then, denoting by P^*_{m-1} the polynomial of best approximation, by (2.1.4), we get

$$\begin{aligned}|\mathcal{R}_m(f,y)| &= \left|\mathcal{R}_m(f - P^*_{m-1}, y)\right|\\ &\le \int_{-1}^{1} \left|\left(f(x) - P^*_{m-1}(x)\right) - \mathcal{L}_m(w, f - P^*_{m-1}, x)\right| k(x,y)w(x)\,dx\\ &\le \mathcal{C} \left\|\left(f - P^*_{m-1}\right)\sigma\right\|_\infty,\end{aligned}$$

and taking into account the iphotesis (2.1.3), we have

$$\sup_{|y|\le 1} |\mathcal{R}_m(f,y)| \le \mathcal{C} E_{m-1}(f)_{\sigma,\infty},$$

i.e. the thesis follows. □

Now we provide some details about the computation of the coefficients of the product rule presented in (2.1.1), for some choices of the kernel functions.

2.1.1 Computation of the 1D-product rule coefficients for *weakly* singular kernels

In this Subsection, we give some details for computing the coefficients in (2.1.2) when the kernel function is *weakly* singular, i.e. we consider integrals of the type

$$I(f,y) = \int_{-1}^{1} f(x)|x-y|^\lambda w(x)dx, \quad y \in (-1,1), \quad -1 < \lambda < 0,$$

where the kernel function is given by

$$k_1(x,y) = |x-y|^\lambda, \; -1 < \lambda < 0. \tag{2.1.7}$$

These type of kernel functions appear, for instance, in one–dimensional weakly singular integral equations (see, for instance, [6, 11, 40, 52, 61, 62, 60, 80] and the references therein) and also in Volterra integral equations with weakly singular kernel (Abel type) (see, for instance [5, 9] and the references therein). Such equations arise from many applications such as reaction-diffusion problems in small cells or from the semidiscretization in space of Volterra-Fredholm integral equations with weakly singular kernel and of partial Abel integral or integro-differential equations.

For these type of kernel functions, the coefficients in (2.1.2), can be computed "exactly" via *modified moment.* More precisely, by (1.1.14), we have

$$A_h(y) = \int_{-1}^{1} \ell_h^w(x)k(x,y)w(x)dx = \lambda_h^w \sum_{j=0}^{m-1} p_j\left(w, \xi_h^w\right) M_j^w(\omega)$$

where the *modified moments*

$$M_j^w(y) = \int_{-1}^{1} p_j(w,x)|x-y|^\lambda w(x)dx$$

satisfy the following recurrence relation (for more details see [52])

$$\begin{aligned}
M_0(y) &= \frac{1}{\sqrt{\mu_0}} \int_{-1}^{1} |x-y|^\lambda w(x)dx, \quad \mu_0 = \int_{-1}^{1} w(x)dx, \quad -1 < \lambda < 0, \\
M_1(y) &= \frac{1}{b_1}\left(\sqrt{\mu_0} - (y-a_1)M_0(y)\right), \\
M_j(y) &= -\frac{1}{b_j}\left[(y+a_j)M_{j-1}(y) + b_{j-1}M_{j-2}(y)\right], \quad j = 2, ..., m,
\end{aligned}$$

where a_j, b_j are the coefficients of the Jacobi three-term recurrence relation

$$\begin{aligned}
&p_0(w,x) = \tfrac{1}{\sqrt{\mu_0}}, \quad p_1(w,x) = \tfrac{(x-a_1)p_0(w,x)}{b_1}, \\
&b_j p_j(w,x) = (x-a_j)p_{j-1}(w,x) - b_{j-1}p_{j-2}(w,x), \quad j = 2, 3, ...
\end{aligned} \tag{2.1.8}$$

and hence

$$a_1 = \tfrac{\beta^2-\alpha^2}{(2+\alpha+\beta)(3+\alpha+\beta)}, \quad b_1 = \sqrt{\tfrac{4(1+\alpha)(1+\beta)}{(2+\alpha+\beta)^2(3+\alpha+\beta)}},$$

$$a_j = \tfrac{\beta^2-\alpha^2}{(2j+\alpha+\beta)(2j+2+\alpha+\beta)}, \quad b_j = \sqrt{\tfrac{4j(j+\alpha)(j+\beta)(j+\alpha+\beta)}{(2j+\alpha+\beta-1)(2j+\alpha+\beta)^2(2j+1+\alpha+\beta)}}, \quad j = 2, 3, ... \,.$$

2.1.2 Computation of the 1D-product rule coefficients for *nearly* singular and/or *highly oscillating* kernels

In this Subsection, we give some details for computing the coefficients in (2.1.2) when the kernel functions are *nearly* singular and/or *highly oscillating*, i.e. we consider integrals of the type

$$I(f,\omega) = \int_{-1}^{1} f(x)k_j(x,\omega)w(x)dx, \quad y \in [-1,1], \quad j \in \{2,3,4\}, \tag{2.1.9}$$

where the kernel functions are given by

$$\begin{aligned} k_2(x,\omega) &= \frac{1}{((x-x_0)^2+\omega^{-1})^{\lambda}}, \ x_0 \in [-1,1] \text{ fixed}, \quad \lambda \in \mathbb{R}^+, \quad 0 \neq \omega \in \mathbb{R}, \\ k_3(x,\omega) &= g(\omega x), \ 0 \neq \omega \in \mathbb{R}, \\ & \quad \text{g is an oscillatory smooth function with frequency } \omega, \\ k_4(x,\omega) &= k_2(x,y,\omega)k_3(x,y,\omega). \end{aligned} \tag{2.1.10}$$

The numerical evaluation of the integrals in (2.1.9) has interested several authors and actually it is a special Chapter of the numerical integration. The interested reader can consult for instance [7, 8, 17, 36, 37, 56, 57, 63, 71] and the references therein.

We underline that the numerical evaluation of integrals with kernels of the types in (2.1.10) presents difficulties for "large"ω, since the kernel k_2 is "close"to be singular, k_3 highly oscillates and k_4 includes both the aforesaid problematic behaviors. In all the above cases, the modulus of the derivatives grows as ω grows.

The following example is useful to point out the difficulties that appear in the computation of (2.1.9) when the kernel function is of the type k_3 (similar conclusions, with respect to the instability, can be deduced when k is of the types k_2 or k_4).

Example 2.1.5. *Consider the following integral (see [17])*

$$I(G) = \int_{-1}^{1} e^x \sin(5000x)dx, \quad G(x) = e^x \sin(5000x).$$

If we apply the Gauss-Legendre quadrature rule, as shown in [17], we get

$$I(G) = \sum_{k=1}^{m} G(\xi_k)\lambda_k + \mathcal{R}_m(G) := \Phi_m(G) + \mathcal{R}_m(G)$$

where $\{\xi_k\}_{k=1}^m$*, are the zeros of the* m*th Legendre polynomial and* $\lambda_k, k = 1, ..., m$*, are the corresponding Christoffel numbers. Since* G *is differentiable, we can write the error as in* (1.1.20)

$$\mathcal{R}_m(G) = \frac{\left\|G^{(2m)}\right\|_\infty}{\gamma_m^2 (2m)!},$$

where γ_m *is the leading coefficient of the* m*th orthonormal Legendre polynomial. Therefore, the following estimate holds (see [17])*

$$|\mathcal{R}_m(G)| = \mathcal{O}\left[\left(\frac{3399}{m}\right)^{2m}\right].$$

Hence, to obtain few exact digits, we need a number of knots greater than 3399. But it is not realistic. On the other hand, working in finite arithmetics, a "small"error in the computation of ξ_k *can produce a "large"error in the evaluation of* $G(\xi_k)$*, with an eventually change of sign. In conclusion, as shown in [17], Gaussian rules cannot give a reasonable approximation in the case of oscillating kernels. This fact is confirmed by the numerical test shown in Table 2.1.*

Table 2.1: $G(x) = e^x \sin(5000x)$

m	$\Phi_m(G)$
8	0.428672207843679
16	-0.192123222625601
32	0.024304986655065
64	0.042348837355750
128	0.078391819539537
256	0.170674839281539
512	-0.013440192353402

On the other hand such kernels are of interest since they appear in many contexts. For instance, k_2-type kernels appear in one–dimensional nearly singular BEM integrals. Highly oscillating kernels of the type k_3 are useful in computational methods for oscillatory phenomena in science and engineering problems. The combination of the two aspects, i.e. integrals with nearly singular and oscillating kernels appear for instance in the solution of problems of propagation in uniform waveguides with nonperfect conductors.

The problem to approximate integrals of the type (2.1.9) with kernel functions of the types in (2.1.10), finds application in the numerical treatment of integral equations, for instance, in Fredholm-type integral equations. Since it is very difficult to compute integral operators of the type (2.1.9) when k belongs to the type (2.1.10), different numerical approaches have been investigated by several authors. Many of them (see, for details, [20, 24, 82, 84]) have considered the kernel functions of the type k_2 with $\omega = 1$, while only few authors [59, 72, 75] have also studied the more interesting case $|\omega| > 1$ because of the numerical difficulties that it involves. In fact, in this case the distance between the complex poles and the real axis becomes too small. In [72, 75], the authors propose a collocation method with cubic splines. In [59], the kernel k is transformed so that the poles move away from the real axis and a Nyström method based on a product-type formula is applied. The method proposed in [59] is better than those examined in [72, 75]; however, in the case where the parameter ω is too large, for example $\omega = 10^3$, the product-type formula gives poor results (see [59, Table II]).

In particular, integrals of the type (2.1.9) with kernel functions of the k_2-type, appear also in Love's integral equation. For this type of integral equation, for instance, in [44] the authors improve the results given in [71] by using the same transformation as in [43] which takes into account the behavior of the integrand function. In Chapter 3 we will present alternative methods for the numerical approximation of the univariate and bivariate Love's integral equation.

In the literature, the standard way in order to compute the corresponding coefficients is to determine the *modified moments* by means of recurrence relations, and to examine the stability of these latter (see for instance [26, 41, 53, 73, 81] and the references therein). These approaches, however, does not appear always feasible for kernels of the type (2.1.10) and, in general, in the literature different strategies according to the kernels are proposed.

Here we present a unique approach for computing the coefficients of the quadrature rule introduced in (2.1.1) when k belongs to the types (2.1.10).

In particular, in order to compute the coefficients in (2.1.2), when the kernel functions are of the type (2.1.10), we propose a common strategy which includes the dilation quadrature method proposed in [71] for nearly singular kernels and in [17] for highly oscillatory functions. Indeed both of them have been considered in the unweighted case. We underline that, in [17, 71] the authors considered the cases *nearly singular* and *highly oscillating* separately, proposing different strategies according to the kernels.

In this thesis, we work in weighted function spaces and this allows to consider also functions with algebraic singularities at the endpoints of $[-1, 1]$. Furthermore we propose a unique approach for *nearly singular* or *highly oscillating* kernels which allows to consider also the new possibility of *nearly singular and highly oscillating* kernels. This strategy, has been recently proposed for the first time in [69] for the bivariate *nearly singular and/or highly oscillating* kernel functions and, successively, in [22] for the univariate and bivariate Love's kernel function (we will give all the details in Chapter 3). Such "dilation"method consists in a preliminary "dilation"of the domain and, by suitable transformations, on the successive reduction of the initial integral to the sum of integrals defined again on $[-1, 1]$. These manipulations "relax"in some sense the "pathological"behavior of the kernels $k_j, j \in \{2, 3, 4\}$.

The 1D-dilation formula

Below, for the convenience of the reader, we will describe the dilation method for a general integral of the type

$$I(F, \omega) = \int_{-1}^{1} F(x)k(x, \omega)w(x)dx, \quad F \in C_\sigma,$$

where $k(x, \omega)$ is one of the kernels in (2.1.10) with the restrictions on ω given there. Succesively, we will apply this tecnique for the special case $F(x) = \ell_h^w(x)$.

By the changes of variable

$$x = \frac{\eta}{\omega}, \qquad \eta \in [-\omega, \omega],$$

we get

$$I(F, \omega) = \frac{1}{\omega} \int_{-\omega}^{\omega} F\left(\frac{\eta}{\omega}\right) k\left(\frac{\eta}{\omega}, \omega\right) w\left(\frac{\eta}{\omega}\right) d\eta$$

and choosing $d \in \mathbb{R}^+$ such that $\boldsymbol{S} = \frac{2\omega}{d} \in \mathbb{N}$, we have

$$I(F, \omega) = \frac{1}{\omega} \sum_{i=1}^{\boldsymbol{S}} \int_{-\omega+(i-1)d}^{-\omega+id} F\left(\frac{\eta}{\omega}\right) k\left(\frac{\eta}{\omega}, \omega\right) w\left(\frac{\eta}{\omega}\right) d\eta. \tag{2.1.11}$$

Now we want to remap each integral into $[-1, 1]$. To this end we introduce the following invertible linear maps

$$\Psi_i : [-\omega + (i-1)d, -\omega + id] \to [-1, 1]$$

defined as, for $i \in N_1^{\boldsymbol{S}}$,

$$x = \Psi_i(\eta) = \frac{2}{d}(\eta + \omega) - (2i - 1)$$

and in (2.1.11) we make the change of variable

$$\eta = \Psi_i^{-1}(x) = \left(\frac{x+1}{2}\right) d - \omega + (i-1)d.$$

In this way, we get

$$\begin{aligned}
I\left(F,\omega\right) &= \frac{d}{2\omega} \sum_{i=1}^{\boldsymbol{S}} \int_{-1}^{1} F_i(x) k_i(x,\omega) w_i(x) dx \\
&= \frac{d}{2\omega} \left\{ \tau_1 \int_{-1}^{1} F_1(x) k_1(x,\omega) U_1(x) u_1(x)\, dx \right. \\
&+ \sum_{j=2}^{\boldsymbol{S}-1} \int_{-1}^{1} F_j(x) k_j(x,\omega) U_j(x) u_2(x) dx \\
&\left. + \tau_2 \int_{-1}^{1} F_{\boldsymbol{S}}(x) k_{\boldsymbol{S}}(x,\omega) U_{\boldsymbol{S}}(x) u_3(x) dx \right\}
\end{aligned}$$

where, for $i = 1, ..., \boldsymbol{S}$ and $j = 2, ..., \boldsymbol{S} - 1$,

$$F_i(x) := F\left(\frac{\Psi_i^{-1}(x)}{\omega}\right), \quad k_i(x,\omega) := k\left(\frac{\Psi_i^{-1}(x)}{\omega}, \omega\right),$$

$$w_i(x) := w\left(\frac{\Psi_i^{-1}(x)}{\omega}\right),$$

$$U_1(x) = v^{\alpha,0}\left(\frac{\Psi_1^{-1}(x_1)}{\omega}\right), \quad U_j(x) = v^{\alpha,\beta}\left(\frac{\Psi_j^{-1}(x_j)}{\omega}\right),$$

$$U_{\boldsymbol{S}} = v^{0,\beta}\left(\frac{\Psi_{\boldsymbol{S}}^{-1}(x_{\boldsymbol{S}})}{\omega}\right) \quad \text{and} \quad \tau_1 = \left(\frac{d}{2\omega}\right)^{\beta}, \quad \tau_2 = \left(\frac{d}{2\omega}\right)^{\alpha},$$

$$u_1(x) = v^{0,\beta}(x), \quad u_2(x) = v^{0,0}(x), \quad u_3(x) = v^{\alpha,0}(x).$$

Finally, according to the notation in (1.1.18), we approximate each integral by the proper Gauss-Jacobi rule depending on the weight functions arising in the integral. To be more precise

$$\begin{aligned}
I(F,\omega) &= \Sigma_m(F,\omega) + \mathcal{R}_m^{\Sigma}(F,\omega) \qquad (2.1.12)\\
&=: \frac{d}{2\omega} \left\{ \tau_1 \mathcal{G}_m^{u_1}\left(F_1 k_1 U_1\right) + \sum_{j=2}^{\boldsymbol{S}-1} \mathcal{G}_m^{u_2}\left(F_j k_j U_j\right) + \tau_2 \mathcal{G}_m^{u_3}\left(F_{\boldsymbol{S}} k_{\boldsymbol{S}} U_{\boldsymbol{S}}\right) \right\} \\
&+ \mathcal{R}_m^{\Sigma}(F,\omega).
\end{aligned}$$

We recall that about the stability and the convergence of the product rule (2.1.1), Theorems 2.1.4 and 2.1.3 hold true and with respect to the quadrature formula $\Sigma_m(F,\omega)$, we get:

Theorem 2.1.6. *Let w be defined in (1.1.1) and let k be defined in* (2.1.10) *with $g \in C^\infty([-\omega,\omega])$. Then, if there exists a σ as in (1.1.2) such that $F \in C_\sigma$ and the following assumption is satisfied*

$$0 \le \gamma < \min\{1, \alpha+1\}, \qquad 0 \le \delta < \min\{1, \beta+1\}, \tag{2.1.13}$$

then

$$|\Sigma_m(F,\omega)| \le \mathcal{C}\|F\sigma\|_\infty, \quad 0 < \mathcal{C} \ne \mathcal{C}(F,m). \tag{2.1.14}$$

Moreover, for any $F \in W^r_{\sigma,\infty}$, for $\boldsymbol{S} \ge 2$, we get

$$|\mathcal{R}^\Sigma_m(F,\omega)| \le \frac{\mathcal{C}}{m^r}\left(\frac{d}{2}\left(\frac{1}{\omega}+1\right)\right)^r \mathcal{N}_r(F,k), \tag{2.1.15}$$

where

$$\mathcal{N}_r(F,k) = \|F\sigma\|_\infty + \max_{s\in N_0^r}\left(\left\|k^{(r-s)}(\cdot,\omega)\right\|_\infty \times \left\|F^{(s)}\right\|_\infty\right) \tag{2.1.16}$$

and $0 < \mathcal{C} \ne \mathcal{C}(F,m)$.

Proof. First we prove (2.1.14). By (2.1.12), we obtain

$$|\Sigma_m(F,\omega)| \le \frac{d}{2\omega}\mathcal{U}\max_{x\in[-1,1]}|F(x)k(x,\omega)\,\sigma(x)|\left\{\tau_1\sum_{i=1}^m \frac{\lambda_i^{u_1}}{\sigma(\xi_i^{u_1})}\right.$$
$$\left. + \sum_{j=2}^{\boldsymbol{S}-1}\sum_{i=1}^m \frac{\lambda_i^{u_2}}{\sigma(\xi_i^{u_2})} + \tau_2\sum_{i=1}^m \frac{\lambda_i^{u_3}}{\sigma(\xi_i^{u_3})}\right\}$$

where

$$\mathcal{U} = \max\left(||U_1||\,, ||U_S||\,, \max_{j\in N_2^{\boldsymbol{S}-1}}||U_j||\right)$$

and taking into account the relationship (see [65, p. 673 (14)])

$$\lambda_i^{u_j} \sim u_j(\xi_i^{u_j})\Delta\xi_i^{u_j}, \quad \Delta\xi_i^{u_j} = \xi_{i+1}^{u_j} - \xi_i^{u_j}, \quad j \in \{1,2,3\}, \quad i = 1,...,m,$$

under the assumptions (2.1.13) it follows,

$$\sum_{i=1}^m \frac{\lambda_i^{u_j}}{\sigma(\xi_i^{u_j})} \le \int_{-1}^1 \frac{u_j(x)}{\sigma(x)}\,dx \le \mathcal{C}, \quad j \in \{1,2,3\}$$

and we have

$$|\Sigma_m(F,\omega)| \le \mathcal{C}\,\mathcal{U}\,||Fk(\cdot,\omega)\sigma||_\infty .$$

Then, in view of the boundedness of $k(\cdot,\omega)$, we can conclude

$$|\Sigma_m(F,\omega)| \le \mathcal{C}\|F\sigma\|_\infty.$$

Now we prove (2.1.15). By (2.1.12), taking into account the Proposition 1.1.4, under the assumption (2.1.13), we have

$$|\mathcal{R}_m^\Sigma(F,\omega)| \le \mathcal{C}\left\{E_{2m-1}(F_1k_1U_1) + \sum_{j=2}^{\boldsymbol{S}-1} E_{2m-1}(F_jk_jU_j) + E_{2m-1}\left(F_{\boldsymbol{S}}k_{\boldsymbol{S}}U_{\boldsymbol{S}}\right)\right\}.$$

By inequality (1.1.4) we get

$$|\mathcal{R}_m^\Sigma(F,\omega)| \le \left\{\widetilde{\mathcal{U}}\sum_{j=1}^{\boldsymbol{S}} E_{m-1}(F_jk_j)_\sigma + \frac{\widetilde{\mathcal{M}}_r^{max}}{m^r}\sum_{j=1}^{\boldsymbol{S}} ||F_jk_j\sigma||_\infty\right\} \tag{2.1.17}$$

where

$$\widetilde{\mathcal{U}} = \max\left(||U_1\sigma||\,, ||U_{\boldsymbol{S}}\sigma||\,, \max_{j\in N_2^{\boldsymbol{S}-1}} ||U_j\sigma||\right) \le \mathcal{C}$$

and

$$\widetilde{\mathcal{M}}_r^{max} := \max\left\{\max_{i=1,\boldsymbol{S}}\left\|U_i^{(r)}\varphi^r\sigma\right\|_\infty, \max_{2\le j\le \boldsymbol{S}-1}\left\|U_j^{(r)}\varphi^r\sigma\right\|_\infty\right\} \le \mathcal{C}\left(\frac{d}{2\omega}\right)^r\widetilde{\mathcal{U}}.$$

Since for $i \in N_1^{\boldsymbol{S}}$

$$\left|(F_i(x)k_i(x,\omega))^{(r)}\right| \le \sum_{s=0}^{r}\binom{r}{s}\left|F_i^{(s)}(x)\right|\left|k_i^{(r-s)}(x,\omega)\right|,$$

we have

$$\begin{aligned}
&\left|(F_i(x)k_i(x,\omega))^{(r)}\right|\varphi(x)^r\sigma(x)\\
&\le \max_{s\in N_0^r}\left\{\left\|F^{(s)}\varphi^r\sigma\right\|_\infty\left\|k^{(r-s)}(\cdot,\omega)\right\|_\infty\right\}\sum_{s=0}^{r}\binom{r}{s}\left(\frac{d}{2\omega}\right)^s\left(\frac{d}{2}\right)^{r-s}\\
&= \max_{s\in N_0^r}\left\{\left\|F^{(s)}\varphi^r\sigma\right\|_\infty\left\|k^{(r-s)}(\cdot,\omega)\right\|_\infty\right\}\left(\frac{d}{2}\right)^r\left(\frac{1}{\omega}+1\right)^r,
\end{aligned}$$

and therefore, taking into account (1.1.3), by (2.1.17) it follows

$$\begin{aligned}
|\mathcal{R}_m^\Sigma(F,\omega)| &\le \frac{\mathcal{C}}{m^r}\left\{\widetilde{\mathcal{U}}\max_{s\in N_0^r}\left(\left\|F^{(s)}\varphi^r\sigma\right\|_\infty \times \left\|k^{(r-s)}(\cdot,\omega)\right\|_\infty\right)\right.\\
&\quad \times \left.\left(\frac{d}{2}\right)^r\left(\frac{1}{\omega}+1\right)^r + \widetilde{\mathcal{M}}_r^{max}\|Fk\sigma\|_\infty\right\}\\
&\le \frac{\mathcal{C}}{m^r}\mathcal{N}_r(F,k)\left(\frac{d}{2}\right)^r\left(\frac{1}{\omega}+1\right)^r,
\end{aligned}$$

where $\mathcal{N}_r(F,k)$ is defined in (2.1.16) and the thesis follows. □

Since in (2.1.1) we need to evaluate integrals of type (2.1.2), we state below the convergence theorem for the formula $\Sigma_m(F,\omega)$ with $F = \ell_h^w$. To this end, we prove the following:

Theorem 2.1.7. *Let k be defined as in* (2.1.10)*. Under the hypotheses of Theorem 2.1.6, for $m > \frac{d}{2}e^{\frac{1}{\omega}}$ and for $d \geq 2$, $\omega \geq 1$, the following error estimate holds*

$$|\mathcal{R}_m^\Sigma(\ell_h^w,\omega)| \leq \mathcal{C}\,\mathcal{T}_{2m}(k)\cdot \begin{cases} \dfrac{1}{m^{m+1-\mu}} & \text{if} \quad \alpha,\beta > -\dfrac{1}{2} \\ \dfrac{\log m}{m^{m+1}} & \text{if} \quad \alpha,\beta \leq -\dfrac{1}{2} \end{cases}$$

where

$$\mathcal{T}_{2m}(k) = \max_{r\in N_{m+1}^{2m}} \left\|k^{(r)}(\cdot,\omega)\right\|_\infty,$$

$$\mu = \max\{\alpha + \frac{1}{2} - 2\gamma, \beta + \frac{1}{2} - 2\delta\}, \tag{2.1.18}$$

and $\mathcal{C} \neq \mathcal{C}(m,\omega)$.

Proof. In order to use Theorem 2.1.6 with $r = 2m$, we have to estimate $\mathcal{N}_{2m}(\ell_h^w, k)$. By iterating the weighted Bernstein inequality (see for instance [47, p.170])

$$\|(\ell_h^w)^{(m-1)}\varphi^{m-1}\sigma\|_\infty \leq \mathcal{C}\ m^{m-1}\|\ell_h^w\sigma\|_\infty$$

and taking into account that under the hypotheses (2.1.3) and (2.1.5) [47, Th.4.3.3, p.274 and p.256]

$$\max_{|x|\leq 1}\sum_{k=1}^m |\ell_k^w(x)|\frac{\sigma(x)}{\sigma(\xi_k^w)} \leq \mathcal{C}\cdot \begin{cases} m^\mu & \text{if} \quad \alpha,\beta > -\frac{1}{2} \\ \log m & \text{if} \quad \alpha,\beta \leq -\frac{1}{2} \end{cases},$$

with μ defined in (2.1.18), we can conclude, in the worst case, that

$$\|(\ell_h^w)^{(m-1)}\varphi^{m-1}\sigma\|_\infty \leq \mathcal{C}\ m^{m-1}\|\ell_h^w\sigma\|_\infty \leq \mathcal{C}\ m^{m-1+\mu}, \quad \mathcal{C} \neq \mathcal{C}(m).$$

Hence,

$$\mathcal{N}_{2m}(\ell_h^w, k) \leq \mathcal{C}\ m^{m-1+\mu} \max_{r\in N_0^{m-1}} \left\|k^{(2m-r)}(\cdot,\omega)\right\|_\infty$$

and by (2.1.15) and using that

$$\left(\frac{d}{2m}\right)^{2m}\left(\frac{1}{\omega}+1\right)^{2m} \leq e^{-2m\log m\left(1-\frac{\log\left(\frac{d}{2}\right)}{\log m}-\frac{1}{\omega_1\log m}\right)} \leq \frac{1}{m^{2m}}$$

for $m > \frac{d}{4}e^{\frac{1}{\omega}}$, the thesis follows. □

Following the previous work-scheme, the evaluation of the coefficient A_h requires $m^2\boldsymbol{S}$ long operations, with $\boldsymbol{S}$ increasing as ω increases. However, as the numerical tests will show, the implementation of the product rule for smooth integrands functions f and independently on the choice of the parameter ω, will give accurate results for "small" values of m.

Remark 2.1.8. *Along the thesis we also consider a special case of Love's kernel functions k_2 when $x_0 = 0$ and $\lambda = 1$, i.e.*

$$k_2(x, \omega) = \frac{1}{x^2 + \omega^{-1}}.$$

In this case, and if in the integral (2.1.9) *w is a Gegenbauer weight, i.e. $\alpha = \beta$ in* (1.1.1)*, we exploit the symmetry of the zeros of $p_m(w, x)$ and the coefficients in (2.1.2) can be computed "exactly" via modified moment (as done in Subsection 2.1.1, mutatis mutandis) and the computational complexity can be drastically reduced. In the next Subsection we will show all the details.*

2.1.3 Cases of complexity reduction

In some special cases the computational complexity can be drastically reduced, for instance, in the case of Love's kernel functions with $w(x) = v^{\alpha,\alpha}(x)$, i.e. w is a Gegenbauer weight. Slightly changing the notation set in Section 1.1, let us denote by

$$\{\xi_i^w\}_{i=-M}^{M}, \; M = \left\lfloor \frac{m}{2} \right\rfloor, \; \xi_0^w = 0, \text{ for } \quad m \quad \text{odd},$$

the zeros of $p_m(w, x)$. Since w is an even weight function, it is $\xi_i^w = -\xi_{-i}^w, \quad i = 1, 2, \ldots, M$, we have

$$\ell_h^w(x) = \begin{cases} \displaystyle\prod_{i=1}^{\lfloor \frac{m}{2} \rfloor} \frac{(\xi_i^w)^2 - x^2}{(\xi_i^w)^2}, & \text{if } h = 0, \quad m \text{ odd,} \\[2ex] \displaystyle\left(\frac{x}{\xi_h^w}\right)^{\frac{1-(-1)^m}{2}} \prod_{\substack{i=1 \\ i \neq h}}^{\lfloor \frac{m}{2} \rfloor} \frac{x^2 - (\xi_i^w)^2}{(\xi_h^w)^2 - (\xi_i^w)^2} \left(\frac{x + \xi_h^w}{2\xi_h^w}\right), & 1 \le h \le m. \end{cases}$$

To compute the coefficients in (2.1.2) when $k \equiv k_2$ with $x_0 = 0$ and $\lambda = 1$, since $A_h(\omega) = A_{m-h+1}(\omega)$, the computation is halved and for $1 \le h \le \left\lfloor \frac{m}{2} \right\rfloor$ we have

$$A_h(t) = \int_{-1}^{1} \frac{\ell_h^w(x)}{x^2+\omega^{-1}}\, w(x)\, dx = \frac{1}{2^{\frac{1}{2}+\alpha}} \int_{-1}^{1} \frac{\Pi_h\left(\frac{x+1}{2}\right)}{x+t}\, v^{\alpha,-\frac{1}{2}}(x)\, dx,$$

where

$$t = 1+2\omega^{-1} \quad \text{and} \quad \Pi_h(z) = \prod_{\substack{i=1\\ i\neq h}}^{\lfloor \frac{m}{2} \rfloor} \frac{z-(\xi_i^w)^2}{(\xi_h^w)^2-(\xi_i^w)^2}.$$

Assume for simplicity m even. Since $\Pi \in \mathbb{P}_{\frac{m}{2}-1}$ we easily deduce

$$\begin{aligned} A_h(t) &= \int_{-1}^{1} \frac{\Pi_h\left(\frac{x+1}{2}\right)}{x+t} v^{\alpha,-\frac{1}{2}}(x)dx \\ &= \sum_{j=1}^{\frac{m}{2}-1} \Pi_h\left(\frac{z_j^{\alpha,-\frac{1}{2}}+1}{2}\right) \lambda_j^{\alpha,-\frac{1}{2}} \sum_{k=0}^{\frac{m}{2}-1} p_k\left(v^{\alpha,-\frac{1}{2}}, z_j^{\alpha,-\frac{1}{2}}\right) M_k^{\alpha,-\frac{1}{2}}(t), \end{aligned}$$

where $\{\lambda_j^{\alpha,-\frac{1}{2}}\}_{j=1}^{\frac{m}{2}}$ are the Christoffel numbers with respect to the Jacobi weight $v^{\alpha,-\frac{1}{2}}$, $\{z_j\}_{j=1}^{\frac{m}{2}}$ are the zeros of $p_{\frac{m}{2}}(v^{\alpha,-\frac{1}{2}}, x)$ and

$$M_k^{\alpha,-\frac{1}{2}}(t) = \int_{-1}^{1} \frac{p_k\left(v^{\alpha,-\frac{1}{2}}, x\right)}{x+t} v^{\alpha,-\frac{1}{2}}(x)dx$$

are the *modified moments*. In this case it is no hard to prove that the sequence $\{M_k^{\alpha,-\frac{1}{2}}(\omega)\}_{k=0}^{\infty}$ satisfy the following recurrence relation

$$\begin{aligned} M_0^{\alpha,-\frac{1}{2}}(t) &= \frac{1}{\sqrt{\mu_0}} \int_{-1}^{1} \frac{v^{\alpha,-\frac{1}{2}}(x)}{x+t} dx,\ \mu_0 = \int_{-1}^{1} v^{\alpha,-\frac{1}{2}}(x)dx = \frac{2\sqrt{\pi}\,\Gamma(\alpha+1)}{(2\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}, \\ M_1^{\alpha,-\frac{1}{2}}(t) &= \frac{1}{b_1}\left(\sqrt{\mu_0}-(t+a_1)M_0(t)\right), \\ M_k^{\alpha,-\frac{1}{2}}(t) &= -\frac{1}{b_k}\left[(t+a_k)M_{k-1}^{\alpha,-\frac{1}{2}}(t)+b_{k-1}M_{k-2}^{\alpha,-\frac{1}{2}}(t)\right], \quad k=2,...,m, \end{aligned}$$

where a_k, b_k are the coefficients of the three-term recurrence relation in (2.1.8) with $w(x) = v^{\alpha,-\frac{1}{2}}(x)$. In the case m odd the coefficients $A_h(t)$ can be computed by similar arguments.

More generally, for all the kernel functions k_j, $j \in \{2,3,4\}$, defined in (2.1.10), we have a reduction of the global computational cost (shortly CC) when w is a Gegenbauer weight. In particular:

- if $k_j(x,\omega)$, $j \in \{2,3,4\}$, is *symmetric through the axis* $x = 0$, i.e.

$$k_j(-x,\omega) = k_j(x,\omega), \quad j \in \{2,3,4\},$$

it is

$$A_h(\omega) \;=\; A_{m-h+1}(\omega), \quad h \in N_1^M,$$

and the CC has a reduction of 75%;

- in the case $k_j(x,\omega)$, $j \in \{2,3,4\}$, is *symmetric with respect to the origin*, i.e.

$$k_j(-x,\omega) = -k_j(x,\omega), \quad j \in \{2,3,4\},$$

it is

$$A_h(\omega) \;=\; -A_{m-h+1}(\omega), \quad h \in N_1^M,$$

and again the CC has a reduction of 75%.

2.2 Product Cubature Rules on the Square $[-1,1] \times [-1,1]$

Consider now the following bivariate integrals

$$\boldsymbol{I}(\boldsymbol{f},\mathbf{y}) = \int_S \boldsymbol{f}(\mathbf{x})\boldsymbol{k}(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x})d\mathbf{x}, \quad \boldsymbol{y} = (y_1,y_2) \in S = [-1,1]^2, \tag{2.2.1}$$

in which we recall that $\boldsymbol{k}$, function of four variables, can be *weakly singular*, *nearly singular* and/or *highly oscillating*.

By replacing the function $\boldsymbol{f}$ with the bivariate Lagrange polynomial $\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w},\mathbf{x})$ defined in (1.2.14), we obtain the following *product cubature rule*

$$\begin{aligned}\boldsymbol{I}(\boldsymbol{f},\mathbf{y}) &= \sum_{h=1}^{m}\sum_{k=1}^{m} A_{h,k}(\mathbf{y})\boldsymbol{f}\left(\xi_{h,k}^{w_1,w_2}\right) + \mathcal{R}_{m,m}(\boldsymbol{f},\mathbf{y}) \\ &=: \boldsymbol{I}_m(\boldsymbol{f},\mathbf{y}) + \mathcal{R}_{m,m}(\boldsymbol{f},\mathbf{y})\end{aligned} \tag{2.2.2}$$

where

$$A_{h,k}(\mathbf{y}) = \int_S \ell_{h,k}^{w_1,w_2}(\mathbf{x})\boldsymbol{k}(\mathbf{x},\mathbf{y})\mathbf{w}(\mathbf{x})d\mathbf{x} \tag{2.2.3}$$

and

$$\mathcal{R}_{m,m}(\boldsymbol{f},\mathbf{y}) =: \boldsymbol{I}(\boldsymbol{f},\mathbf{y}) - \boldsymbol{I}_m(\boldsymbol{f},\mathbf{y})$$

is the remainder term. We recall that the cubature rule is exact for bivariate algebraic polynomials of degree $m-1$ in each variable, i.e.

$$\mathcal{R}_{m,m}(\boldsymbol{P},\mathbf{y})=0,\quad \forall \boldsymbol{P}\in\mathbb{P}_{m-1,m-1},\quad \forall \mathbf{y}\in S.$$

In order to prove the stability and the convergence of the proposed product formula, we recall a result needed in the successive proof.

Let $\mathcal{S}_m$ be the univariate mth Fourier sum defined in (1.1.8) and let $\mathcal{S}_{m,m}$ be the bivariate m-th Fourier sum defined in (1.2.9).

For $1<p<\infty$, denoting by $L^p(S)$ the usual L^p space on S, under the assumptions (see for instance [66, p.2332])

$$\frac{\boldsymbol{\sigma}}{\sqrt{\mathbf{w}\varphi_1\varphi_2}}\in L^p(S),\quad \frac{\mathbf{w}}{\boldsymbol{\sigma}}\in L^q(S),\quad \frac{1}{\boldsymbol{\sigma}}\sqrt{\frac{\mathbf{w}}{\varphi_1\varphi_2}}\in L^q(S),\quad q=\frac{p}{p-1},$$

then, for any $\boldsymbol{f}\in C_{\boldsymbol{\sigma}}$ holds true

$$\|\mathcal{S}_{m,m}(\boldsymbol{f},\mathbf{w})\boldsymbol{\sigma}\|_p\le \mathcal{C}\|\boldsymbol{f}\boldsymbol{\sigma}\|_\infty,\quad \mathcal{C}\neq\mathcal{C}(m,\boldsymbol{f}). \tag{2.2.4}$$

Now, we are able to prove the following.

Theorem 2.2.1. *Let $\mathbf{w}(\mathbf{x})=w_1(x_1)w_2(x_2)$ the product of two Jacobi weight. If there exist a $\boldsymbol{\sigma}$ defined as in (1.2.2) such that $\boldsymbol{f}\in C_{\boldsymbol{\sigma}}$ and the following assumptions are satisfied*

$$\boldsymbol{k}(\cdot,\mathbf{y})\sqrt{\mathbf{w}}\in L^2(S), \tag{2.2.5}$$

$$\frac{\mathbf{w}}{\boldsymbol{\sigma}},\qquad \frac{\boldsymbol{\sigma}}{\sqrt{\mathbf{w}\varphi_1\varphi_2}},\quad \frac{1}{\sigma}\sqrt{\frac{\mathbf{w}}{\varphi_1\varphi_2}}\in L^2(S), \tag{2.2.6}$$

then we have

$$\sup_{\mathbf{y}\in S}|\boldsymbol{I}_m(\boldsymbol{f},\mathbf{y})|\le\mathcal{C}\|\boldsymbol{f}\boldsymbol{\sigma}\|_\infty, \tag{2.2.7}$$

where $\mathcal{C}\neq\mathcal{C}(m,\boldsymbol{f})$. Moreover, the following error estimate holds true

$$\sup_{\mathbf{y}\in S}|\mathcal{R}_{m,m}(\boldsymbol{f},\mathbf{y})|\le\mathcal{C}E_{m-1,m-1}(\boldsymbol{f})_{\boldsymbol{\sigma}}, \tag{2.2.8}$$

where $\mathcal{C}\neq\mathcal{C}(m,\boldsymbol{f})$.

Remark 2.2.2. *Let us remark that if the parameters of the weight $\mathbf{w}$ are such that $\alpha_i,\beta_i<\frac{1}{2}$, $i\in\{1,2\}$, then the parameters of the weight $\boldsymbol{\sigma}$ could also be chosen equal to zero.*

Proof. First we prove

$$||\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w})\boldsymbol{k}(\cdot,\mathbf{y})\mathbf{w}||_1 \le \mathcal{C}\|\boldsymbol{f}\boldsymbol{\sigma}\|_\infty, \tag{2.2.9}$$

which implies (2.2.7).

For any fixed $\mathbf{y} \in S$, let $\boldsymbol{g}_m = sgn(\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w})\boldsymbol{k}(\mathbf{x},\mathbf{y}))$.

