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Rational curves on Calabi-Yau fiber spaces and Twisted cotangent bundles of Hyperkähler manifolds

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and

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Tesi di dottorato in matematica di

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Introduction

The original results that I present in this thesis can be mainly divided in two parts. In the first one I study rational curves on varieties which admits a fibration such that the canonical bundle of a generic fiber is trivial. In the second part I study the twists of the cotangent sheaf of a Hyperkähler manifold. More precisely I look for conditions such that the twists have some positivity properties. The second part is a joint work with Andreas Höring.

Rational curves on Calabi–Yau fiber spaces

Finding rational curves in a projective variety X is useful to understand the geometry of X because these curves are strongly related to many invariants. Rational curves on Calabi–Yau varieties are particularly useful but the existence of such curves in full generality on these varieties is proven only in dimension two by Bogomolov–Mumford [**MM83**]. On K3 surfaces there are rational curves in any ample linear series. This leads to define Beauville–Voisin class as the zero-cycle class of a point on a rational curve [**BV04**]. In higher dimension doing this is more difficult because it is hard to find an ample divisor $H \xrightarrow{i} X$ with $i_*(CH_0(H)) = \mathbb{Z}$, but many things can be said in the case of Hyperkähler manifolds [**CMP19**]. Let me briefly give a couple of other motivations: finding a rational curve on a variety implies that the variety is not hyperbolic in the sense of Kobayashi in a very strong way; a rational morphism to a manifold without rational curves is everywhere defined; rational curves play an important role in many parts of theoretical physics, for example see [**CXGP91**] [**AM93**]. This list can be made much longer.

In some sense a variety with no rational curves has very special properties. Indeed if one suppose that X is a projective variety with mild singularities and no rational curves, then by the Cone Theorem the canonical bundle of X is numericallly effective. In this situation Abundance Conjecture predicts that the canonical bundle is semiample, hence some power of the canonical bundle gives a well-defined fibration $X \xrightarrow{f} B$ such that $\dim(B) = k(X)$ is the Kodaira dimension of X, B is of log-general type and $K_X \sim_{\mathbb{Q}} f^*L$ for some ample line bundle L on B. By adjunction formula the general fiber of f has trivial canonical bundle.

Assuming that X has no rational curves has some strong consequences on the geometry of the fibration. Clearly there are some cases where one cannot say anything only with this fibration: if the canonical bundle is ample then f is an isomorphism and if the Kodaira dimension is zero then B is a point. In the second case one can say something for example studying other fibrations. For this reasons it is very natural to focus on the rational curves that are vertical for a fibration $X \xrightarrow{f} B$ such that $K_X \sim_{\mathbb{Q}} f^*L$.

The experience with minimal model program suggests that even if one is mainly interested in smooth varieties, the natural setting is to allow at least log-terminal singularities. With this idea in mind I started my thesis trying to extend the results proven in [**DFM19**] in a singular setting typical of the minimal model program. In their paper, Diverio, Fontanari and Martinelli, proved, among the others, the following theorem.

0.1. **Theorem.** *[**DFM19**, Theorem 1.1.] Let X be a smooth projective manifold with finite fundamental group. Suppose there exists a projective variety B and a morphism $f : X \to B$ such that the general fiber has dimension one. Suppose, moreover, that there exists a line bundle L on B such that $K_X \sim f^*L$. Then X does contain a rational curves.*

The first positive result in this direction is the following theorem.

0.2. **Theorem.** *Let X be a normal projective variety of dimension n with at most log terminal singularities and vanishing augmented irregularity,* i.e. *the irregularity of any quasi-étale cover of X is zero. Suppose that there exists a surjective morphism $X \xrightarrow{f} B$ to a variety of dimension $n-1$. If there exists a Cartier divisor L on B such that $f^*L \sim K_X$, then there exists a subvariety of codimension one in X that is covered by rational curves contracted by f.*

The assumption on the fundamental group made in [**DFM19**] is replaced by the condition on the augmented irregularity. Also in the smooth case, the finiteness of the fundamental group is stronger than the vanishing of the augmented irregularity.

The condition $f^*L \sim K_X$ is quite unpleasant, but in the case where the canonical bundle of X is itself trivial it is automatically satisfied. When the Kodaira dimension is zero, passing to an index one cover, this result can be easily generalized to the following:

0.3. **Theorem.** *Let X be a normal projective variety of dimension n with at most log terminal singularities, numerically trivial canonical bundle and vanishing augmented irregularity. Suppose that there exists a morphism $f : X \to B$ whose general fiber is a curve. Then there* *exists a uniruled subvariety of codimension one in X that is covered by rational curves contracted by f.*

This results led me, following the ideas in [**DFM19**], to study also the case of a fibration over a curves. More precisely starting from a rational point p in the Néron–Severi group with some numerical properties and assuming there are no rational curves on X, I prove the existence of another rational point in the Néron–Severi that is the class of a line bundle that induces a genus one fibration. The existence of such fibration and no rational curves gives a contradiction to Theorem 0.3.

Genus one fibrations can be divided in two main families, formed by generically isotrivial fibrations and the others. If f is not generically isotrivial then it is well-known to the experts that X does contain rational curves. Roughly speaking because the moduli space of elliptic curves is $\mathbb{A}^1$ and over the boundary of its compactification there are always rational curves. In the case of a generically isotrivial genus one fibration $X \xrightarrow{f} B$ one needs to add some hypothesis in order to have some vertical rational curves in X.