Then,

$$\begin{aligned}
&||\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w})\boldsymbol{k}(\cdot,\mathbf{y})\mathbf{w}||_1 \\
&= \int_S \mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w},\mathbf{x})\,\boldsymbol{k}(\mathbf{x},\mathbf{y})(\mathbf{x})\,\boldsymbol{g}_m(\mathbf{x})\,\mathbf{w}(\mathbf{x})\,d\mathbf{x} \\
&= \Bigg|\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i^{w_1}\lambda_j^{w_2}\boldsymbol{f}(\xi_{i,j}^{w_1,w_2})\sum_{k=0}^{m-1} p_k(w_1,\xi_i^{w_1})\sum_{r=0}^{m-1} p_r(w_2,\xi_j^{w_2}) \\
&\times \int_S p_k(w_1,x_1)\,p_r(w_2,x_2)\,\boldsymbol{k}(\mathbf{x},\mathbf{y})\,\boldsymbol{g}_m(\mathbf{x})\,\mathbf{w}(\mathbf{x})\,d\mathbf{x}\Bigg| \\
&= \left|\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i^{w_1}\lambda_j^{w_2}\boldsymbol{f}(\xi_{i,j}^{w_1,w_2})\mathcal{S}_{m,m}(\boldsymbol{k}(\cdot,\mathbf{y})\boldsymbol{g}_m,\mathbf{w},\xi_{i,j}^{w_1,w_2})\right|.
\end{aligned}$$

By Hölder inequality

$$\begin{aligned}
&||\mathcal{L}_{m,m}(\boldsymbol{f},\mathbf{w})\boldsymbol{k}(\cdot,\mathbf{y})\mathbf{w}||_1 \le \sum_{i=1}^{m}\lambda_i^{w_1}\left(\sum_{j=1}^{m}\lambda_j^{w_2}\boldsymbol{f}^2(\xi_{i,j}^{w_1,w_2})\right)^{\frac{1}{2}} \\
&\times \left(\sum_{j=1}^{m}\lambda_j^{w_2}\mathcal{S}_{m,m}^2\left(\boldsymbol{k}(\cdot,\mathbf{y})\boldsymbol{g}_m,\mathbf{w},\xi_{i,j}^{w_1,w_2}\right)\right)^{\frac{1}{2}} \\
&\le \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i^{w_1}\lambda_j^{w_2}\boldsymbol{f}^2(\xi_{i,j}^{w_1,w_2})\right)^{\frac{1}{2}} \\
&\times \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i^{w_1}\lambda_j^{w_2}\mathcal{S}_{m,m}^2\left(\boldsymbol{k}(\cdot,\mathbf{y})\boldsymbol{g}_m,\mathbf{w},\xi_{i,j}^{w_1,w_2}\right)\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.2.10}$$

Now, taking into account (2.2.4) and assumptions (2.2.6), we get

$$\begin{aligned}
&\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i^{w_1}\lambda_j^{w_2}\mathcal{S}_{m,m}^2(\boldsymbol{k}(\cdot,\mathbf{y})\boldsymbol{g}_m,\mathbf{w},\xi_{i,j}^{w_1,w_2})\right)^{\frac{1}{2}} \\
&= \left(\int_S \mathcal{S}_{m,m}^2\left(\boldsymbol{k}(\cdot,\mathbf{y})\boldsymbol{g}_m,\mathbf{w},\mathbf{x}\right)\mathbf{w}(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{2}} \\
&= \|\mathcal{S}_{m,m}\left(\boldsymbol{k}(\cdot,\mathbf{y}),\mathbf{w}\right)\sqrt{\mathbf{w}}\|_2 \le \mathcal{C}\|\boldsymbol{k}(\cdot,\mathbf{y})\sqrt{\mathbf{w}}\|_2.
\end{aligned} \tag{2.2.11}$$

Moreover,

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\boldsymbol{f}^2(\xi_{i,j}^{w_1,w_2})\right)^{\frac{1}{2}} \le ||\boldsymbol{f}\boldsymbol{\sigma}||_\infty \left(\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\boldsymbol{\sigma}(\xi_{i,j}^{w_1,w_2})}\right)^{\frac{1}{2}}$$

and taking into account the relationship (see [65, p. 673 (14)])

$$\lambda_i^{w_j} \sim w_j(\mathbf{x}_i^{w_j})\Delta\xi_i^{w_j}, \quad \Delta\xi_i^{w_j} = \xi_{i+1}^{w_j} - \xi_i^{w_j}, \quad j \in \{1,2\},$$

it follows

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\lambda_i^{w_1}\lambda_j^{w_2}}{\boldsymbol{\sigma}(\xi_{i,j}^{w_1,w_2})} &\le \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\Delta\xi_i^{w_1}w_1(\xi_i^{w_1})}{\sigma_1(\xi_i^{w_1})}\frac{\Delta\xi_j^{w_2}w_2(\xi_j^{w_2})}{\sigma_2(\xi_j^{w_2})} \\ &\le \int_S \frac{\mathbf{w}(\mathbf{x})}{\boldsymbol{\sigma}(\mathbf{x})}\, d\mathbf{x} \le \mathcal{C}. \end{aligned} \tag{2.2.12}$$

Combining the last inequality and (2.2.11) with (2.2.10), (2.2.9) follows.

In order to prove (2.2.8), start from

$$\begin{aligned}\left|\mathcal{R}_{m,m}(\boldsymbol{f},\mathbf{y})\right| &\le \int_S \left|[\boldsymbol{f}(\mathbf{x}) - \boldsymbol{P}^*_{m-1,m-1}(\mathbf{x})]\boldsymbol{k}(\mathbf{x},\mathbf{y})\right| \mathbf{w}(\mathbf{x})\, d\mathbf{x} \\ &+ \int_S \left|\mathcal{L}_{m,m}(\boldsymbol{f} - \boldsymbol{P}^*_{m-1,m-1}, \mathbf{w}, \mathbf{x})\boldsymbol{k}(\mathbf{x},\mathbf{y})\right| \mathbf{w}(\mathbf{x})\, d\mathbf{x} \\ &=: A_1(\mathbf{y}) + A_2(\mathbf{y}), \end{aligned} \tag{2.2.13}$$

where $\boldsymbol{P}^*_{m-1,m-1}(\mathbf{x})$ is the best approximation polynomial of $\boldsymbol{f} \in C_{\boldsymbol{\sigma}}$.

By Hölder inequality and taking into account (2.2.5) and (2.2.6) it follows

$$A_1(\mathbf{y}) \le \mathcal{C}E_{m-1,m-1}(\boldsymbol{f})_{\boldsymbol{\sigma}} \int_S |\boldsymbol{k}(\mathbf{x},\mathbf{y})| \frac{\mathbf{w}(\mathbf{x})}{\boldsymbol{\sigma}(\mathbf{x})}\, d\mathbf{x} \le \mathcal{C}E_{m-1,m-1}(\boldsymbol{f})_{\boldsymbol{\sigma}}. \tag{2.2.14}$$

Since by (2.2.9)

$$A_2(\mathbf{y}) \le \mathcal{C}E_{m-1,m-1}(\boldsymbol{f})_{\boldsymbol{\sigma}}, \tag{2.2.15}$$

(2.2.8) follows combining (2.2.14) and (2.2.15) with (2.2.13). □

Remark 2.2.3. *From* (2.2.8) *it follows that for* $m \to \infty$, *the error rate of decay of the product rule is bounded by that of the error of the best polynomial approximation of the only function* $\boldsymbol{f}$. *This appealing speed of convergence holds under the "exact"computation of the coefficients in* $\boldsymbol{I}_m(\boldsymbol{f},\mathbf{y})$. *Their (approximate) evaluation is however not a simple task; only for kernels having special properties it can be performed with a low computational cost. Details on the computation of the coefficients in* (2.2.3) *for some kind kernels will be given in the next Sections.*

Remark 2.2.4. *In* (2.2.1)*, we can also consider all the combination of two one–dimensional kernel functions presented in Subsections 2.1.1 and 2.1.2. To be more precise, we can also consider bivariate separable kernel functions of the type*

$$\boldsymbol{k}(\mathbf{x},\omega) = k_i(x_1,\omega)k_j(x_2,\omega), \quad i,j = \{1,2,3,4\}$$

where k_i *and* k_j *are given in* (2.1.7) *and* (2.1.10) *and the coefficients in* (2.2.3) *take the form*

$$A_{h,k}(\omega) = A_h(\omega)A_k(\omega), \quad (h,k) \in N_1^m \times N_m^1,$$

where

$$A_h(\omega) = \int_{-1}^{1} \ell_h^{w_1}(x_1)k_i(x_1,\omega)w_1(x_1)dx_1, \quad h \in N_m^1, \quad i = \{1,2,3,4\},$$
$$A_k(\omega) = \int_{-1}^{1} \ell_k^{w_2}(x_2)k_j(x_2,\omega)w_2(x_2)dx_2, \quad k \in N_m^1, \quad j = \{1,2,3,4\}.$$

In these case, the computation effort is drastically reduced, since the coefficients above can be approximated by implementing the one–dimensional dilation method or the modified moments (see Subsections 2.1.1 and 2.1.2).

2.2.1 Computation of the 2D-product rule coefficients for *weakly* singular kernels

In this Subsection, we give some details for computing the coefficients in (2.2.3) when the kernel functions is a *weakly* singular, i.e. we consider integrals of the type

$$\boldsymbol{I}(\boldsymbol{f},\mathbf{y}) = \int_S \boldsymbol{f}(\mathbf{x})|x_1-y_1|^{\lambda_1}|x_2-y_2|^{\lambda_2}\mathbf{w}(\mathbf{x})d\mathbf{x}, \quad \boldsymbol{y} = (y_1,y_2) \in \dot{S} = (-1,1)^2$$

where $-1 < \lambda_1, \lambda_2 < 0$, that means that the kernel function is given by

$$\boldsymbol{k}_1(\mathbf{x},\mathbf{y}) = |x_1-y_1|^{\lambda_1}|x_2-y_2|^{\lambda_2}. \tag{2.2.16}$$

In this case, the coefficients in (2.2.3), can be computed "exactly" via *modified moments*, as done in Subsection 2.1.1. Infact, the kernel function $\boldsymbol{k}_1$ is separable, i.e. is a product of two univariate *weakly singular* kernel fuctions.

A special case: $x_2 = x_1$

In (2.2.16), in the special case $x_2 = x_1$, i.e.

$$\mathfrak{k}_1(x_1, \mathbf{y}) = |x_1 - y_1|^{\lambda_1} |x_1 - y_2|^{\lambda_2}, \ -1 < \lambda_1, \lambda_2 < 0, \tag{2.2.17}$$

the cubature rule proposed in (2.2.2), it is reduced to

$$\boldsymbol{I}(\boldsymbol{f}, \mathbf{y}) = \sum_{h=1}^{m} \sum_{k=1}^{m} A_h(\mathbf{y}) \lambda_k^{w_2} \boldsymbol{f}\left(\xi_{h,k}^{w_1,w_2}\right) + \mathcal{R}_{m,m}(\boldsymbol{f}, \mathbf{y}, \omega)$$

where

$$A_h(\mathbf{y}) = \int_{-1}^{1} \ell_h^{w_1}(x_1) |x_1 - y_1|^{\lambda_1} |x_1 - y_2|^{\lambda_2} w_1(x_1) dx_1. \tag{2.2.18}$$

To compute the coefficients $A_h(\mathbf{y})$, we assume, for instance, $y_1 < y_2$. The integral (2.2.18) can be split in the sum of three integrals. To be more precise:

$$\begin{aligned}
A_h(\mathbf{y}) &= \int_{-1}^{y_1} \ell_h^{w_1}(x_1) \left(x_1 - y_1\right)^{\lambda_1} \left(x_1 - y_2\right)^{\lambda_2} w_1(x_1) dx_1 \\
&+ \int_{y_1}^{y_2} \ell_h^{w_1}(x_1) \left(y_1 - x_1\right)^{\lambda_1} \left(x_1 - y_2\right)^{\lambda_2} w_1(x_1) dx_1 \\
&+ \int_{y_2}^{1} \ell_h^{w_1}(x_1) \left(y_1 - x_1\right)^{\lambda_1} \left(y_2 - x_1\right)^{\lambda_2} w_1(x_1) dx_1 \\
&= \nu_1 \int_{-1}^{1} \ell_h^{w_1}\left(\Omega_1(z, y_1)\right) \left(1 - \Omega_1(z, y_1)\right)^{\alpha_1} \left(y_2 - \Omega_1(z, y_1)\right)^{\lambda_2} v^{\lambda_1, \beta_1}(z) dz \\
&+ \nu_2 \int_{-1}^{1} \ell_h^{w_1}\left(\Omega_2(z, y_1, y_2)\right) \left(1 - \Omega_2(z, y_1, y_2)\right)^{\alpha_1} \left(1 + \Omega_2(z, y_1, y_2)\right)^{\beta_1} \\
&\times v^{\lambda_2, \lambda_1}(z) dz \\
&+ \nu_3 \int_{-1}^{1} \ell_h^{w_1}\left(\Omega_3(z, y_2)\right) \left(\Omega_3(z, y_2) - y_1\right)^{\lambda_1} \left(1 + \Omega_3(z, y_2)\right)^{\beta_1} v^{\alpha_1, \lambda_2}(z) dz,
\end{aligned}$$

where

$$\Omega_1(z, y_1) = \frac{(z+1)\,(y_1+1)}{2} - 1, \quad \Omega_2(z, y_1, y_2) = \frac{(z+1)\,(y_2 - y_1)}{2} + y_1,$$

$$\Omega_3(z, y_2) = \frac{(z+1)\,(1 - y_2)}{2} + y_2, \quad \nu_1 = \left(\frac{1 + y_1}{2}\right)^{\lambda_1 + \beta_1 + 1},$$

$$\nu_2 = \left(\frac{y_2 - y_1}{2}\right)^{\lambda_1 + \lambda_2 + 1}, \quad \nu_3 = \left(\frac{1 - y_2}{2}\right)^{\lambda_2 + \alpha_1 + 1}.$$

At last, we approximate the integrals by using proper Gauss-Jacobi rules and all the above integrals can be computed with high accuracy with few Gaussian knots, since the involved integrand functions are very smooth.

Remark 2.2.5. *Let us note that, by the linear transformation that maps the unit square in the unit triangle, we can transform the kernels in* (2.2.16) *and* (2.2.17) *in not-degenerate kernels. To be more precise: let us denote by* T *the triangle defined as* $T = \{(u_1, u_2) : u_1 \geq 0, u_2 \geq 0, u_1 + u_2 \leq 1\}$. *For any* $\mathbf{x} = (x_1, x_2) \in S$ *and* $\mathbf{u} = (u_1, u_2) \in T$, *we can consider the following transformations between the square* S *and the triangle* T

$$x_1 = 2(u_1 + u_2) - 1, \quad x_2 = \frac{u_1 - u_2}{u_1 + u_2}$$

or equivalently

$$u_1 = \frac{1}{4}(1 + x_1)(1 + x_2), \quad u_2 = \frac{1}{4}(1 + x_1)(1 - x_2).$$

By these transformations (so called Duffy's transformation), the edge connecting the vertices $(-1, 0)$ *and* $(-1, 1)$ *of the square* S, *reduced into the vertex* $(0, 0)$ *of the triangle* T, *while the remaining three edges of the square* S, *are mapped into an edge of the triangle* T. *Then, with respect to the kernel* (2.2.16) *we obtain integrals of this type*

$$\boldsymbol{I}(\boldsymbol{f}, \mathbf{v}) = \int_T \boldsymbol{f}(\mathbf{u}) \, |u_1 + u_2 - v_1 - v_2|^{\lambda_1} \, |v_1 u_2 - v_2 u_1|^{\lambda_2} \, \widetilde{\mathbf{w}}(\mathbf{u}) \, d\mathbf{u},$$

where $-1 < \lambda_1, \lambda_2 < 0$ *and the transformed kernel is given by*

$$\widetilde{\boldsymbol{k}}_1(\mathbf{u}, \mathbf{v}) = |u_1 + u_2 - v_1 - v_2|^{\lambda_1} \, |v_1 u_2 - v_2 u_1|^{\lambda_2},$$

where

$$\mathbf{u} = (u_1, u_2) \in T, \quad \mathbf{v} = (v_1, v_2) \in T$$

and

$$\widetilde{\mathbf{w}}(\mathbf{u}) = (1 - u_1 - u_2)^b (u_1 + u_2)^a u_1^{p-1} u_2^{q-1},$$

with $p, q > 0, \; p + q + a > 0, \; b > -1$.

With respect to the kernel (2.2.17), *we obtain integrals of this type*

$$\boldsymbol{I}(\boldsymbol{f}, \mathbf{v}) = \int_T \boldsymbol{f}(\mathbf{u}) \, |u_1 + u_2 - v_1|^{\lambda_1} \, |u_1 + u_2 - v_2|^{\lambda_2} \, \widetilde{\mathbf{w}}(\mathbf{u}) d\mathbf{u},$$

where $-1 < \lambda_1, \lambda_2 < 0$ *and the transformed kernel is given by*

$$\widetilde{\mathfrak{k}}_1(\mathbf{u}, \mathbf{v}) = |u_1 + u_2 - v_1|^{\lambda_1} \, |u_1 + u_2 - v_2|^{\lambda_2},$$

where

$$\mathbf{u} = (u_1, u_2) \in T, \quad \mathbf{v} = (v_1, v_2) \in T$$

and $\widetilde{\mathbf{w}}(\mathbf{u})$ *is given above.*

2.2.2 Computation of the 2D-product rule coefficients for *nearly* and/or *highly oscillating* kernels

In this Subsection, we give some details for computing the coefficients in (2.2.3) when the kernel functions are *nearly* singular and/or *highly oscillating*, i.e. we consider, in general, integrals of the type

$$\boldsymbol{I}(\boldsymbol{f},\omega)=\int_S \boldsymbol{f}(\mathbf{x})\boldsymbol{k}_j(\mathbf{x},\omega)\mathbf{w}(\mathbf{x})d\mathbf{x},\quad j\in\{2,3,4\}$$

where the kernel functions can be

$$\begin{aligned}
\boldsymbol{k}_2(\mathbf{x},\omega) &= \frac{1}{(|\mathbf{x}-\mathbf{x}_0|^2+\omega^{-1})^{\lambda}},\quad \mathbf{x}_0=(s_0,t_0)\in S \text{ fixed},\\
&\quad \text{with } \lambda\in\mathbb{R}^+,\quad 0\neq\omega\in\mathbb{R},\\
\boldsymbol{k}_3(\mathbf{x},\omega) &= \boldsymbol{g}(\omega\mathbf{x}),\ 0\neq\omega\in\mathbb{R} \qquad (2.2.19)\\
&\quad \boldsymbol{g} \text{ is a bivariate oscillatory "smooth" function}\\
&\quad \text{with frequency } \omega,\\
\boldsymbol{k}_4(\mathbf{x},\omega) &= \boldsymbol{k}_2(\mathbf{x},\omega)\boldsymbol{k}_3(\mathbf{x},\omega).
\end{aligned}$$

The graphs in Fig. 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7 show the behavior of some kernels of the types (2.2.19) for some choices of the parameter ω.

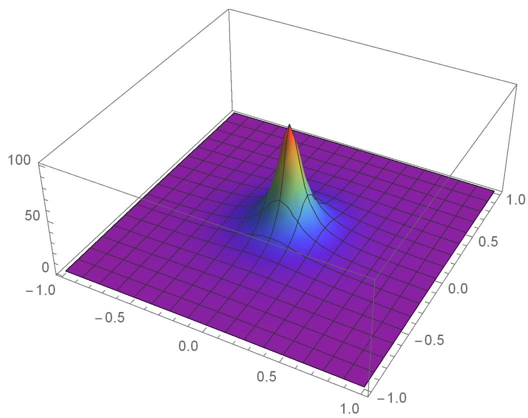

Figure 2.1: Kernel $\boldsymbol{k}_2(\mathbf{x},\omega)=(x_1^2+x_2^2+\omega^{-1})^{-1}$ with $\omega=10^2$.

The numerical evaluation of these integrals, as seen for the univariate case, presents difficulties for "large" ω, since $\boldsymbol{k}_2$ is "close" to be singular, $\boldsymbol{k}_3$ highly oscillates, while $\boldsymbol{k}_4$ includes both the aforesaid problematic behaviors. In all the cases, for these kernels the modulus of the derivatives grows as ω grows.

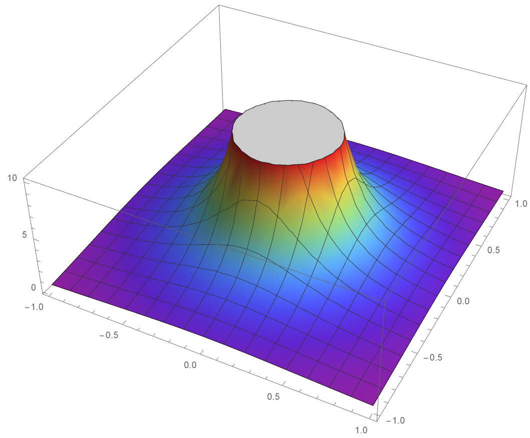

Figure 2.2: Section of the Kernel $\boldsymbol{k}_2(\mathbf{x},\omega) = (x_1^2 + x_2^2 + \omega^{-1})^{-1}$ with $\omega = 10^2$.

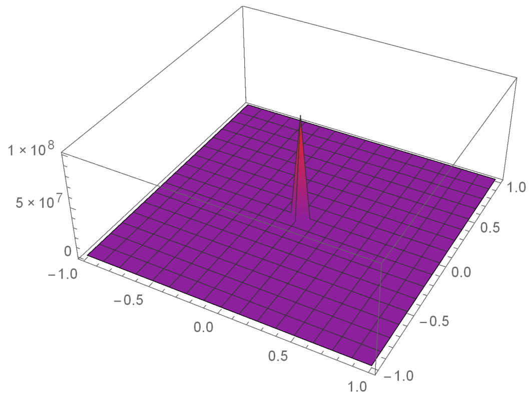

Figure 2.3: Kernel $\boldsymbol{k}_2(\mathbf{x},\omega) = (x_1^2 + x_2^2 + \omega^{-1})^{-1}$ with $\omega = 10^8$.

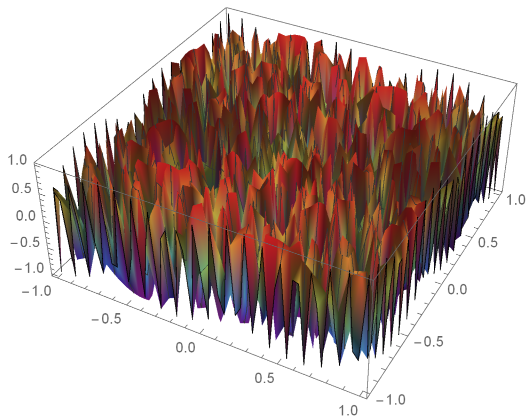

Figure 2.4: Kernel $\boldsymbol{k}_3(\mathbf{x},\omega) = \sin(\omega x_1 x_2)$, with $\omega = 10^8$.

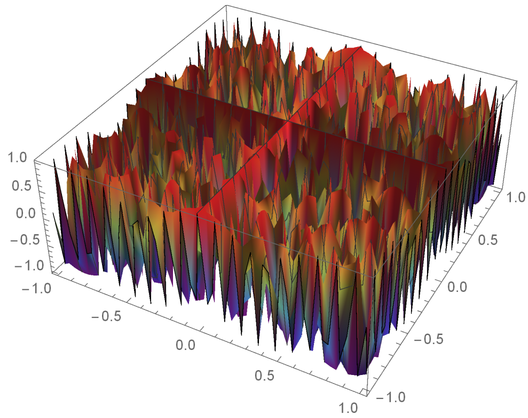

Figure 2.5: Kernel $\boldsymbol{k}_3(\mathbf{x}, \omega) = \cos(\omega x_1 x_2)$, with $\omega = 10^8$.

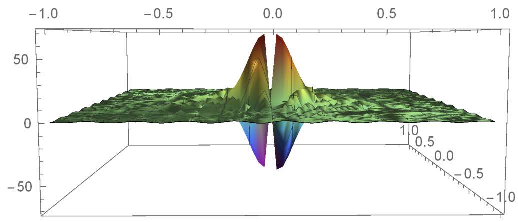

Figure 2.6: Kernel $\boldsymbol{k}_4(\mathbf{x}, \omega) = \sin(\omega x_1 x_2)(x_1^2 + x_2^2 + \omega^{-1})^{-1}$ with $\omega = 10^4$.

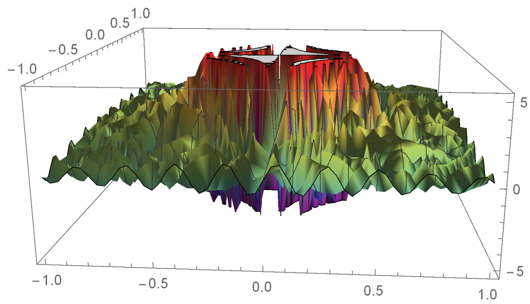

Figure 2.7: Section of the Kernel $\boldsymbol{k}_4(\mathbf{x}, \omega) = \sin(\omega x_1 x_2)(x_1^2 + x_2^2 + \omega^{-1})^{-1}$ with $\omega = 10^4$.

$\boldsymbol{k}_2$-type kernels appear, for instance, in two–dimensional nearly singular BEM integrals on quadrilateral elements (see, for instance, [31, 62, 76]). Highly oscillating kernels of the type $\boldsymbol{k}_3$ are useful in computational methods for oscillatory phenomena in science and engineering problems, including wave scattering, wave propagation, quantum mechanics, signal processing and image recognition (see [30] and references therein). The combination of the two aspects, i.e. integrals with nearly singular and oscillating kernels appear for instance in the solution of problems of propagation in uniform waveguides with nonperfect conductors (see [19] and the references therein).

Here we propose a *product cubature formula* obtained by replacing the "regular" function $\boldsymbol{f}$ by a bivariate Lagrange polynomial based on a set of knots chosen such that the stability and the convergence of the rule are assured. Despite the simplicity of these formulas, the computation of their coefficients is not yet an easy task. Analogously to the univariate case, in order to compute the corresponding coefficients one needs to determine *modified moments* by means of recurrence relations, and to examine the stability of these latter (see, for instance, [26, 41, 53, 73, 81]). This approach, however, does not appear feasible for multivariate not degenerate kernels.

Here we present a unique approach for computing the coefficients of the aforesaid cubature rule when $\boldsymbol{k}$ belongs to the types (2.2.19). Such method, that we call *2D-dilation method*, is based on a preliminary "dilation" of the domain and, by suitable transformations, on the successive reduction of the initial integral to the sum of ones on S again. These manipulations "relax" in some sense the "too fast" behavior of $\boldsymbol{k}$ as ω grows. For a correct use of the 2D-dilation method, which could be also applied directly for computing integrals with kernels of the kind (2.2.19), we determine conditions under which the rule is stable and convergent.

Both the rules have advantages and drawbacks. The product integration rule requires a smaller number of evaluations of the integrand function $\boldsymbol{f}$, while the number of samples involved in the 2D-dilation rule increases as ω increases. On the other hand, the product rule involves the computation of m^2 coefficients, which are integrals, and for this reason, in general its computational cost can be excessively high. However, as we will show, this cost can be drastically reduced when the kernels present some symmetries.

We point out that many of the existing methods on the approximation of multivariate integrals are reliable for very smooth functions (see, for instance, [4, 13, 14, 30, 78, 83] and references therein). Some of them treat degenerate kernels [77], others require changes of variable generally not right for weighted integrands [30, 31]. Our procedure allows to compute not degenerate weighted integrals, with oscillating and/or nearly singular kernels.

To perform the evaluation of integral (2.2.1), the main idea is to dilate the integration domain S by a change of variables in order to relax in some sense the "too fast" behavior of $\boldsymbol{k}$ when ω grows. Successively the new domain Ω is divided into $\boldsymbol{S}^2$ squares $\{S_{i,j}\}_{(i,j)\in N_1^S\times N_1^S}$ and each integral is reduced into S one more time. At last, the integrals are approximated by suitable Gauss-Jacobi rules. For one–dimensional unweighted integrals with a nearly singular kernel in Love's equation [71] and for highly oscillating kernels in

[17], a “dilation” technique has been developed.

Here we describe a dilation method for weighted bivariate integrals having nearly singular kernels, highly oscillating kernels and also for their composition.

The 2D-dilation formula

Below, for the convenience of the reader, we will give the details of the computation of the coefficients $A_{h.k}(\mathbf{y},\omega)$, for a general integral of the type

$$\boldsymbol{I}(\boldsymbol{F},\omega)=\int_S \boldsymbol{F}(\mathbf{x})\boldsymbol{k}(\mathbf{x},\omega)\mathbf{w}(\mathbf{x})d\mathbf{x},\quad \omega\in\mathbb{R},\quad \boldsymbol{F}\in C_{\boldsymbol{\sigma}},$$

where $\boldsymbol{k}(\mathbf{x},\omega)$ is one of the kernels in (2.2.19).

In what follows we assume $\omega>0$. Setting $\omega_1=\omega^{\frac{1}{2}}$, by the changes of variables

$$x_1=\frac{\eta_1}{\omega_1},\quad x_2=\frac{\eta_2}{\omega_1},$$

and assuming that $\boldsymbol{\eta}=(\eta_1,\eta_2)\in[-\omega_1,\omega_1]^2$, we get

$$\boldsymbol{I}\left(\boldsymbol{F},\omega\right)\;=\;\frac{1}{\omega_1^2}\int_{[-\omega_1,\omega_1]^2}\boldsymbol{F}\left(\frac{\boldsymbol{\eta}}{\omega_1}\right)\boldsymbol{k}\left(\frac{\boldsymbol{\eta}}{\omega_1},\omega\right)\mathbf{w}\left(\frac{\boldsymbol{\eta}}{\omega_1}\right)d\boldsymbol{\eta}.$$

and choosing $d\in\mathbb{R}^+$ such that $\boldsymbol{S}=\frac{2\omega_1}{d}\in\mathbb{N}$, we have

$$\boldsymbol{I}\left(\boldsymbol{F},\omega\right)\;=\;\tau_0\sum_{i=1}^{\boldsymbol{S}}\sum_{j=1}^{\boldsymbol{S}}\int_{S_{i,j}}\boldsymbol{F}\left(\frac{\boldsymbol{\eta}}{\omega_1}\right)\boldsymbol{k}\left(\frac{\boldsymbol{\eta}}{\omega_1},\omega\right)\mathbf{w}\left(\frac{\boldsymbol{\eta}}{\omega_1}\right)d\boldsymbol{\eta},\qquad(2.2.20)$$

where $\tau_0=\frac{1}{\omega_1^2}$ and

$$\begin{aligned}&S_{i,j}:[-\omega_1+(i-1)d,-\omega_1+id]\times[-\omega_1+(j-1)d,-\omega_1+jd]\,,\\&\qquad\forall(i,j)\in N_1^{\boldsymbol{S}}\times N_1^{\boldsymbol{S}}.\end{aligned}$$

Then, by using the following invertible linear maps

$$\boldsymbol{\Psi}_{ij}:S_{i,j}\to S$$

defined as

$$\begin{aligned}\mathbf{x}&=\boldsymbol{\Psi}_{ij}(\boldsymbol{\eta})=\boldsymbol{\Psi}_{ij}(\eta_1,\eta_2)\\&:=(\Psi_i(x_1),\Psi_j(x_2))\\&=\left(\frac{2}{d}(\eta_1+\omega_1)-(2i-1),\;\frac{2}{d}(\eta_2+\omega_1)-(2j-1)\right),\end{aligned}$$

we can remap each integral into the unit square S. In fact, by making in (2.2.20) the following change of variables

$$\begin{aligned}\boldsymbol{\eta} &= \boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x}) = \boldsymbol{\Psi}_{ij}^{-1}(x_1,\, x_2) \\ &:= \left(\Psi_i^{-1}(x_1), \Psi_j^{-1}(x_2)\right) \\ &= \left(\left(\frac{x_1+1}{2}\right) d - \omega_1 + (i-1)d,\; \left(\frac{x_2+1}{2}\right) d - \omega_1 + (j-1)d\right)\end{aligned}$$

we have

$$\begin{aligned}\boldsymbol{I}(\boldsymbol{F},\omega) &= \frac{d^2\tau_0}{4}\sum_{i=1}^{\boldsymbol{S}}\sum_{j=1}^{\boldsymbol{S}}\int_S \boldsymbol{F}_{i,j}(\mathbf{x})\boldsymbol{k}_{i,j}(\mathbf{x},\omega)\mathbf{w}_{i,j}(\mathbf{x})d\mathbf{x} \\ &= \frac{d^2\tau_0}{4}\Bigg\{\tau_1\int_S \boldsymbol{F}_{1,1}(\mathbf{x})\boldsymbol{k}_{1,1}(\mathbf{x},\omega)U_1(\mathbf{x})u_2(x_1)u_4(x_2)d\mathbf{x} \\ &+ \tau_2\int_S \boldsymbol{F}_{1,\boldsymbol{S}}(\mathbf{x})\boldsymbol{k}_{1,\boldsymbol{S}}(\mathbf{x},\omega)U_2(\mathbf{x})u_2(x_1)u_3(x_2)d\mathbf{x} \\ &+ \tau_3\int_S \boldsymbol{F}_{\boldsymbol{S},1}(\mathbf{x})\boldsymbol{k}_{\boldsymbol{S},1}(\mathbf{x},\omega)U_3(\mathbf{x})u_1(x_1)u_4(x_2)d\mathbf{x} \\ &+ \tau_4\int_S \boldsymbol{F}_{\boldsymbol{S},\boldsymbol{S}}(\mathbf{x})\boldsymbol{k}_{\boldsymbol{S},\boldsymbol{S}}(\mathbf{x},\omega)U_4(\mathbf{x})u_1(x_1)u_3(x_2)d\mathbf{x} \\ &+ \tau_1\sum_{j=2}^{\boldsymbol{S}-1}\int_S \boldsymbol{F}_{1,j}(\mathbf{x})\boldsymbol{k}_{1,j}(\mathbf{x},\omega)U_{5,j}(\mathbf{x})u_2(x_1)d\mathbf{x} \\ &+ \tau_2\sum_{i=2}^{\boldsymbol{S}-1}\int_S \boldsymbol{F}_{i,\boldsymbol{S}}(\mathbf{x})\boldsymbol{k}_{i,\boldsymbol{S}}(\mathbf{x},\omega)U_{6,i}(\mathbf{x})u_3(x_2)d\mathbf{x} \\ &+ \tau_1\sum_{i=2}^{\boldsymbol{S}-1}\int_S \boldsymbol{F}_{i,1}(\mathbf{x})\boldsymbol{k}_{i,1}(\mathbf{x},\omega)U_{7,i}(\mathbf{x})u_4(x_2)d\mathbf{x} \\ &+ \tau_3\sum_{j=2}^{\boldsymbol{S}-1}\int_S \boldsymbol{F}_{\boldsymbol{S},j}(\mathbf{x})\boldsymbol{k}_{\boldsymbol{S},j}(\mathbf{x},\omega)U_{8,j}(\mathbf{x})u_1(x_1)d\mathbf{x} \\ &+ \tau_1\sum_{i=2}^{\boldsymbol{S}-1}\sum_{j=2}^{\boldsymbol{S}-1}\int_S \boldsymbol{F}_{i,j}(\mathbf{x})\boldsymbol{k}_{i,j}(\mathbf{x},\omega)U_{9,i,j}(\mathbf{x})d\mathbf{x}\Bigg\},\end{aligned}$$

where

$$\boldsymbol{F}_{i,j}(\mathbf{x}) := \boldsymbol{F}\left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega_1}\right), \quad \boldsymbol{k}_{i,j}(\mathbf{x},\omega) := \boldsymbol{k}\left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega_1},\omega\right),$$

$$\mathbf{w}_{i,j}(\mathbf{x}) := \mathbf{w}\left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega_1}\right),$$

$$U_1(\mathbf{x}) = v^{\alpha_1,0}\left(\frac{\Psi_1^{-1}(x_1)}{\omega_1}\right) v^{\alpha_2,0}\left(\frac{\Psi_1^{-1}(x_2)}{\omega_1}\right),$$

$$U_2(\mathbf{x}) = v^{\alpha_1,0}\left(\frac{\Psi_1^{-1}(x_1)}{\omega_1}\right) v^{0,\beta_2}\left(\frac{\Psi_{\boldsymbol{S}}^{-1}(x_2)}{\omega_1}\right),$$

$$U_3(\mathbf{x}) = v^{0,\beta_1}\left(\frac{\Psi_{\boldsymbol{S}}^{-1}(x_1)}{\omega_1}\right) v^{\alpha_2,0}\left(\frac{\Psi_1^{-1}(x_2)}{\omega_1}\right),$$

$$U_4(\mathbf{x}) = v^{0,\beta_1}\left(\frac{\Psi_{\boldsymbol{S}}^{-1}(x_1)}{\omega_1}\right) v^{0,\beta_2}\left(\frac{\Psi_{\boldsymbol{S}}^{-1}(x_2)}{\omega_1}\right),$$

$$U_{5,j}(\mathbf{x}) = v^{\alpha_1,0}\left(\frac{\Psi_1^{-1}(x_1)}{\omega_1}\right) w_2\left(\frac{\Psi_j^{-1}(x_2)}{\omega_1}\right),$$

$$U_{6,i}(\mathbf{x}) = v^{0,\beta_2}\left(\frac{\Psi_{\boldsymbol{S}}^{-1}(x_2)}{\omega_1}\right) w_1\left(\frac{\Psi_i^{-1}(x_1)}{\omega_1}\right),$$

$$U_{7,i}(\mathbf{x}) = v^{\alpha_2,0}\left(\frac{\Psi_1^{-1}(x_2)}{\omega_1}\right) w_1\left(\frac{\Psi_i^{-1}(x_1)}{\omega_1}\right),$$

$$U_{8,j}(\mathbf{x}) = v^{0,\beta_1}\left(\frac{\Psi_{\boldsymbol{S}}^{-1}(x_1)}{\omega_1}\right) w_2\left(\frac{\Psi_j^{-1}(x_2)}{\omega_1}\right),$$

$$U_{9,i,j}(\mathbf{x}) = w_1\left(\frac{\Psi_i^{-1}(x_1)}{\omega_1}\right) w_2\left(\frac{\Psi_j^{-1}(x_2)}{\omega_1}\right),$$

and

$$\tau_1 = \left(\frac{d}{2\omega_1}\right)^{\beta_1+\beta_2}, \tau_2 = \left(\frac{d}{2\omega_1}\right)^{\beta_1+\alpha_2}, \tau_3 = \left(\frac{d}{2\omega_1}\right)^{\alpha_1+\beta_2}, \tau_4 = \left(\frac{d}{2\omega_1}\right)^{\alpha_1+\alpha_2},$$

$$u_0 = v^{0,0}, \quad u_1 = v^{\alpha_1,0}, \quad u_2 = v^{0,\beta_1}, \quad u_3 = v^{\alpha_2,0}, \quad u_4 = v^{0,\beta_2}.$$

Then, approximating each integral by the proper Gauss-Jacobi rule depending on the couple of weight functions arising in the integral, according to the notation in (1.2.16), we get

$$\begin{aligned}\boldsymbol{I}(\boldsymbol{F},\omega) &= \frac{d^2\tau_0}{4}\Big\{\tau_1\mathcal{G}_{m,m}^{(u_2,u_4)}\left(\boldsymbol{F}_{1,1}\boldsymbol{k}_{1,1}U_1\right) + \tau_2\mathcal{G}_{m,m}^{(u_2,u_3)}\left(\boldsymbol{F}_{1,\boldsymbol{S}}\boldsymbol{k}_{1,\boldsymbol{S}}U_2\right) \\ &+ \tau_3\mathcal{G}_{m,m}^{(u_1,u_4)}\left(\boldsymbol{F}_{\boldsymbol{S},1}\boldsymbol{k}_{\boldsymbol{S},1}U_3\right) + \tau_4\mathcal{G}_{m,m}^{(u_1,u_3)}\left(\boldsymbol{F}_{\boldsymbol{S},\boldsymbol{S}}\boldsymbol{k}_{\boldsymbol{S},\boldsymbol{S}}U_4\right)\end{aligned}$$

$$
\begin{aligned}
&+ \quad \tau_1 \sum_{j=2}^{\boldsymbol{S}-1} \mathcal{G}_{m,m}^{(u_2,u_0)} \left(\boldsymbol{F}_{1,j}\boldsymbol{k}_{1,j}U_{5,j}\right) + \tau_2 \sum_{i=2}^{\boldsymbol{S}-1} \mathcal{G}_{m,m}^{(u_0,u_3)} \left(\boldsymbol{F}_{i,\boldsymbol{S}}\boldsymbol{k}_{i,\boldsymbol{S}}U_{6,i}\right) \\
&+ \quad \tau_1 \sum_{i=2}^{\boldsymbol{S}-1} \mathcal{G}_{m,m}^{(u_0,u_4)} \left(\boldsymbol{F}_{i,1}\boldsymbol{k}_{i,1}U_{7,i}\right) + \tau_3 \sum_{j=2}^{\boldsymbol{S}-1} \mathcal{G}_{m,m}^{(u_1,u_0)} \left(\boldsymbol{F}_{\boldsymbol{S},j}\boldsymbol{k}_{\boldsymbol{S},j}U_{8,j}\right) \\
&+ \quad \tau_1 \sum_{i=2}^{\boldsymbol{S}-1} \sum_{j=2}^{\boldsymbol{S}-1} \mathcal{G}_{m,m}^{(u_0,u_0)} \left(\boldsymbol{F}_{i,j}\boldsymbol{k}_{i,j}U_{9,i,j}\right) \Bigg\} \\
&=: \quad \boldsymbol{\Sigma}_m(\boldsymbol{F},\omega) + \mathcal{R}_m^{\boldsymbol{\Sigma}}(\boldsymbol{F},\omega) \qquad (2.2.21)
\end{aligned}
$$

where the cubature formula $\boldsymbol{\Sigma}_m(\boldsymbol{F},\omega)$ has been obtained by applying suitable Gauss-Jacobi cubature rules in order to evaluate the $\boldsymbol{S}^2$ integrals in (2.2.21) and

$$
\mathcal{R}_m^{\boldsymbol{\Sigma}}(\boldsymbol{F},\omega) = \boldsymbol{I}(\boldsymbol{F},\omega) - \boldsymbol{\Sigma}_m(\boldsymbol{F},\omega)
$$

is the remainder term.

Remark 2.2.6. *To compute the coefficients $A_{h,k}(\omega)$, we use suitable Gaussian cubature rules according to the different couples of Jacobi weigths appearing in the integrals in* (2.2.21)*. All the details are reported in Table 2.2.*

Table 2.2: Different couples of weigths appearing in the integrals in (2.2.21).

Squares	Couples of weigths
$S_{1,1}$	$(u_2(x_1), u_4(x_2)) = \left(v^{0,\beta_1}(x_1), v^{0,\beta_2}(x_2)\right)$
$S_{1,\boldsymbol{S}}$	$(u_2(x_1), u_3(x_2)) = \left(v^{0,\beta_1}(x_1), v^{\alpha_2,0}(x_2)\right)$
$S_{\boldsymbol{S},1}$	$(u_1(x_1), u_4(x_2)) = \left(v^{\alpha_1,0}(x_1), v^{0,\beta_2}(x_2)\right)$
$S_{\boldsymbol{S},\boldsymbol{S}}$	$(u_1(x_1), u_3(x_2)) = (v^{\alpha_1,0}(x_1), v^{\alpha_2,0}(x_2))$
$\{S_{1,j}\}_{j=2}^{\boldsymbol{S}-1}$	$(u_2(x_1), u_0(x_2)) = \left(v^{0,\beta_1}(x_1), v^{0,0}(x_2)\right)$
$\{S_{i,\boldsymbol{S}}\}_{i=2}^{\boldsymbol{S}-1}$	$(u_0(x_1), u_3(x_2)) = (v^{0,0}(x_1), v^{\alpha_2,0}(x_2))$
$\{S_{i,1}\}_{i=2}^{\boldsymbol{S}-1}$	$(u_0(x_1), u_4(x_2)) = \left(v^{0,0}(x_1), v^{0,\beta_2}(x_2)\right)$
$\{S_{\boldsymbol{S},j}\}_{j=2}^{\boldsymbol{S}-1}$	$(u_1(x_1), u_0(x_2)) = (v^{\alpha_1,0}(x_1), v^{0,0}(x_2))$
$\{S_{i,j}\}_{i,j=2}^{\boldsymbol{S}-1}$	$(u_0(x_1), u_0(x_2)) = (v^{0,0}(x_1), v^{0,0}(x_2))$

We state now a result about the stability and the convergence of the rule $\boldsymbol{\Sigma}_m(\boldsymbol{F},\omega)$ defined in (2.2.21).

Theorem 2.2.7. *Let* $\mathbf{w}$ *be defined in* (1.2.1) *and let* $\boldsymbol{k}$ *be defined in* (2.2.19) *with* $\boldsymbol{g} \in C^{\infty}(\Omega)$ *and* $\Omega \equiv [-\omega_1, \omega_1]^2$.

Then, if there exists a $\boldsymbol{\sigma}$ *as in* (1.2.2) *such that* $\boldsymbol{F} \in C_{\boldsymbol{\sigma}}$ *and the following assumption is satisfied*

$$0 \le \gamma_i < \min\{1, \alpha_i + 1\}, \quad 0 \le \delta_i < \min\{1, \beta_i + 1\}, \quad i \in \{1, 2\}, \tag{2.2.22}$$

then

$$|\boldsymbol{\Sigma}_m(\boldsymbol{F}, \omega)| \le \mathcal{C} \|\boldsymbol{F}\boldsymbol{\sigma}\|_{\infty}, \quad 0 < \mathcal{C} \ne \mathcal{C}(\boldsymbol{F}, m). \tag{2.2.23}$$

Moreover, for any $\boldsymbol{F} \in \boldsymbol{W}^r_{\boldsymbol{\sigma},\infty}$, *for* $\boldsymbol{S} \ge 2$, *we get*

$$|\mathcal{R}^{\boldsymbol{\Sigma}}_m(\boldsymbol{F}, \omega)| \le \mathcal{C}\left(\frac{d}{2}\left(\frac{1}{\omega} + 1\right)\right)^r \frac{\mathcal{N}_r(\boldsymbol{F}, \boldsymbol{k})}{m^r} \tag{2.2.24}$$

where

$$\mathcal{N}_r(\boldsymbol{F}, \boldsymbol{k}) = \|\boldsymbol{F}\boldsymbol{\sigma}\|_{\infty} + \max_{h \in N_1^2} \max_{s \in N_0^r} \left(\left\| \frac{\partial^{r-s} \boldsymbol{k}(\cdot, \omega)}{\partial x_h^{r-s}} \right\|_{\infty} \times \left\| \frac{\partial^s \boldsymbol{F}(\cdot, \omega)}{\partial x_h^s} \right\|_{\infty} \right) \tag{2.2.25}$$

and $0 < \mathcal{C} \ne \mathcal{C}(\boldsymbol{F}, m)$.

Proof. First we prove (2.2.23). Starting from expression (2.2.21), we obtain the following bound:

$$\begin{aligned}
|\boldsymbol{\Sigma}_m(\boldsymbol{F}, \omega)| \;\le\; & \frac{d^2 \tau_0}{4} \mathcal{U}_1 \max_{\mathbf{x} \in S} |\boldsymbol{F}(\mathbf{x}) \boldsymbol{k}(\mathbf{x}, \omega) \boldsymbol{\sigma}(\mathbf{x})| \left\{ \tau_1 \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_2} \lambda_s^{u_4}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_2,u_4})} \right. \\
+ \; & \tau_2 \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_2} \lambda_s^{u_3}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_2,u_3})} + \tau_3 \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_1} \lambda_s^{u_4}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_1,u_4})} \\
+ \; & \tau_4 \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_1} \lambda_s^{u_3}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_1,u_3})} + \tau_1 \sum_{j=2}^{\boldsymbol{S}-1} \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_2} \lambda_s^{u_0}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_2,u_0})} \\
+ \; & \tau_2 \sum_{i=2}^{\boldsymbol{S}-1} \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_0} \lambda_s^{u_3}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_0,u_3})} + \tau_1 \sum_{i=2}^{\boldsymbol{S}-1} \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_0} \lambda_s^{u_4}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_0,u_4})} \\
+ \; & \left. \tau_3 \sum_{j=2}^{\boldsymbol{S}-1} \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_1} \lambda_s^{u_0}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_1,u_0})} + \tau_1 \sum_{i=2}^{\boldsymbol{S}-1} \sum_{j=2}^{\boldsymbol{S}-1} \sum_{r=1}^{m} \sum_{s=1}^{m} \frac{\lambda_r^{u_0} \lambda_s^{u_0}}{\boldsymbol{\sigma}(\xi_{r,s}^{u_0,u_0})} \right\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{U}_1 \;=\; & \max \Bigg(\|U_1\|, \|U_2\|, \|U_3\|, \|U_4\|, \max_{j \in N_1^S} (\|U_{5,j}\|, \|U_{6,j}\|, \|U_{7,j}\|, \|U_{8,j}\|), \\
& \max_{i \in N_1^S, j \in N_1^S} \|U_{9,i,j}\| \Bigg).
\end{aligned}$$

Using (2.2.12), we have

$$|\boldsymbol{\Sigma}_m(\boldsymbol{F},\omega)| \leq \mathcal{C}\mathcal{U}_1\|\boldsymbol{F}\boldsymbol{k}(\cdot,\omega)\boldsymbol{\sigma}\|_\infty \left\{ \sum_{k=0,1,2} \sum_{j=0,3,4} \int_S \frac{u_k(x_1)u_j(x_2)}{\boldsymbol{\sigma}(x_1,x_2)}\, dx_1 dx_2 \right\}$$

and taking into account the assumption (2.2.22) it follows

$$|\boldsymbol{\Sigma}_m(\boldsymbol{F},\omega)| \leq \mathcal{C}\|\boldsymbol{F}\boldsymbol{\sigma}\|_\infty,$$

and therefore (2.2.23) follows.

Now we prove (2.2.24). By (2.2.21), taking into account the Proposition 1.2.4, under the assumption (2.2.22), we have

$$\begin{aligned}
|\mathcal{R}_m^{\boldsymbol{\Sigma}}(\boldsymbol{F},\omega)| \leq \mathcal{C}\Bigg\{ & E_{2m-1,2m-1}(\boldsymbol{F}_{1,1}\boldsymbol{k}_{1,1}U_1)_{\boldsymbol{\sigma}} + E_{2m-1,2m-1}(\boldsymbol{F}_{1,1}\boldsymbol{k}_{1,\boldsymbol{S}}U_2)_{\boldsymbol{\sigma}} \\
& + E_{2m-1,2m-1}(\boldsymbol{F}_{\boldsymbol{S},1}\boldsymbol{k}_{\boldsymbol{S},1}U_3)_{\boldsymbol{\sigma}} + \sum_{j=2}^{\boldsymbol{S}-1} E_{2m-1,2m-1}(\boldsymbol{F}_{1,j}\boldsymbol{k}_{1,j}U_{5,j})_{\boldsymbol{\sigma}} \\
& + \sum_{i=2}^{\boldsymbol{S}-1} E_{2m-1,2m-1}(\boldsymbol{F}_{i,\boldsymbol{S}}\boldsymbol{k}_{i,\boldsymbol{S}}U_{6,i})_{\boldsymbol{\sigma}} + \sum_{i=2}^{\boldsymbol{S}-1} E_{2m-1,2m-1}(\boldsymbol{F}_{i,1}\boldsymbol{k}_{i,1}U_{7,i})_{\boldsymbol{\sigma}} \\
& + \sum_{j=2}^{\boldsymbol{S}-1} E_{2m-1,2m-1}(\boldsymbol{F}_{\boldsymbol{S},j}\boldsymbol{k}_{\boldsymbol{S},j}U_{8,j})_{\boldsymbol{\sigma}} + \sum_{i=2}^{\boldsymbol{S}-1}\sum_{j=2}^{\boldsymbol{S}-1} E_{2m-1,2m-1}(\boldsymbol{F}_{i,j}\boldsymbol{k}_{i,j}U_{9,i,j})_{\boldsymbol{\sigma}} \\
& + E_{2m-1,2m-1}(\boldsymbol{F}_{\boldsymbol{S},\boldsymbol{S}}\boldsymbol{k}_{\boldsymbol{S},\boldsymbol{S}}U_4)_{\boldsymbol{\sigma}} \Bigg\}.
\end{aligned}$$

By inequality (1.2.5) we get

$$\begin{aligned}
|\mathcal{R}_m^{\boldsymbol{\Sigma}}(\boldsymbol{F},\omega)| \leq \mathcal{C}\Bigg\{ & \mathcal{U} \sum_{j=1}^{\boldsymbol{S}} \sum_{i=1}^{\boldsymbol{S}} E_{m-1,m-1}(\boldsymbol{F}_{i,j}\boldsymbol{k}_{i,j})_{\boldsymbol{\sigma}} \\
& + \frac{\widetilde{\mathcal{M}}_r^{max}}{m^r} \sum_{j=1}^{\boldsymbol{S}} \sum_{i=1}^{\boldsymbol{S}} \|\boldsymbol{F}_{i,j}\boldsymbol{k}_{i,j}\boldsymbol{\sigma}\|_\infty \Bigg\}
\end{aligned} \tag{2.2.26}$$

where

$$\begin{aligned}
\mathcal{U} \;=\; \max\Bigg(& \|U_1\boldsymbol{\sigma}\|, \|U_2\boldsymbol{\sigma}\|, \|U_3\boldsymbol{\sigma}\|, \|U_4\boldsymbol{\sigma}\|, \\
& \max_{j\in N_1^m}(\|U_{5,j}\boldsymbol{\sigma}\|, \|U_{6,j}\boldsymbol{\sigma}\|, \|U_{7,j}\boldsymbol{\sigma}\|, \|U_{8,j}\boldsymbol{\sigma}\|), \max_{i\in N_1^m, j\in N_1^m} \|U_{9,i,j}\boldsymbol{\sigma}\| \Bigg) \leq \mathcal{C}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathcal{M}}_r^{max} &:= \max\left\{\max_{1\le k\le 4}\mathcal{M}_r(U_k), \max_{2\le i\le \boldsymbol{S}-1}\left[\mathcal{M}_r(U_{5,i}), \mathcal{M}_r(U_{6,i}), \mathcal{M}_r(U_{7,i}),\right.\right.\\
&\qquad \left.\left.\mathcal{M}_r(U_{8,i}), \max_{2\le j\le \boldsymbol{S}-1}\mathcal{M}_r(U_{9,i,j})\right]\right\}\\
&\le \mathcal{C}\left(\frac{d}{2\omega_1}\right)^r\mathcal{U}.
\end{aligned}$$

Since for $h\in\{1,2\}$ and $(i,j)\in N_1^{\boldsymbol{S}}\times N_1^{\boldsymbol{S}}$

$$\left|\frac{\partial^r}{\partial x_h^r}\boldsymbol{F}_{i,j}(x_1,x_2)\boldsymbol{k}_{i,j}(x_1,x_2,\omega)\right| \le \sum_{s=0}^r\binom{r}{s}\left|\frac{\partial^s}{\partial x_h^s}\boldsymbol{F}_{i,j}(x_1,x_2)\right|\left|\frac{\partial^{r-s}}{\partial x_h^{r-s}}\boldsymbol{k}_{i,j}(x_1,x_2,\omega)\right|,$$

we have

$$\begin{aligned}
&\left|\frac{\partial^r}{\partial x_h^r}\boldsymbol{F}_{i,j}(x_1,x_2)\boldsymbol{k}_{i,j}(x_1,x_2,\omega)\right|\varphi_h(x_h)^r\boldsymbol{\sigma}(x_1,x_2)\\
&\le \max_{s\in N_0^r}\left\{\left\|\frac{\partial^s\boldsymbol{F}}{\partial x_h^s}\varphi_h^r\sigma\right\|_\infty\left\|\frac{\partial^{r-s}\boldsymbol{k}(\cdot,\omega)}{\partial x_h^{r-s}}\right\|_\infty\right\}\sum_{s=0}^r\binom{r}{s}\left(\frac{d}{2\omega_1}\right)^s\left(\frac{d}{2}\right)^{r-s}\\
&= \max_{s\in N_0^r}\left\{\left\|\frac{\partial^s\boldsymbol{F}}{\partial x_h^s}\varphi_h^r\boldsymbol{\sigma}\right\|_\infty\left\|\frac{\partial^{r-s}\boldsymbol{k}(\cdot,\omega)}{\partial x_h^{r-s}}\right\|_\infty\right\}\left(\frac{d}{2}\right)^r\left(\frac{1}{\omega_1}+1\right)^r,
\end{aligned}$$

and therefore, taking into account (1.2.4), by (2.2.26) it follows

$$\begin{aligned}
|\mathcal{R}^{\boldsymbol{\Sigma}}(\boldsymbol{F},\omega)| &\le \frac{\mathcal{C}}{m^r}\left\{\mathcal{U}\max_{h\in N_1^2}\max_{s\in N_0^r}\left(\left\|\frac{\partial^s\boldsymbol{F}}{\partial x_h^s}\varphi_h^r\boldsymbol{\sigma}\right\|_\infty\left\|\frac{\partial^{r-s}\boldsymbol{k}(\cdot,\omega)}{\partial x_h^{r-s}}\right\|_\infty\right)\right.\\
&\times \left.\left(\frac{d}{2}\right)^r\left(\frac{1}{\omega_1}+1\right)^r+\widetilde{\mathcal{M}}_r^{max}\|\boldsymbol{F}\boldsymbol{k}\boldsymbol{\sigma}\|_\infty\right\}\\
&\le \frac{\mathcal{C}}{m^r}\mathcal{N}_r(\boldsymbol{F},\boldsymbol{k})\left(\frac{d}{2}\right)^r\left(\frac{1}{\omega_1}+1\right)^r
\end{aligned}$$

where $\mathcal{N}_r(\boldsymbol{F},\boldsymbol{k})$ is defined in (2.2.25) and the thesis follows. □

By using the 2D-dilation formula in (2.2.21) of degree m with $\boldsymbol{F}=\ell_{h,k}^{w_1,w_2}$ where we remember that $\ell_{h,k}^{w_1,w_2}=\ell_h^{w_1}\ell_k^{w_2}$, we have

$$\begin{aligned}
A_{h,k}(\omega) &= \int_S \ell_{h,k}^{w_1,w_2}(\mathbf{x})\boldsymbol{k}_{i,j}(\mathbf{x},\omega)\boldsymbol{w}(\mathbf{x})d\mathbf{x} \qquad (2.2.27)\\
&= \boldsymbol{\Sigma}_m(\ell_{h,k}^{w_1,w_2},\omega)+\mathcal{R}^{\boldsymbol{\Sigma}}(\ell_{h,k}^{w_1,w_2},\omega).
\end{aligned}$$

About the rate of convergence of (2.2.27) we state the following

Theorem 2.2.8. *Under the hypotheses of Theorem 2.2.7, for* $m > \frac{d}{2} e^{\frac{1}{\omega_1}}$ *and for* $d \geq 2$, $\omega_1 \geq 1$, *the following error estimate holds*

$$|\mathcal{R}^{\mathbf{\Sigma}}_{m,m}(\ell^{w_1,w_2}_{h,k}, \omega)| \leq \mathcal{C}\ \mathcal{T}_{2m}(\boldsymbol{k}) \cdot \begin{cases} \dfrac{1}{m^{m+1-\mu}} & \text{if} \quad \alpha_i, \beta_i > -\dfrac{1}{2} \\ \dfrac{\log m}{m^{m+1}} & \text{if} \quad \alpha_i, \beta_i \leq -\dfrac{1}{2} \end{cases}$$

where $\mathcal{C} \neq \mathcal{C}(m, \omega)$, $i \in \{1, 2\}$ *and*

$$\mathcal{T}_{2m}(\boldsymbol{k}) = \max_{h \in N_1^2} \max_{s \in N_{m+1}^{2m}} \left\| \frac{\partial^s \boldsymbol{k}(\cdot, \omega)}{\partial x_h^s} \right\|_\infty,$$

$$\mu = \max\{\alpha_i + \frac{1}{2} - 2\gamma_i, \beta_i + \frac{1}{2} - 2\delta_i\}. \tag{2.2.28}$$

Proof. In order to use Theorem 2.2.7 with $r = 2m$, we have to estimate $\mathcal{N}_{2m}(\ell^{w_1,w_2}_{h,k}, \boldsymbol{k})$. By iterating the weighted Bernstein inequality (see for instance [47, p.170]) with $i \in \{1, 2\}$

$$\|(\ell_h^{w_i})^{(m-1)} \varphi_i^{m-1} \sigma_i\|_\infty \leq \mathcal{C}\ m^{m-1} \|\ell_h^{w_i} \sigma_i\|_\infty$$

and taking into account that under the hypotheses (2.2.6) [47, Th.4.3.3, p.274 and p.256]

$$\max_{|x| \leq 1} \sum_{s=1}^{m} |\ell_s^{w_i}(x)| \frac{\sigma_i(x)}{\sigma_i(\xi_s^{w_i})} \leq \mathcal{C} \cdot \begin{cases} m^\mu & \text{if} \quad \alpha_i, \beta_i > -\frac{1}{2} \\ \log m & \text{if} \quad \alpha_i, \beta_i \leq -\frac{1}{2} \end{cases},$$

with μ defined in (2.2.28), we can conclude, in the worst case, that

$$\|(\ell_h^{w_i})^{(m-1)} \varphi_i^{m-1} \sigma_i\|_\infty \leq \mathcal{C}\ m^{m-1} \|\ell_h^{w_i} \sigma_i\|_\infty \leq \mathcal{C}\ m^{m-1+\mu}, \quad \mathcal{C} \neq \mathcal{C}(m).$$

Hence,

$$\mathcal{N}_{2m}(\ell^{w_1,w_2}_{h,k}, \boldsymbol{k}) \leq \mathcal{C}\ m^{m-1+\mu} \max_{h \in N_1^2} \max_{s \in N_0^{m-1}} \left\| \frac{\partial^{2m-s} \boldsymbol{k}(\cdot, \omega)}{\partial x_h^{2m-s}} \right\|_\infty$$

and by (2.2.24) and using

$$\left(\frac{d}{2m}\right)^{2m} \left(\frac{1}{\omega_1} + 1\right)^{2m} \leq e^{-2m \log m \left(1 - \frac{\log(d/2)}{\log m} - \frac{1}{\omega_1 \log m}\right)} \leq \frac{1}{m^{2m}}$$

for $m > \frac{d}{4} e^{\frac{1}{\omega_1}}$, the thesis follows. □

Following the previous work-scheme, the evaluation of the coefficient $A_{h,k}$ requires $(m^2 \boldsymbol{S})^2$ long operations, with $\boldsymbol{S}$ increasing as ω increases. However, as the numerical tests will show, the implementation of the product rule for smooth integrands functions $\boldsymbol{f}$ and independently on the choice of the parameter ω, will give accurate results for "small" values of m.

2.2.3 Cases of complexity reduction

In some cases the computational complexity in computing the product rule can be drastically reduced. For all the kernel functions $\boldsymbol{k}_j$, $j \in \{2,3,4\}$, defined in (2.2.19), when $\mathbf{w}$ is a product of two Gegenbauer weights, it is possible to make the similar complexity reduction as shown in Subsection 2.1.3, mutatis mutandis. In particular:

- if $\boldsymbol{k}_j(\mathbf{x},\omega)$, $j \in \{2,3,4\}$, is *symmetric through the axes* $x_1 = 0$ *and* $x_2 = 0$, i.e.

  $$\boldsymbol{k}_j(-\mathbf{x},\omega) = \boldsymbol{k}_j(\mathbf{x},\omega), \quad j \in \{2,3,4\},$$

  and $\mathbf{w}(\mathbf{x}) = v^{\alpha_1,\alpha_1}(x_1)v^{\alpha_2,\alpha_2}(x_2)$, it is

  $$\begin{aligned} A_{h,k}(\omega) &= A_{h,m-k+1}(\omega), \quad h \in N_1^m,\ k \in N_1^M, \\ A_{h,k}(\omega) &= A_{m-h+1,k}(\omega), \quad h \in N_1^m,\ k \in N_1^M, \end{aligned}$$

  and the global computational cost (shortly CC) has a reduction of 75%. If in addition $\alpha_1 = \alpha_2$, i.e. $\mathbf{w}(\mathbf{x}) = v^{\alpha_1,\alpha_1}(x_1)v^{\alpha_1,\alpha_1}(x_2)$, since it is also

  $$A_{h,k}(\omega) = A_{h,k}(\omega), \quad (h,k) \in N_1^m \times N_1^m,$$

  a reduction of 87.5% is achieved;

- in the case $\boldsymbol{k}_j(\mathbf{x},\omega)$, $j \in \{2,3,4\}$, is *odd with respect to both the coordinate axes*, i.e.

  $$\boldsymbol{k}_j(-\mathbf{x},\omega) = -\boldsymbol{k}_j(\mathbf{x},\omega), \quad j \in \{2,3,4\},$$

  and $\mathbf{w}(\mathbf{x}) = v^{\alpha_1,\alpha_1}(x_1)v^{\alpha_2,\alpha_2}(x_2)$, it is

  $$\begin{aligned} A_{h,k}(\omega) &= -A_{h,m-k+1}(\omega), \quad h \in N_1^m,\ k \in N_1^M, \\ A_{h,k}(\omega) &= -A_{m-h+1,k}(\omega), \quad h \in N_1^m,\ k \in N_1^M, \end{aligned}$$

  and the CC has a reduction of 75%. If, in addition, $\alpha_1 = \alpha_2$, the following additional relations hold

  $$A_{h,k}(\omega) = A_{h,k}(\omega), \quad h,k \in N_1^m,$$

  and the CC has a reduction of 87,5%.

2.2.4 The choice of the parameter d

Now we want to discuss briefly how to choose the number $\boldsymbol{S}^2$ of the domain subdivisions in the 2D-dilation rule, or equivalently how to set the length d of the squares side, since $\boldsymbol{S} = \frac{2\omega_1}{d}$. By the error estimate (2.2.24), assuming negligible the contribute of $\mathcal{N}_r(\boldsymbol{F}, \boldsymbol{k})$ and fixing the desired computational accuracy *toll*, m and $\boldsymbol{S}$ are inversely proportional. Therefore, whenever let be useful to have m as small as possible, we have to take larger $\boldsymbol{S}$. We point out that this behavior depends on the slower rate of convergence of the involved Gauss-Jacobi cubature rules when the "stretching" parameter $\boldsymbol{S}$ is "too small" or d is too large.

Of course, the previous considerations are not yet conclusive on the choice of $\boldsymbol{S}$. However, by numerical evidence, a good "compromise" to reduce m seems to be $\boldsymbol{S} = \lfloor \omega_1 \rfloor$ and therefore $d = \frac{2\omega_1}{\boldsymbol{S}} \sim 2$. To show this behavior, we propose the graphic of the relative errors achieved for some values of d chosen between 2 and ω_1, referred to the first two numerical tests produced in the Section 2.3 (see Figures 2.8,2.9).

2.2.5 A comparison between product and 2D-dilation formula

In order to explain the advantage of the product integration rule with respect to the straightforward approach by 2D-dilation method, consider the following bivariate Fredholm equation

$$\boldsymbol{F}(\mathbf{y}) - \mu \int_S \boldsymbol{F}(\mathbf{x}) \boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega) \mathbf{w}(\mathbf{x}) d\mathbf{x} = \boldsymbol{g}(\mathbf{y}), \quad \mu \in \mathbb{R}, \quad \mathbf{y} = (y_1, y_2) \in S, \tag{2.2.29}$$

where $\boldsymbol{g}$ is a known function, $\boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)$ is one of the kernels in (2.2.19) appropriately readjusted, now also dependent on $\mathbf{y}$. If we approximate the solution $\boldsymbol{F}$ by a Nyström method based on the product cubature rule proposed in (2.2.2), we have to solve an m^2-system of linear equation, where m doesn't depend on ω. The situation is quite different whenever we follow a dilation procedure as well as in Subsection 2.2.2. Indeed, by mapping the integration domain into $\Omega \equiv [-\omega_1, \omega_1]^2$, by partitioning it into $\boldsymbol{S}^2$ squares $\{S_{i,j}\}_{(i,j) \in N_1^{\boldsymbol{S}} \times N_1^{\boldsymbol{S}}}$ and by the successive reduction to square S one more time, the following system of integral equations is obtained

$$\boldsymbol{F}_{h,k}(\mathbf{y}) \quad - \quad \tau \sum_{i=1}^{\boldsymbol{S}} \sum_{j=1}^{\boldsymbol{S}} \int_S \boldsymbol{F}_{i,j}(\mathbf{x}) \boldsymbol{k}_{i,j,h,k}(\mathbf{x}, \mathbf{y}, \omega) \boldsymbol{w}_{i,j}(\mathbf{x}) d\mathbf{x} = \boldsymbol{G}_{h,k}(\mathbf{y}),$$

$$(h,k) \in N_1^{\boldsymbol{S}} \times N_1^{\boldsymbol{S}},$$

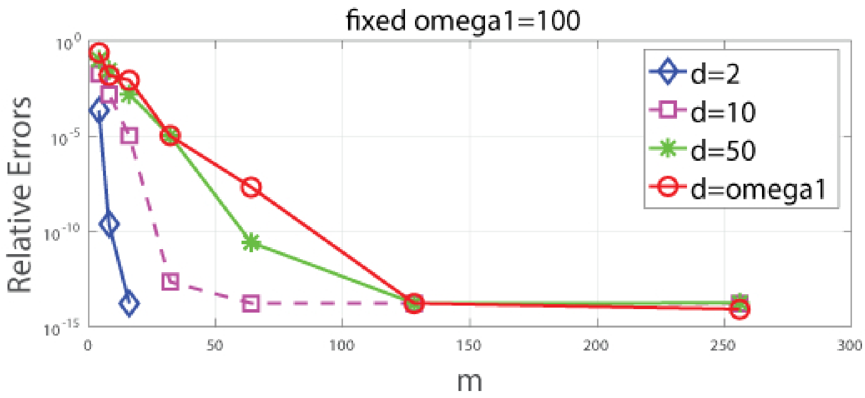

Figure 2.8: Errors behaviors for different choices of d in Example 2.3.8.

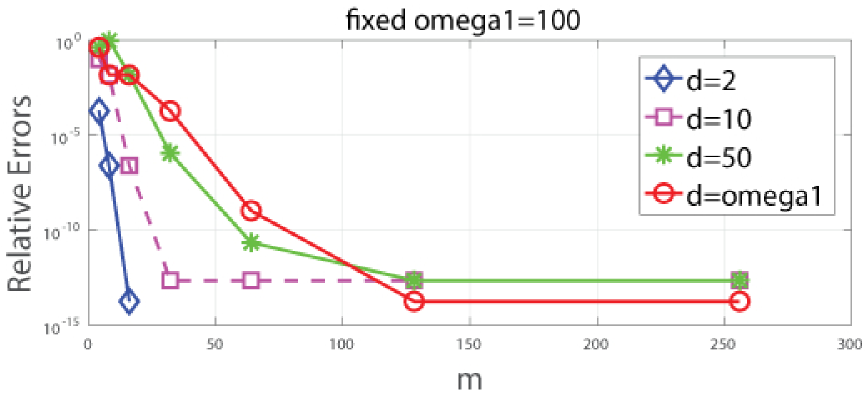

Figure 2.9: Errors behaviors for different choices of d in Example 2.3.9.

where, for $(h,k) \in N_1^{\boldsymbol{S}} \times N_1^{\boldsymbol{S}}$ and $(i,j) \in N_1^{\boldsymbol{S}} \times N_1^{\boldsymbol{S}}$,

$$\psi_\ell^{-1}(\cdot) = \left(\frac{\cdot + 1}{2}\right) d - \omega_1 + (\ell - 1)d, \quad \ell = 1, 2, \ldots, \boldsymbol{S},$$

$$\boldsymbol{F}_{h,k}(\mathbf{x}) := \boldsymbol{F}\left(\frac{\psi_h^{-1}(x_1)}{\omega_1}, \frac{\psi_k^{-1}(x_2)}{\omega_1}\right), \quad \boldsymbol{G}_{h,k}(\mathbf{y}) := \boldsymbol{g}\left(\frac{\psi_h^{-1}(y_1)}{\omega_1}, \frac{\psi_k^{-1}(y_2)}{\omega_1}\right),$$

$$\boldsymbol{k}_{i,j,h,k}(\mathbf{x}, \mathbf{y}, \omega) := \boldsymbol{k}\left(\frac{\psi_i^{-1}(x_1)}{\omega_1}, \frac{\psi_j^{-1}(x_2)}{\omega_1}, \frac{\psi_h^{-1}(y_1)}{\omega_1}, \frac{\psi_k^{-1}(y_2)}{\omega_1}, \omega\right),$$

$$\boldsymbol{w}_{i,j}(\mathbf{x}) := w_1\left(\frac{\psi_i^{-1}(x_1)}{\omega_1}\right) w_2\left(\frac{\psi_j^{-1}(x_2)}{\omega_1}\right), \quad \tau = \frac{d^2\mu}{4\omega_1^2}.$$

The system of $\boldsymbol{S}^2$ integral equations is equivalent to the equation (2.2.29). Hence, if we use a Nyström method based on 2D-dilation formula, a system of linear equation of order $(\boldsymbol{S}m)^2$ is generated, where $\boldsymbol{S}$ increases with increasing ω. We point out that it is required the collectively compactness of the involved sequences of discrete operators to assure the convergence of both methods.

On the other hand, as we will see in Chapter 3 in a particular case (Love's integral equation), applying the product cubature formula proposed in (2.2.2) and computing the coefficients as done in Subsection 2.2.2, we obtain a classical Nyström method and solve only one finite dimensional equation instead of a system of finite dimensional equations (in the one-dimensional case see [71]).

We conclude by proposing a comparison between the proposed rules with respect to the time complexity. The following Tables 2.3, 2.4 contain the computational times (in seconds) obtained by implementing the *product rule* $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$ and the *2D-dilation rule* $\boldsymbol{\Sigma}_m(\boldsymbol{f}, \mathbf{y}, \omega)$ for the integrals given in Examples 2.3.8 and 2.3.9 of the next Section 2.3. For each of them the times have been computed for $\omega = 10^2, 10^4, 10^6$, by implementing both the algorithms in Matlab version R2016a, on a PC with a Intel Core i7-2600 CPU 3.40GHz and 8GB of memory. We point out that times related to the product formula include those spent for computing the coefficients $\{A_{h,k}\}_{(h,k)\in N_1^m \times N_1^m}$.

Table 2.3: Times for $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$ and $\boldsymbol{\Sigma}_m(\boldsymbol{f}, \mathbf{y}, \omega)$ in Example 2.3.8.

	$\omega = 10^2$		$\omega = 10^4$		$\omega = 10^6$	
m	$\boldsymbol{I}_m$	$\boldsymbol{\Sigma}_m$	$\boldsymbol{I}_m$	$\boldsymbol{\Sigma}_m$	$\boldsymbol{I}_m$	$\boldsymbol{\Sigma}_m$
4	0.016	0.015	0.85	0.63	76.19	65.44
8	0.023	0.020	1.22	0.81	120.74	89.66
16	0.036	0.029	2.22	1.98	220.61	167.47
32	0.055	0.047	4.49	2.96	451.06	295.76

Table 2.4: Times for $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$ and $\boldsymbol{\Sigma}_m(\boldsymbol{f}, \mathbf{y}, \omega)$ in Example 2.3.9.

	$\omega = 10^2$		$\omega = 10^4$		$\omega = 10^6$	
m	$\boldsymbol{I}_m$	$\boldsymbol{\Sigma}_m$	$\boldsymbol{I}_m$	$\boldsymbol{\Sigma}_m$	$\boldsymbol{I}_m$	$\boldsymbol{\Sigma}_m$
4	0.023	0.018	0.852	0.669	79.33	65.08
8	0.025	0.024	1.208	1.124	123.81	109.97
16	0.036	0.033	2.219	2.155	229.61	217.20
32	0.068	0.072	4.449	4.630	458.91	461.75

As one can see, the timings required by the product rule are a little bit longer, but not too much, than those required by the 2D-dilation formula, till m is small. However, in the Example 2.3.8, with $m = 32$ and for all the values of ω, the timings required by the product rule are a little bit smaller than those required by the 2D-dilation rule . Indeed, 2D-dilation formula requires $(m\boldsymbol{S})^2$ samples of the integrand function $\boldsymbol{f}$, where $\boldsymbol{S}$ increases on ω. Thus the global time strongly depend on the computing time of the function. In Example 2.3.9 the time complexity for evaluating $\boldsymbol{f}(\mathbf{x}) = \log^{\frac{15}{2}}(x_1 + x_2 + 4)$ is longer than the time for computing $\boldsymbol{f}(\mathbf{x}) = e^{x_1 x_2}$ in Example 2.3.8. This variability cannot happen in the product rule, where the number m of function samples is independent of ω. In any case, since in the product rule the main effort is mainly due to the computation of its coefficients, it should be preferable to use it when the kernels present some symmetry properties, by virtue of them, the number of the coefficients is drastically reduced (see Subsection 2.2.3).

2.3 Numerical Tests

Before concluding this Chapter, we present some examples to test the quadrature and cubature rules proposed in Sections 2.1 and 2.2, for different choices of kernel functions.

We point out that all the computations were performed in double-machine precision ($eps \approx 2.22044e - 16$) and in the tables the symbol "–"will mean that machine precision has been achieved.

Univariate case

For the univariate kernel functions presented in (2.1.7) and (2.1.10), we compare our results with those obtained by other methods. To be more precise, we approximate each integral by the product quadrature rule $I_m(f, y)$ presented in (2.1.1), for increasing values of m, choosing three different values of y or ω and computing the coefficients via 1D-dilation rule. In each example we state also the numerical results obtained by the univariate Gauss-Jacobi quadrature rule (shortly 1D-GJ-rule). Furthermore, for the kernel functions presented in (2.1.10), we report also the results achieved by the straightforward application of the 1D-dilation rule $\Sigma_m(f, \omega)$ (shortly 1D-d-rule).

Example 2.3.1. *Let us consider the integral*

$$I(f, y) = \int_{-1}^{1} e^x \, |x - y|^{-\frac{1}{3}} \; v^{\frac{1}{2},0}(x) \, dx$$

where

$$f(x) = e^x, \quad k_1(x,y) = |x-y|^{-\frac{1}{3}}, \quad \lambda = -\frac{1}{3}, \quad w(x) = v^{\frac{1}{2},0}(x).$$

The integral contains a weakly singular-type kernel and the function $f \in \mathcal{W}^r_{\sigma,\infty}$ *for any* $r \geq 1$*, with* $\sigma = 1$*. In Table 2.5, the results obtained by implementing the product quadrature rule presented in* (2.1.1)*, show that the machine precision is attained at* $m = 16$ *for any choice of* y*. As we can expect, by using the 1D-GJ-rule, we have poor results (Table 2.6).*

Table 2.5: Example 2.3.1: results by the product rule $I_m(f,y)$.

m	$y = \frac{3}{20}$	$y = \frac{17}{60}$	$y = \frac{1}{10}$
4	$2.808e+00$	$2.828e+00$	$2.7946e+00$
8	$2.80894332e+00$	$2.8282151e+00$	$2.79564352e+00$
16	$2.80894332230038e+00$	$2.828215104216676e+00$	$2.79564352017323e+00$

Table 2.6: Example 2.3.1: results by 1D-GJ-rule.

m	$y = \frac{3}{20}$	$y = \frac{17}{60}$	$y = \frac{1}{10}$
4	$2.7e+00$	$3.3e+00$	$2.5e+00$
8	$3.3e+00$	$2.5e+00$	$2.9e+00$
16	$2.6e+00$	$2.8e+00$	$2.7e+00$
32	$2.7e+00$	$2.7e+00$	$2.7e+00$
64	$2.7e+00$	$2.7e+00$	$2.7e+00$
128	$2.7e+00$	$2.7e+00$	$2.7e+00$
256	$2.8e+00$	$2.8e+00$	$2.7e+00$
512	$2.80e+00$	$2.8e+00$	$2.8e+00$
1024	$2.80e+00$	$2.81e+00$	$2.7e+00$

Example 2.3.2. *Let us consider the integral*

$$I(f,\omega) = \int_{-1}^{1} \frac{\log(x+121)}{x^2+\omega^{-1}} v^{\frac{1}{4},\frac{1}{4}}(x)\, dx$$

where

$$f(x) = \log(x+121), \quad k_2(x,\omega) = \frac{1}{x^2+\omega^{-1}}, \quad \lambda = 1, \quad w(x) = v^{\frac{1}{4},\frac{1}{4}}(x).$$

The integral contains a nearly singular-type kernel and the function $f \in \mathcal{W}^r_{\sigma,\infty}$ *for any* $r \geq 1$*, with* $\sigma = 1$*. In Table 2.7, the results obtained by implementing the product quadrature rule presented in* (2.1.1)*, show that the machine precision is attained at* $m = 16$ *for any choice of* ω*. A similar*

Table 2.7: Example 2.3.2: results by the product rule $I_m(f, y)$.

m	$\omega = 10^2,\ \boldsymbol{S} = 10^1$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$1.49e+03$	$1.506e+05$	$1.506e+07$
8	$1.4941049e+03$	$1.5065162e+05$	$1.5066407e+07$
16	$1.49410501136664e+03$	$1.50651628940186e+05$	$1.50664077712722e+07$

Table 2.8: Example 2.3.2: results by 1D-d-rule.

n	$\omega = 10^2,\ \boldsymbol{S} = 10^1$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$1.49e+03$	$1.506e+05$	$1.506e+07$
8	$1.49410e+03$	$1.5065162e+05$	$1.5066407e+07$
16	$1.49410501136664e+03$	$1.50651628940186e+05$	$1.50664077712722e+07$

Table 2.9: Example 2.3.2: results by 1D-GJ-rule.

m	$\omega = 10^2$	$\omega = 10^4$	$\omega = 10^6$
4	$5.5e+01$	$5.5e+01$	$5.5e+01$
8	$1.1e+02$	$1.1e+02$	$1.1e+02$
16	$2.3e+02$	$2.3e+02$	$2.3e+02$
32	$4.6e+02$	$4.7e+02$	$4.7e+02$
64	$8.4e+02$	$9.5e+02$	$9.5e+02$
128	$1.2e+03$	$1.9e+03$	$1.9e+03$
256	$1.4e+03$	$3.8e+03$	$3.8e+03$
512	$1.4939e+03$	$7.7e+03$	$7.7e+03$
1024	$1.4941050e+03$	$1.5e+04$	$1.5e+04$

behavior presents the 1D-d-rule presented in (2.1.12)*, whose results are set in Table 2.8. Finally, as we can expect, by using the 1D-GJ-rule, as ω increases a progressive loss of precision is detected, until results become very poor (Table 2.9).*

Example 2.3.3. *Let us consider the integral*

$$I(f, \omega) = \int_{-1}^{1} \sinh(x) \sin(\omega x)\, dx$$

where

$$f(x) = \sinh(x), \quad k_3(x, \omega) = \sin(\omega x), \quad w(x) = v^{0,0}(x).$$

The integral contains a highly oscillating-type kernel and the function $f \in \mathcal{W}^r_{\sigma,\infty}$ for any $r \geq 1$, with $\sigma = 1$. In Table 2.10, the results obtained by implementing the product quadrature rule presented in (2.1.1)*, show that the machine precision is attained at $m = 16$ for any choice of ω. A similar behavior presents the 1D-d-rule presented in* (2.1.12)*, whose results are set in Table 2.11. Finally, as we can expect, by using the 1D-GJ-rule, as ω increases*

Table 2.10: Example 2.3.3: results by the product rule $I_m(f,y)$.

n	$\omega=10^2,\ \boldsymbol{S}=10^1$	$\omega=10^4,\ \boldsymbol{S}=10^2$	$\omega=10^6,\ \boldsymbol{S}=10^3$
4	$-2.04e-02$	$2.23e-04$	$-2.19e-06$
8	$-2.042219e-02$	$2.237853e-04$	$-2.201745e-06$
16	$-2.04221937438933e-02$	$2.23785391071713e-04$	$-2.20174551699787e-06$

Table 2.11: Example 2.3.3: results by 1D-d-rule.

n	$\omega=10^2,\ \boldsymbol{S}=10^1$	$\omega=10^4,\ \boldsymbol{S}=10^2$	$\omega=10^6,\ \boldsymbol{S}=10^3$
4	$-2.042219e+02$	$2.237853e+04$	$-2.201745e-06$
8	$-2.04221937438933e+02$	$2.237853910717e+04$	$-2.201745516997e-06$
16	$-2.04221937438933e+02$	$2.23785391071713e+04$	$-2.20174551699787e-06$

Table 2.12: Example 2.3.3: results by 1D-GJ-rule.

m	$\omega=10^2$	$\omega=10^4$	$\omega=10^6$
4	$-4.0e-01$	$8.5e-02$	$1.2e-02$
8	$6.10-02$	$1.6e-01$	$2.0e-01$
16	$-1.9e-01$	$3.1e-01$	$-6.8e-02$
32	$-1.0e-01$	$2.3e-02$	$-2.3e-01$
64	$-2.0422e-02$	$1.2e-01$	$1.3e-01$
128	$-2.04221937438e-02$	$-2.9e-02$	$-3.8e-02$
256	$-2.04221937438e-02$	$-4.9e-02$	$2.1e-02$
512	$-2.04221937438e-02$	$-4.9e-02$	$-6.0e-03$
1024	$-2.042219374389e-02$	$-3.2e-02$	$5.1e-02$

a progressive loss of precision is detected, until results become completely wrong (Table 2.12).

Example 2.3.4. *Let us consider the integral*

$$I(f,\omega)=\int_{-1}^{1}\cosh(x)\cos(\omega x)\,dx$$

where

$$f(x)=\cosh(x),\quad k_3(x,\omega)=\cos(\omega x),\quad w(x)=v^{0,0}(x).$$

Also in this case, the integral contains a highly oscillating-type kernel and the function $f\in\mathcal{W}^r_{\sigma,\infty}$ *for any* $r\geq 1$*, with* $\sigma=1$*. In Table 2.13, the results obtained by implementing the product quadrature rule presented in* (2.1.1)*, show that the machine precision is attained at* $m=16$ *for any choice of* ω*. A similar behavior presents the 1D-d-rule presented in* (2.1.12)*, whose results are set in Table 2.14. Finally, as we can expect, by using the 1D-GJ-rule, as* ω *increases a progressive loss of precision is detected, until results become completely wrong (Table 2.15).*

Table 2.13: Example 2.3.4: results by the product rule $I_m(f, y)$.

n	$\omega = 10^2,\ \boldsymbol{S} = 10^1$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$-1.54e-02$	$-9.4e-05$	$-1.0e-06$
8	$-1.542303e-02$	$-9.43399e-05$	$-1.08013e-06$
16	$-1.54230383612065e-02$	$-9.43399075819722e-05$	$-1.08013418928421e-06$

Table 2.14: Example 2.3.4: results by 1D-d-rule.

n	$\omega = 10^2,\ \boldsymbol{S} = 10^1$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$-1.542303e+02$	$-9.4339e+03$	$-1.0801e-06$
8	$-1.54230383612065e+02$	$-9.43399075819e+03$	$-1.08013e-06$
16	$-1.54230383612065e+02$	$-9.43399075819722e+03$	$-1.08013418928421e-06$

Table 2.15: Example 2.3.4: results by 1D-GJ-rule.

m	$\omega = 10^2$	$\omega = 10^4$	$\omega = 10^6$
4	$-1.4e+00$	$1.9e-01$	$6.9e-02$
8	$-1.4e-01$	$4.5e-01$	$-3.7e-01$
16	$-1.2e-01$	$7.5e-02$	$-6.4e-01$
32	$-4.8e-01$	$-7.5e-02$	$-1.4e-02$
64	$-1.5423e-02$	$-1.3e-01$	$1.8e-01$
128	$-1.5423038361e-02$	$-1.4e-02$	$-2.7e-01$
256	$-1.54230383612e-02$	$-1.2e-01$	$7.7e-03$
512	$-1.54230383612e-02$	$4.5e-02$	$9.7e-02$
1024	$-1.54230383612e-02$	$-1.4e-03$	$-5.5e-03$

Example 2.3.5. *Let us consider the integral*

$$I(f, \omega) = \int_{-1}^{1} e^x \frac{\sin(\omega x)}{x^2 + \omega^{-1}}\, dx$$

where

$$f(x) = e^x, \quad k_4(x, \omega) = k_2(x, \omega) k_3(x, \omega) = \frac{\sin(\omega x)}{x^2 + \omega^{-1}}, \quad w(x) = v^{0,0}(x).$$

In this case, the integral contains a mixed-type kernel (i.e. a nearly singular and highly oscillating) and the function $f \in \mathcal{W}^r_{\sigma,\infty}$ *for any* $r \geq 1$, *with* $\sigma = 1$. *In Table 2.16, the results obtained by implementing the product quadrature rule presented in* (2.1.1), *show that the machine precision is attained at* $m = 16$ *for any choice of* ω. *The 1D-d-rule presented in* (2.1.12) *obtained the machine precision at* $m = 32$, *whose results are set in Table 2.17. Finally, as we can expect, by using the 1D-GJ-rule, as* ω *increases a progressive loss of precision is detected, until results become completely wrong (Table 2.18).*

Table 2.16: Example 2.3.5: results by the product rule $I_m(f, y)$.

n	$\omega = 10^2,\ \boldsymbol{S} = 10^1$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$1.135e+00$	$1.15e+00$	$1.155e+00$
8	$1.135527e+00$	$1.1559511e+00$	$1.155725e+00$
16	$1.13552735537749e+00$	$1.15595114761890e+00$	$1.15572510946973e+00$

Table 2.17: Example 2.3.5: results by 1D-d-rule.

n	$\omega = 10^2,\ \boldsymbol{S} = 10^1$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$1.13e+00$	$1.15e+00$	$1.15e+00$
8	$1.135527e+00$	$1.155951e+00$	$1.155725e+00$
16	$1.135527355377e+00$	$1.155951147618e+00$	$1.155725109469e+00$
32	$1.13552735537749e+00$	$1.15595114761890e+00$	$1.15572510946973e+00$

Table 2.18: Example 2.3.5: results by 1D-GJ-rule.

m	$\omega = 10^2$	$\omega = 10^4$	$\omega = 10^6$
4	$1.1e+00$	$1.9e+00$	$-2.8e+00$
8	$-1.3e+00$	$-7.2e-01$	$2.8e-02$
16	$-8.3e-02$	$5.4e+00$	$-4.8e+00$
32	$-3.1e+00$	$-3.4e+00$	$1.6e+00$
64	$2.7e+00$	$-1.4e+00$	$2.7e+00$
128	$1.6e+00$	$2.8e+00$	$3.7e+00$
256	$1.1e+00$	$-3.1e+00$	$-3.3e+00$
512	$1.135e+00$	$-3.3e+00$	$-3.8e+00$
1024	$1.1355273e+00$	$3.0e+00$	$2.7e+00$

Bivariate case

For the bivariate kernel functions presented in (2.2.16), (2.2.17) and (2.2.19), we compare our results with those obtained by other methods. To be more precise, we approximate each integral by the product cubature rule $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$ presented in (2.2.2), for increasing values of m, choosing three different values of $\mathbf{y} = (y_1, y_2)$ or ω and computing the coefficients via 2D-dilation rule. In each example we state also the numerical results obtained by the bivariate Gauss-Jacobi cubature rule (shortly 2D-GJ-rule) and those achieved by the straightforward application of the 2D-dilation rule $\boldsymbol{\Sigma}_m(\boldsymbol{f}, \omega)$ (shortly 2D-d-rule). About the first two tests involving nearly singular kernels $\boldsymbol{k}_2(\cdot, \omega)$, we provide also the results obtained by the iterated sinh transformation proposed by Johnston & Johnston & Elliott in [31] (shortly JJE-method). The integrals in Examples 2.3.10 and 2.3.11 involve oscillatory kernels of the type $\boldsymbol{k}_3(\cdot, \omega)$. In Example 2.3.10 our results are compared with those achieved by the method proposed by Huybrechs & Vandevalle in [30] (shortly HV-method), since the function $\boldsymbol{f}$ satisfies their assumptions of convergence. The last two tests involve a mixed-type kernels and for them we compare our results with those achieved by the JJE-method related to the kernel $\boldsymbol{k}_2(\cdot,\omega)$ with the function $\boldsymbol{f}$ replaced by $\boldsymbol{f}\boldsymbol{k}_3(\cdot,\omega)$.

Example 2.3.6. *Let us consider the integral*

$$\boldsymbol{I}(\boldsymbol{f},\mathbf{y}) = \int_S \log(x_1x_2^2{+}121)\,|x_1-y_1|^{-\frac{1}{2}}\,|x_2-y_2|^{\frac{1}{5}}\; v^{\frac{1}{3},-\frac{1}{2}}(x_1)\,v^{-\frac{3}{4},\frac{3}{10}}(x_2)\,dx_1\,dx_2$$

where

$$\boldsymbol{f}(\mathbf{x}) = \log(x_1x_2^2+121), \quad \boldsymbol{k}_1(\mathbf{x},\mathbf{y}) = |x_1-y_1|^{-\frac{1}{2}}\,|x_2-y_2|^{\frac{1}{5}},$$

$$\mathbf{w} = w_1w_2, \quad w_1(x_1) = v^{\frac{1}{3},-\frac{1}{2}}(x_1), \quad w_2(x_2) = v^{-\frac{3}{4},\frac{3}{10}}(x_2).$$

The integral contains a weakly-type kernel and the function $\boldsymbol{f} \in W^r_{\boldsymbol{\sigma},\infty} \forall r$, with $\boldsymbol{\sigma} = \sigma_1\sigma_2$ and $\sigma_1 = \sigma_2 = 1$. As you can see in Table 2.19 applying the product cubature rule proposed in (2.2.2), *for each choice of parameters $\mathbf{y} = (y_1, y_2)$, we get the machine precision in double arithmetic for $m = 16$. As you can see in table 2.20, the 2D-GJ-rule cannot give a reasonable numerical approximation. In fact, for each choice of the couple of values $\mathbf{y}$, the 2D-GJ-rule takes only the first digit. This fact is confirmed by the numerical test in double arithmetics shown in Table 2.20.*

Table 2.19: Example 2.3.6: results by product rule $\boldsymbol{I}_m(\boldsymbol{f},\mathbf{y})$.

m	$\mathbf{y} = (-\frac{1}{2}, \frac{1}{5})$	$\mathbf{y} = (\frac{1}{10}, \frac{1}{3})$	$\mathbf{y} = (-\frac{3}{10}, -\frac{1}{9})$
4	$1.9022744e+02$	$1.7218204e+02$	$1.5148342e+02$
8	$1.902274466223e+02$	$1.721820496209e+02$	$1.5148342305516e+02$
16	$1.902274466223350e+02$	$1.721820496209833e+02$	$1.514834230551673e+02$

Table 2.20: Example 2.3.6: results by 2D-GJ-rule.

m	$\mathbf{y} = (-\frac{1}{2}, \frac{1}{5})$	$\mathbf{y} = (\frac{1}{10}, \frac{1}{3})$	$\mathbf{y} = (-\frac{3}{10}, -\frac{1}{9})$
32	$1.3e+02$	$1.0e+02$	$1.2e+02$
64	$1.6e+02$	$1.0e+02$	$1.3e+02$
128	$1.4e+02$	$1.0e+02$	$1.3e+02$
256	$1.5e+02$	$1.0e+02$	$1.4e+02$
512	$1.4e+02$	$1.1e+02$	$1.5e+02$
1024	$1.5e+02$	$1.0e+02$	$1.3e+02$

Example 2.3.7. *Let us consider the integral*

$$\boldsymbol{I}(\boldsymbol{f},\mathbf{y}) = \int_S e^{x_1x_2}(x_1x_2^2{+}1)\;|y_1-x_1|^{-\frac{1}{4}}\,|y_2-x_1|^{-\frac{1}{2}}\;v^{\frac{3}{2},\frac{1}{2}}(x_1)\,v^{-\frac{1}{10},\frac{4}{5}}(x_2)\,dx_1\,dx_2$$

where

$$\boldsymbol{f}(\mathbf{x}) = e^{x_1x_2}(x_1x_2^2+1), \quad \mathfrak{k}_1(x_1,\mathbf{y}) = |y_1-x_1|^{-\frac{1}{4}}\,|y_2-x_1|^{-\frac{1}{2}},$$

$$\mathbf{w} = w_1w_2, \quad w_1(x_1) = v^{\frac{3}{2},\frac{1}{2}}(x_1), \quad w_2(x_2) = v^{-\frac{1}{10},\frac{4}{5}}(x_2).$$

The integral contains a special weakly-type kernel and the function $\boldsymbol{f} \in W^r_{\boldsymbol{\sigma},\infty}\forall r$*, with* $\boldsymbol{\sigma} = \sigma_1\sigma_2$ *and* $\sigma_1 = \sigma_2 = 1$*. As you can see in Table 2.21 applying the cubature rule proposed in* (2.2.2)*, for each choice of parameters* $\mathbf{y} = (y_1, y_2)$*, we get the machine precision in double arithmetic for* $m = 16$*. As you can see in table 2.22, also for this example, the 2D-GJ-rule cannot give a reasonable numerical approximation. This fact is confirmed by the numerical test in double arithmetics shown in Table 2.22.*

Table 2.21: Example 2.3.7: results by the product rule $\boldsymbol{I}_m(\boldsymbol{f},\mathbf{y})$.

m	$\mathbf{y} = (\frac{1}{2}, \frac{5}{6})$	$\mathbf{y} = (-\frac{1}{2}, \frac{1}{5})$	$\mathbf{y} = (-\frac{1}{7}, \frac{1}{3})$
4	$5.17429e+00$	$8.28350e+00$	$8.45589e+00$
8	$5.174290263e+00$	$8.28350210e+00$	$8.455891224e+00$
16	$5.174290263908625e+00$	$8.283502100542043e+00$	$8.455891224593199e+00$

Table 2.22: Example 2.3.7: results by 2D-GJ-rule.

m	$\mathbf{y} = (\frac{1}{2}, \frac{5}{6})$	$\mathbf{y} = (-\frac{1}{2}, \frac{1}{5})$	$\mathbf{y} = (-\frac{1}{7}, \frac{1}{3})$
32	$5.0e+00$	$8.3e+00$	$7.7e+00$
64	$5.0e+00$	$8.8e+00$	$8.2e+00$
128	$5.0e+00$	$8.7e+00$	$8.1e+00$
256	$5.1e+00$	$8.0e+00$	$1.0e+01$
512	$5.1e+00$	$8.1e+00$	$8.2e+00$
1024	$5.1e+00$	$8.1e+00$	$8.3e+00$

Example 2.3.8. *Let us consider the integral*

$$\boldsymbol{I}(\boldsymbol{f},\omega) = \int_S \frac{e^{x_1x_2}}{x_1^2+x_2^2+\omega^{-1}}\,dx_1\,dx_2$$

where

$$\boldsymbol{f}(\mathbf{x}) = e^{x_1x_2}, \quad \boldsymbol{k}_2(\mathbf{x},\omega) = \frac{1}{x_1^2+x_2^2+\omega^{-1}}, \quad \lambda = 1,$$

$$\mathbf{w} = w_1w_2, \quad w_1(x_1) = v^{0,0}(x_1), \quad w_2(x_2) = v^{0,0}(x_2).$$

In this case, the integral contains a nearly-type kernel and the function $\boldsymbol{f} \in W^r_{\boldsymbol{\sigma},\infty}$ *for any* $r \geq 1$*, with* $\boldsymbol{\sigma} = \sigma_1\sigma_2$ *and* $\sigma_1 = \sigma_2 = 1$*. In Table 2.23 the results obtained by implementing the product rule presented in* (2.2.2) *show that the machine precision is attained at* $m = 16$ *for any choice of* ω. *A similar behavior presents the 2D-d-rule presented in* (2.2.21), *whose results are set in Table 2.24. Also the JJE-method (Table 2.25) fastly converges, achieving almost satisfactory results, even if it is required the use of the Gauss-Laguerre cubature rule of order* $m = 1024$ *in order to obtain* 13 *digits. Finally, as we can expect, by using the 2D-GJ-rule, as* ω *increases a progressive loss of precision is detected, until results become completely wrong (Table 2.26).*

Table 2.23: Example 2.3.8: results by the product rule $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$1.540e+01$	$2.984e+01$	$4.43e+01$
8	$1.54013067e+01$	$2.984630059e+01$	$4.43136435e+01$
16	$1.54013067981755e+01$	$2.98463005967465e+01$	$4.43136435598934e+01$

Table 2.24: Example 2.3.8: results by 2D-d-rule.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$1.5e+1$	$2.984e+1$	$4.431e+1$
8	$1.54013067e+1$	$2.98463005e+1$	$4.43136435e+1$
16	$1.5401306798175e+1$	$2.9846300596746e+1$	$4.4313643559893e+1$
32	$1.5401306798175e+1$	$2.9846300596746e+1$	$4.43136435598934e+1$

Table 2.25: Example 2.3.8: results by JJE-method.

m	$\omega = 10^2$	$\omega = 10^4$	$\omega = 10^6$
4	$1.5e+1$	$3.5e+1$	$6.5e+1$
8	$1.540e+1$	$3.0e+1$	$4.7e+1$
16	$1.540130e+1$	$2.984e+1$	$4.44e+1$
32	$1.540130679817e+1$	$2.9846300e+1$	$4.4313e+1$
64	$1.540130679817e+1$	$2.984630059674e+1$	$4.43136435e+1$
128	$1.540130679817e+1$	$2.984630059674e+1$	$4.431364355989e+1$
256	$1.540130679817e+1$	$2.984630059674e+1$	$4.431364355989e+1$
512	$1.540130679817e+1$	$2.984630059674e+1$	$4.431364355989e+1$
1024	$1.540130679817e+1$	$2.98463005967465e+1$	$4.431364355989e+1$

Example 2.3.9. *Let us consider the integral*

$$\boldsymbol{I}(\boldsymbol{f}, \omega) = \int_S \frac{\log^{\frac{15}{2}}(x_1 + x_2 + 4)}{x_1^2 + x_2^2 + \omega^{-1}} v^{\frac{1}{2},\frac{1}{2}}(x_1) v^{\frac{1}{2},\frac{1}{2}}(x_2)\, dx_1\, dx_2$$

where

$$\boldsymbol{f}(\mathbf{x}) = \log^{\frac{15}{2}}(x_1 + x_2 + 4), \quad \boldsymbol{k}_2(\mathbf{x}, \omega) = \frac{1}{x_1^2 + x_2^2 + \omega^{-1}}, \quad \lambda = 1,$$

Table 2.26: Example 2.3.8: results by 2D-GJ-rule.

m	$\omega = 10^2$	$\omega = 10^4$	$\omega = 10^6$
16	$1.49e+1$	$1.82e+1$	$1.82e+1$
32	$1.53e+1$	$2.23e+1$	$2.25e+1$
64	$1.5401e+1$	$2.61e+1$	$2.68e+1$
128	$1.54013067981e+1$	$2.88e+1$	$3.11e+1$
256	$1.5401306798175e+1$	$2.97e+1$	$3.53e+1$
512	$1.54013067981755e+1$	$2.98e+1$	$3.94e+1$

$$\mathbf{w} = w_1 w_2, \quad w_1(x_1) = v^{\frac{1}{2},\frac{1}{2}}(x_1), \quad w_2(x_2) = v^{\frac{1}{2},\frac{1}{2}}(x_2).$$

Also in this case the integral contains a nearly-type kernel and the function $\boldsymbol{f} \in W^r_{\boldsymbol{\sigma},\infty}$ *for any* $r \geq 1$*, with* $\boldsymbol{\sigma} = \sigma_1\sigma_2$ *and* $\sigma_1 = \sigma_2 = v^{\frac{1}{4},\frac{1}{4}}$*. In Table 2.27 the results obtained by implementing the product rule proposed in* (2.2.2) *show that the machine precision is attained for* $m = 32$*. In this case the value of the seminorm growth faster than the previous example. For instance,* $\mathcal{M}_{10}(\boldsymbol{f}) \sim 2.5 \times 10^4$*. A similar behavior presents the 2D-d-rule presented in* (2.2.21)*, whose results are given in Table 2.28. In this case the JJE-method in Table 2.29 converges lower than the previous example, achieving* $8-9$ *exact digits. In this case the changes of variables are applied to the whole integrand, including two Chebyshev weights, and this explains this bad behavior. Similar to the previous test, by the 2D-GJ-rule a progressive loss of precision occurs as* ω *increases, till* $\omega = 10^6$ *for which the values are completely wrong (Table 2.30).*

Table 2.27: Example 2.3.9: results by the product rule $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$1.677e+2$	$3.35e+2$	$5.02e+2$
8	$1.6772623e+2$	$3.350653e+2$	$5.026790e+2$
16	$1.67726234163080e+2$	$3.3506538134727e+2$	$5.0267905399542e+2$
32	$-$	$3.35065381347276e+2$	$5.02679053995422e+2$

Table 2.28: Example 2.3.9: results by 2D-d-rule.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$	$\omega = 10^6,\ \boldsymbol{S} = 10^3$
4	$1.677e+2$	$3.350e+2$	$5.026e+2$
8	$1.6772623e+2$	$3.35065381e+2$	$5.02679053e+2$
16	$1.67726234163080e+2$	$3.3506538134727e+2$	$5.026790539954e+2$
32	$1.67726234163080e+2$	$3.3506538134727e+2$	$5.026790539954e+2$

Table 2.29: Example 2.3.9: results by JJE-method.

m	$\omega = 10^2$	$\omega = 10^4$	$\omega = 10^6$
16	$1.677e+2$	$3.35e+2$	$5.03e+2$
32	$1.677e+2$	$3.350e+2$	$5.02e+2$
64	$1.6772e+2$	$3.3506e+2$	$5.026e+2$
128	$1.67726e+2$	$3.35065e+2$	$5.02679e+2$
256	$1.677262e+2$	$3.35065e+2$	$5.02679e+2$
512	$1.6772623e+2$	$3.3506538e+2$	$5.026790e+2$
1024	$1.67726234e+2$	$3.35065381e+2$	$5.0267905e+2$

Table 2.30: Example 2.3.9: results by 2D-GJ-rule.

m	$\omega = 10^2$	$\omega = 10^4$	$\omega = 10^6$
16	$1.62e+2$	$2.02e+2$	$2.03e+2$
32	$1.67e+2$	$2.49e+2$	$2.51e+2$
64	$1.67726e+2$	$2.92e+2$	$3.00e+2$
128	$1.6772623416e+2$	$3.23e+2$	$3.50e+2$
256	$1.677262341630e+2$	$3.34e+2$	$3.99e+2$
512	$1.677262341630e+2$	$3.3506e+2$	$4.45e+2$
1024	$1.677262341630e+2$	$3.35065381e+2$	$4.82e+2$

Example 2.3.10. *Let us consider the integral*

$$\boldsymbol{I}(\boldsymbol{f},\omega) = \int_S \sinh(x_1x_2)\, e^{i\omega_1(x_1+x_2)}\, dx_1\, dx_2$$

where

$$\boldsymbol{f}(\mathbf{x}) = \sinh(x_1x_2), \quad \boldsymbol{k}_3(\mathbf{x},\omega) = e^{i\omega_1(x_1+x_2)},$$
$$\mathbf{w} = w_1w_2, \quad w_1(x) = v^{0,0}(x_1), \quad w_2(x_2) = v^{0,0}(x_2).$$

In this example, the integral contains a highly oscillating-type kernel and the function $\boldsymbol{f} \in W^r_{\boldsymbol{\sigma},\infty}$ *for any* $r \geq 1$, *with* $\boldsymbol{\sigma} = \sigma_1\sigma_2$ *and* $\sigma_1 = \sigma_2 = 1$. *By Table 2.31, containing the results of the product rule proposed in* (2.2.2), *the machine precision is attained with* $m = 16$ *for* $\omega_1 = 10, 10^2$, *while for greater values of* ω_1 *the convergence is slower. Similar is the behavior of the 2D-d-rule proposed in* (2.2.21) *whose results are in Table 2.32, where, as well as in other examples, 1-2 final digits are lost with respect to the product rule. HV-method in Table 2.33 gives very good results and this is not surprising, since, according to the convergence hypotheses of the HV-method, the* oscillator (x_1+x_2) *and the function* $\boldsymbol{f}$ *are both analytic. Finally, for large* ω_1, *the 2D-GJ-rule doesn't give any correct result till* $m \leq 512$, *achieving acceptable results only for* $m = 1024$ *(see Table 2.34).*

Table 2.31: Example 2.3.10: results by the product rule $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$.

m	$\omega_1 = 10,\ \boldsymbol{S} = 10$	$\omega_1 = 10^2,\ \boldsymbol{S} = 10^2$	$\omega_1 = 10^3,\ \boldsymbol{S} = 10^3$
4	$-2.73e-2$	$-3.53e-4$	$-1.47e-6$
8	$-2.73295580e-2$	$-3.54895e-4$	$-1.480988e-6$
16	$-2.73295580076672e-2$	$-3.54895314058265e-4$	$-1.4809885630938e-6$
32	$-$	$-$	$-1.4809885630938e-6$
64	$-$	$-$	$-1.48098856309385e-6$

Table 2.32: Example 2.3.10: results by 2D-d-rule.

n	$\omega_1 = 10,\ \boldsymbol{S} = 10$	$\omega_1 = 10^2,\ \boldsymbol{S} = 10^2$	$\omega_1 = 10^3,\ \boldsymbol{S} = 10^3$
4	$-2.73295e-2$	$-3.54895e-4$	$-1.480988e-6$
8	$-2.7329558007667e-2$	$-3.548953140582e-4$	$-1.480988563093e-6$
16	$-2.7329558007667e-2$	$-3.548953140582e-4$	$-1.480988563093e-6$
32	$-2.7329558007667e-2$	$-3.5489531405826e-4$	$-1.480988563093e-6$
64	$-2.73295580076672e-2$	$-3.54895314058265e-4$	$-1.4809885630938e-6$

Table 2.33: Example 2.3.10: results by HV-method.

n	$\omega_1 = 10$	$\omega_1 = 10^2$	$\omega_1 = 10^3$
4	$-2.732955800e-02$	$-3.5489531405826e-04$	$-1.4809885630938e-06$
8	$-2.7329558007667e-02$	$-3.5489531405826e-04$	$-1.4809885630938e-06$
16	$-2.73295580076672e-02$	$-3.5489531405826e-04$	$-1.4809885630938e-06$
32	$-2.73295580076672e-02$	$-3.5489531405826e-04$	$-1.48098856309385e-06$
64	$-3.46e+36$	$-3.54895314058265e-04$	$-$

Table 2.34: Example 2.3.10: results by 2D-GJ-rule.

n	$\omega_1 = 10$	$\omega_1 = 10^2$	$\omega_1 = 10^3$
16	$-2.73295580076e-2$	$-3.2e-2$	$-1.91e-2$
32	$-2.732955800766e-2$	$-8.9e-3$	$-4.81e-2$
64	$-2.732955800766e-2$	$-3.5489e-4$	$-4.67e-2$
128	$-2.732955800766e-2$	$-3.54895314058e-4$	$-3.30e-4$
256	$-2.732955800766e-2$	$-3.54895314058e-4$	$-6.33e-3$
512	$-2.732955800766e-2$	$-3.54895314058e-4$	$-2.64e-7$
1024	$-2.732955800766e-2$	$-3.54895314058e-4$	$-1.48098856309e-6$

Example 2.3.11. *Let us consider the integral*

$$\boldsymbol{I}(\boldsymbol{f}, \omega) = \int_S |\sinh(x_1x_2)|^{11.5} \sin(\omega x_1 x_2)\, v^{-\frac{1}{4},\frac{1}{4}}(x_1)\, v^{-\frac{1}{4},\frac{1}{4}}(x_2)\, dx_1\, dx_2$$

where

$$\boldsymbol{f}(\mathbf{x}) = |\sinh(x_1x_2)|^{11.5}, \quad \boldsymbol{k}_3(\mathbf{x}, \omega) = \sin(\omega x_1 x_2),$$

$$\mathbf{w} = w_1 w_2, \quad w_1(x_1) = v^{-\frac{1}{4},\frac{1}{4}}(x_1), \quad w_2(x_2) = v^{-\frac{1}{4},\frac{1}{4}}(x_2).$$

Also in this example the integral contains a highly oscillating-type kernel, but in thic case the function $\boldsymbol{f} \in W^{11}_{\boldsymbol{\sigma},\infty}$ with $\boldsymbol{\sigma} = \sigma_1\sigma_2$ and $\sigma_1 = \sigma_2 = 1$. By Table

2.35 which contains the results of the product rule proposed in (2.2.2)*, the machine precision is attained with* $m = 512$ *for* $\omega = 10^2$*, while for greater values of* ω *the convergence is slower, but* 14 *digits are taken. However, the results are coherent with the theoretical estimate (2.2.8) combined with (1.2.4), since the seminorm* $\mathcal{M}_{11}(f) \sim 10^{11}$*. Similar is the behavior of the 2D-d-rule presented in* (2.2.21) *whose results are in Table 2.36, where, as well as in other examples, 1-2 final digits are lost with respect to the product rule. Since the assumptions of the HV-method are not satisfied, we didn't implement it. Finally, for large* ω*, the 2D-GJ-rule doesn't give any correct result till* $m \leq 512$*, achieving acceptable results for* $m = 1024$ *only (see Table 2.37).*

Table 2.35: Example 2.3.11: results by the product rule $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$.

m	$\omega = 10^2,\ S = 10$	$\omega = 10^3,\ S = 32$	$\omega = 10^4,\ S = 10^2$
8	$-4.478551724e-3$	$-2.10e-4$	$9.45e-6$
16	$-6.436821087e-3$	$-2.98e-4$	$1.20e-5$
32	$-6.439284731e-3$	$-2.98928017714e-4$	$1.20606902036e-5$
64	$-6.4392847317303e-3$	$-2.989280177142e-4$	$1.2060690203683e-5$
128	$-6.4392847317303e-3$	$-2.989280177142e-4$	$1.2060690203683e-5$
256	$-6.4392847317303e-3$	$-2.9892801771422e-4$	$1.2060690203683e-5$
512	$-6.43928473173037e-3$	$-2.9892801771422e-4$	$1.2060690203683e-5$

Table 2.36: Example 2.3.11: results by 2D-d-rule.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^3,\ \boldsymbol{S} = 32$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$
16	$-6.43928473173e-3$	$3.47e-3$	$8.58e-4$
32	$-6.439284731730e-3$	$-2.98928017714e-4$	$7.78e-4$
64	$-6.439284731730e-3$	$-2.98928017714e-4$	$1.2060e-5$
128	$-6.439284731730e-3$	$-2.989280177142e-4$	$1.2060690203e-5$
256	$-6.439284731730e-3$	$-2.989280177142e-4$	$1.2060690203e-5$

Table 2.37: Example 2.3.11: results by 2D-GJ-rule.

m	$\omega = 10^2$	$\omega = 10^3$	$\omega = 10^4$
64	$-6.4392847e-3$	$-1.34e-2$	$-2.78e-3$
128	$-6.4392847317e-3$	$-2.30e-3$	$-5.70e-5$
256	$-6.4392847317e-3$	$-1.43e-3$	$6.82e-3$
512	$-6.4392847317e-3$	$-4.08e-4$	$3.13e-3$
1024	$-6.4392847317e-3$	$-4.08e-4$	$-8.56e-4$

Example 2.3.12. *Let us consider the integral*

$$\boldsymbol{I}(\boldsymbol{f}, \omega) = \int_S (x_1 + x_2)^{20} \, \frac{\sin(\omega x_1 x_2)}{x_1^2 + x_2^2 + \omega^{-1}} \, dx_1 \, dx_2$$

where

$$\boldsymbol{f}(\mathbf{x}) = (x_1 + x_2)^{20}, \quad \boldsymbol{k}_4(\mathbf{x}, \omega) = \frac{\sin(\omega x_1 x_2)}{x_1^2 + x_2^2 + \omega^{-1}}, \quad \lambda = 1,$$

$$\mathbf{w} = w_1 w_2, \quad w_1(x_1) = v^{0,0}(x_1), \quad w_2(x_2) = v^{0,0}(x_2).$$

In this case, the integral contains a mixed-type kernel (i.e. a nearly singular and highly oscillating) and the function $\boldsymbol{f} \in W^r_{\boldsymbol{\sigma},\infty}$ *for any* r*, with* $\boldsymbol{\sigma} = \sigma_1\sigma_2$ *and* $\sigma_1 = \sigma_2 = 1$*. The results of the product rule proposed in* (2.2.2) *given in Table 2.38 are coherent with the theoretical estimates, since the values of the seminorms are too large. For instance for* $r = 20$*, it is* $\mathcal{M}_r(f) \sim 10^{18}$*. Comparing our results with those obtained with the 2D-d-rule presented in* (2.2.21) *given in Table 2.39, we observe that more or less* 2 *digits are lost with respect to the product rule. In absence of other procedures, we have forced the use of the JJE-method, by which for* $\omega = 10^2$ *the results present* 12 *correct digits, while with larger* ω *the results are completely wrong (see Table 2.40). However this bad behavior is to be expected, since the oscillating factor is not covered within their method. Finally, the results in Table 2.41 evidence that 2D-GJ-rule is unreliable for* ω *large.*

Table 2.38: Example 2.3.12: results by the product rule $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^3,\ \boldsymbol{S} = 32$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$
16	$3.666247e+1$	$-3.0625e-1$	$3.22e-3$
32	$3.666247509043e+1$	$-3.06250405322e-1$	$3.2214048203e-3$
64	$3.666247509043e+1$	$-3.06250405322e-1$	$3.22140482036e-3$
128	$3.666247509043e+1$	$-3.06250405322e-1$	$3.22140482036e-3$
256	$3.6662475090432e+1$	$-3.06250405322e-1$	$3.221404820367e-3$
512	$3.66624750904321e+1$	$-3.0625040532207e-1$	$3.2214048203672e-3$

Table 2.39: Example 2.3.12: results by 2D-d-rule.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^3,\ \boldsymbol{S} = 32$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$
16	$3.6662475090e+1$	$1.29e+2$	$8.98e+0$
32	$3.66624750904e+1$	$-3.062504053e-01$	$1.60e+1$
64	$3.66624750904e+1$	$-3.0625040532e-01$	$3.2214e-03$
128	$3.66624750904e+1$	$-3.0625040532e-01$	$3.221404820e-03$

Example 2.3.13. *Let us consider the integral*

$$\boldsymbol{I}(\boldsymbol{f}, \omega) = \int_S |x_1 - x_2|^{7.1} \, \frac{\sin(\omega x_1 x_2)}{x_1^2 + x_2^2 + \omega^{-1}} \, v^{\frac{1}{2},\frac{1}{2}}(x_1) \, v^{-\frac{1}{4},-\frac{1}{4}}(x_2) \, dx_1 \, dx_2$$

Table 2.40: Example 2.3.12: results by JJE-method.

m	$\omega = 10^2$	$\omega = 10^3$	$\omega = 10^4$
64	$3.73e+1$	$2.92e+3$	$1.17e+3$
128	$3.6662475090e+01$	$-2.66e+2$	$-4.38e+2$
256	$3.6662475090e+01$	$1.02e+2$	$-8.48e+1$
512	$3.6662475090e+01$	$7.92e+0$	$-1.68e+2$
1024	$3.66624750904e+01$	$-8.76e-1$	$-5.98e+1$

Table 2.41: Example 2.3.12: results by 2D-GJ-rule.

m	$\omega = 10^2$	$\omega = 10^3$	$\omega = 10^4$
64	$3.666247e+1$	$3.93e+2$	$-8.40e+2$
128	$3.6662475090e+1$	$4.21e+2$	$4.01e+2$
256	$3.66624750904e+1$	$2.10e+0$	$-3.36e+1$
512	$3.66624750904e+1$	$-8.76e-01$	$4.89e+1$
1024	$3.66624750904e+1$	$-8.76e-01$	$-3.68e+1$

$$\boldsymbol{f}(\mathbf{x}) = |x_1 - x_2|^{7.1}, \quad \boldsymbol{k}_4(\mathbf{x}, \omega) = \frac{\sin(\omega x_1 x_2)}{x_1^2 + x_2^2 + \omega^{-1}}, \quad \lambda = 1,$$

$$\mathbf{w} = w_1 w_2, \quad w_1(x_1) = v^{\frac{1}{2},\frac{1}{2}}(x_1), \quad w_2(x_2) = v^{-\frac{1}{4},-\frac{1}{4}}(x_2).$$

We conclude with another test on a mixed-type kernel. Here the function $\boldsymbol{f} \in W^7_{\boldsymbol{\sigma},\infty}$ *with* $\boldsymbol{\sigma} = \sigma_1\sigma_2$ *and* $\sigma_1 = v^{\frac{1}{4},\frac{1}{4}}$, $\sigma_2 = 1$. *Since the seminorm* $\mathcal{M}_r(f) \sim 6 \times 10^3$, *according to the theoretical estimate,* 15 *exact (not always significant) digits are computed for* $m = 512$ *(Table 2.42). The results are comparable with those achieved by the 2D-d-rule presented in* (2.2.21) *reported in Table 2.43, while the 2D-GJ-rule results in Table 2.45, as well as those achieved by the JJE-method in Table 2.44, give poor approximations.*

Table 2.42: Example 2.3.12: results by the product rule $\boldsymbol{I}_m(\boldsymbol{f}, \mathbf{y})$.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^3,\ \boldsymbol{S} = 32$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$
16	$-4.23e-3$	$-1.83e-4$	$-4.42e-8$
32	$-4.23634e-3$	$-1.8313e-4$	$1.29e-8$
64	$-4.23634393e-3$	$-1.831311e-4$	$1.44e-8$
128	$-4.2363439329e-3$	$-1.8313118e-4$	$1.4448e-8$
256	$-4.2363439329106e-3$	$-1.8313118400e-4$	$1.444854e-8$
512	$-4.23634393291069e-3$	$-1.831311840010e-4$	$1.44485497e-8$

Table 2.43: Example 2.3.13: results by 2D-d-rule.

m	$\omega = 10^2,\ \boldsymbol{S} = 10$	$\omega = 10^3,\ \boldsymbol{S} = 32$	$\omega = 10^4,\ \boldsymbol{S} = 10^2$
16	$-4.2363439329e-3$	$-9.75e-3$	$1.75e-2$
32	$-4.23634393291e-3$	$-1.8313118400e-4$	$-7.41e-3$
64	$-4.23634393291e-3$	$-1.8313118400e-4$	$1.44e-8$
128	$-4.23634393291e-3$	$-1.8313118400e-4$	$1.44485e-8$
256	$-4.23634393291e-3$	$-1.8313118400e-4$	$1.444854e-8$
512	$-4.23634393291e-3$	$-1.8313118400e-4$	$1.4448549e-8$

Table 2.44: Example 2.3.13: results by JJE-method.

m	$\omega = 10^2$	$\omega = 10^3$	$\omega = 10^4$
64	$-4.53e-3$	$-7.21e-1$	$-2.71e-1$
128	$-4.26e-3$	$4.73e-1$	$4.18e-1$
256	$-4.24e-3$	$4.22e-2$	$7.40e-2$
512	$-4.23e-3$	$-2.36e-2$	$7.23e-2$
1024	$-4.23e-3$	$1.72e-5$	$1.28e-2$

Table 2.45: Example 2.3.13: results by 2D-GJ-rule.

m	$\omega = 10^2$	$\omega = 10^3$	$\omega = 10^4$
64	$-4.236e-3$	$3.46e-1$	$3.83e-1$
128	$-4.2363439329e-3$	$-1.18e-1$	$4.03e-2$
256	$-4.2363439329e-3$	$-1.96e-2$	$-4.86e-2$
512	$-4.2363439329e-3$	$-1.35e-4$	$6.52e-2$
1024	$-4.2363439329e-3$	$1.74e-5$	$3.43e-3$

Chapter 3

Numerical Treatment of the Generalized Univariate and Bivariate Love Integral Equation

In this Chapter we consider the generalized univariate and bivariate Love's integral equations. In both cases, in order to approximate the solution, we propose a Nyström method based on a mixed quadrature and cubature rule, respectively. Such rules are a combination of a product and a "dilation" quadrature/cubature formula presented in Chapter 2 in a revisiting form. We prove the stability and convergence of the described numerical procedures in suitable weighted spaces and we show the efficiency of the two methods by some numerical tests.

In 1949 Love investigated for the first time on a mathematical model describing the capacity of a circular plane condenser consisting of two identical coaxial discs placed at a distance q and having a common radius r. In his paper [45], he proved that the capacity of each disk is given by

$$C = \frac{r}{\pi} \int_{-1}^{1} f(x)dx,$$

where f is the solution of the following integral equation of the second kind

$$f(y) - \frac{1}{\pi} \int_{-1}^{1} \frac{\omega^{-1}}{(x-y)^2 + \omega^{-2}} f(x)dx = 1 \qquad (3.0.1)$$

with $\omega = q/r$ a real positive parameter. Then, he proved that equation (3.0.1) has a unique, continuous, real and even solution which analitically has the following form

$$f(y) = 1 - \sum_{j=1}^{\infty} (-1)^j \int_{-1}^{1} K_j(x,y)\, dx,$$

where the iterated kernels are given by

$$K_1(x,y) = \frac{1}{\pi} \frac{\omega^{-1}}{(x-y)^2 + \omega^{-2}},$$
$$K_j(x,y) = \int_{-1}^{1} K_{j-1}(x,s) K_1(s,y) ds, \quad j = 2, \ldots .$$

From a numerical point of view, the developed methods [44, 43, 59, 71, 75] for the undisputed most interesting case (i.e. when $\omega^{-1} \to 0$) have followed the very first methods [20, 24, 72, 82, 84], and the most recent ones [58], proposed for the case when $\omega = 1$.

If $\omega^{-1} \to 0$ the kernel function is "close" to be singular on the bisector $x = y$ and as Phillips noted in [72],

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\omega^{-1}}{(x-y)^2 + \omega^{-2}} f(x) dx \to f(y) \qquad \text{if} \qquad \omega^{-1} \to 0. \tag{3.0.2}$$

Hence for ω sufficiently large the left hand side of equation (3.0.1) becomes approximately zero which does not coincide with the right-hand side of (3.0.1).

In [59] the authors presented a numerical approach based on a suitable transformation, in order to move away the poles $x = y \pm \omega^{-1} i$ from the real axis. The numerical method produced very accurate results in the case when ω^{-1} is not so small but they are poor if $\omega^{-1} \to 0$.

Then, in order to get satisfactory errors also in this latter case, in [71] the author proposed to dilate the integration interval and to decompose it into N subintervals. Hence, the equation was reduced to an equivalent system of N integral equations and a Nyström method based on a Gauss-Legendre quadrature formula was proposed for its numerical approximation. The approach produces satisfactory order of convergence even if ω^{-1} is small. However, the dimension of the structured linear system that one needs to solve is very large as ω^{-1} decreases.

In [44] the authors improve the results given in [71] by using the same transformation as in [43] which takes into account the behavior of the unknown function showed in (3.0.2). Then they follow the approach given in [71] i.e. they write the integral as the sum of m new integrals which are approximated by means of a n-point Gauss-Legendre quadrature rule. In this way they get a linear system of size nm that, multipled by suitable diagonal matrices, is equivalent to a new linear system which is solved by using a preconditioned conjugate gradient method, being the matrix of coefficients symmetric, positive definitive and having a Toeplitz block structure.

In this Chapter we consider the more general equation

$$f(y) - \frac{1}{\pi} \int_{-1}^{1} \frac{\omega^{-1}}{(x-y)^2 + \omega^{-2}} f(x) w(x) dx = g(y), \qquad |y| < 1, \tag{3.0.3}$$

where w is a Jacobi weight defined as (1.1.1), f is the unknown function, g is a known right-hand side, and $0 < \omega \in \mathbb{R}$.

Such equation includes equation (3.0.1) (in the case when $g \equiv w \equiv 1$) and, at the same time, the presence of the weight w leads to the case when the unknown function has algebraic singularities at the endpoints of $[-1, 1]$.

The method we propose is a Nyström method based on a mixed quadrature formula. This is a *product rule* whose coefficients are computed by using a revisiting form of the quadrature scheme proposed in Subsection 2.1.2. In fact, following an idea presented in [17, 71], we approximate such coefficients by using a "dilation" quadrature formula that we prove to be stable and convergent. Such idea consists in a preliminary dilation of the domain that "relax" in some sense the pathological behavior of the kernel of the integral.

The proposed method, for which convergence and stability are proved in suitable weighted spaces, allow us to get very accurate results with respect to those in [44, 71] by solving a well-conditioned linear system whose dimension is greatly reduced with respect to the ones involved in [44, 71].

We also extend the procedure to the case of the bivariate Love equation for which, according to our knowledge, no numerical methods exist. It takes the form

$$\boldsymbol{f}(\mathbf{y}) - \frac{1}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} \frac{\omega^{-1}}{|\mathbf{x}-\mathbf{y}|^2 + \omega^{-2}} \boldsymbol{f}(\mathbf{x}) \mathbf{w}(\mathbf{x}) d\mathbf{x} = \boldsymbol{g}(\mathbf{y}), \qquad 0 < \omega \in \mathbb{R}, \tag{3.0.4}$$

where $\mathbf{x} = (x_1, x_2) \in [-1, 1]^2$, $\mathbf{y} = (y_1, y_2) \in [-1, 1]^2$, $\mathbf{w}$ is a product of two Jacobi weights defined as (1.2.1), $\boldsymbol{f}$ is the unknown function and $\boldsymbol{g}$ is the known right-hand side. Also in this case we propose a Nyström method which is based on a cubature mixed formula, namely a combination of a suitable product cubature formula and a cubature dilation formula. Specifically we use the product cubature formula proposed in Section 2.2 and we generalize the "dilation"cubature rule given in Subsection 2.2.2.

The outline of this Chapter is as follows. Section 3.1 is completely devoted to the one-dimensional Love equation. At first we study a "dilation" quadrature formula in a revisiting form (Subsection 3.1.1) and we propose a new mixed quadrature scheme (Subsection 3.1.2) which is used in the Nyström method (Subsection 3.1.3). Section 3.2 is dedicated to the two-dimensional Love equation: once introduced a “dilation” cubature formula in a revisiting form (Subsections 3.2.1), we propose a new mixed cubature formula (Subsection 3.2.2) and we describe the related Nyström method (Subsection 3.2.3). Section 4.4 shows the efficiency of the proposed procedures by means of several numerical tests, both, in 1D and 2D.

3.1 One–dimensional Love's Integral Equation

In this Section we present a numerical method for approximating the solution of Love's univariate equation (3.0.3) defined on $[-1, 1]$.

3.1.1 The 1D-dilation formula: a revisiting

We present a quadrature formula in order to approximate the integrals of the type

$$\mathcal{I}(F, y, \omega) = \int_{-1}^{1} k(x, y, \omega) F(x) w(x) dx, \tag{3.1.1}$$

where F is a given function, w is as in (1.1.1) and $k(x, y, \omega)$ is a known kernel which is close to be singular if $\omega^{-1} \to 0$. This is the case of the kernel function appearing in the Love equation (3.0.3).

In Subsection 2.1.2 we proposed a “dilation”quadrature rule for approximating integrals of the type

$$I(F, \omega) = \int_{-1}^{1} k(x, \omega) F(x) w(x) dx,$$

namely, to integrals in which the kernel function is not a function of two variables. Our idea is to “generalize”in some sense the approach proposed in Subsection 2.1.2 and provide new error estimates.

In order to construct such kind of formula, we follow the approach proposed in [17, 71] for the unweighted case.

First, in order to “relax” the “too fast” behaviour of the kernel function when ω grows, we introduce in (3.1.1) the change of variables

$$x = \frac{\eta}{\omega}, \qquad y = \frac{\theta}{\omega}$$

with $\eta, \theta \in [-\omega, \omega]$.

In this way (3.1.1) is equivalent to the following integral having a dilated domain of integration $[-\omega, \omega]$

$$\begin{aligned}\mathcal{I}(F,y,\omega) &= \frac{1}{\omega}\int_{-\omega}^{\omega} k\left(\frac{\eta}{\omega},\frac{\theta}{\omega},\omega\right)\, F\left(\frac{\eta}{\omega}\right) w\left(\frac{\eta}{\omega}\right) d\eta \\ &=: \frac{1}{\omega}\int_{-\omega}^{\omega} \kappa\left(\eta,\theta,\omega\right)\, F\left(\frac{\eta}{\omega}\right) w\left(\frac{\eta}{\omega}\right) d\eta \\ &= \frac{1}{\omega}\int_{-\omega}^{\omega} \kappa\left(\eta,\omega y,\omega\right)\, F\left(\frac{\eta}{\omega}\right) w\left(\frac{\eta}{\omega}\right) d\eta.\end{aligned}$$

Then, we split the new integration interval $[-\omega,\, \omega]$ into $\boldsymbol{S}$ subintervals of size $2 \leq d \in \mathbb{R}$ such that $\boldsymbol{S} = \frac{2\omega}{d} \in \mathbb{N}$, namely

$$[-\omega,\, \omega] = \bigcup_{i=1}^{\boldsymbol{S}} [-\omega + (i-1)d,\ -\omega + id],$$

getting

$$\mathcal{I}\left(F,y,\omega\right) = \frac{1}{\omega}\sum_{i=1}^{\boldsymbol{S}} \int_{-\omega+(i-1)d}^{-\omega+id} \kappa\left(\eta,\omega y,\omega\right)\, F\left(\frac{\eta}{\omega}\right)\, w\left(\frac{\eta}{\omega}\right) d\eta. \tag{3.1.2}$$

Now, we want to remap each integral into $[-1,\, 1]$. To this end we introduce the invertible linear maps

$$\Psi_i : [-\omega + (i-1)d, -\omega + id] \to [-1,\, 1]$$

defined as

$$x = \Psi_i(\eta) = \frac{2}{d}(\eta + \omega) - (2i - 1)$$

and in (3.1.2) we make the change of variable

$$\eta = \Psi_i^{-1}(x) = \left(\frac{x+1}{2}\right) d - \omega + (i-1)d. \tag{3.1.3}$$

In this way we get

$$\mathcal{I}\left(F,y,\omega\right) = \frac{d}{2\omega}\sum_{i=1}^{\boldsymbol{S}} \int_{-1}^{1} k_i(x,\omega y,\omega)\, F_i\left(x\right)\, u_i(x) dx, \tag{3.1.4}$$

where $F_i(x) := F\left(\frac{\Psi_i^{-1}(x)}{\omega}\right)$, u_i are the new weight functions

$$u_i(x) := \begin{cases} v^{0,\beta}(x), & i = 1 \\ v^{0,0}(x), & 2 \leq i \leq \boldsymbol{S} - 1, \\ v^{\alpha,0}(x), & i = \boldsymbol{S} \end{cases} \tag{3.1.5}$$

and k_i are the new kernel functions

$$k_i(x, \omega y, \omega) := \begin{cases} \left(\frac{d}{2\omega}\right)^{\beta} \kappa\left(\Psi_i^{-1}(x), \omega y, \omega\right) v^{\alpha,0}\left(\frac{\Psi_i^{-1}(x)}{\omega}\right), & i = 1 \\ \kappa\left(\Psi_i^{-1}(x), \omega y, \omega\right) v^{\alpha,\beta}\left(\frac{\Psi_i^{-1}(x)}{\omega}\right), & 2 \leq i \leq \boldsymbol{S} - 1, \\ \left(\frac{d}{2\omega}\right)^{\alpha} \kappa\left(\Psi_i^{-1}(x), \omega y, \omega\right) v^{0,\beta}\left(\frac{\Psi_i^{-1}(x)}{\omega}\right), & i = \boldsymbol{S} \end{cases}$$

or, equivalently, in terms of the original kernel k,

$$k_i(x, \omega y, \omega) := \begin{cases} \left(\frac{d}{2\omega}\right)^{\beta} k\left(\frac{\Psi_i^{-1}(x)}{\omega}, y, \omega\right) v^{\alpha,0}\left(\frac{\Psi_i^{-1}(x)}{\omega}\right), & i = 1 \\ k\left(\frac{\Psi_i^{-1}(x)}{\omega}, y, \omega\right) v^{\alpha,\beta}\left(\frac{\Psi_i^{-1}(x)}{\omega}\right), & 2 \leq i \leq \boldsymbol{S} - 1 \,. \\ \left(\frac{d}{2\omega}\right)^{\alpha} k\left(\frac{\Psi_i^{-1}(x)}{\omega}, y, \omega\right) v^{0,\beta}\left(\frac{\Psi_i^{-1}(x)}{\omega}\right), & i = \boldsymbol{S} \end{cases} \tag{3.1.6}$$

By approximating each integral appearing in (3.1.4) by means of the Gauss-Jacobi quadrature rule (1.1.18) with u_i in place of w and $k_i F_i$ instead of f, we have the following "dilation" quadrature formula

$$\mathcal{I}(F, y, \omega) = \frac{d}{2\omega} \sum_{i=1}^{\boldsymbol{S}} \sum_{j=1}^{n} \lambda_j^{u_i} k_i(\xi_j^{u_i}, \omega y, \omega) F_i(\xi_j^{u_i}) + \Lambda_n(F, \omega y, \omega),$$

where Λ_n is the remainder term.

Next results state the stability of the previous formula and give an error estimate for Λ_n in the case when $F \in \mathcal{W}_{\sigma,\infty}^r$ or $F \in C^{2n}([-1, 1])$.

Theorem 3.1.1. *Let $F \in C_\sigma$ be with σ as in* (1.1.2) *and let w be as in* (1.1.1). *If*

$$0 \leq \gamma < \min\{1, \alpha + 1\}, \quad 0 \leq \delta < \min\{1, \beta + 1\}$$

and k is such that

$$\max_{|y|\le 1} \|k(\cdot,\omega y,\omega)\|_\infty < \infty$$

then

$$\sup_{|y|\le 1} \frac{d}{2\omega}\left|\sum_{i=1}^{\boldsymbol{S}}\sum_{j=1}^{n} \lambda_j^{u_i} k_i(\xi_j^{u_i},\omega y,\omega) F_i(\xi_j^{u_i})\right| \le \mathcal{C}\|F\sigma\|_\infty, \qquad \mathcal{C}\ne\mathcal{C}(F,n). \tag{3.1.7}$$

Moreover, for any $F\in\mathcal{W}^r_{\sigma,\infty}$, if

$$\max_{|y|\le 1}\max_{|x|\le 1}\left|\frac{\partial^r k(x,y,\omega)}{\partial x^r}\varphi^r(x)\right| < \infty \tag{3.1.8}$$

we have

$$\sup_{|y|\le 1} |\Lambda_n(F,\omega y,\omega)| \le \frac{\mathcal{C}}{n^r}\left(\frac{d}{\omega}\right)^r \|F\|_{\mathcal{W}^r_{\sigma,\infty}}, \qquad \mathcal{C}\ne\mathcal{C}(F,n). \tag{3.1.9}$$

Proof. First, let us prove the stability of the formula, i.e estimate (3.1.7). We can write

$$\begin{aligned}
&\frac{d}{2\omega}\left|\sum_{i=1}^{\boldsymbol{S}}\sum_{j=1}^{n}\frac{\lambda_j^{u_i}}{\sigma(\xi_j^{u_i})}(F_i\sigma)(\xi_j^{u_i})k_i(\xi_j^{u_i},\omega y,\omega)\right| \\
&\le \frac{d}{2\omega}\|F\sigma\|_\infty \sum_{i=1}^{\boldsymbol{S}} \|k_i(\cdot,\omega y,\omega)\|_\infty \sum_{j=1}^{n}\frac{\lambda_j^{u_i}}{\sigma(\xi_j^{u_i})}.
\end{aligned}$$

Then (3.1.7) follows taking into account the definition of k_i given in (3.1.6), the first assumption on the kernel, and by considering that in virtue on the assumptions on the parameters of the weights we have

$$\sum_{j=1}^{n}\frac{\lambda_j^{u_i}}{\sigma(\xi_j^{u_i})} \le \int_{-1}^{1}\frac{u_i(x)}{\sigma(x)}dx \le \mathcal{C}. \tag{3.1.10}$$

In order to prove (3.1.9), we can note that by (1.2.17), we have

$$|\Lambda_n(F,\omega y,\omega)| \le \sum_{i=1}^{\boldsymbol{S}} |\mathcal{R}_n(F_i\, k_i,\omega y,\omega)| \le \mathcal{C}\sum_{i=1}^{\boldsymbol{S}} E_{2n-1}(F_i\, k_i)_\sigma,$$

so that by using the well-known estimate [47]

$$E_{2n-1}(h_1\, h_2)_\sigma \le \|h_1\sigma\|_\infty E_{[\frac{2n-1}{2}]}(h_2) + 2\|h_2\|_\infty E_{[\frac{2n-1}{2}]}(h_1)_\sigma,$$

we can write

$$|\Lambda_n(F,\omega y,\omega)| \leq \mathcal{C}\sum_{i=1}^{\boldsymbol{S}} \Big(\|F_i\sigma\|_\infty E_{[\frac{2n-1}{2}]}(k_i) + \|k_i(\cdot,\omega y,\omega)\|_\infty \; E_{[\frac{2n-1}{2}]}(F_i)_\sigma \Big) .$$

Then, taking into account that by the assumptions $\|k_i(\cdot,\omega y,\omega)\|_\infty < \mathcal{C}$ and by applying the Favard inequality [47]

$$E_n(h)_v \leq \frac{\mathcal{C}}{n^r}\left\|h^{(r)}\varphi^r v\right\|_\infty, \qquad \mathcal{C}\neq\mathcal{C}(n,h), \qquad \forall h\in\mathcal{W}^r_{v,\infty}, \tag{3.1.11}$$

once with the Jacobi weight $v=1$, and then with $v=\sigma$, we deduce

$$|\Lambda_n(F,\omega y,\omega)| \leq \frac{\mathcal{C}}{n^r}\sum_{i=1}^{\boldsymbol{S}} \left(\|F_i\sigma\|_\infty \sup_{|x|\leq 1}\left|\frac{\partial^r}{\partial x^r}k_i(x,\omega y,\omega)\varphi^r(x)\right| + \|F_i^{(r)}\varphi^r\sigma\|_\infty \right) .$$

Now, let us note that the functions k_i defined in (3.1.6) can be rewritten as

$$k_i(x,\omega y,\omega) = k\left(\frac{\Psi_i^{-1}(x)}{\omega},y,\omega\right) U_i(x,\omega,d)$$

where the functions U_i, defined as

$$U_i(x,\omega,d) := \begin{cases} \left(\dfrac{d}{2\omega}\right)^{\beta} v^{\alpha,0}\left(\dfrac{\Psi_i^{-1}(x)}{\omega}\right), & i=1 \\ v^{\alpha,\beta}\left(\dfrac{\Psi_i^{-1}(x)}{\omega}\right), & 2\leq i\leq \boldsymbol{S}-1, \\ \left(\dfrac{d}{2\omega}\right)^{\alpha} v^{0,\beta}\left(\dfrac{\Psi_i^{-1}(x)}{\omega}\right), & i=\boldsymbol{S} \end{cases} \tag{3.1.12}$$

are bounded functions such that

$$\sup_{|x|\leq 1}|U_i^{(r)}(x,\omega,d)\varphi^r(x)| \leq \mathcal{C}\left(\frac{d}{2\omega}\right)^r, \qquad \forall i=1,\ldots\boldsymbol{S}.$$

Hence, being for each $i=1,\ldots,\boldsymbol{S}$

$$\left|\frac{\partial^r k_i(x,\omega y,\omega)}{\partial x^r}\varphi^r(x)\right| = \left|\sum_{j=0}^{r}\binom{r}{j}\frac{\partial^j}{\partial x^j}k\left(\frac{\Psi_i^{-1}(x)}{\omega},y,\omega\right)\varphi^j(x)\,U_i^{(r-j)}(x,\omega,d)\varphi^{r-j}(x)\right|,$$

by using (3.1.8) we get

$$\sup_{|y|\le 1}\sup_{|x|\le 1}\left|\frac{\partial^r k_i(x,\omega y,\omega)}{\partial x^r}\varphi^r(x)\right| \le \mathcal{C}\sum_{j=0}^{r}\binom{r}{j}\left(\frac{d}{2\omega}\right)^{r-j}\left(\frac{d}{2\omega}\right)^{j} = \mathcal{C}\left(\frac{d}{\omega}\right)^{r}$$

and therefore

$$\sup_{|y|\le 1}|\Lambda_n(F,\omega y,\omega)| \le \frac{\mathcal{C}}{n^r}\left(\frac{d}{\omega}\right)^{r}\|F\|_{\mathcal{W}^r_{\sigma,\infty}}.$$

□

Corollary 3.1.2. *Let $F, k \in C^{2n}([-1,\, 1])$ with respect to the variable x. Then*

$$\sup_{|y|\le 1}|\Lambda_n(F,\omega y,\omega)| \le \frac{\mathcal{C}}{n^{2n+\frac{1}{2}}}\left(\frac{d}{\omega}\right)^{2n} e^{\frac{48n^2+1}{24n}}\, 2^{n-1}\,\left[\|F\|_\infty + \|F^{(2n)}\|_\infty\right] \tag{3.1.13}$$

with $\mathcal{C} \neq \mathcal{C}(n,\omega,d)$.

Remark 3.1.3. *We outline that the quantity $\dfrac{d}{\omega}$ appearing in both the estimates* (3.1.9) *and* (3.1.13) *is a quantity $\ll 1$, since we are considering the case ω large.*

Proof. Taking into account the errore estimate (1.1.20), we have

$$\begin{aligned}|\Lambda_n(F,\omega y,\omega)| &\le \sum_{i=1}^{\boldsymbol{s}}|\mathcal{R}_n(F_i\, k_i,\omega y,\omega)|\\ &\le \frac{1}{(2n)!\,\gamma_n(w)}\sum_{i=1}^{\boldsymbol{s}}\sup_{|x|\le 1}\left|\frac{\partial^{2n}}{\partial x^{2n}}[F_i(x)k_i(x,\,\omega y,\,\omega)]\right|.\end{aligned}$$

Then by applying the Leibnitz rule we get

$$\begin{aligned}&\left|\frac{\partial^{2n}}{\partial x^{2n}}[F_i(x)k_i(x,\,\omega y,\,\omega)]\right|\\ &\le \sum_{j=0}^{2n}\binom{2n}{j}\left|\left[F\left(\frac{\Psi_i^{-1}(x)}{\omega}\right)\right]^{(2n-j)}\right|\left|\frac{\partial^j}{\partial x^j}k_i\left(x,\omega y,\omega\right)\right|\\ &\le \sum_{j=0}^{2n}\binom{2n}{j}\left(\frac{d}{2\omega}\right)^{2n-j}\|F^{(2n-j)}\|_\infty\sup_{|x|\le 1}\left|\frac{\partial^j}{\partial x^j}k_i\left(x,\omega y,\omega\right)\right|,\end{aligned}$$

from which being [18] $\|F^{(2n-j)}\|_\infty \le \mathcal{C}\left[\frac{\|F\|_\infty}{2^{2n-j}} + 2^j\|F^{(2n)}\|_\infty\right]$ we get

$$\left|\frac{\partial^{2n}}{\partial x^{2n}}[F_i(x)k_i(x,\,\omega y,\,\omega)]\right| \le \|F\|_\infty \sum_{j=0}^{2n}\binom{2n}{j}\left(\frac{d}{4\omega}\right)^{2n-j}\left|\frac{\partial^j}{\partial x^j}k_i\left(x,\omega y,\omega\right)\right|$$
$$+\|F^{(2n)}\|_\infty \sum_{j=0}^{2n}\binom{2n}{j}\left(\frac{d}{2\omega}\right)^{2n-j}2^j\left|\frac{\partial^j}{\partial x^j}k_i\left(x,\omega y,\omega\right)\right|. \tag{3.1.14}$$

By the definitions (3.1.6) of the kernels k_i and taking into account the form of the functions U_i given in (3.1.12), we can write

$$\left|\frac{\partial^j}{\partial x^j}k_i\left(x,\omega y,\omega\right)\right| \le \sum_{\ell=0}^{j}\binom{j}{\ell}\left|\frac{\partial^{j-\ell}}{\partial x^{j-\ell}}\left[k\left(\frac{\Psi_i^{-1}(x)}{\omega},y,\omega\right)\right]\right|\,\left|[U_i(x,\omega,d)]^{(\ell)}\right|$$
$$\le \mathcal{C}\sum_{\ell=0}^{j}\binom{j}{\ell}\left(\frac{d}{2\omega}\right)^{j-\ell}\left(\frac{d}{2\omega}\right)^{\ell}\sup_{|x|\le 1}\left|\frac{\partial^{j-\ell}}{\partial x^{j-\ell}}k\left(\frac{\Psi_i^{-1}(x)}{\omega},y,\omega\right)\right|$$

and being [18]

$$\sup_{|x|\le 1}\left|\frac{\partial^{j-\ell}}{\partial x^{j-\ell}}k\left(\frac{\Psi_i^{-1}(x)}{\omega},y,\omega\right)\right| \le \mathcal{C}\left[\left(\frac{1}{2}\right)^{j-\ell}\sup_{|x|\le 1}\left|k\left(\frac{\Psi_i^{-1}(x)}{\omega},y,\omega\right)\right|\right.$$
$$\left.+2^{2n-j+\ell}\sup_{|x|\le 1}\left|\frac{\partial^{2n}}{\partial x^{2n}}k\left(\frac{\Psi_i^{-1}(x)}{\omega},y,\omega\right)\right|\right],$$

in virtue of the assumptions on the kernel k, we have

$$\left|\frac{\partial^j}{\partial x^j}k_i\left(x,\omega y,\omega\right)\right| \le \mathcal{C}\,2^{2n}\left(\frac{3d}{4\omega}\right)^j.$$

Thus by replacing the above estimate in (3.1.14) we have

$$\left|\frac{\partial^{2n}}{\partial x^{2n}}[F_i(x)k_i(x,\,\omega y,\,\omega)]\right| \le \mathcal{C}\left(\frac{2d}{\omega}\right)^{2n}\left[\|F\|_\infty + \|F^{(2n)}\|_\infty\right],$$

from which we deduce

$$|\Lambda_n(F,\omega y,\omega)| \le \frac{1}{(2n)!\,\gamma_n(w)}\left(\frac{2d}{\omega}\right)^{2n}\left[\|F\|_\infty + \|F^{(2n)}\|_\infty\right].$$

Therefore, by using the well-known Stirling formula

$$\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\,e^{-\frac{1}{12n}} \le n! \le \left(\frac{n}{e}\right)^n\sqrt{2\pi n}\,e^{-\frac{1}{12n+1}},$$

and, taking into account that [47] $\gamma_n(w)\sim 2^n$, we get the thesis. □

3.1.2 A new mixed quadrature formula

In this Subsection we want to propose a mixed quadrature rule which will be essential for our method. It consists in applying an m-point *product rule* (2.1.1) in order to approximate the integral

$$\int_{-1}^{1} k(x,y,\omega) f(x) w(x) dx$$

and hence in computing the coefficients A_j of such a product rule (defined as in (2.1.2) with $k(x,y) = k(x,y,\omega)$) by means of the n-point *dilation quadrature formula* (2.1.12).

Then, the mixed quadrature formula is the following

$$\begin{aligned}\int_{-1}^{1} k(x,y,\omega) f(x) w(x) dx &= \sum_{j=1}^{m} A_j^n(y,\omega) f\left(\xi_j^w\right) + \mathcal{E}_m^n(f,y,\omega) \\ &=: K_m^n(f,y,\omega) + \mathcal{E}_m^n(f,y,\omega) \qquad (3.1.15)\end{aligned}$$

where $\mathcal{E}_m^n$ is the remainder term and

$$A_j^n(y,\omega) = \frac{d}{2\omega} \sum_{i=1}^{\boldsymbol{S}} \sum_{\nu=1}^{n} \lambda_\nu^{u_i} k_i(\xi_\nu^{u_i}, \omega y, \omega)\, \ell_{j,i}^w(\xi_\nu^{u_i}),$$

with k_i and u_i as in (3.1.6) and (3.1.5), respectively, and
$\ell_{j,i}^w\left(\xi_\nu^{u_i}\right) := \ell_j^w\left(\Psi_i^{-1}\left(\frac{\xi_\nu^{u_i}}{\omega}\right)\right)$ being ℓ_j^w and Ψ_i^{-1} defined as in (1.1.13) and (3.1.3), respectively.

Next theorem gives an error estimate for $\mathcal{E}_m^n$ in the case when $n = m$.

Theorem 3.1.4. *Let w and σ be defined in* (1.1.1) *and* (1.1.2)*, respectively with*

$$\max\left\{0, \frac{\alpha}{2} + \frac{1}{4}\right\} < \gamma < \min\left\{1, \alpha+1, \frac{\alpha}{2} + \frac{5}{4}\right\}, \qquad (3.1.16)$$

$$\max\left\{0, \frac{\beta}{2} + \frac{1}{4}\right\} < \delta < \min\left\{1, \beta+1, \frac{\beta}{2} + \frac{5}{4}\right\}. \qquad (3.1.17)$$

If $f \in C_\sigma$ and the kernel function k satisfies the conditions (2.1.3), (2.1.5) *and the assumptions given in Theorem 3.1.1, the following error estimate holds true*

$$|\mathcal{E}_m^m(f,\omega)| \;\leq\; \mathcal{C}\left[E_m(f)_\sigma + \left(\frac{d}{\omega}\right)^{m-1} \log m \|f\sigma\|_\infty\right], \qquad \mathcal{C} \neq \mathcal{C}(m,\omega).$$

Remark 3.1.5. *Let us remark that if $\alpha, \beta < -\frac{1}{2}$, then the parameters of the weight σ could also be chosen equal to zero. Moreover, in Theorem 3.1.4, for the sake of simplicity, we considered the case $m = n$. Nevertheless in practice in the numerical test we can use n fixed. Indeed according with* (3.1.13) *the error decreases exponentially and, for instance, for $n = 20$, $d = 2$ and $\omega = 10^2$, the quantity before the square brackets is of the order 10^{-98}. Hence, the error of the mixed quadrature formula is, in practice, of the same order of the error of best approximation of f.*

Proof. By (3.1.15) we can write

$$\begin{aligned}|\mathcal{E}_m^m(f,y,\omega)| &\le \left|\int_{-1}^{1} k(x,y,\omega)(fw)(x)dx - \sum_{j=1}^{m} A_j(y) f(\xi_j^w)\right| \\ &+ \left|\sum_{j=1}^{m} \left((A_j - A_j^m)(y)\right) f(\xi_j^w)\right| \\ &\le |\mathcal{E}_m(f,y,\omega)| + \|f\sigma\|_\infty \sum_{j=1}^{m} \frac{|\Lambda_m(\ell_j^w, \omega y, \omega)|}{\sigma(\xi_j^w)}.\end{aligned}$$

The first term can be estimated by using (2.1.6) since (3.1.16) and (3.1.17) include (2.1.5). Let us now estimate the last one. By using (3.1.9) with $r = m - 1$ we can have

$$|\Lambda_m(\ell_j^w, \omega y, \omega)| \le \frac{\mathcal{C}}{m^{m-1}} \left(\frac{d}{\omega}\right)^{m-1} \|\ell_j^w\|_{\mathcal{W}^r_{\sigma,\infty}}$$

and thus, by applying the weighted Bernstein inequality (see, for instance [47, p. 170]) which leads to state that $\|\ell_j^w\|_{\mathcal{W}^r_{\sigma,\infty}} \le \mathcal{C} m^{m-1} \|\ell_j^w \sigma\|_\infty$, we get

$$|\Lambda_m(\ell_j^w, \omega y, \omega)| \le \mathcal{C} \left(\frac{d}{\omega}\right)^{m-1} \|\ell_j^w \sigma\|_\infty.$$

Therefore

$$\sum_{j=1}^{m} \frac{|\Lambda_m(\ell_j^w, \omega y, \omega)|}{\sigma(\xi_j^w)} \le \left(\frac{d}{\omega}\right)^{m-1} \sum_{j=1}^{m} \frac{\|\ell_j^w \sigma\|_\infty}{\sigma(\xi_j^w)} \le \mathcal{C} \left(\frac{d}{\omega}\right)^{m-1} \log m$$

being [47], in virtue of (2.1.5)

$$\max_{|x|\le 1} \sum_{j=1}^{m} \frac{|\ell_j^w(x)|}{\sigma(\xi_j^w)} \sigma(x) \simeq \log m, \tag{3.1.18}$$

and the proof is completed. □

3.1.3 The numerical method

In this Subsection we propose a numerical method for the univariate Love integral equation (3.0.3) which can be rewritten in operatorial form as

$$(I - K)\, f = g, \tag{3.1.19}$$

where I is the identity operator and

$$(Kf)(y,\omega) = \frac{1}{\pi}\int_{-1}^{1} k(x,y,\omega)\, f(x)\, w(x) dx \tag{3.1.20}$$

with

$$k(x,y,\omega) = \frac{\omega^{-1}}{(x-y)^2 + \omega^{-2}}. \tag{3.1.21}$$

The next proposition shows the mapping properties of the operator K.

Proposition 3.1.6. *Let σ and w be defined in* (1.1.2) *and* (1.1.1)*, respectively such that the parameters γ, δ, α and β satisfy*

$$0 \le \gamma < 1 + \alpha, \quad 0 \le \delta < 1 + \beta.$$

Then

$$K : C_\sigma \to C_\sigma$$

is continuous, bounded and compact. Moreover,

$$\forall f \in C_\sigma, \quad Kf \in \mathcal{W}^r_{\sigma,\infty}, \forall r \in \mathbb{N}.$$

Remark 3.1.7. *We remark that according to Proposition 3.1.6 and in virtue of the Fredholm Alternative Theorem, under the assumption $Ker\{I+K\} = \{0\}$, equation (3.1.19) has a unique solution $f \in C_\sigma$.*

Proof. First, let us note that the kernel k given in (3.1.21), satisfies the following conditions

$$\max_{|x|\le 1} \|k(x,\cdot,\omega)\sigma\|_\infty < \infty, \qquad \max_{|x|\le 1} \left\| \frac{\partial^r}{\partial y^r} k(x,\cdot,\omega)\varphi^r \sigma \right\|_\infty < \infty, \qquad r \ge 1. \tag{3.1.22}$$

By the definition (3.1.20), and taking into account the conditions on the parameters of the weights, we have

$$\begin{aligned} |(Kf)(y)\sigma(y)| &\le \|f\sigma\|_\infty \int_{-1}^{1} |k(x,y,\omega)\sigma(y)| \frac{w(x)}{\sigma(x)}\, dx \\ &\le \mathcal{C}\, \|f\sigma\|_\infty \max_{|x|\le 1} \|k(x,\cdot,\omega)\sigma\|_\infty \end{aligned}$$

from which, by using (3.1.22), we can deduce that the operator K is continuous and bounded. In order to prove its compactness, we remind that [79] if K satisfies the following condition

$$\lim_{m\to\infty} \sup_{\|f\sigma\|_\infty=1} E_m(Kf)_\sigma = 0 \tag{3.1.23}$$

then K is compact. We note that

$$\begin{aligned}\left|(Kf)^{(r)}(y)(\varphi^r\sigma)(y)\right| &\le \int_{-1}^{1} |f(x)\sigma(x)| \left|\frac{\partial^r}{\partial y^r}k(x,y,\omega)\varphi^r(y)\sigma(y)\right| \frac{w(x)}{\sigma(x)}\,dx \\ &\le \|f\sigma\|_\infty \max_{|x|\le 1}\left\|\frac{\partial^r}{\partial y^r}k(x,\cdot,\omega)\varphi^r\sigma\right\|_\infty \int_{-1}^{1}\frac{w(x)}{\sigma(x)}\,dx.\end{aligned}$$

Hence, $Kf \in \mathcal{W}^r_{\sigma,\infty}$ for each $f \in C_\sigma$, and by using the Favard inequality (3.1.11) with m instead of n, Kf in place of H and σ in place of v, we deduce (3.1.23). □

The proposed numerical strategy is a Nyström method based on the mixed quadrature formula $K_m^n f$ introduced in (3.1.15). Then, we consider the functional equation

$$(I - K_m^n)\, f_m^n = g, \tag{3.1.24}$$

where f_m^n is unknown and we included the constant $\frac{1}{\pi}$ in the definition of K_m^n. We multiply both sides of (3.1.24) by the weight function σ and we collocate each equation at the points $\{\xi_i^w\}_{i=1}^m$. In this way we have that the quantities $a_i = (f_m^n\sigma)(\xi_i^w)$ are the unknowns of the following $m \times m$ linear system

$$\sum_{j=1}^{m}\left[\delta_{ij} - \frac{1}{\pi}\sigma(\xi_i^w)\frac{A_j^n(\xi_i^w,\omega)}{\sigma(\xi_j^w)}\right] a_j = (g\sigma)(\xi_i^w), \qquad i = 1,...,m, \tag{3.1.25}$$

where δ_{ij} is the Kronecker symbol. In terms of matrices the system is

$$[\mathbb{I} - \mathbb{A}_m]\mathbb{a} = \mathbb{b},$$

where $\mathbb{I}$ is the identity matrix of order m and

$$[\mathbb{A}_m]_{i,j=1}^m = -\frac{1}{\pi}\frac{\sigma(\xi_i^w)}{\sigma(\xi_j^w)}A_j^n(\xi_i^w,\omega), \qquad [\mathbb{b}]_{i=1}^n = (g\sigma)(\xi_i^w), \qquad [\mathbb{a}]_{i=1}^m = a_i.$$

Once solved, its solution $[\mathbb{a}^*]_{i=1}^m = a_i^*$ allows us to construct the following weighted Nyström interpolant

$$(f_m^n\sigma)(y) = (g\sigma)(y) + \frac{1}{\pi}\sigma(y)\sum_{j=1}^{m}\frac{A_j^n(y,\omega)}{\sigma(\xi_j^w)}a_j^*, \tag{3.1.26}$$

which will approximate the unknown solution $f \in C_\sigma$.

Next theorem states that the above described Nyström method is stable and convergent, as well as, that the condition number in infinity norm of the matrix $\mathbb{A}_m$ i.e. $cond(\mathbb{A}_m) = \|\mathbb{A}_m\|_\infty \|\mathbb{A}_m^{-1}\|_\infty$ is bounded by a constant which does not depend on m.

Theorem 3.1.8. *Let w and σ be defined in* (1.1.1) *and* (1.1.2)*, respectively with parameters satisfying* (3.1.16) *and* (3.1.17)*, and let us assume that $Ker\{I-K\} = \{0\}$ in C_σ.*

Then, if $g \in \mathcal{W}^r_{\sigma,\infty}$, $r > 1$, for m sufficiently large, the operators $(I - K_m^m)^{-1}$ exist and are uniformly bounded. Moreover, system (3.1.25) is well conditioned, since $cond(\mathbb{A}_m) \le \mathcal{C}$ with $\mathcal{C} \ne \mathcal{C}(m)$ and the following estimate holds true

$$\|[f - f_m^m]\sigma\|_\infty \le \mathcal{C}\left[\frac{1}{m^r} + \left(\frac{d}{\omega}\right)^{m-1} \log m\right] \|f\|_{\mathcal{W}^r_{\sigma,\infty}}, \qquad \mathcal{C} \ne \mathcal{C}(m,f). \tag{3.1.27}$$

Proof. The goal of the proof is to prove that

1. $\|(K - K_m^m) f\sigma\|$ tends to zero for any $f \in C_\sigma$

2. The set of the operators $\{K_m\}_m$ is collectively compact.

In fact, by condition 1., in virtue of the principle of uniform boundedness, we can deduce that $\sup_m \|K_m^m\| < \infty$ and, by condition 2. we can deduct that $\|(K - K_m^m)K_m^m\|$ tends to zero [2, Lemma 4.1.2]. Consequently, under all these conditions, we can claim that for m sufficiently large, the operator $(I - K_m^m)^{-1}$ exists and it is uniformly bounded since

$$\|(I - K_m^m)^{-1}\| \le \frac{1 + \|(I-K)^{-1}\| \|K_m^m\|}{1 - \|(I-K)^{-1}\| \|(K - K_m^m)K_m^m\|},$$

i.e. the method is stable.
Condition 1. follows by Theorem 3.1.4. Condition 2. can be deducted by [35, Theorem 12.8] for the case $\gamma = \delta = 0$. Concerning the general case it is sufficient to prove that [79]

$$\lim_{m\to\infty} \sup_{\|f\sigma\|_\infty = 1} E_m(K_m^m f)_\sigma = 0. \tag{3.1.28}$$

To this end let us introduce $\boldsymbol{S}$ polynomials $q_{m,i}(x,y)$ with $i = 1, ..., \boldsymbol{S}$ of degree m in each variable, and for any $f \in C_\sigma$, let us define the univariate polynomial

$$(Q_m f)(y,\omega) = \frac{d}{2\omega}\sum_{j=1}^{m}\sum_{i=1}^{\boldsymbol{S}}\sum_{\nu=1}^{m}\lambda_\nu^{u_i}\ell_{j,i}^w\left(\xi_\nu^{u_i}\right) q_{m,i}(\xi_\nu^{u_i},y) f(\xi_j^w),$$

where we recall that $\ell_{j,i}^w\left(\xi_\nu^{u_i}\right) := \ell_j^w\left(\Psi_i^{-1}\left(\frac{\xi_\nu^{u_i}}{\omega}\right)\right)$.

Then, in virtue of the definition (3.1.15), by applying (3.1.10), (3.1.18) and taking into account the assumptions on the parameters of the weights, we can write

$$\begin{aligned}
&|[(K_m^m f - Q_m f)(y,\omega)]\sigma(y)| \\
&\le \frac{d}{2\omega}\sum_{j=1}^{m}\left| f(\xi_j^w)\sigma(\xi_j^w)\sum_{i=1}^{\boldsymbol{S}}\sum_{\nu=1}^{m}\lambda_\nu^{u_i}\frac{\ell_{j,i}^w(\xi_\nu^{u_i})}{\sigma(\xi_j^w)}\left(k_i\left(\xi_\nu^{u_i},\omega y,\omega\right) - q_{m,i}\left(\xi_\nu^{u_i},y\right)\right)\sigma(y)\right| \\
&\le \frac{d}{2\omega}\,\|f\sigma\|_\infty\sum_{j=1}^{m}\sum_{i=1}^{\boldsymbol{S}}\sum_{\nu=1}^{m}\lambda_\nu^{u_i}\left|\frac{\ell_{j,i}^w(\xi_\nu^{u_i})}{\sigma(\xi_j^w)}\left(k_i\left(\xi_\nu^{u_i},\omega y,\omega\right) - q_{m,i}\left(\xi_\nu^{u_i},y\right)\right)\sigma(y)\right| \\
&= \frac{d}{2\omega}\|f\sigma\|_\infty\sum_{j=1}^{m}\sum_{i=1}^{\boldsymbol{S}}\sum_{\nu=1}^{m}\frac{\lambda_\nu^{u_i}}{\sigma\left(\xi_\nu^{u_i}\right)}\left|\frac{\ell_{j,i}^w(\xi_\nu^{u_i})}{\sigma(\xi_j^w)}\sigma\left(\xi_\nu^{u_i}\right)\left(k_i\left(\xi_\nu^{u_i},\omega y,\omega\right) - q_{m,i}\left(\xi_\nu^{u_i},y\right)\right)\sigma(y)\right| \\
&\le \frac{d}{2\omega}\,\|f\sigma\|_\infty\sum_{j=1}^{m}\max_{|x|\le 1}\left|\frac{\ell_j^w(x)}{\sigma(\xi_j^w)}\right|\sigma(x)\sum_{i=1}^{\boldsymbol{S}}\max_{|x|\le 1}\left|\left(k_i\left(x,\omega y,\omega\right) - q_{m,i}\left(x,y\right)\right)\sigma(y)\right| \\
&\times\sum_{\nu=1}^{m}\frac{\lambda_\nu^{u_i}}{\sigma\left(\xi_\nu^{u_i}\right)} \\
&= \frac{d}{2\omega}\,\|f\sigma\|_\infty\sum_{j=1}^{m}\max_{|x|\le 1}\left|\frac{\ell_j^w(x)}{\sigma(\xi_j^w)}\right|\sigma(x)\sum_{i=1}^{\boldsymbol{S}}\max_{|x|\le 1}E_m\left(k_i(x,\cdot,\omega)\right)_\sigma\sum_{\nu=1}^{m}\frac{\lambda_\nu^{u_i}}{\sigma\left(\xi_\nu^{u_i}\right)} \\
&\le \mathcal{C}\frac{d}{2\omega}\log m\,\|f\sigma\|_\infty\sum_{i=1}^{\boldsymbol{S}}\max_{|x|\le 1}E_m\left(k_i(x,\cdot,\omega)\right)_\sigma
\end{aligned}$$

The only point remaining is to estimate the quantity $E_m\left(k_i(x,\cdot,\omega)\right)_\sigma$. To this end, taking into account the definition of k_i given in (3.1.6) and (3.1.22), by using the Favard inequality (3.1.11), we get

$$E_m\left(k_i(x,\cdot,\omega)\right)_\sigma \le \frac{\mathcal{C}}{m^r}\left(\frac{d}{\omega}\right)^r,$$

i.e. (3.1.28). About the well-conditioning of the matrix $\mathbb{A}_m$, it is sufficient to prove that

$$cond(\mathbb{A}_m) \le cond(I - K_m^m) = \|I - K_m^m\|\,\|(I - K_m^m)^{-1}\|.$$

To this end we can use the same arguments in [2, p.113] only by replacing the usual infinity norm with the weighted uniform norm of C_σ. Finally, estimate (3.1.27) follows taking into account that

$$\|(f - f_m^m)\sigma\|_\infty \leq \|(I + K_m^m)^{-1}\| \, \|(K - K_m^m)f\|_\infty$$

and by applying Theorem 3.1.4 to the last term. □

3.2 Two–dimensional Love's Integral Equation

In this Section we present a numerical method for approximating the solution of Love's bivariate equation (3.0.4) defined on the square $S := [-1, 1]^2$.

3.2.1 The 2D-dilation formula: a revisiting

We focus our attention to the approximation of the integrals of the form

$$\mathcal{I}(\boldsymbol{F}, \mathbf{y}, \omega) = \int_S \boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)\boldsymbol{F}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x}, \tag{3.2.1}$$

where $\boldsymbol{F}$ is a given function, $\mathbf{w}$ is as in (1.2.1) and $\boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)$ is a known kernel which is close to be singular if $\omega^{-1} \to 0$. This is the case of the kernel function appearing in the bivariate Love equation (3.0.4).

In Subsection 2.2.2 has been presented a "dilation" cubature formula for approximating integrals of the type

$$\boldsymbol{I}(\boldsymbol{F}, \omega) = \int_S \boldsymbol{k}(\mathbf{x}, \omega)\boldsymbol{F}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x},$$

namely, to integrals in which the kernel function $\boldsymbol{k}$ is not a function of four variables. Our idea is to "generalize" in some sense the approach proposed in Subsection 2.2.2 and provide new error estimates. Then, firstly we restate what it is given in Subsection 2.2.2, in order to have a dilation cubature formula for the case when the kernel is a function of four variables as (3.2.1).

Similarly to the one-dimensional case, we aim to dilate the domain of integration from the square $S = [-1, 1] \times [-1, 1]$ into the dilated square $S_\omega = [-\omega, \omega] \times [-\omega, \omega]$. Thus, in (3.2.1) we make the following change of variables

$$\mathbf{x} = \frac{\boldsymbol{\eta}}{\omega}, \qquad \mathbf{y} = \frac{\boldsymbol{\theta}}{\omega},$$

with $\boldsymbol{\eta} = (\eta_1, \eta_2)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2)$ in S_ω.

Then, by partitioning the new domain S_ω into $\boldsymbol{S}^2$ squares of area d^2 with d such that $\boldsymbol{S} = \frac{2\omega}{d} \in \mathbb{N}$, i.e.

$$S_\omega = \bigcup_{i=1}^{\boldsymbol{S}} S_i \times \bigcup_{j=1}^{\boldsymbol{S}} S_j \quad \text{with} \quad S_\ell = [-\omega + (\ell - 1)d, \, -\omega + \ell d], \;\; \ell \in \{i, j\},$$

we get

$$\begin{aligned}
\mathcal{I}(\boldsymbol{F}, \mathbf{y}, \omega) &= \frac{1}{\omega^2} \int_{S_\omega} \boldsymbol{k}\left(\frac{\boldsymbol{\eta}}{\omega}, \frac{\boldsymbol{\theta}}{\omega}, \omega\right) (\boldsymbol{F}\mathbf{w}) \left(\frac{\boldsymbol{\eta}}{\omega}\right) d\boldsymbol{\eta} \\
&=: \frac{1}{\omega^2} \int_{S_\omega} \boldsymbol{\kappa}\left(\boldsymbol{\eta}, \boldsymbol{\theta}, \omega\right) (\boldsymbol{F}\mathbf{w}) \left(\frac{\boldsymbol{\eta}}{\omega}\right) d\boldsymbol{\eta} \\
&= \frac{1}{\omega^2} \sum_{i=1}^{\boldsymbol{S}} \sum_{j=1}^{\boldsymbol{S}} \int_{S_i \times S_j} \boldsymbol{\kappa}\left(\boldsymbol{\eta}, \omega\mathbf{y}, \omega\right) (\boldsymbol{F}\mathbf{w}) \left(\frac{\boldsymbol{\eta}}{\omega}\right) d\boldsymbol{\eta}. \qquad (3.2.2)
\end{aligned}$$

Then, by using the invertible linear maps

$$\boldsymbol{\Psi}_{ij} : S_i \times S_j \to [-1, \, 1] \times [-1, \, 1]$$

defined as

$$\mathbf{x} = \boldsymbol{\Psi}_{ij}(\boldsymbol{\eta}) = (\Psi_i(\eta_1), \Psi_j(\eta_2)),$$

where Ψ_ℓ with $\ell \in \{i, j\}$, is the map introduced in the previous Section (or, equivalently, in Subsections 2.1.2 and 2.2.2), we can remap each integral into the unit square S. In fact, by making in (3.2.2) the change of variables

$$\boldsymbol{\eta} = \boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x}) = \left(\Psi_i^{-1}(x_1), \Psi_j^{-1}(x_2)\right)$$

we have

$$\mathcal{I}(\boldsymbol{F}, \mathbf{y}, \omega) = \frac{d^2}{4\omega^2} \sum_{i=1}^{\boldsymbol{S}} \sum_{j=1}^{\boldsymbol{S}} \int_S \boldsymbol{k}_{ij}(\mathbf{x}, \omega\mathbf{y}, \omega) \, \boldsymbol{F}_{ij}(\mathbf{x}) \, \boldsymbol{u}_{ij}(\mathbf{x}) d\mathbf{x} \qquad (3.2.3)$$

where $\boldsymbol{F}_{ij}(\mathbf{x}) := \left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega}\right)$, $\boldsymbol{u}_{ij}(\mathbf{x}) := u_{1,i}(x_1) u_{2,j}(x_2)$ with

$$u_{1,i}(x_1) := \begin{cases} v^{0,\beta_1}(x_1), & i = 1 \\ v^{0,0}(x_1), & 2 \le i \le \boldsymbol{S} - 1, \\ v^{\alpha_1,0}(x_1), & i = \boldsymbol{S} \end{cases} \quad u_{2,j}(x_2) := \begin{cases} v^{0,\beta_2}(x_2), & j = 1 \\ v^{0,0}(x_2), & 2 \le j \le \boldsymbol{S} - 1, \\ v^{\alpha_2,0}(x_2), & j = \boldsymbol{S} \end{cases} \qquad (3.2.4)$$

and $\boldsymbol{k}_{ij}$ the new kernel functions defined as

$$\boldsymbol{k}_{ij}(\mathbf{x}, \omega\mathbf{y}, \omega) := \boldsymbol{k}\left(\frac{\boldsymbol{\Psi}_{ij}^{-1}(\mathbf{x})}{\omega}, \mathbf{y}, \omega\right) U_{1,i}\,(x_1, \omega, d)\, U_{2,j}\,(x_2, \omega, d) \tag{3.2.5}$$

with

$$U_{p,\ell}(x_1, \omega, d) := \begin{cases} \left(\frac{d}{2\omega}\right)^{\beta_p} v^{\alpha_p, 0}\left(\frac{\Psi_\ell^{-1}(x_p)}{\omega}\right), & \ell = 1 \\ v^{\alpha_p, \beta_p}\left(\frac{\Psi_\ell^{-1}(x_p)}{\omega}\right), & 2 \le \ell \le \boldsymbol{S} - 1, \\ \left(\frac{d}{2\omega}\right)^{\alpha_p} v^{0, \beta_p}\left(\frac{\Psi_\ell^{-1}(x_p)}{\omega}\right), & \ell = \boldsymbol{S} \end{cases} \qquad \ell \in \{i,\, j\},$$

being $p = 1$ if $\ell = i$ and $p = 2$ if $\ell = j$. By approximating each integral appearing in (3.2.3) by means of the Gauss-Jacobi cubature rule (1.2.16) with $\boldsymbol{u}_{ij}$ in place of $\mathbf{w}$ and $\boldsymbol{k}_{ij}\boldsymbol{F}_{ij}$ instead of $\boldsymbol{f}$, we have the following "dilation" cubature formula

$$\begin{aligned} \mathcal{I}(\boldsymbol{F}, \mathbf{y}, \omega) = &\frac{d^2}{4\omega^2} \sum_{i=1}^{\boldsymbol{S}} \sum_{j=1}^{\boldsymbol{S}} \sum_{h=1}^{n} \sum_{\nu=1}^{n} \lambda_h^{u_{1,i}} \lambda_\nu^{u_{2,j}} \boldsymbol{k}_{ij}(\xi_{h,\nu}^{u_{1,i}, u_{2,j}}, \omega\mathbf{y}, \omega) \boldsymbol{F}_{ij}(\xi_{h,\nu}^{u_{1,i}, u_{2,j}}) \\ &+ \Lambda_{n,n}(\boldsymbol{F}, \omega\mathbf{y}, \omega), \end{aligned}$$

where $\Lambda_{n,n}$ denotes the remainder term.

The given rule is stable and convergent as the following theorem shows.

Theorem 3.2.1. *Let $\boldsymbol{F} \in C_{\boldsymbol{\sigma}}$ with $\boldsymbol{\sigma}$ as in* (1.2.2) *such that*

$$0 \le \gamma_i < \min\{1, 1 + \alpha_i\}, \quad 0 \le \delta_i < \min\{1, 1 + \beta_i\}, \quad i \in \{1, 2\}$$

and let us assume that the kernel function $\boldsymbol{k}$ is such that

$$\max_{\mathbf{x}, \mathbf{y} \in S} |\boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)| < \infty.$$

Then the above formula is stable and for any $\boldsymbol{F} \in W^r_{\boldsymbol{\sigma}, \infty}$, if

$$\sup_{\boldsymbol{y} \in S} \max\left\{ \max_{x_1 \in [-1,1]} \left| \frac{\partial^r \boldsymbol{k}(\mathbf{x}, \boldsymbol{y}, \omega)}{\partial x_1^r} \varphi_1^r(x_1) \right|, \max_{x_2 \in [-1,1]} \left| \frac{\partial^r \boldsymbol{k}(\mathbf{x}, \boldsymbol{y}, \omega)}{\partial x_2^r} \varphi_2^r(x_2) \right| \right\} < \infty \tag{3.2.6}$$

we get

$$\sup_{\mathbf{y} \in S} |\Lambda_{n,n}(\boldsymbol{F}, \omega\mathbf{y}, \omega)| \le \frac{\mathcal{C}}{n^r} \left(\frac{d}{\omega}\right)^r \|\boldsymbol{F}\|_{W^r_{\boldsymbol{\sigma}, \infty}}, \qquad \mathcal{C} \ne \mathcal{C}(n). \tag{3.2.7}$$

Proof. First, we prove the stability of the formula. By definition

$$|\Sigma_{n,n}(\boldsymbol{F}, \omega\mathbf{y}, \omega)| = \frac{d^2}{4\omega^2}\left|\sum_{i=1}^{\boldsymbol{S}}\sum_{j=1}^{\boldsymbol{S}}\sum_{h=1}^{n}\sum_{\nu=1}^{n}\lambda_h^{u_{1,i}}\lambda_\nu^{u_{2,j}}\boldsymbol{k}_{ij}(\xi_{h,\nu}^{u_{1,i},u_{2,j}}, \omega\mathbf{y}, \omega)\boldsymbol{F}_{ij}(\xi_{h,\nu}^{u_{1,i},u_{2,j}})\right|$$

$$\leq \frac{d^2}{4\omega^2}\|\boldsymbol{F}\boldsymbol{\sigma}\|_\infty \sum_{i=1}^{\boldsymbol{S}}\sum_{j=1}^{\boldsymbol{S}} \sup_{\xi_{h,\nu}^{u_{1,i},u_{2,j}}\in S}|\boldsymbol{k}_{ij}(\xi_{h,\nu}^{u_{1,i},u_{2,j}}, \omega\mathbf{y}, \omega)| \sum_{h=1}^{n}\frac{\lambda_h^{u_{1,i}}}{\sigma(\xi_h^{u_{1,i}})}\sum_{\nu=1}^{n}\frac{\lambda_\nu^{u_{2,j}}}{\sigma(\xi_\nu^{u_{2,j}})}$$

from which taking into account the assumption on the kernel and on the weights and by applying (3.1.10) we can deduce the stability of the formula.

In order to prove (3.2.7), we proceede as done for the proof of (3.1.9). Taking into account (1.2.17) we can write with $N = [\frac{2n-1}{2}]$

$$|\Lambda_{n,n}(\boldsymbol{F}, \omega y, \omega)| \leq \mathcal{C}\sum_{i=1}^{\boldsymbol{S}}\sum_{j=1}^{\boldsymbol{S}}\Bigg(\|\boldsymbol{F}_{ij}\boldsymbol{\sigma}\|_\infty E_{N,N}(\boldsymbol{k}_{ij}) + \sup_{\mathbf{x}\in S}|\boldsymbol{k}_{ij}(\mathbf{x}, \omega y, \omega)|E_{N,N}(\boldsymbol{F}_{ij})_{\boldsymbol{\sigma}}\Bigg).$$

Then, by applying the Favard inequality according to which for any bivariate function $\boldsymbol{h} \in W^r_{\boldsymbol{\sigma},\infty}$ we have $E_{m,m}(h)_{\boldsymbol{\sigma}} \leq \mathcal{C}\frac{\mathcal{M}_r(\boldsymbol{h},\boldsymbol{\sigma})}{m^r}$ with $\mathcal{M}_r(\boldsymbol{h}, \boldsymbol{\sigma})$ as in (1.2.8), we get

$$|\Lambda_{n,n}(\boldsymbol{F}, \omega\mathbf{y}, \omega)| \leq \frac{\mathcal{C}}{n^r}\sum_{i=1}^{\boldsymbol{S}}\sum_{j=1}^{\boldsymbol{S}}\Bigg(\|\boldsymbol{F}_{ij}\boldsymbol{\sigma}\|_\infty\mathcal{N}_r(\boldsymbol{k}_{ij}, \mathbf{y}) + \sup_{\mathbf{x}\in S}|\boldsymbol{k}_{ij}(\mathbf{x}, \omega\mathbf{y}, \omega)|\ \mathcal{M}_r(\boldsymbol{F}_{ij}, \boldsymbol{\sigma})\Bigg)$$

with

$$\mathcal{N}_r(\boldsymbol{k}_{ij}, \boldsymbol{y}) := \max\left\{\max_{\mathbf{x}\in S}\left|\frac{\partial^r\boldsymbol{k}_{ij}(\mathbf{x}, \omega\mathbf{y})}{\partial x_1^r}\varphi_1^r(x_1)\right|, \max_{\mathbf{x}\in S}\left|\frac{\partial^r\boldsymbol{k}_{ij}(\mathbf{x}, \omega\mathbf{y})}{\partial x_2^r}\varphi_2^r(x_2)\right|\right\}.$$

By definition (3.2.5), being

$$\left|\frac{\partial^r\boldsymbol{U}_{1,i}(x_1, x_2, \omega, d)}{\partial x_1^r}\varphi_1^r(x_1)\right| \leq \mathcal{C}\left(\frac{d}{2\omega}\right)^r,$$

and by using (3.2.6), we have

$$\left|\frac{\partial^r\boldsymbol{k}_{ij}(x_1, x_2, \mathbf{y}, \omega)}{\partial x_1^r}\varphi_1^r(x_1)\right| \leq \mathcal{C}\sum_{k=0}^{r}\binom{r}{k}\left(\frac{d}{2\omega}\right)^{r-k}\left(\frac{d}{2\omega}\right)^k = \mathcal{C}\left(\frac{d}{\omega}\right)^r$$

and therefore

$$\sup_{\mathbf{y}\in D} |\Lambda_{n,n}(\boldsymbol{F}, \omega\mathbf{y}, \omega)| \leq \frac{\mathcal{C}}{n^r} \left(\frac{d}{\omega}\right)^r \|\boldsymbol{F}\|_{W^r_{\boldsymbol{\sigma},\infty}}.$$

□

Corollary 3.2.2. *Let $\boldsymbol{F}(x_1, x_2)$ and $\boldsymbol{k}(x_1, x_2, y_1, y_2, \omega)$ be two continuous functions having $2m$ continuous partial derivatives with respect to the variable x_1 and x_2. Then*

$$\sup_{\mathbf{y}\in S} |\Lambda_{n,n}(\boldsymbol{F}, \omega\mathbf{y}, \omega)| \leq \frac{\mathcal{C}}{n^{2n+\frac{1}{2}}} \left(\frac{d}{\omega}\right)^{2n} e^{\frac{48n^2+1}{24n}}\, 2^{2n-1} \left[\|\boldsymbol{F}\|_\infty + \Gamma(\boldsymbol{F})\right]$$

with $\mathcal{C} \neq \mathcal{C}(n, \omega, d)$ and $\Gamma(\boldsymbol{F}) = \max\left\{\left\|\frac{\partial^{2m}\boldsymbol{F}}{\partial x_1^{2m}}\right\|_\infty, \left\|\frac{\partial^{2m}\boldsymbol{F}}{\partial x_2^{2m}}\right\|_\infty\right\}$.

Proof. The thesis can be proved by proceeding mutandis mutandis as in Corollary 3.1.2 taking into account Proposition 1.2.5. □

3.2.2 A new mixed cubature formula

The aim of this Subsection is to propose a cubature formula that we will use in the next Section in order to approximate the solution of the Love bivariate integral equation (3.0.4). To this end, let us approximate the generic integral

$$\int_S \boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)\boldsymbol{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x}$$

by using the product rule (2.2.2) that is

$$\int_S \boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)\boldsymbol{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x} = \sum_{h=1}^{m}\sum_{\nu=1}^{m} A_{h,\nu}(\mathbf{y}, \omega)\boldsymbol{f}\left(\xi_{h,\nu}^{w_1,w_2}\right) + \mathcal{E}_{m,m}(\boldsymbol{f}, \mathbf{y}, \omega)$$

and by approximating the coefficients $A_{h,\nu}$ (defined as in (2.2.3) with $\boldsymbol{k}(\mathbf{x}, \mathbf{y}) = \boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)$)

$$A_{h,\nu}(\mathbf{y}, \omega) = \int_S \boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)\ell_{h,\nu}^{w_1,w_2}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x}$$

by using the "dilation" cubature formula (2.2.21).

In this way we get the following mixed cubature rule

$$\begin{aligned}\int_S \boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)\boldsymbol{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x} &= \sum_{h=1}^{m}\sum_{\nu=1}^{m} A_{h,\nu}^{n,n}(\mathbf{y}, \omega)\boldsymbol{f}\left(\xi_{h,\nu}^{w_1,w_2}\right) + \mathcal{E}_{m,m}^{n,n}(\boldsymbol{f}, \mathbf{y}, \omega)\\ &=: \boldsymbol{K}_{m,m}^{n,n}(\boldsymbol{f}, \mathbf{y}, \omega) + \mathcal{E}_{m,m}^{n,n}(\boldsymbol{f}, \mathbf{y}, \omega) \qquad (3.2.8)\end{aligned}$$

where $\xi_{h,\nu}^{w_1,w_2} := (\xi_h^{w_1}, \xi_\nu^{w_2})$ with $\{\xi_h^{w_1}\}_{h=1}^m$ and $\{\xi_\nu^{w_2}\}_{\nu=1}^m$ the zeros of $p_m(w_1, x_1)$ and $p_m(w_2, x_2)$, respectively, $\mathcal{E}_{m,m}^{n,n}$ the remainder term and

$$\begin{aligned} &A_{h,\nu}^{n,n}(\mathbf{y}, \omega) \\ &= \frac{d^2}{4\omega^2} \sum_{i=1}^{\boldsymbol{S}} \sum_{j=1}^{\boldsymbol{S}} \sum_{p=1}^{n} \sum_{q=1}^{n} \lambda_p^{u_{1,i}} \lambda_q^{u_{2,j}} \ell_{h,\nu}^{w_1,w_2} \left(\boldsymbol{\Psi}_{ij}^{-1} \left(\frac{\xi_{p,q}^{u_{1,i},u_{2,j}}}{\omega} \right) \right) \boldsymbol{k}_{ij}(\xi_{p,q}^{u_{1,i},u_{2,j}}, \omega\mathbf{y}, \omega) \end{aligned}$$

with $\ell_{h,\nu}^{w_1,w_2} \left(\boldsymbol{\Psi}_{ij}^{-1} \left(\frac{\xi_{p,q}^{u_{1,i},u_{2,j}}}{\omega} \right) \right) = \ell_h^{w_1} \left(\Psi_i^{-1} \left(\frac{\xi_p^{u_{1,i}}}{\omega} \right) \right) \ell_\nu^{w_2} \left(\Psi_j^{-1} \left(\frac{\xi_q^{u_{2,j}}}{\omega} \right) \right)$, $\{\lambda_p^{u_{1,i}}\}_{p=1}^n$ and $\{\lambda_q^{u_{2,i}}\}_{q=1}^n$ the Christoffel numbers with respect to the weights $u_{1,i}$ and $u_{2,i}$ given in (3.2.4) and $\boldsymbol{k}_{ij}$ defined as in (3.2.5).

Next theorem gives the conditions on the kernel k and on the weights which ensure the convergence of the above formula by providing an error estimate in the case when $n \equiv m$.

Theorem 3.2.3. *Let $\boldsymbol{f} \in C_{\boldsymbol{\sigma}}$. Then if conditions* (2.2.5) *and* (2.2.6) *are satisfied and the assumptions stated in Theorem 3.2.1 are verified, then the following error estimate holds true*

$$|\mathcal{E}_{m,m}^{m,m}(\boldsymbol{f}, \omega)| \quad \leq \quad \mathcal{C} \left[E_{m,m}(\boldsymbol{f})_{\boldsymbol{\sigma}} + \left(\frac{d}{\omega} \right)^{m-1} m^{2\mu} \|\boldsymbol{f\sigma}\|_\infty \right],$$

where $\mathcal{C} \neq \mathcal{C}(m, \omega)$ and $\mu = \max\{\alpha_i + \frac{1}{2} - 2\gamma_i, \, \beta_i + \frac{1}{2} - 2\delta_i\}$, $i \in \{1, 2\}$.

Remark 3.2.4. *Let us remark that the previous theorem gives the error estimate for $n = m$. In practice we can apply our method with n fixed since in virtue of Corollary 3.2.2 the coefficients of the mixed formula are approximated with an error which decreases exponentially.*

Proof. By (3.2.8) we can write

$$\begin{aligned} |\mathcal{E}_{m,m}^{m,m}(\boldsymbol{f}, \mathbf{y}, \omega)| &\leq \left| \int_S \boldsymbol{k}(\boldsymbol{x}, \mathbf{y}, \omega) \boldsymbol{f}(\boldsymbol{x}) \mathbf{w}(\boldsymbol{x}) d\boldsymbol{x} - \sum_{h=1}^m \sum_{\nu=1}^m A_{h,\nu}(\mathbf{y}) \boldsymbol{f}(\xi_{h,\nu}^{w_1,w_2}) \right| \\ &+ \left| \sum_{h=1}^m \sum_{\nu=1}^m \left(A_{h,\nu}(\mathbf{y}) - A_{h,\nu}^{m,m}(\mathbf{y}) \right) \boldsymbol{f}(\xi_{h,\nu}^{w_1,w_2}) \right| \\ &\leq |\mathcal{E}_{m,m}(\boldsymbol{f}, \mathbf{y}, \omega)| + \|\boldsymbol{f\sigma}\|_\infty \sum_{h=1}^m \sum_{\nu=1}^m \frac{|\Lambda_{m,m}(\ell_{h,\nu}^{w_1,w_2}, \omega\mathbf{y}, \omega)|}{\boldsymbol{\sigma}(\xi_{h,\nu}^{w_1,w_2})}. \end{aligned}$$

By (3.2.7) with $r = m - 1$ we can write

$$|\Lambda_{m,m}(\ell_{h,\nu}^{w_1,w_2}, \omega\mathbf{y}, \omega)| \leq \frac{\mathcal{C}}{m^{m-1}} \left(\frac{d}{\omega} \right)^{m-1} \|\ell_{h,\nu}^{w_1,w_2}\|_{W_{\boldsymbol{\sigma},\infty}^r}$$

and then, by using the weighted Bernstein inequality (see, for instance [47, p. 170]), we get

$$|\Lambda_{m,m}(\ell_{h,\nu}^{w_1,w_2}, \omega\mathbf{y}, \omega)| \leq \mathcal{C}\left(\frac{d}{\omega}\right)^{m-1} \|\ell_{h,\nu}^{w_1,w_2}\boldsymbol{\sigma}\|_\infty,$$

from which [47, Th.4.3.3, p.274 and p.256] we deduce

$$|\mathcal{E}_{m,m}^{m,m}(\boldsymbol{f}, \mathbf{y}, \omega)| \leq |\mathcal{E}_{m,m}(\boldsymbol{f}, \mathbf{y}, \omega)| + \mathcal{C}\left(\frac{d}{\omega}\right)^{m-1} m^{2\mu} \|\boldsymbol{f}\boldsymbol{\sigma}\|_\infty,$$

and consequently the thesis taking into account (2.2.8). □

3.2.3 The numerical method

The goal of this Subsection is to propose a Nyström method for the bivariate Love integral equation (3.0.4) rewritten as

$$(\boldsymbol{I} - \boldsymbol{K})\,\boldsymbol{f} = \boldsymbol{g}, \tag{3.2.9}$$

where $\boldsymbol{I}$ is the identity bivariate operator and

$$(\boldsymbol{K}\boldsymbol{f})(\mathbf{y}, \omega) = \frac{1}{\pi^2}\int_S \boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega)\boldsymbol{f}(\mathbf{x})\,\mathbf{w}(\mathbf{x})d\mathbf{x}$$

with

$$\boldsymbol{k}(\mathbf{x}, \mathbf{y}, \omega) = \frac{\omega^{-1}}{|\mathbf{x} - \mathbf{y}|^2 + \omega^{-2}}. \tag{3.2.10}$$

Before describing such a method, let us investigate on the mapping properties of the operator $\mathbf{K}$.

Proposition 3.2.5. *Let* $\boldsymbol{\sigma}$ *and* $\mathbf{w}$ *be as in* (1.2.2) *and* (1.2.1)*, respectively such that*

$$0 \leq \gamma_i < 1 + \alpha_i, \quad 0 \leq \delta_i < 1 + \beta_i, \quad i \in \{1, 2\}.$$

Then

$$\boldsymbol{K} : C_{\boldsymbol{\sigma}} \to C_{\boldsymbol{\sigma}}$$

is continuous, bounded and compact. Moreover,

$$\forall \boldsymbol{f} \in C_{\boldsymbol{\sigma}}, \quad \boldsymbol{K}\boldsymbol{f} \in W_{\boldsymbol{\sigma},\infty}^r, \forall r \in \mathbb{N}.$$

Remark 3.2.6. *We remark that according to Proposition 3.2.5 and in virtue of the Fredholm Alternative Theorem, under the assumption* $Ker\{\boldsymbol{I} + \boldsymbol{K}\} = \{0\}$*, equation (3.2.9) has a unique solution* $\mathbf{f} \in C_{\boldsymbol{\sigma}}$*.*

Proof. The thesis can be proved by proceeding as done in the proof of Proposition 3.1.6 mutandis mutandis, by noting that the kernel $\boldsymbol{k}$ given in (3.2.10), satisfies the following conditions

$$\max_{\mathbf{x}\in S} \|\boldsymbol{k}(\mathbf{x},\cdot,\omega)\boldsymbol{\sigma}\|_\infty < +\infty,$$

$$\max_{\mathbf{x}\in S} \left\{ \left\| \frac{\partial^r \boldsymbol{k}(\mathbf{x},\cdot,\omega)}{\partial y_1^r} \varphi_1^r \boldsymbol{\sigma} \right\|, \left\| \frac{\partial^r \boldsymbol{k}(\mathbf{x},\cdot,\omega)}{\partial y_2^r} \varphi_2^r \boldsymbol{\sigma} \right\| \right\}_\infty < +\infty, \quad r \geq 1.$$

□

In order to approximate the solution of (3.2.9) let us consider the functional equation

$$\left(\boldsymbol{I} - \boldsymbol{K}_{m,m}^{n,n}\right) \boldsymbol{f}_{m,m}^{n,n} = \boldsymbol{g}, \tag{3.2.11}$$

where $\boldsymbol{f}_{m,m}^{n,n}$ is unknown and $\boldsymbol{K}_{m,m}^{n,n}$ is the mixed cubature operator introduced in (3.2.8), and in the definition of which we included the constant $\frac{1}{\pi^2}$.

Then, we multiply both sides of equation (3.2.11) by the weight function $\boldsymbol{\sigma}$ and we collocate it on the pairs $\xi_{i,j}^{w_1,w_2} := \left(\xi_i^{w_1}, \xi_j^{w_2}\right)$, $i,j = 1,...,m$. In this way we have, for $i,j = 1,...,m$, the following $m^2 \times m^2$ linear system

$$a_{ij} - \frac{1}{\pi^2} \boldsymbol{\sigma}(\xi_{i,j}^{w_1,w_2}) \sum_{h=1}^{m} \sum_{\nu=1}^{m} \frac{A_{h,\nu}^{n,n}(\xi_{i,j}^{w_1,w_2},\omega)}{\boldsymbol{\sigma}(\xi_{h,\nu}^{w_1,w_2})} a_{h\nu} = (\boldsymbol{g}\boldsymbol{\sigma})(\xi_{i,j}^{w_1,w_2}), \tag{3.2.12}$$

where the unknowns $a_{ij} = (\boldsymbol{f}_{m,m}^{n,n}\boldsymbol{\sigma})(\xi_{i,j}^{w_1,w_2})$, $i,j = 1,...,m$ allow us to construct the weighted bivariate Nyström interpolant

$$(\boldsymbol{f}_{m,m}^{n,n}\boldsymbol{\sigma})(\mathbf{y}) = \frac{1}{\pi^2}\boldsymbol{\sigma}(\mathbf{y}) \sum_{h=1}^{m} \sum_{\nu=1}^{m} \frac{A_{h,\nu}^{n,n}(\mathbf{y},\omega)}{\boldsymbol{\sigma}(\xi_{h,\nu}^{w_1,w_2})} a_{h\nu}^* + (\boldsymbol{g}\boldsymbol{\sigma})(\mathbf{y}). \tag{3.2.13}$$

Next theorem states that the above described Nyström method is stable, convergent and the condition number of the system we solve does not depend on m.

Theorem 3.2.7. *Let* $\mathbf{w}$ *and* $\boldsymbol{\sigma}$ *be defined in* (1.2.1) *and* (1.2.2), *respectively with parameters satisfying* (2.2.6), *and let us assume that* $Ker\{\boldsymbol{I}-\boldsymbol{K}\} = \{0\}$ *in* $C_{\boldsymbol{\sigma}}$.

Then if $\boldsymbol{g} \in W_{\boldsymbol{\sigma},\infty}^r$, $r > 2$, *for m sufficiently large, the operators* $\left(\boldsymbol{I} - \boldsymbol{K}_{m,m}^{m,m}\right)^{-1}$ *exist and are uniformly bounded. Moreover, system (3.2.12) is well conditioned, and the following estimate holds true*

$$\|[\boldsymbol{f} - \boldsymbol{f}_{m,m}^{m,m}]\boldsymbol{\sigma}\|_\infty \leq \mathcal{C} \left[\frac{1}{m^r} + \left(\frac{d}{\omega}\right)^{m-1} m^{2\mu} \right] \|\boldsymbol{f}\|_{W_{\boldsymbol{\sigma}}^r}, \tag{3.2.14}$$

with $\mathcal{C} \neq \mathcal{C}(m,\boldsymbol{f})$ *and* $\mu = \max\{\alpha_i + \frac{1}{2} - 2\gamma_i,\ \beta_i + \frac{1}{2} - 2\delta_i\}$, $i \in \{1,2\}$.

Proof. The proof follows in the same line as that of Theorem 3.1.8. □

3.3 Numerical Tests

In this Section we show by some numerical tests the performace of the methods described in the previous Sections. Specifically, for the univariate case, we first test the proposed approach on the classical Love integral equations (Example 3.3.1) and then we show its effectiveness on other two generalized Love's equation (Examples 3.3.2 and 3.3.3). Similarly, for the bivariate case, we apply the method described in Section 3.2 to the classical Love bivariate equation (Example 3.3.4) and finally we test the described method to a specific generalized Love equation defined on the square (Example 3.3.5).

In all the numerical tests the solution f (respectively $\mathbf{f}$) is very smooth and we expect a fast convergence according to estimate (3.1.27) (respectively (3.2.14)).

Example 3.3.1. *Let us consider the classical Love integral equation* (3.0.3) *in the space* C_σ *with* $\sigma \equiv 1$. *We approximate its solution by means of the Nyström interpolant* (3.1.26) *and we compute the absolute errors*

$$err^n_{M,m}(x) = |(f^n_M(x) - f^n_m(x))\sigma(x)|, \tag{3.3.1}$$

in different points $x \in [-1, 1]$. *In* (3.3.1) f^n_M *is the solution assumed to be exact which is obtained with a fixed value* $m = M$. *In Table 3.1 we report the results we get for different choises of* ω. *By comparing them with those presented in [44, Table 1, Table 3 and Table 5], we can see that, in the case when* $\omega = 10^2$ *by solving a square system of* $m = 256$ *equations we get an error of the order* 10^{-16}, *instead of* 10^{-5} *as shown in [44, Table 1]. If* $\omega = 10^3$, *by solving a system of order* 700 *we get the machine precision, accuracy that in [44, Table 3] is reached with a system of* 16384 *equations. Similarly, the method gives accurate results also in the case when* $\omega = 10^4$.

Example 3.3.2. *Let us test our method on the equation*

$$f(y) - \frac{1}{\pi}\int_{-1}^{1} \frac{10^{-2}}{(x-y)^2 + 10^{-4}}\, f(x)\, v^{\frac{1}{2},\frac{1}{2}}(x)\, dx = e^y,$$

namely, a generalized Love integral equation with $\omega = 10^2$. *Table 3.2 shows the errors* (3.3.1) *that we get with* $\sigma(x) = v^{\frac{1}{2},\frac{1}{2}}(x)$, $n = 20$ *and* $M = 350$ *for increasing value of* m. *As we can see by solving a linear system of order* $m = 256$ *we get the machine precision.*

Example 3.3.3. *Let us consider the following generalized Love's integral equation with* $\omega = 10^3$

$$f(y) - \frac{1}{\pi}\int_{-1}^{1} \frac{10^{-3}}{(x-y)^2 + 10^{-6}}\, f(x)\, v^{-\frac{1}{2},-\frac{1}{2}}(x)\, dx = \frac{1+y^3}{y^2+9}$$

Table 3.1: Example 3.3.1: results by 1D-Nyström method.

ω	n	M	m	$err^n_{M,m}(0)$	$err^n_{M,m}(0.5)$	$err^n_{M,m}(0.9)$	$err^n_{M,m}(1)$
10^2	20	350	16	$2.884e-04$	$2.881e-04$	$1.807e-03$	$9.363e-03$
			32	$4.723e-05$	$1.590e-05$	$7.419e-05$	$1.983e-04$
			64	$6.739e-07$	$6.199e-07$	$2.088e-06$	$3.670e-07$
			128	$2.190e-10$	$6.364e-11$	$2.478e-10$	$9.129e-13$
			256	$3.330e-16$	0	$1.110e-16$	0
10^3	20	750	16	$3.215e-05$	$3.249e-05$	$2.390e-04$	$3.783e-02$
			32	$2.422e-05$	$8.784e-06$	$5.220e-05$	$2.554e-02$
			64	$1.375e-05$	$1.382e-05$	$1.011e-04$	$3.233e-03$
			128	$4.872e-07$	$1.758e-07$	$2.617e-06$	$1.785e-05$
			256	$6.148e-09$	$5.999e-09$	$8.002e-09$	$1.633e-08$
			512	$5.880e-13$	$2.007e-13$	$1.246e-12$	$8.881e-16$
			700	$2.886e-16$	$1.110e-16$	$2.220e-15$	$2.220e-16$
10^4	20	750	16	$3.136e-06$	$3.186e-06$	$2.341e-05$	$4.236e-02$
			32	$2.259e-06$	$8.284e-07$	$4.873e-06$	$4.085e-02$
			64	$1.669e-06$	$1.690e-06$	$1.317e-05$	$3.532e-02$
			128	$1.254e-06$	$4.631e-07$	$8.017e-06$	$1.827e-02$
			256	$4.966e-07$	$5.014e-07$	$8.856e-07$	$1.050e-04$
			512	$1.475e-08$	$5.302e-09$	$5.738e-08$	$6.362e-06$
			700	$8.386e-10$	$7.958e-10$	$3.767e-09$	$5.386e-08$

Table 3.2: Example 3.3.2: results by 1D-Nyström method.

m	$err^{20}_{350,m}(0)$	$err^{20}_{350,m}(0.1)$	$err^{20}_{350,m}(0.3)$	$err^{20}_{350,m}(0.7)$
16	$1.099e-04$	$1.617e-05$	$6.341e-05$	$1.650e-04$
32	$7.196e-07$	$7.714e-07$	$7.547e-07$	$1.405e-06$
64	$2.792e-09$	$2.966e-09$	$2.094e-09$	$6.056e-09$
128	$5.410e-13$	$5.526e-13$	$3.153e-14$	$8.726e-13$
256	$8.881e-16$	0	0	$3.330e-16$

in the space C_σ *with* $\sigma(x) = v^{\frac{1}{4},\frac{1}{4}}(x)$. *Table 3.3 shows the accurate results we get also in this case.*

Example 3.3.4. *Let us consider the classical bivariate integral equation* (3.0.4) *in the space* $C_{\boldsymbol{\sigma}}$ *with* $\boldsymbol{\sigma} \equiv 1$. *Table 3.4 shows the values that the weigthed Nyström interpolant* (3.2.13) *(*$n = 20$*) has in different points of the square for increasing values of* m. *Specifically, we test our method for two different values of* ω. *As we can see, by solving a system of order* m^2 *with* $m = 64$ *we get,* 16 *correct decimal digits if* $\omega = 10$ *and about* 10 *correct decimal digits if* $\omega = 10^2$.

Example 3.3.5. *Let us consider the generalized Love bivariate integral equa-*

Table 3.3: Example 3.3.3: results by 1D-Nyström method.

m	$err^{20}_{750,m}(-0.5)$	$err^{20}_{750,m}(0)$	$err^{20}_{750,m}(0.5)$	$err^{20}_{750,m}(0.9)$
16	$1.914e-05$	$5.700e-05$	$6.142e-05$	$4.520e-04$
32	$2.288e-06$	$6.526e-06$	$7.033e-06$	$2.685e-05$
64	$2.639e-07$	$7.536e-07$	$7.980e-07$	$9.026e-06$
128	$6.095e-07$	$1.749e-06$	$1.845e-06$	$8.765e-06$
256	$2.383e-08$	$6.975e-08$	$7.242e-08$	$5.407e-07$
512	$2.914e-12$	$8.866e-12$	$8.802e-12$	$8.087e-13$
700	$5.627e-16$	$1.613e-15$	$1.666e-15$	$9.159e-16$

Table 3.4: Example 3.3.4: results by 2D-Nyström method.

ω	m	$\boldsymbol{f}^{20,20}_{m,m}(0.5, 0.5)$	$\boldsymbol{f}^{20,20}_{m,m}(0.3, 0.99)$	$\boldsymbol{f}^{20,20}_{m,m}(0, 0)$
10	8	$1.16085e+00$	$1.0988e+00$	$1.1796e+00$
	16	$1.160854e+00$	$1.0988559e+00$	$1.179642e+00$
	32	$1.16085413981e+00$	$1.09885600758e+00$	$1.17964277689e+00$
	64	$1.160854139816865e+00$	$1.098856007581168e+00$	$1.179642776903225e+00$
	128	$1.160854139816865e+00$	$1.098856007581168e+00$	$1.179642776903225e+00$
ω	m	$\boldsymbol{f}^{20,20}_{m,m}(0.9, 0.7)$	$\boldsymbol{f}^{20,20}_{m,m}(0.1, 0.6)$	$\boldsymbol{f}^{20,20}_{m,m}(0.5, 0.2)$
10^2	8	$1.02453e+00$	$1.02961e+00$	$1.030007e+00$
	16	$1.024539e+00$	$1.029616e+00$	$1.030007e+00$
	32	$1.02453917e+00$	$1.02961640e+00$	$1.03000799e+00$
	64	$1.0245391724e+00$	$1.029616409104e+00$	$1.03000799184e+00$
	128	$1.024539172475126e+00$	$1.029616409104135e+00$	$1.030007991847319e+00$

tion.

$$\boldsymbol{f}(\mathbf{y}) - \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{\omega^{-1}}{|\mathbf{x}-\mathbf{y}|^2+\omega^{-2}}\boldsymbol{f}(\mathbf{x})\mathbf{w}(\mathbf{x})d\mathbf{x} = \log(10 - y_1 - y_2),$$

with $\mathbf{w}(\mathbf{x}) = v^{\frac{1}{2},\frac{1}{2}}(x_1)v^{\frac{1}{2},\frac{1}{2}}(x_2)$. *Table 3.5 shows the weighted Nyström interpolant with* $\boldsymbol{\sigma}(\mathbf{x}) = v^{\frac{1}{4},\frac{1}{4}}(x_1)v^{\frac{1}{4},\frac{1}{4}}(x_2)$ *for two different values of* ω *in three different points of the square. Once again, we get very accurate results.*

Table 3.5: Example 3.3.5: results by 2D-Nyström method.

ω	m	$(\boldsymbol{f}_{m,m}^{20,20}\boldsymbol{\sigma})(-0.5,-0.2)$	$(\boldsymbol{f}_{m,m}^{20,20}\boldsymbol{\sigma})(0,0)$	$(\boldsymbol{f}_{m,m}^{20,20}\boldsymbol{\sigma})(0.9,-0.9)$
10	8	$2.449868e+00$	$2.64698e+00$	$1.038961e+00$
	16	$2.449868e+00$	$2.6469851e+00$	$1.038961260e+00$
	32	$2.4498685945e+00$	$2.6469851445e+00$	$1.0389612603577e+00$
	64	$2.449868594590459e+00$	$2.646985144594663e+00$	$1.038961260357739e+00$
	128	$2.449868594590459e+00$	$2.646985144594663e+00$	$1.038961260357739e+00$
ω	m	$(\boldsymbol{f}_{m,m}^{20,20}\boldsymbol{\sigma})(-0.9,-0.3)$	$(\boldsymbol{f}_{m,m}^{20,20}\boldsymbol{\sigma})(0.1,0)$	$(\boldsymbol{f}_{m,m}^{20,20}\boldsymbol{\sigma})(0.9,0.9)$
10^2	8	$1.57643e+00$	$2.35163e+00$	$9.22751e-01$
	16	$1.5764310e+00$	$2.351633e+00$	$9.2275158e-01$
	32	$1.576431062e+00$	$2.35163329669e+00$	$9.22751584e-01$
	64	$1.576431062245e+00$	$2.351633296691e+00$	$9.227515843740e-01$
	128	$1.576431062245818e+00$	$2.351633296691469e+00$	$9.2275158437401 88e-01$

Chapter 4

Numerical Methods for Cauchy Bisingular Integral Equations of the First Kind on the Square

In this Chapter we investigate the numerical solution of Cauchy bisingular integral equations of the first kind on the square.

Singular integral equations with Cauchy kernels arise in the mathematical modelling of several problems of the Applied Sciences like aerodynamics, elasticity, fluid flow problems and crack theory [1, 27, 70].

For the univariate case, a general theory on such type of equations is well developed and described in the monographs [25, 55, 64, 74] and several numerical methods have been extensively investigated [3, 12, 16, 28, 32, 33, 38, 39, 40, 49] in terms of stability, convergence, well-conditioning and accuracy of the results.

Concerning the multivariate case, the theoretical analysis of these equations is well studied in the books [42, 54] and several authors focus their research on bisingular integral equations arising from the 3D Helmholtz equations. An example is the following bivariate singular integral equation of the first kind which is strictly related to the stationary problem of a flow past a rectangular airfoil of large span [21]

$$\frac{1}{\pi^2} \oint_{-1}^{1} \oint_{-1}^{1} \frac{\boldsymbol{F}(x,y)}{(x-t)(y-s)}\, dx\, dy = \boldsymbol{g}(t,s),$$

where here and in the sequel the symbol $\oint$ means that the integral has to be interpreted in the Cauchy Principal Value sense, i.e.

$$\oint_{-1}^{1} \oint_{-1}^{1} \frac{\boldsymbol{F}(x,y)}{(x-t)(y-s)}\, dx\, dy := \lim_{\epsilon_1,\epsilon_2 \to 0^+} \int_{|x-t|\geq \epsilon_1} \int_{|y-s|\geq \epsilon_2} \frac{\boldsymbol{F}(x,y)}{(x-t)(y-s)}\, dx\, dy,$$

$(t, s) \in \dot{S}$, ϵ_1, $\epsilon_2 > 0$.

However, even if these equations are of applicative nature, according to our knowledge, very few numerical methods are disposable in the literature [29, 34].

In this Chapter, we propose two different methods based on a global polynomial approximation of the unknown solution.

We underline that, all the results in this Chapter, are new and have recently been presented in [23] and can also be used elsewhere.

The principal aim of this Chapter is to investigate on the numerical treatment of the more general bisingular integral equation of the first kind defined on the square $S = [-1, 1] \times [-1, 1]$

$$\frac{1}{\pi^2} \oint_S \frac{\boldsymbol{F}(x, y)}{(x-t)(y-s)} \, dx \, dy + \int_S \boldsymbol{k}(x, y, t, s) \boldsymbol{F}(x, y) \, dx \, dy = \boldsymbol{g}(t, s), \quad (4.0.1)$$

where $\boldsymbol{F}$ is the bivariate unknown function and $\boldsymbol{k}$ and $\boldsymbol{g}$ are given functions defined on S^2 and S, respectively.

According to [21, 28], the solution of the above equation can be singular along two o more edges of the square S and the behavior of the singularities is known.

In this thesis we consider the case when the solution turns to be unbounded at $x = y = -1$ and thus [21, 28] it can be expressed as

$$\boldsymbol{F}(x, y) = \boldsymbol{f}(x, y) \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-y}{1+y}},$$

where $\boldsymbol{f}$ has to be determined. In a nutshell, the function $\boldsymbol{F}$ has a behaviour similar to that of the solution of the airfoil equation in the univariate case [74]. We remark that the other cases (i.e. unboundedness at $x = 1, y = 1$, or at $x = 1, y = -1$, or at $x = -1, y = 1$) can be treated similarly.

Hence, equation (4.0.1) can be rewritten as

$$(D + K)\boldsymbol{f} = \boldsymbol{g}, \quad (4.0.2)$$

where D is the dominant operator

$$D\boldsymbol{f}(t, s) = \frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x, y)}{(x-t)(y-s)} \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-y}{1+y}} \, dx \, dy \quad (4.0.3)$$

and K is the perturbation operator

$$K\boldsymbol{f}(t, s) = \int_S \boldsymbol{k}(x, y, t, s) \boldsymbol{f}(x, y) \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-y}{1+y}} \, dx \, dy. \quad (4.0.4)$$

In this thesis, for the numerical treatment of (4.0.2), we propose two different approaches, both based on a global polynomial approximation of the unknown bivariate function $\boldsymbol{f}$. The first one is a *direct method* since we act directly on the equation, while the second one is an *indirect procedure*, since we go to solve an equivalent *regularized Fredholm equation*.

In both cases, by using a suitable Lagrange interpolating operator, we project the considered equation into the subspace of polynomials and we discretize the integrals by using a suitable Gaussian cubature formula and by applying the fundamental invariance property of D on the orthogonal polynomials. Then, by collocation on suitable nodes, we end up with a linear system whose unknowns are the coefficients of the polynomial approximating the exact solution.

For both methods, we give a complete analysis in suitable weighted L^2 spaces. In details, we examine the stability, show the related convergence results and error estimates, and discuss the condition numbers of the systems we get.

Comparing the presented two procedures, they are equivalent in terms of convergence order and computational costs, at least when in the indirect approach we can compute exactly the involved integrals. Otherwise the indirect procedure is more expensive. Nevertheless the strategy of using the Fredholm equation equivalent to the Cauchy singular one, can be much easier extended to other functional spaces.

We underline that in order to achieve such theoretical analysis, we needed to prove some auxiliary results concerning the mapping properties of the involved integral operators and the bivariate Lagrange and Fourier operators. Also these auxiliary results are new and can also be used elsewhere. With respect to the preliminary results about bivariate Lagrange and Fourier operators, these have been already reported in Subsection 1.2.2.

This Chapter is structured into four Sections. In Section 4.1 we state the mapping properties of the integral operators D and K. Sections 4.2 and 4.3 are devoted to the two different methods we propose and whose numerical tests are showed in Section 4.4.

4.1 Mapping properties of the dominant and perturbation operators

In this Section we investigate on the mapping properties of the dominant operator D and the perturbation operator K, involved in equation (4.0.2). To this end, let $\boldsymbol{v}$ be the product of two fourth kind Chebyshev weight functions, i.e.

$$\boldsymbol{v}(x,y) = u(x)u(y), \quad \text{with} \quad u(z) = \sqrt{\frac{1-z}{1+z}}. \tag{4.1.1}$$

According to the above notation, we rewrite the dominant operator D introduced in (4.0.3) as

$$D\boldsymbol{f}(t,s) = \frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)} \, \boldsymbol{v}(x,y) dx\, dy. \tag{4.1.2}$$

By using standard arguments, it is not hard to prove that the adjoint operator of D has the following form

$$\widehat{D}\boldsymbol{f}(t,s) = \frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)} \boldsymbol{v}^{-1}(x,y) \, dx\, dy. \tag{4.1.3}$$

Now we recall the explicit expression for $p_m(u,z)$ and $p_m(u^{-1},z)$ (the fourth and third kind Chebyshev orthonormal polynomials with respect to the weights u and u^{-1}, respectively), namely [26, 46]

$$p_m(u,z) = \frac{\sin\left(\left(m+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}, \qquad z = \cos\theta, \quad 0 \le \theta \le \pi, \tag{4.1.4}$$

and

$$p_m(u^{-1},z) = \frac{\cos\left(\left(m+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{1}{2}\theta\right)}, \qquad z = \cos\theta, \quad 0 \le \theta \le \pi. \tag{4.1.5}$$

Next results state useful properties of the operators D, $\widehat{D}$ and K which are basic for our methods.

Lemma 4.1.1. *Let u be defined in* (4.1.1), $q_m(t,s) = p_m(u,t)p_m(u,s)$ *and* $r_m(t,s) = p_m(u^{-1},t)p_m(u^{-1},s)$*. Then,*

$$Dq_m(t,s) = r_m(t,s) \tag{4.1.6}$$

and

$$\widehat{D}r_m(t,s) = q_m(t,s). \tag{4.1.7}$$

Proof. Taking into account the definition of the dominant operator D, we write

$$\begin{aligned} Dq_m(t,s) &= \frac{1}{\pi^2} \oint_S \frac{q_m(x,y)}{(x-t)(y-s)} \, u(x)\, u(y) \, dx\, dy \\ &= \left[\frac{1}{\pi} \oint_{-1}^{1} \frac{p_m(u,x)}{(x-t)} \, u(x) \, dx\right] \left[\frac{1}{\pi} \oint_{-1}^{1} \frac{p_m(u,y)}{(y-s)} \, u(y) \, dy\right] \\ &= p_m(u^{-1},t) \, p_m(u^{-1},s) \\ &= r_m(t,s) \end{aligned}$$

being [55, 74]

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u,z)}{(z-\eta)}u(z)\,dz = p_m(u^{-1},\eta).$$

Analogously,

$$\begin{aligned}
\widehat{D}r_m(t,s) &= \frac{1}{\pi^2}\oint_S \frac{r_m(x,y)}{(x-t)(y-s)}\,u^{-1}(x)\,u^{-1}(y)\,dx\,dy\\
&= \left[-\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u^{-1},x)}{(x-t)}\,u^{-1}(x)\,dx\right]\left[-\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u^{-1},y)}{(y-s)}\,u^{-1}(y)\,dy\right]\\
&= p_m(u,t)\,p_m(u,s)\\
&= q_m(t,s)
\end{aligned}$$

since [55, 74]

$$-\frac{1}{\pi}\oint_{-1}^{1}\frac{p_m(u^{-1},z)}{(z-\eta)}\,u^{-1}(z)\,dz = p_m(u,\eta).$$

□

For brevity, from now on we set $W^r_{\boldsymbol{v}} := W^r_{\boldsymbol{v},2}$, where $W^r_{\boldsymbol{v},2}$ is defined in (1.2.7) with $\boldsymbol{\sigma} = \boldsymbol{v}$.

Proposition 4.1.2. *Let D and $\widehat{D}$ be the operators defined in* (4.1.2) *and* (4.1.3)*, respectively. Then*

$$D : W^r_{\boldsymbol{v}} \to W^r_{\boldsymbol{v}^{-1}} \tag{4.1.8}$$

is continuous and invertible and its two-sided inverse is the continuous operator

$$\widehat{D} : W^r_{\boldsymbol{v}^{-1}} \to W^r_{\boldsymbol{v}}. \tag{4.1.9}$$

In order to prove Proposition 4.1.2, let us note that the dominant operator D can be rewritten in terms of the *Hilbert transform* of a univariate function h

$$H(h,t) = \frac{1}{\pi}\oint_{-1}^{1}\frac{h(x)}{(x-t)}\,u(x)\,dx$$

as follows

$$\begin{aligned}
D\boldsymbol{f}(t,s) &= \frac{1}{\pi^2}\oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)}\,\boldsymbol{v}(x,y)\,dx\,dy = \frac{1}{\pi}\oint_{-1}^{1}\frac{H(f_x,s)}{(x-t)}\,u(x)\,dx\\
&= \frac{1}{\pi}\oint_{-1}^{1}\frac{H(f_y,t)}{(y-s)}\,u(y)\,dy = H\left(H(\boldsymbol{f})\right)(t,s)
\end{aligned}$$

where f_x and f_y denote the function f as a univariate function of the variable y and x, respectively. Let us also remind that for a univariate function h the following estimates hold true [50, 55]

$$\|(Hh)^{(r)}\varphi^r\|_{L^2_{u^{-1}}} \le \|h\|_{\mathcal{W}^r_u}, \qquad \text{and} \qquad \|Hh\|_{L^2_{u^{-1}}} \le \|h\|_{L^2_u}. \tag{4.1.10}$$

Proof. At first we note that, by definition, the operator D is a linear operator. Moreover, by (1.2.12) we have

$$\|D\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}^{-1}}} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c^2_{ij}(D\boldsymbol{f}, \boldsymbol{v}^{-1}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c^2_{ij}(\boldsymbol{f}, \boldsymbol{v}) = \|\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}}} < \infty$$

being, in virtue of (4.1.7)

$$\begin{aligned}
&c^2_{ij}(D\boldsymbol{f}, \boldsymbol{v}^{-1}) = \left(\int_S D\boldsymbol{f}(x,y)\, p_i(u^{-1},x)\, p_j(u^{-1},y)\, \boldsymbol{v}^{-1}(x,y)\, dx\, dy\right)^2 \\
&= \left(\int_S \frac{1}{\pi^2} \oint_S \left[\frac{\boldsymbol{f}(\eta,\xi)}{(\eta-x)(\xi-y)} \boldsymbol{v}(\eta,\xi) d\eta\, d\xi\right] p_i(u^{-1},x) p_j(u^{-1},y)\, \boldsymbol{v}^{-1}(x,y)\, dx\, dy\right)^2 \\
&= \left(\int_S \boldsymbol{f}(\eta,\xi) \left[\frac{1}{\pi^2} \oint_S \frac{p_i(u^{-1},x)\, p_j(u^{-1},y)}{(x-\eta)(y-\xi)}\, \boldsymbol{v}^{-1}(x,y)\, dx\, dy\right] \boldsymbol{v}(\eta,\xi)\, d\eta\, d\xi\right)^2 \\
&= \left(\int_S \boldsymbol{f}(\eta,\xi)\, p_i(u,\eta)\, p_j(u,\xi)\, \boldsymbol{v}(\eta,\xi)\, d\eta\, d\xi\right)^2 = c^2_{ij}(\boldsymbol{f}, \boldsymbol{v}).
\end{aligned}$$

Moreover, by applying (4.1.10) and taking into account that

$$(a+b)^2 \le 2(a^2+b^2), \quad \forall a, b \in \mathbb{R}$$

and

$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b}, \quad \forall a, b \in \mathbb{R}^+,$$

we have

$$\begin{aligned}
&\left(\int_{-1}^1 \int_{-1}^1 \left|\frac{\partial^r}{\partial t^r} D\boldsymbol{f}(t,s)\, \varphi^r(t)\right|^2 \boldsymbol{v}^{-1}(t,s)\, dt\, ds\right)^{\frac{1}{2}} \\
&= \left(\int_{-1}^1 \int_{-1}^1 \left|H\left(\frac{\partial^r}{\partial t^r} H(\boldsymbol{f})\right)(t,s)\, \varphi^r(t)\right|^2 u^{-1}(t)\, u^{-1}(s)\, dt\, ds\right)^{\frac{1}{2}} \\
&\le \mathcal{C}\left(\int_{-1}^1 u(s) \int_{-1}^1 \left|\frac{\partial^r}{\partial t^r} H(\boldsymbol{f})(t,s)\, \varphi^r(t)\right|^2 u^{-1}(t)\, dt\, ds\right)^{\frac{1}{2}} \\
&\le \mathcal{C}\left(\int_{-1}^1 u(s) \left[\int_{-1}^1 \left|\frac{\partial^r}{\partial t^r} \boldsymbol{f}(t,s)\, \varphi^r(t)\right|^2 u(t)\, dt + \int_{-1}^1 |\boldsymbol{f}(t,s)|^2\, u(t)\, dt\right]^2 ds\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \mathcal{C}\left\{\left[\int_{-1}^{1} u(s)\left(\int_{-1}^{1}\left|\frac{\partial^r}{\partial t^r}\boldsymbol{f}(t,s)\,\varphi^r(t)\right|^2 u(t)\,dt\right)^2 ds\right]^{\frac{1}{2}} + \left[\int_{-1}^{1} u(s)\left(\int_{-1}^{1}|\boldsymbol{f}(t,s)|^2\,u(t)\,dt\right)^2 ds\right]^{\frac{1}{2}}\right\} < \infty$$

which prove the boundedness of $D : W^r_{\boldsymbol{v}} \to W^r_{\boldsymbol{v}^{-1}}$ and consequently its continuity.
Now we show that $\widehat{D}(D\boldsymbol{f}) = \boldsymbol{f}$ and $D(\widehat{D}\boldsymbol{f}) = \boldsymbol{f}$. Let $\boldsymbol{f} \in L^2_{\boldsymbol{v}}(S)$. Taking into account the linearity of the operators D and $\widehat{D}$, and applying firstly (4.1.6) and then the fact that $c_{ij}\left(Df, v^{-1}\right) \equiv c_{ij}\left(f, v\right)$, we have

$$\begin{aligned}\widehat{D}\left(Df\right) &= \widehat{D}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}(Df, v^{-1})p_i(u^{-1})p_j(u^{-1})\right)\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij}(f,v)p_i(u)p_j(u) = f.\end{aligned}$$

Proceeding in the same way, we can show also that $D(\widehat{D}\boldsymbol{f}) = \boldsymbol{f}$ and hence $\widehat{D} \equiv D^{-1}$. As regards to the mapping property (4.1.9) of $\widehat{D}$, this can be proved as done for the property (4.1.8). □

From now on we denote by $\boldsymbol{k}_{(x,y)}$ and $\boldsymbol{k}_{(t,s)}$ the kernel function $\boldsymbol{k}(x,y,t,s)$ in (4.0.4) as a function of the only variables (t,s) and (x,y), respectively.

Proposition 4.1.3. *Let K be defined in* (4.0.4) *and let us assume that the kernel function $\boldsymbol{k}$ satisfies the following conditions*

$$\sup_{(t,s)\in S}\|\boldsymbol{k}_{(t,s)}\|_{W^r_{\boldsymbol{v}}} < \infty, \qquad \sup_{(x,y)\in S}\|\boldsymbol{k}_{(x,y)}\|_{W^{r_1}_{\boldsymbol{v}^{-1}}} < \infty, \tag{4.1.11}$$

for some positive integers numbers r and r_1. Then the perturbation operator

$$K : L^2_{\boldsymbol{v}}(S) \to W^{r_1}_{\boldsymbol{v}^{-1}}$$

is linear and bounded if $r_1 \leq r$. Moreover, K is a compact operator for all $r_1 < r$.

Proof. The linearity of the operator K is a trivial consequence of its definition (4.0.4) while the boundedness follows by

$$\|K\boldsymbol{f}\|_{W^{r_1}_{\boldsymbol{v}^{-1}}} = \|K\boldsymbol{f}\|_{L^2_{\boldsymbol{v}^{-1}}} + \mathcal{M}_{r_1}(K\boldsymbol{f}, \boldsymbol{v}^{-1}) \leq \mathcal{C}\,\|f\|_{L^2_{\boldsymbol{v}}}. \tag{4.1.12}$$

In fact, by applying Schwarz's inequality and taking into account the first hypotheses on the kernel function $\boldsymbol{k}$, we have

$$\begin{aligned}
\|K\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}^{-1}}} &= \int_S |K\boldsymbol{f}(t,s)|^2 \boldsymbol{v}^{-1}(t,s)\,dt\,ds \\
&= \int_S \left| \int_S \boldsymbol{k}(x,y,t,s)\boldsymbol{f}(x,y)\boldsymbol{v}(x,y)dx\,dy \right|^2 \boldsymbol{v}^{-1}(t,s)\,dt\,ds \\
&\leq \|\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}}} \sup_{(t,s)\in S} \|\boldsymbol{k}_{(t,s)}\|^2_{W^r_{\boldsymbol{v}}} \int_S \boldsymbol{v}^{-1}(t,s)\,dt\,ds \\
&\leq \mathcal{C}\|\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}}}.
\end{aligned}$$

Moreover, by using again the Schwarz inequality we can write

$$\begin{aligned}
\left| \frac{\partial^{r_1}(K\boldsymbol{f})(t,s)}{\partial t^{r_1}} \right|^2 &= \left| \frac{\partial^{r_1}}{\partial t^{r_1}} \int_S \boldsymbol{k}(x,y,t,s)\boldsymbol{f}(x,y)\,\boldsymbol{v}(x,y)\,dx\,dy \right|^2 \\
&= \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}} \left(\int_S \left| \frac{\partial^{r_1}\boldsymbol{k}(x,y,t,s)}{\partial t^{r_1}} \right|^2 \boldsymbol{v}(x,y)\,dx\,dy \right)
\end{aligned}$$

from which we can deduce

$$\begin{aligned}
&\int_S \left| \frac{\partial^{r_1}(K\boldsymbol{f})(t,s)}{\partial t^{r_1}} \varphi^{r_1}(t) \right|^2 \boldsymbol{v}^{-1}(t,s)\,dt\,ds \\
&\leq \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}} \int_S \left(\int_S \left| \frac{\partial^{r_1}\boldsymbol{k}(x,y,t,s)}{\partial t^{r_1}} \right|^2 \boldsymbol{v}(x,y)dxdy \right) \varphi^{2r_1}(t)\boldsymbol{v}^{-1}(t,s)\,dt\,ds \\
&= \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}} \int_S \left(\int_S \left| \frac{\partial^{r_1}\boldsymbol{k}(x,y,t,s)}{\partial t^{r_1}} \varphi^{r_1}(t) \right|^2 \boldsymbol{v}^{-1}(t,s)dtds \right) \boldsymbol{v}(x,y)\,dx\,dy \\
&\leq \mathcal{C}\|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}} \sup_{(x,y)\in S} \|\boldsymbol{k}_{(x,y)}\|_{W^{r_1}_{\boldsymbol{v}^{-1}}}.
\end{aligned}$$

Analogously

$$\int_S \left| \frac{\partial^{r_1}(K\boldsymbol{f})(t,s)}{\partial s^{r_1}} \varphi^{r_1}(s) \right|^2 \boldsymbol{v}^{-1}(t,s)\,dt\,ds \leq \mathcal{C}\|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}} \sup_{(x,y)\in S} \|\boldsymbol{k}_{(x,y)}\|_{W^{r_1}_{\boldsymbol{v}^{-1}}}.$$

The only point remaining concerns the compactness. To this end let us note that we have

$$\begin{aligned}
E_{m,m}(K\boldsymbol{f})_{W^{r_1}_{\boldsymbol{v}^{-1}}} \leq \|K\boldsymbol{f} - \mathcal{S}_{m,m}(K\boldsymbol{f},\boldsymbol{v})\|_{W^{r_1}_{\boldsymbol{v}^{-1}}} &\leq \frac{\mathcal{C}}{m^{r-r1}}\|K\boldsymbol{f}\|_{W^{r_1}_{\boldsymbol{v}^{-1}}} \\
&\leq \frac{\mathcal{C}}{m^{r-r1}}\|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}}.
\end{aligned}$$

Therefore, setting $T = \{\boldsymbol{f} \in L^2_{\boldsymbol{v}} : \|\boldsymbol{f}\sqrt{\boldsymbol{v}}\|_2 \leq 1\}$, we have

$$\lim_m \sup_{\boldsymbol{f} \in T} E_{m,m}(\boldsymbol{f})_{\boldsymbol{v}} = 0$$

from which we deduce [79] that $K : L^2_{\boldsymbol{v}} \to W^{r_1}_{\boldsymbol{v}^{-1}}$ is a compact operator for all $r_1 < r$. □

Let us remark that, in virtue of Proposition 4.1.2 and 4.1.3, we can claim that under the assumptions (4.1.11) and if the null space $\text{Ker}\{D + K\}$ is trivial in $L^2_{\boldsymbol{v}}(S)$, then the operator

$$D + K : W^{r_1}_{\boldsymbol{v}} \to W^{r_1}_{\boldsymbol{v}^{-1}}$$

is an invertible linear bounded operator for all $0 \leq r_1 < r$. Hence, equation (4.0.2) has a unique solution $\boldsymbol{f} \in W^{r_1}_{\boldsymbol{v}}$, for each given right-hand side $\boldsymbol{g} \in W^{r_1}_{\boldsymbol{v}^{-1}}$.

4.2 A direct numerical method

In this Section we present a *direct numerical approach* for the solution of equation (4.0.2). Inspired by the discrete collocation method proposed for the univariate case [39, 49], we first approximate operator K by means of

$$K_m\boldsymbol{f}(t, s) = \int_S \mathcal{L}_{m,m}\left(\boldsymbol{k}_{(t,s)}, \boldsymbol{v}, x, y\right) \boldsymbol{f}(x, y)\, \boldsymbol{v}(x, y)\, dx\, dy. \tag{4.2.1}$$

Hence we project equation (4.0.2) with K_m instead of K by means of the interpolating operator $\mathcal{L}_{m,m}(\boldsymbol{v}^{-1})$ and we search for a polynomial solution $\boldsymbol{f}_m \in \mathbb{P}_{m-1,m-1}$, i.e. we solve the finite dimensional equation

$$\mathcal{L}_{m,m}\left((D + K_m)\boldsymbol{f}_m, \boldsymbol{v}^{-1}, t, s\right) = \mathcal{L}_{m,m}(\boldsymbol{g}, \boldsymbol{v}^{-1}, t, s),$$

namely

$$\mathcal{L}_{m,m}\left((D + K_m)\boldsymbol{f}_m - \boldsymbol{g}, \boldsymbol{v}^{-1}, t, s\right) = 0. \tag{4.2.2}$$

Equation (4.2.2) is equivalent in the weighted space $L^2_{\boldsymbol{v}^{-1}}$ to

$$\|\mathcal{L}_{m,m}\left((D + K_m)\boldsymbol{f}_m - \boldsymbol{g}, \boldsymbol{v}^{-1}\right)\|_{L^2_{\boldsymbol{v}^{-1}}} = 0$$

that is

$$\int_S \left|\mathcal{L}_{m,m}\left((D + K_m)\boldsymbol{f}_m - \boldsymbol{g}, \boldsymbol{v}^{-1}, t, s\right)\right|^2 \boldsymbol{v}^{-1}(t, s)\, dt\, ds = 0.$$

Thus, by approximating the integral by means of the Gaussian cubature rule (1.2.16) with $w_i = u^{-1}$, $i \in \{1, 2\}$, that in this case turns out to be exact, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i(u^{-1})\lambda_j(u^{-1}) \left|\mathcal{L}_{m,m}\left((D + K_m)\boldsymbol{f}_m - \boldsymbol{g}, \boldsymbol{v}^{-1}, t_i, t_j\right)\right|^2 = 0 \quad (4.2.3)$$

where [26, 46]

$$t_i = \cos\left(\frac{\left(m - i + \frac{1}{2}\right)\pi}{m + \frac{1}{2}}\right), \qquad i = 1, \ldots, m$$

are the nodes of the mth third kind Chebyshev polynomial $p_m(u^{-1})$ defined in (4.1.5) and

$$\lambda_i(u^{-1}) = \frac{\pi}{m + \frac{1}{2}}(1 + t_i), \qquad i = 1, \ldots, m$$

are the corresponding Christoffel numbers.

From (4.2.3) we deduce

$$\begin{aligned}\sqrt{\lambda_i(u^{-1})\lambda_j(u^{-1})}\,[D\boldsymbol{f}_m(t_i, t_j) \quad &+ \quad K_m\boldsymbol{f}_m(t_i, t_j)] \qquad (4.2.4)\\ &= \quad \sqrt{\lambda_i(u^{-1})\lambda_j(u^{-1})}\boldsymbol{g}(t_i, t_j), \quad i, j = 1, \ldots, m.\end{aligned}$$

Now we develop the terms $D\boldsymbol{f}_m(t_i, t_j)$ and $K_m\boldsymbol{f}_m(t_i, t_j)$ involved in the previous equations, in order to construct the approximated polynomial solution $\boldsymbol{f}_m$ in the form

$$\boldsymbol{f}_m(t, s) = \mathcal{L}_{m,m}(\boldsymbol{f}_m, \boldsymbol{v}, t, s). \quad (4.2.5)$$

About the second term $K_m\boldsymbol{f}_m(t_i, t_j)$, by using again the cubature formula (1.2.16) now with $w_i = u$, $i \in \{1, 2\}$, which is once again exact, we have

$$K_m\boldsymbol{f}_m(t_i, t_j) = \sum_{h=1}^{m}\sum_{k=1}^{m} \lambda_h(u)\lambda_k(u)\boldsymbol{k}(x_h, x_k, t_i, t_j)\boldsymbol{f}_m(x_h, x_k), \quad i, j = 1, \ldots, m, \quad (4.2.6)$$

where [26, 46]

$$x_h = \cos\left(\frac{(m - h + 1)\pi}{m + \frac{1}{2}}\right), \qquad h = 1, \ldots, m \quad (4.2.7)$$

are the nodes of the mth fourth kind Chebyshev polynomial $p_m(u)$ defined in (4.1.4) and

$$\lambda_h(u) = \frac{\pi}{m+\frac{1}{2}}(1-x_h), \qquad h = 1, \ldots, m \tag{4.2.8}$$

are the corresponding Christoffel numbers.

Concerning to the first term $D\boldsymbol{f}_m(t_i, t_j)$, we have the following proposition.

Proposition 4.2.1. *Let* $\boldsymbol{f}_m$ *be the polynomial defined in* (4.2.5) *and let* $\{t_i\}_{i=1}^m$ *and* $\{x_h\}_{h=1}^m$ *be the zeros of* $p_m(u^{-1})$ *and* $p_m(u)$*, respectively. Then,*

$$D\boldsymbol{f}_m(t_i, t_j) = \frac{1}{\pi^2}\sum_{h=1}^{m}\sum_{k=1}^{m}\lambda_h(u)\lambda_k(u)\frac{\boldsymbol{f}_m(x_h, x_k)}{(x_h - t_i)(x_k - t_j)} \tag{4.2.9}$$

for $i, j = 1, ..., m$.

Proof. By the definitions of the operator D and the function $\boldsymbol{f}_m$, we get

$$\begin{aligned} D\boldsymbol{f}_m(t_i, s_j) &= \frac{1}{\pi^2}\oint_S \frac{\boldsymbol{f}_m(x,y)}{(x-t_i)(y-s_j)}\boldsymbol{v}(x,y)\,dx\,dy \\ &= \frac{1}{\pi^2}\sum_{h=1}^{m}\sum_{k=1}^{m}\boldsymbol{f}_m(x_h, y_k)\oint_S \frac{\ell_h(u,x)\ell_k(u,y)}{(x-t_i)(y-s_j)}u(x)u(y)\,dx\,dy. \end{aligned}$$

Moreover, by (1.1.13) we have

$$\begin{aligned} \frac{\ell_h(u,x)}{(x-t_i)}u(x) &= \frac{p_m(u,x)u(x)}{p'_m(u,x_h)(x-x_h)(x-t_i)} \\ &= \frac{p_m(u,x)u(x)}{p'_m(u,x_h)(x_h-t_i)}\left[\frac{1}{x-x_h} - \frac{1}{x-t_i}\right], \end{aligned}$$

and similarly

$$\begin{aligned} \frac{\ell_k(u,y)}{(y-s_j)}u(y) &= \frac{p_m(u,y)u(y)}{p'_m(u,y_k)(y-y_k)(y-s_j)} \\ &= \frac{p_m(u,y)u(y)}{p'_m(u,y_k)(y_k-s_j)}\left[\frac{1}{y-y_k} - \frac{1}{y-s_j}\right]. \end{aligned}$$

Then, setting $q_m(t,s) = p_m(u,t)p_m(u,s)$, $r_m(t,s) = p_m(u^{-1},t)p_m(u^{-1},s)$ and taking into account Lemma 4.1.1, we can write

$$
\begin{aligned}
&D\boldsymbol{f}_m(t_i, s_j) \\
&= \sum_{h=1}^{m} \sum_{k=1}^{m} \left[\frac{\boldsymbol{f}_m(x_h, y_k) \{Dq_m(x_h, y_k) - Dq_m(x_h, s_j) - Dq_m(t_i, y_k)\}}{q_m'(x_h, y_k)(x_h - t_i)(y_k - s_j)} \right. \\
&\left. + \frac{\boldsymbol{f}_m(x_h, y_k)\, Dq_m(t_i, s_j)}{q_m'(x_h, y_k)(x_h - t_i)(y_k - s_j)} \right] \\
&= \sum_{h=1}^{m} \sum_{k=1}^{m} \frac{\boldsymbol{f}_m(x_h, y_k) \{r_m(x_h, y_k) - r_m(x_h, s_j) - r_m(t_i, y_k) + r_m(t_i, s_j)\}}{q_m'(x_h, y_k)(x_h - t_i)(y_k - s_j)}
\end{aligned}
$$

and consequently,

$$
D\boldsymbol{f}_m(t_i, s_j) = \sum_{h=1}^{m} \sum_{k=1}^{m} \frac{\boldsymbol{f}_m(x_h, y_k)\, r_m(x_h, y_k)}{q_m'(x_h, y_k)(x_h - t_i)(y_k - s_j)}.
$$

Thus, the thesis can be deduced by observing that by using property (4.1.6), we have

$$
\begin{aligned}
r_m(x_h, y_k) &= Dq_m(x_h, y_k) \\
&= \frac{1}{\pi^2} q_m'(x_h, y_k) \oint_{-1}^{1} \ell_h(u, x) u(x) dx \oint_{-1}^{1} \ell_k(u, y) u(y) dy \\
&= \frac{1}{\pi^2} q_m'(x_h, y_k) \lambda_h(u) \lambda_k(u)
\end{aligned}
$$

where $\lambda_h(u)$ denotes the h-th Christoffel number with respect to the weight u. $\square$

Hence, by replacing (4.2.9) and (4.2.6) in (4.2.4), we get

$$
\begin{aligned}
&\sqrt{\lambda_i(u^{-1})\lambda_j(u^{-1})} \sum_{h=1}^{m} \sum_{k=1}^{m} \sqrt{\lambda_h(u)\lambda_k(u)} \left[\frac{\pi^{-2}}{(x_h - t_i)(x_k - t_j)} + \boldsymbol{k}(x_h, x_k, t_i, t_j) \right] a_{hk} \\
&\qquad = \sqrt{\lambda_i(u^{-1})\lambda_j(u^{-1})}\, \boldsymbol{g}(t_i, t_j), \qquad i, j = 1, \dots, m, \quad (4.2.10)
\end{aligned}
$$

where we set $a_{hk} = \sqrt{\lambda_h(u)\lambda_k(u)}\, \boldsymbol{f}_m(x_h, x_k)$.

This is a linear system of m^2 equations in the m^2 unknown a_{hk} that, once solved, allow us to approximate the solution we are looking for

$$
\boldsymbol{f}_m(t, s) = \sum_{h=1}^{m} \sum_{k=1}^{m} \frac{\ell_h(u, t)}{\sqrt{\lambda_h(u)}} \frac{\ell_k(u, s)}{\sqrt{\lambda_k(u)}}\, a_{hk}. \qquad (4.2.11)
$$

Let us remark that system (4.2.10) is well-defined, since $\min|x_h - t_i| = \mathcal{O}(1/m)$, $h, i = 1, \dots, m$, [49] , and that it can be rewritten in a matrix form as

$$\mathbf{P}_m\left(\mathbf{D}_m + \mathbf{K}_m\right)\mathbf{P}_m\mathbf{a} = \mathbf{P}_m\left(\widetilde{\mathbf{g}}\mathbf{P}_m\right)^T. \tag{4.2.12}$$

Here $\mathbf{P}_m$ is a m-blocks matrix in which each block is given by

$$\mathbf{P} = diag\left(\sqrt{\lambda_1(u^{-1})}, \cdots, \sqrt{\lambda_m(u^{-1})}\right),$$

the matrices $\mathbf{D}_m$ and $\mathbf{K}_m$ are the m-blocks matrix defined as

$$\mathbf{D}_m = \begin{pmatrix} \mathbf{D}^{(1,1)} & \mathbf{D}^{(1,2)} & \dots & \mathbf{D}^{(1,m)} \\ \mathbf{D}^{(2,1)} & \mathbf{D}^{(2,2)} & \dots & \mathbf{D}^{(2,m)} \\ \dots & \dots & \dots & \\ \mathbf{D}^{(m,1)} & \mathbf{D}^{(m,2)} & \dots & \mathbf{D}^{(m,m)} \end{pmatrix},$$

$$\mathbf{K}_m = \begin{pmatrix} \mathbf{K}^{(1,1)} & \mathbf{K}^{(1,2)} & \dots & \mathbf{K}^{(1,m)} \\ \mathbf{K}^{(2,1)} & \mathbf{K}^{(2,2)} & \dots & \mathbf{K}^{(2,m)} \\ \dots & \dots & \dots & \\ \mathbf{K}^{(m,1)} & \mathbf{K}^{(m,2)} & \dots & \mathbf{K}^{(m,m)} \end{pmatrix}$$

with

$$\mathbf{D}^{(h,k)} = \left[\mathbf{D}^{(h,k)}\right]_{i,j=1}^m = \sqrt{\lambda_h(u)\lambda_k(u)}\frac{\pi^{-2}}{(x_h - t_i)(x_k - t_j)},$$
$$\mathbf{K}^{(h,k)} = \left[\mathbf{K}^{(h,k)}\right]_{i,j=1}^m = \sqrt{\lambda_h(u)\lambda_k(u)}k(x_h, x_k, t_i, t_j),$$

and $\mathbf{a} \in \mathbb{R}^{m^2}$ and $\widetilde{\mathbf{g}} \in \mathbb{R}^{m^2}$ are the arrays of the unknown function and the right-hand side which have been obtained by reordering column by column the matrices $\mathbf{G}$ and $\mathbf{A}$, respectively defined as

$$\mathbf{G} = \left[\boldsymbol{g}_{ij}\right]_{i,j=1}^m = \boldsymbol{g}(t_i, t_j) \in \mathbb{R}^{m\times m}, \quad \mathbf{A} = [a_{hk}]_{h,k=1}^m = \boldsymbol{f}_m(x_h, x_k) \in \mathbb{R}^{m\times m}$$

namely,

$$\widetilde{\mathbf{g}}_{(j-1)m+i} = \boldsymbol{g}_{ij}, \quad \mathbf{a}_{(k-1)m+h} = a_{hk}.$$

Next proposition, concerning with the operator introduced in (4.2.1), is essential for the analysis of the method.

Proposition 4.2.2. *Under the assumptions* (4.1.11)*, the estimate*

$$\|K\boldsymbol{f} - \mathcal{L}_{m,m}(K_m\boldsymbol{f}, \boldsymbol{v}^{-1})\|_{L^2_{\boldsymbol{v}^{-1}}} \leq \frac{\mathcal{C}}{m^{r_1}}\|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}}$$

holds true with $\mathcal{C} \neq \mathcal{C}(m, \boldsymbol{f})$.

Proof. We start by writing

$$\|K\boldsymbol{f} - \mathcal{L}_{m,m}(K_m\boldsymbol{f}, \boldsymbol{v}^{-1})\|_{L^2_{\boldsymbol{v}^{-1}}} \leq \left\|K\boldsymbol{f} - \mathcal{L}_{m,m}(K\boldsymbol{f}, \boldsymbol{v}^{-1})\right\|_{L^2_{\boldsymbol{v}^{-1}}} + \left\|\mathcal{L}_{m,m}((K - K_m)\boldsymbol{f}, \boldsymbol{v}^{-1})\right\|_{L^2_{\boldsymbol{v}^{-1}}} := A + B.$$

By using Proposition 1.2.3 and (4.1.12) we can deduce that

$$A \leq \frac{\mathcal{C}}{m^{r_1}} \|K\boldsymbol{f}\|_{W^{r_1}_{\boldsymbol{v}^{-1}}} \leq \frac{\mathcal{C}}{m^{r_1}} \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}}.$$

Moreover, by using the Gaussian cubature rule (1.2.16) with $w_i = u$, $i \in \{1, 2\}$, we have

$$B = \left(\int_S \left|\mathcal{L}_{m,m}\left((K - K_m)\boldsymbol{f}, \boldsymbol{v}^{-1}, t, s\right)\right|^2 \boldsymbol{v}^{-1}(t, s)\, dt\, ds\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i(u^{-1})\lambda_j(u^{-1}) \left|(K - K_m)\, \boldsymbol{f}(t_i, t_j)\right|^2\right)^{\frac{1}{2}}.$$

Since one has

$$\left|(K - K_m)\, \boldsymbol{f}(t, s)\right|^2 \leq \|\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}}} \int_S \left|\boldsymbol{k}(x, y, t, s) - \mathcal{L}_{m,m}\left(\boldsymbol{k}_{(t,s)}, \boldsymbol{v}, x, y\right)\right|^2 \boldsymbol{v}(x, y)\, dx\, dy = \|\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}}} \left\|\boldsymbol{k}_{(t,s)} - \mathcal{L}_{m,m}\left(\boldsymbol{k}_{(t,s)}\right)\right\|^2_{L^2_{\boldsymbol{v}}} \leq \frac{\mathcal{C}}{m^{2r}} \|\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}}} \left\|\boldsymbol{k}_{(t,s)}\right\|^2_{W^r_{\boldsymbol{v}}},$$

from the first assumption in (4.1.11), it follows

$$B \leq \frac{\mathcal{C}}{m^r} \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}} \left(\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i(u^{-1})\lambda_j(u^{-1}) \left\|\boldsymbol{k}_{(t_i,t_j)}\right\|^2_{W^r_{\boldsymbol{v}}}\right)^{\frac{1}{2}} \leq \frac{\mathcal{C}}{m^r} \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}} \sup_{(t_i,t_j)\in S} \left\|\boldsymbol{k}_{(t_i,t_j)}\right\|^2_{W^r_{\boldsymbol{v}}} \left(\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i(u^{-1})\lambda_j(u^{-1})\right)^{\frac{1}{2}} \leq \frac{\mathcal{C}}{m^r} \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}} \left(\int_S \boldsymbol{v}^{-1}(x, y)\, dx\, dy\right)^{\frac{1}{2}} \leq \frac{\mathcal{C}}{m^r} \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}}.$$

□

Next theorem assures that the proposed discrete collocation method is stable and convergent. It also states that, in the case when the right-hand side $\boldsymbol{g}$ belongs to a certain class of functions, namely the Sobolev-type space $W^{r_1}_{\boldsymbol{v}^{-1}}$, then the solution $\boldsymbol{f}$ of (4.0.2) belongs to $W^{r_1}_{\boldsymbol{v}}$. Moreover the theorem gives an estimate of the error of the approximate solution. Finally it shows that the condition number in the spectral norm of system (4.2.12)

$$cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m) = \|\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m\| \, \|(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)^{-1}\|$$

is independent of the dimension of the matrix and uniformly bounded by the condition number of the operator $D + K$.

Theorem 4.2.3. *Assume that equation* (4.0.2) *has a unique solution* $\boldsymbol{f} \in L^2_{\boldsymbol{v}}$ *and the kernel function* $\boldsymbol{k}$ *satisfies* (4.1.11).

Then, for sufficiently large m*, say* $m \geq m_0$*, the system of equations* (4.2.12) *has a unique solution* $\boldsymbol{f}_m$*. Moreover if the right-hand side* $\boldsymbol{g} \in W^{r_1}_{\boldsymbol{v}^{-1}}$ *then the solution* $\boldsymbol{f} \in W^{r_1}_{\boldsymbol{v}}$ *and the following estimate holds true*

$$\|\boldsymbol{f} - \boldsymbol{f}_m\|_{L^2_{\boldsymbol{v}}} \leq \frac{\mathcal{C}}{m^{r_1}} \|\boldsymbol{f}\|_{W^{r_1}_{\boldsymbol{v}}} \tag{4.2.13}$$

with $\mathcal{C} \neq \mathcal{C}(m, \boldsymbol{f}, \boldsymbol{g})$*. Furthermore,*

$$\limsup_m cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m) \leq \mathcal{C}\, cond(D + K), \tag{4.2.14}$$

where here $\mathcal{C} \neq \mathcal{C}(m)$*.*

Proof. Taking into account Proposition 4.2.2, by standard arguments (see, for instance, Theorem 3.3.1 in [2]), it follows that for sufficiently large m, say $m \geq m_0$, the operators $D + \mathcal{L}_{m,m} K_m : L^2_{\boldsymbol{v}} \to L^2_{\boldsymbol{v}^{-1}}$ exist and are uniformly bounded being

$$\|(D + \mathcal{L}_{m,m} K_m)^{-1}\| \leq \frac{\|(D + K)^{-1}\|}{1 - \|(D + K)^{-1}\| \sup\limits_{m \geq m_0} \|K - \mathcal{L}_{m,m} K_m\|} < \infty$$

(where the notation $\|\cdot\|$ denotes the norm of the operators), i.e. the method is stable. In order to prove the convergence estimate (4.2.13), we note that

$$\boldsymbol{f} - \boldsymbol{f}_m = (D + \mathcal{L}_{m,m} K_m)^{-1} \left[\left(\boldsymbol{g} - \mathcal{L}_{m,m} \left(\boldsymbol{g}, \boldsymbol{v}^{-1} \right) \right) - \left(K\boldsymbol{f} - \mathcal{L}_{m,m} \left(K_m \boldsymbol{f}, \boldsymbol{v}^{-1} \right) \right) \right]$$

from which we deduce

$$\|\boldsymbol{f} - \boldsymbol{f}_m\|_{L^2_{\boldsymbol{v}}} \leq \mathcal{C} \, \|\boldsymbol{g} - \mathcal{L}_{m,m} \left(\boldsymbol{g}, \boldsymbol{v}^{-1} \right) \|_{L^2_{\boldsymbol{v}^{-1}}} + \|K\boldsymbol{f} - \mathcal{L}_{m,m} \left(K_m \boldsymbol{f}, \boldsymbol{v}^{-1} \right) \|_{L^2_{\boldsymbol{v}^{-1}}}.$$

Then, by applying Proposition 1.2.3 to the first term and Proposition 4.2.2 to the second one we get (4.2.13). Let us now prove (4.2.14). To this end let us introduce an arbitrary array $\mathbf{c} = [c_{11}, \dots, c_{1m}, \dots, c_{m1}, \dots c_{mm}]^T$ of length m^2, and let us denote by $\|\mathbf{c}\|_2 = \left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}^2\right)^{1/2}$ its Euclidean norm. Then, the vector $\mathbf{b} = [b_{11}, \dots, b_{1m}, \dots, b_{m1}, \dots b_{mm}]^T$ satisfies the system $\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m\mathbf{c} = \mathbf{b}$ if and only if $(D + \mathcal{L}_{m,m}K_m)\boldsymbol{f}_m = \boldsymbol{g}_m$ where $\boldsymbol{f}_m$ and $\boldsymbol{g}_m$ are the bivariate polynomials defined as

$$\boldsymbol{f}_m(t,s) = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\ell_i(u,t)}{\sqrt{\lambda_j(u)}}\frac{\ell_j(u,s)}{\sqrt{\lambda_j(u)}} c_{ij}$$

and

$$\boldsymbol{g}_m(t,s) = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\ell_i(u^{-1},t)}{\sqrt{\lambda_i(u^{-1})}}\frac{\ell_j(u^{-1},s)}{\sqrt{\lambda_j(u^{-1})}} b_{ij}.$$

Being

$$\begin{aligned}\|\boldsymbol{g}_m\|^2_{L^2_{\boldsymbol{v}^{-1}}} = \int_S |\boldsymbol{g}_m(t,s)|^2 \boldsymbol{v}^{-1}(t,s)\,dt\,ds &= \sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i(u^{-1})\lambda_j(u^{-1})|\boldsymbol{g}_m(t_i,t_j)|^2 \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} b_{ij}^2 = \|\mathbf{b}\|_2\end{aligned}$$

and analogously $\|\boldsymbol{f}_m\|_{L^2_{\boldsymbol{v}}} = \|\mathbf{c}\|_2$, we have

$$\begin{aligned}\|\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m\| &= \sup_{\substack{\mathbf{c}\in\mathbb{R}^{m^2} \\ \mathbf{c}\neq 0}} \frac{\|\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m\mathbf{c}\|_2}{\|\mathbf{c}\|_2} \\ &= \sup_{\substack{\boldsymbol{f}_m\in\mathbb{P}_{m-1,m-1} \\ \boldsymbol{f}_m\neq 0}} \frac{\|(D + \mathcal{L}_{m,m}K_m)\boldsymbol{f}_m\|_{L^2_{\boldsymbol{v}^{-1}}}}{\|\boldsymbol{f}_m\|_{L^2_{\boldsymbol{v}}}} \\ &= \|D + \mathcal{L}_{m,m}K_m\|_{L^2_{\boldsymbol{v}} \to L^2_{\boldsymbol{v}^{-1}}}.\end{aligned}$$

Then, in virtue of Proposition 4.2.2, for m sufficiently large,

$$\|\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m\| \leq \mathcal{C}\,\|D + K\|_{L^2_{\boldsymbol{v}} \to L^2_{\boldsymbol{v}^{-1}}}. \tag{4.2.15}$$

In the same way we can prove that

$$\|(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)^{-1}\| = \|(D + \mathcal{L}_{m,m}K_m)^{-1}\|_{L^2_{\boldsymbol{v}^{-1}} \to L^2_{\boldsymbol{v}}}$$

from which, by applying again Proposition 4.2.2, we deduce that, for m sufficiently large,

$$\|(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)^{-1}\| \le \mathcal{C}\, \|(D+K)^{-1}\|_{L^2_{\boldsymbol{v}^{-1}} \to L^2_{\boldsymbol{v}}}. \tag{4.2.16}$$

Hence, the thesis (4.2.14) follows from (4.2.15) and (4.2.16). □

4.3 An indirect numerical method

In this Section we propose an alternative numerical method still based on a polynomial approximation of the unknown solution written in the form

$$\boldsymbol{f}_m(t,s) = \mathcal{L}_{m,m}(\boldsymbol{f}_m, \boldsymbol{v}, t, s), \qquad \boldsymbol{f}_m \in \mathbb{P}_{m-1,m-1}.$$

The method takes advantages of the smoothness properties of the operators D and K stated in Section 4.1. In fact, thanks to the compactness of K and the invertibility of D, following [16], we can move from the equation (4.0.2) into the equivalent regularized Fredholm equation

$$(I + \widehat{D}K)\boldsymbol{f} = \widehat{D}\boldsymbol{g}, \tag{4.3.1}$$

where I is the identity operator in $L^2_{\boldsymbol{v}}$.

Then, if we assume that the null space $\mathrm{Ker}\{I + \widehat{D}K\}$ is trivial, by applying the Fredholm Alternative Theorem, equation (4.3.1) has a unique solution for each given right hand side $\widehat{D}\boldsymbol{g} \in L^2_{\boldsymbol{v}}$.

For our convenience, let us rewrite (4.3.1) as

$$(I + \mathcal{K})\boldsymbol{f} = \mathcal{G}, \tag{4.3.2}$$

where $\mathcal{G} = \widehat{D}\boldsymbol{g}$ and $\mathcal{K} = \widehat{D}K$ i.e.

$$\mathcal{K}\boldsymbol{f}(t,s) = \int_S \boldsymbol{\phi}(\xi,\eta,t,s)\boldsymbol{f}(\xi,\eta)\boldsymbol{v}(\xi,\eta)\, d\xi\, d\eta,$$

with

$$\boldsymbol{\phi}(\xi,\eta,t,s) = \widehat{D}\boldsymbol{k}_{(\xi,\eta)}(t,s). \tag{4.3.3}$$

In order to approximate the solution of (4.3.2), let us project the equation on the finite dimensional space $\mathbb{P}_{m-1,m-1}$ by means of the interpolating operator $\mathcal{L}_{m,m}(\boldsymbol{v})$ and then let us consider the following finite dimensional equation

$$\mathcal{L}_{m,m}\left((I + \mathcal{K}_m)\,\boldsymbol{f}_m, \boldsymbol{v}, t, s\right) = \mathcal{L}_{m,m}\left(\mathcal{G}, \boldsymbol{v}, t, s\right), \tag{4.3.4}$$

where

$$\mathcal{K}_m \boldsymbol{f}(t,s) = \int_S \mathcal{L}_{m,m}\left(\boldsymbol{\phi}(t,s), \boldsymbol{v}, \xi, \eta\right) \boldsymbol{f}(\xi,\eta)\boldsymbol{v}(\xi,\eta)\, d\xi\, d\eta.$$

Equation (4.3.4), considered in $L^2_{\boldsymbol{v}}$, means that

$$\int_S \left|\mathcal{L}_{m,m}\left((I+\mathcal{K}_m)\boldsymbol{f}_m - \mathcal{G}, \boldsymbol{v}, t, s\right)\right|^2 \boldsymbol{v}(t,s)\, dt\, ds = 0$$

that is, for $i, j = 1, \ldots, m$,

$$\sqrt{\lambda_i(u)\lambda_j(u)}\left[\boldsymbol{f}_m(x_i,x_j) + \mathcal{K}_m \boldsymbol{f}_m(x_i,x_j)\right] = \sqrt{\lambda_i(u)\lambda_j(u)}\mathcal{G}(x_i,x_j),$$

where x_i and $\lambda_i(u)$ were introduced in (4.2.7) and (4.2.8), respectively. Hence by approximating the operator $\mathcal{K}_m$ by means of the Gaussian cubature rule (1.2.16) we get the following linear system

$$\begin{aligned}\sqrt{\lambda_i(u)\lambda_j(u)}\sum_{h=1}^{m}\sum_{k=1}^{m}\Big[\delta_{hk}^{ij} &+ \sqrt{\lambda_h(u)\,\lambda_k(u)}\,\boldsymbol{\phi}(x_h,x_k,x_i,x_j)\Big]\, a_{hk} \quad (4.3.5)\\ &= \sqrt{\lambda_i(u)\lambda_j(u)}\,\mathcal{G}(x_i,x_j), \quad i,j = 1,\ldots,m,\end{aligned}$$

where $a_{hk} = \sqrt{\lambda_h(u)\,\lambda_k(u)}\,\boldsymbol{f}_m(x_h,x_k)$ and $\delta_{hk}^{ij} = \begin{cases} 1, & i = h \quad \text{and} \quad j = k \\ 0, & otherwise \end{cases}$.

Once solved (4.3.5), the solution allows us to compute the approximate solution

$$\boldsymbol{f}_m(t,s) = \sum_{h=1}^{m}\sum_{k=1}^{m} \frac{\ell_h(u,t)}{\sqrt{\lambda_h(u)}}\frac{\ell_k(u,s)}{\sqrt{\lambda_k(u)}}\; a_{hk}. \qquad (4.3.6)$$

Note that the polynomial solution $\boldsymbol{f}_m$ just defined has the same expression of the solution $\boldsymbol{f}_m$ given in (4.2.11), obtained applying the method described in Section 4.2.

Let us also remark that in order to implement system (4.3.5) we need to evaluate the integrals

$$\boldsymbol{\phi}(\xi,\eta,t,s) = \frac{1}{\pi^2}\oint_S \frac{\boldsymbol{k}(x,y,\xi,\eta)}{(x-t)(y-s)}\boldsymbol{v}^{-1}(x,y)\, dx\, dy$$

$$\mathcal{G}(t,s) = \frac{1}{\pi^2}\oint_S \frac{\boldsymbol{g}(x,y)}{(x-t)(y-s)}\boldsymbol{v}^{-1}(x,y)\, dx\, dy$$

whose analytical expressions are not always known. Then, in the case when we do not have such expressions, we propose to approximate the known involved functions $\boldsymbol{k}$ and $\boldsymbol{g}$ with

$$\boldsymbol{k}(x,y,\xi,\eta) \simeq \mathcal{L}_{m,m}\left(\boldsymbol{k}_{(\xi,\eta)},\boldsymbol{v},x,y\right), \qquad \boldsymbol{g}(x,y) \simeq \mathcal{L}_{m,m}\left(\boldsymbol{g},\boldsymbol{v},x,y\right)$$

and then by proceeding as in the proof of Proposition 4.2.1, in virtue of Lemma 4.1.1, we end up to approximate $\boldsymbol{\phi}(\xi,\eta,t,s)$ and $\mathcal{G}(t,s)$ with

$$\boldsymbol{\phi}_m(x_h,x_k,x_i,x_j) = \frac{1}{\pi^2}\sum_{\iota=1}^{m}\sum_{\zeta=1}^{m}\lambda_\iota(u^{-1})\lambda_\zeta(u^{-1})\frac{\boldsymbol{k}(t_\iota,t_\zeta,x_h,x_k)}{(t_\iota-x_i)(t_\zeta-x_j)},$$

and

$$\mathcal{G}_m(x_i,x_j) = \frac{1}{\pi^2}\sum_{\iota=1}^{m}\sum_{\zeta=1}^{m}\lambda_\iota(u^{-1})\lambda_\zeta(u^{-1})\frac{\boldsymbol{g}(t_\iota,t_\zeta)}{(t_\iota-x_i)(t_\zeta-x_j)}.$$

Let us now rewrite (4.3.5) in a matrix form as

$$\boldsymbol{\mathcal{P}}_m\left(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m\right)\boldsymbol{\mathcal{P}}_m\mathbf{a} = \boldsymbol{\mathcal{P}}_m\left(\mathfrak{g}\,\boldsymbol{\mathcal{P}}_m\right)^T, \tag{4.3.7}$$

where $\boldsymbol{\mathcal{P}}_m$ is a m-blocks matrix in which each block is given by

$$\boldsymbol{\mathcal{P}} = diag\left(\sqrt{\lambda_1(u)},\,\cdots,\,\sqrt{\lambda_m(u)}\right),$$

the matrices $\mathbf{I}_m$ and $\mathbf{K}_m$ are the m-blocks matrix defined as

$$\mathbf{I}_m = \begin{pmatrix} \mathbf{I} & 0 & \ldots 0 \\ 0 & \mathbf{I} & \ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & \ldots \mathbf{I} \end{pmatrix},$$

$$\boldsymbol{\mathcal{K}}_m = \begin{pmatrix} \boldsymbol{\mathcal{K}}^{(1,1)} & \boldsymbol{\mathcal{K}}^{(1,2)} & \ldots \boldsymbol{\mathcal{K}}^{(1,m)} \\ \boldsymbol{\mathcal{K}}^{(2,1)} & \boldsymbol{\mathcal{K}}^{(2,2)} & \ldots \boldsymbol{\mathcal{K}}^{(2,m)} \\ \ldots & \ldots & \ldots \\ \boldsymbol{\mathcal{K}}^{(m,1)} & \boldsymbol{\mathcal{K}}^{(m,2)} & \ldots \boldsymbol{\mathcal{K}}^{(m,m)} \end{pmatrix},$$

where $\mathbf{I}$ denotes the identity matrix of order m,

$$\boldsymbol{\mathcal{K}}^{(h,k)} = \left[\boldsymbol{\mathcal{K}}^{(h,k)}\right]_{i,j=1}^{m} = \sqrt{\lambda_h(u)\lambda_k(u)}\boldsymbol{\phi}(x_h,x_k,x_i,x_j)$$

and $\mathbf{a} \in \mathbb{R}^{m^2}$ and $\mathfrak{g} \in \mathbb{R}^{m^2}$ are the arrays of the unknown function and the right-hand side which have been obtained by reordering column by column the matrices $\boldsymbol{\mathcal{G}}$ and $\mathbf{A}$, respectively

$$\boldsymbol{\mathcal{G}} = [\mathcal{G}_{ij}]_{i,j=1}^{m} = \mathcal{G}(x_i,x_j) \in \mathbb{R}^{m\times m}, \quad \mathbf{A} = [a_{hk}]_{h,k=1}^{m} = \boldsymbol{f}_m(x_h,x_k)\mathbb{R}^{m\times m}$$

namely,

$$\mathfrak{g}_{(j-1)m+i} = \mathcal{G}_{ij}, \quad \mathbf{a}_{(k-1)m+h} = a_{hk}.$$

Next proposition is essential for the stability and the convergence of the described method stated in Theorem 4.3.2.

Proposition 4.3.1. *Assume that kernel* $\boldsymbol{k}$ *satisfies the conditions* (4.1.11). *Then*

$$\|\mathcal{K}\boldsymbol{f} - \mathcal{L}_{m,m}(\mathcal{K}_m\boldsymbol{f}, \boldsymbol{v})\|_{L^2_{\boldsymbol{v}}} \leq \frac{\mathcal{C}}{m^r} \|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}}$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

Proof. We can proceed analogously to the proof of Proposition 4.2.2. Therefore we only give the main sketch. We have

$$\begin{aligned}\|\mathcal{K}\boldsymbol{f} - \mathcal{L}_{m,m}\left(\mathcal{K}_m\boldsymbol{f}, \boldsymbol{v}\right)\|_{L^2_{\boldsymbol{v}}} &\leq \|\mathcal{K}\boldsymbol{f} - \mathcal{L}_{m,m}\left(\mathcal{K}\boldsymbol{f}, \boldsymbol{v}\right)\|_{L^2_{\boldsymbol{v}}} \\ &\quad + \|\mathcal{L}_{m,m}\left((\mathcal{K} - \mathcal{K}_m)\boldsymbol{f}, \boldsymbol{v}\right)\|_{L^2_{\boldsymbol{v}}}.\end{aligned}$$

By noting that in virtue of Proposition 4.1.2 one has $\mathcal{K}\boldsymbol{f} = (\widehat{D}K)(\boldsymbol{f}) \in W^{r_1}_{\boldsymbol{v}}$ and taking into account (1.2.15) and (4.1.11), we get

$$\|\mathcal{K}\boldsymbol{f} - \mathcal{L}_{m,m}\left(\mathcal{K}\boldsymbol{f}, \boldsymbol{v}\right)\|_{L^2_{\boldsymbol{v}}} \leq \frac{\mathcal{C}}{m^{r_1}}\|\mathcal{K}\boldsymbol{f}\|_{W^{r_1}_{\boldsymbol{v}^{-1}}} \leq \frac{\mathcal{C}}{m^{r_1}}\|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}}.$$

Moreover,

$$\left|(\mathcal{K} - \mathcal{K}_m)\,\boldsymbol{f}(t,s)\right|^2 \leq \frac{\mathcal{C}}{m^{2r}}\,\|\boldsymbol{f}\|^2_{L^2_{\boldsymbol{v}}}\,\left\|\boldsymbol{\phi}_{(t,s)}\right\|^2_{W^r_{\boldsymbol{v}}},$$

and by (4.3.3) and Proposition 4.1.2, we can write

$$\left\|\boldsymbol{\phi}_{(t,s)}\right\|^2_{W^r_{\boldsymbol{v}}} = \left\|\widehat{D}\boldsymbol{k}_{(\xi,\eta)}\right\|^2_{W^r_{\boldsymbol{v}}} \leq \|\boldsymbol{k}_{(\xi,\eta)}\|^2_{W^r_{\boldsymbol{v}^{-1}}}.$$

Consequently, from the assumption (4.1.11), we can deduce

$$\begin{aligned}\|\mathcal{L}_{m,m}\left((\mathcal{K} - \mathcal{K}_m)\boldsymbol{f}, \boldsymbol{v}\right)\|_{L^2_{\boldsymbol{v}}} &\leq \frac{\mathcal{C}}{m^r}\,\|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}}\left(\sum_{i=1}^m\sum_{j=1}^m \lambda_i(u)\lambda_j(u)\left\|\boldsymbol{k}_{(x_i,x_j)}\right\|^2_{W^r_{\boldsymbol{v}^{-1}}}\right)^{\frac{1}{2}} \\ &\leq \frac{\mathcal{C}}{m^r}\,\|\boldsymbol{f}\|_{L^2_{\boldsymbol{v}}}\end{aligned}$$

from which the thesis follows. □

Theorem 4.3.2. *Assume that* $Ker\{I + \widehat{D}K\} = \{0\}$, *the assumptions of Proposition 4.3.1 are satisfied and the function* $\boldsymbol{g}$ *belongs to* $W^{r_1}_{\boldsymbol{v}^{-1}}$.

Then, for sufficiently large m, *say* $m \geq m_0$, *system* (4.3.7) *has a unique solution* $\boldsymbol{f}_m$ *and the following estimate holds true*

$$\|\boldsymbol{f} - \boldsymbol{f}_m\|_{L^2_{\boldsymbol{v}}} \leq \frac{\mathcal{C}}{m^{r_1}} \|\boldsymbol{f}\|_{W^{r_1}_{\boldsymbol{v}}} \tag{4.3.8}$$

with $\mathcal{C} \neq \mathcal{C}(m, \boldsymbol{f}, \boldsymbol{g})$. *Moreover*

$$\limsup_m cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m + \boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m) \leq \mathcal{C}\, cond(I + \mathcal{K}),$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

Proof. In order to prove this theorem it is sufficient to proceed as in the proof of Theorem 4.2.3 with I, $\mathcal{K}$, $\mathcal{G}$ in place of D, K and $\boldsymbol{g}$, respectively. Moreover the thesis on the condition number can be proved as done for (4.2.14). □

4.4 Numerical Tests

In this Section, by means of some numerical tests, we show the performance of the methods described in the previous Sections. In each example, for the direct method, we solve system (4.2.10) and compute the approximate solution $\boldsymbol{f}_m$ given in (4.2.11). For the indirect method through the unique solution of system (4.3.5) we compute $\boldsymbol{f}_m$ defined in (4.3.6).

Since the exact solutions of the equations we will consider are unknown, we assume as exact those obtained for a fixed value of $m = M$ that we will specify in each test and we compute the relative errors

$$\epsilon_{M,m}(t, s) = \frac{|\boldsymbol{f}_M(t, s) - \boldsymbol{f}_m(t, s)|}{|\boldsymbol{f}_M(t, s)|}$$

in different points $(t, s) \in S$.

Example 4.4.1. *Let us consider the equation*

$$\frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x, y)}{(x - t)(y - s)} \boldsymbol{v}(x, y) dx dy + \int_S \log(4 + sx + ty) \boldsymbol{f}(x, y) \boldsymbol{v}(x, y) dx dy = e^{ts}.$$

In Tables 4.1 and 4.2 we report, for increasing value of m, *the relative errors we get in three different points of the square and the condition number in the spectral norm of the systems we solve. As we can see the convergence is very fast in virtue of the smoothness properties of the kernel and right-hand side. Moreover, the sequence* $\{cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)\}_m$ *as well as* $\{cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m + \boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)\}_m$ *is convergent as* m *goes to infinity.*

Table 4.1: Example 4.4.1: results by the direct method.

m	$\epsilon_{64,m}(0.5, 0.8)$	$\epsilon_{64,m}(0.1, -0.5)$	$\epsilon_{64,m}(-0.6, 0.7)$	$cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)$
4	2.89e-03	1.27e-03	6.73e-03	1.3931498886229416e+01
8	1.19e-08	1.24e-07	2.53e-08	1.3931550518318879e+01
16	6.24e-15	4.88e-15	4.08e-15	1.3931550518335689e+01
32	8.73e-16	5.75e-15	4.80e-16	1.3931550518335690e+01

Table 4.2: Example 4.4.1: results by the indirect method.

m	$\epsilon_{64,m}(0.5, 0.8)$	$\epsilon_{64,m}(0.1, -0.5)$	$\epsilon_{64,m}(-0.6, 0.7)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m + \boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	2.89e-03	1.27e-03	6.73e-03	1.3931498886229420e+01
8	1.19e-08	1.24e-07	2.53e-08	1.3931550518318886e+01
16	1.25e-16	3.50e-15	3.12e-15	1.3931550518335696e+01
32	2.87e-15	2.38e-15	3.72e-15	1.3931550518335680e+01

Example 4.4.2. *Let us apply our methods to the following equation*

$$\frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)} \boldsymbol{v}(x,y) dx\, dy + \int_S \frac{xt}{5+y^2+s^2} \boldsymbol{f}(x,y)\boldsymbol{v}(x,y) dxdy = \log(10-s-t).$$

Table 4.3 and 4.4 show the numerical results we get. As in the previous example, in virtue of the presence of a kernel and a right-hand side very smooth, by solving a system with $m = 32$, we get very accurate results.

Table 4.3: Example 4.4.2: results by the direct method.

m	$\epsilon_{64,m}(0.7, 0.2)$	$\epsilon_{64,m}(0.1, -0.5)$	$\epsilon_{64,m}(-0.6, 0.7)$	$cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)$
4	2.14e-04	1.63e-06	4.76e-04	2.2455715459596859e+00
8	2.57e-06	1.62e-06	9.27e-07	2.2455977174082378e+00
16	6.13e-12	4.14e-16	3.58e-12	2.2455977175654063e+00
32	3.45e-16	0.00e+00	3.20e-16	2.2455977175654054e+00

Example 4.4.3. *Let us consider again an equation which present a kernel and a right-hand side very smooth*

$$\frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)} \boldsymbol{v}(x,y) dx\, dy + \int_S e^{tsxy} \boldsymbol{f}(x,y)\boldsymbol{v}(x,y) dxdy = \sin(3+st).$$

In Tables 4.5 and 4.6 we give the relative errors and the condition number in the spectral norm. Once again, we get very accurate results.

Table 4.4: Example 4.4.2: results by the indirect method.

m	$\epsilon_{64,m}(0.7, 0.2)$	$\epsilon_{64,m}(0.1, -0.5)$	$\epsilon_{64,m}(-0.6, 0.7)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m + \boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	2.14e-04	1.63e-06	4.76e-04	2.2455715459596877e+00
8	2.57e-06	1.62e-06	9.27e-07	2.2455977174082391e+00
16	6.13e-12	3.73e-15	3.58e-12	2.2455977175654072e+00
32	5.17e-15	1.24e-15	7.99e-16	2.2455977175654058e+00

Table 4.5: Example 4.4.3: results by the direct method.

m	$\epsilon_{64,m}(0.1, -0.4)$	$\epsilon_{64,m}(0.3, -0.6)$	$\epsilon_{64,m}(-0.1, 0.5)$	$cond(\mathbf{P}_m(\mathbf{D}_m + \mathbf{K}_m)\mathbf{P}_m)$
4	5.71e-04	1.33e-03	2.96e-04	9.4647134096191934e+01
8	1.02e-08	1.38e-08	1.25e-08	9.4646712492247204e+01
16	3.24e-15	3.21e-15	3.76e-16	9.4646712492247048e+01
32	1.80e-16	5.13e-16	1.25e-16	9.4646712492247090e+01

Table 4.6: Example 4.4.3: results by the indirect method.

m	$\epsilon_{64,m}(0.1, -0.4)$	$\epsilon_{64,m}(0.3, -0.6)$	$\epsilon_{64,m}(-0.1, 0.5)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m + \boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	3.95e-04	2.62e-04	1.55e-03	9.4647134096192175e+01
8	6.74e-09	1.23e-08	5.76e-09	9.4646712492247545e+01
16	3.77e-15	3.58e-15	7.43e-16	9.4646712492246621e+01
32	2.32e-15	1.73e-15	8.91e-16	9.4646712492247204e+01

Example 4.4.4. *Let us test the performance of our methods to the equation which present a convolution kernel*

$$\frac{1}{\pi^2} \oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)}\, \boldsymbol{v}(x,y)dx\, dy + \int_S |x-t|^3\, |y-s|^4 \boldsymbol{f}(x,y)\boldsymbol{v}(x,y)dxdy = \sqrt{\frac{e^{ts}}{9+ts}}.$$

As we can see through Tables 4.7 and 4.8, the numerical results confirm the theoretical estimates given in (4.2.13) *and* (4.3.8)*.*

Example 4.4.5. *Let us test the performance of our method to the following equation in which the kernel* $\boldsymbol{k}(x,y,t,s) = |\sin(xs)|^{\frac{11}{2}} + yt$ *belongs to the Sobolev-type space of index* $r = 5$*,*

Table 4.7: Example 4.4.4: results by the direct method.

m	$\epsilon_{175,m}(0.4,-0.4)$	$\epsilon_{175,m}(0.2,-0.6)$	$\epsilon_{175,m}(-0.1,0.8)$	$cond(\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m)$
4	2.84e-01	7.73e-02	3.49e-01	5.8341767720850817e+02
8	4.36e-04	1.75e-04	8.60e-04	5.4307032442099785e+02
16	1.77e-05	9.38e-06	1.89e-05	5.4309967621958026e+02
32	1.05e-06	6.28e-07	8.45e-07	5.4310149342166687e+02
64	6.44e-08	3.99e-08	4.73e-08	5.4310161017935911e+02
128	2.95e-09	1.84e-09	2.11e-09	5.4310161764433553e+02

Table 4.8: Example 4.4.4: results by the indirect method.

m	$\epsilon_{175,m}(0.4,-0.4)$	$\epsilon_{175,m}(0.2,-0.6)$	$\epsilon_{175,m}(-0.1,0.8)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	2.85e-01	7.76e-02	3.50e-01	5.8341767720850714e+02
8	4.37e-04	1.75e-04	8.62e-04	5.4307032442101217e+02
16	1.78e-05	9.38e-06	1.90e-05	5.4309967621958094e+02
32	1.05e-06	6.28e-07	8.51e-07	5.4310149342166756e+02
64	6.46e-08	3.98e-08	4.77e-08	5.4310161017936002e+02
128	2.96e-09	1.84e-09	2.13e-09	5.4310161764433508e+02

$$\frac{1}{\pi^2}\oint_S \frac{\boldsymbol{f}(x,y)}{(x-t)(y-s)}\,\boldsymbol{v}(x,y)dx\,dy + \int_S \left(|\sin(xs)|^{\frac{11}{2}} + yt\right)\,\boldsymbol{f}(x,y)\boldsymbol{v}(x,y)dxdy = \cos(ts).$$

As shown in Tables 4.9 and 4.10, the two methods are equivalent in terms of order of convergence and the numerical evidence confirms our theoretical estimates.

Table 4.9: Example 4.4.5: results by the direct method.

m	$\epsilon_{175,m}(0.5,-0.7)$	$\epsilon_{175,m}(0.3,0.6)$	$\epsilon_{175,m}(0,0)$	$cond(\mathbf{P}_m(\mathbf{D}_m+\mathbf{K}_m)\mathbf{P}_m)$
4	1.33e-02	3.74e-03	2.06e-02	1.3576451986839258e+01
8	2.31e-04	6.08e-04	8.62e-04	1.3584012702947833e+01
16	5.45e-07	1.21e-06	4.92e-06	1.3584041062960397e+01
32	5.93e-09	1.46e-09	8.09e-08	1.3584041246139085e+01
64	2.18e-10	2.17e-12	1.69e-09	1.3584041247052387e+01
128	2.74e-12	8.31e-13	4.15e-11	1.3584041247056279e+01

Table 4.10: Example 4.4.5: results by the indirect method.

m	$\epsilon_{175,m}(0.5,-0.7)$	$\epsilon_{175,m}(0.3,0.6)$	$\epsilon_{175,m}(0,0)$	$cond(\boldsymbol{\mathcal{P}}_m(\mathbf{I}_m+\boldsymbol{\mathcal{K}}_m)\boldsymbol{\mathcal{P}}_m)$
4	1.33e-02	3.74e-03	2.06e-02	1.3576451986839267e+01
8	2.31e-04	6.08e-04	8.62e-04	1.3584012702947835e+01
16	5.45e-07	1.21e-06	4.92e-06	1.3584041062960395e+01
32	5.93e-09	1.46e-09	8.09e-08	1.3584041246139078e+01
64	2.18e-10	3.11e-12	1.69e-09	1.3584041247052406e+01
128	2.13e-12	4.18e-12	4.23e-11	1.3584041247056248e+01

Publications

S1. G. Serafini, *Numerical Approximation of Weakly Singular Integrals on a Triangle*, **The American Institute of Physics (AIP)** Conference Proceedings dedicated to the NUMTA-2016 Conference, 1776, 010001 (2016); doi: 10.1063/1.4965306; indice scopus: 2-s2.0-84995404837.

C1. F.A. Costabile, M.I. Gualtieri and G. Serafini, *Cubic Lidstone-Spline for numerical solution of BVPs*, **Mathematics and Computer in Simulation**, 141 (2017) pp. 56-64; doi: 10.1016/j.matcom.2017.01.006; indice scopus: 2-s2.0-85014009431.

O1. D. Occorsio and G. Serafini, *Cubature formulae for nearly singular and highly oscillating integrals*, **Calcolo** 55, Vol. 1, Art. 4, ISSN:0008-0624 (2018); doi: 10.1007/s10092-018-0243-x; indice scopus: 2-s2.0-85041903908.

F1. L. Fermo, M. G. Russo and G. Serafini, *Numerical Methods for Cauchy Bisingular Integral Equations of the First Kind on the Square*, **Journal of Scientific Computing**, first Online: 3 October 2018 (in press); ISSN: 0885-7474 (2018); doi: 10.1007/s10915-018-0842-3; indice scopus: 2-s2.0-85054509172.

F2. L. Fermo, M. G. Russo and G. Serafini, *Numerical Treatment of the Generalized Univariate and Bivariate Love Integral Equation* (2019, submitted).

C2. F.A. Costabile, M.I. Gualtieri and G. Serafini, *Cubic Lidstone-Spline for numerical solution of BVPs* - abstract in Atti di Convegno in Volume "International Conference New Trends in Numerical Analysis. Theory, Methods, Algorithms and Applications - Falerna (CZ), Italy, 18-21 June 2015", **Luigi Pellegrini Editore (Cosenza)**, ISBN: 978-88-6822-299-4 (2015).

Future Works

- In the future, we will consider Fredholm integral equations of the second kind with kernel functions of the type presented in Chapter 2, both, in univariate and bivariate case. In particular, we will consider the possibility to solve Fredholm integral equations, as done in Chapter 3, but with *weakly singular*, *highly oscillating* and *nearly singular and highly oscillating* kernel functions. The idea is to adopt the quadrature and cubature formulas introduced in Chapter 2 in order to approximate the involved integral operators.

- Another possible future development, concerns Chapter 4. In particular, we will try to obtain a generalization of Cauchy bisingular integral equations of the second kind, considering, in the *dominant operator* and in the *perturbation operator*, the product of two general Jacobi weights of the type

  $$v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta \quad \text{with} \quad \alpha,\beta : \alpha+\beta \in \{-1,0,1\}.$$

  In [F1] it was considered just the case

  $$\alpha+\beta = 0 \quad \text{with} \quad \alpha = \frac{1}{2}, \ \beta = -\frac{1}{2}.$$

  Furthermore, we will consider the possibility to study the Cauchy bisingular integral equation in spaces of weighted continuous functions.

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