As explained before, a reasonable condition is that f is relatively minimal, *i.e.* $K_X \sim_{\mathbb{Q}} f^*L$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on B. Under this assumption I proved the following result that gives a precise description of relatively minimal genus one fibrations which contains divisors covered by vertical rational curves, and also characterizes the case where there are no vertical rational curves at all.

0.4. **Theorem.** *Let X be a projective variety with at most log-terminal singularities. Suppose there is a fibration $X \to B$ such that the general fiber is a genus one curve and there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on B such that $f^*L \sim_{\mathbb{Q}} K_X$. Then*

- *The variety X does not contain a divisor covered by vertical rational curves if and only if X is isomorphic in codimension one to a variety Y which has a finite cover, étale in codimension two, isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*
- *The variety X does not contain vertical rational curves if and only if there is a finite globally étale cover of X isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*

This theorem is a satisfactory answer in the case of relatively minimal genus one fibration. At this point it is natural to ask how the condition to be relatively minimal may fail and what happens in this case. A genus one fibration kay fail to be relatively minimal for two reasons: the presence of some exceptional components in the canonical bundle of X, or the canonical bundle does not exists at all as a $\mathbb{Q}$-Cartier divisor. The first solution I found for these problems is the following:

0.5. **Proposition.** *Let (X, Δ) be a klt pair such that there exists a surjective morphism $f : X \to B$ to a variety of dimension $n-1$ that is not a quasi-product over B,* i.e. *isomorphic in codimension one to the quotient of a product, see Definition 3.21. Suppose moreover $K_X + \Delta \sim_{\mathbb{Q}} f^*L + \sum a_i E_i$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on B, some f-exceptional divisor E_i and whose coefficients are not all strictly negative. Then, X does contain rational curves.*

Although not optimal, this proposition shows when and how one can apply a result of Kawamata on the uniruledness of the exceptional locus to control the exceptional divisors. Then I prove some statements strictly linked to the previous one, obtained mixing the result of Kawamata with some tricks in birational geometry. After that, I figured out that I could use some results by Hacon and McKernan to control in the general case the exceptional locus. After some technical work I proved the following theorem that has the same spirit of Theorem 0.4, but where I relax some of the hypothesis.

0.6. **Theorem.** *Let $X \xrightarrow{f} B$ be a genus one fibration. Assume that there exists an effective Weil $\mathbb{Q}$-divisor Δ on B such that (B, Δ) is klt. Then*

- *If X is not a quasi-product, then X contains a uniruled divisor.*
- *If X is not an orbibundle, then X contains a vertical rational curve.*

More precisely:

- *The variety X does not contain a divisor covered by vertical rational curves if and only if X is isomorphic in codimension one to a variety which has a finite cover, étale in codimension two, isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*
- *The variety X does not contain vertical rational curves if and only if there is a finite globally étale cover of X isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*

In the case the relative dimension is more than one, I have some quite satisfactory results in the generically isotrivial case as the following:

0.7. **Theorem.** *Let $X \xrightarrow{f} B$ be a generically isotrivial fibration and denote by F the generic fiber. Suppose that X is smooth and does not contain rational curves. Suppose moreover that there exists a $\mathbb{Q}$-Cartier divisor on B such that $K_X \sim_{\mathbb{Q}} f^*L$. Then X is a finite étale quotient of $\tilde{B} \times F$, for some cover $\tilde{B}$ of B.*

The twisted cotangent sheaf of a Hyperkähler manifold (joint work with Andreas Höring)

Let X be a compact Kähler manifold, and let Ω_X be the cotangent bundle of X. If the canonical bundle $K_X = \det \Omega_X$ is positive (e.g. pseudoeffective or nef) we can use stability theory to describe the positivity of Ω_X. The most famous result in this direction is Miyaoka's theorem [**Miy87**] which says that for a projective manifold that is not uniruled, the restriction $\Omega_X|_C$ to a general complete intersection curve C of sufficiently ample general divisors is nef. However this result only captures a part of the picture: denote by $\zeta \to \mathbb{P}(\Omega_X)$ the tautological class on the projectivised cotangent bundle $\pi : \mathbb{P}(\Omega_X) \to X$. If X is Calabi-Yau or a projective Hyperkähler manifold the tautological class ζ is not pseudoeffective [**HP19**, Thm.1.6]. In particular X is covered by curves C such that $\Omega_X|_C$ is not nef.

Our goal is to measure this defect of positivity by considering polarised manifolds (X, H). This has been accomplished for infinitely many families of projective K3 surfaces in a beautiful paper of Gounelas and Ottem:

0.8. **Theorem.** *[**GO18**, Thm.B] Let (X, H) be a primitively polarised K3 surface of degree d and Picard number one. Denote by $\pi : \mathbb{P}(\Omega_X) \to X$ the projectivisation of the cotangent bundle, and by $\zeta \to \mathbb{P}(\Omega_X)$ the tautological class.*

Suppose that $\frac{d}{2}$ is a square and the Pell equation

$$x^2 - 2dy^2 = 5$$

has no integer solution. Then $\zeta + \frac{2}{\sqrt{\frac{d}{2}}}\pi^* H$ is pseudoeffective and $\zeta + (\frac{2}{\sqrt{\frac{d}{2}}} - \varepsilon)\pi^* H$ is not pseudoeffective for any $\varepsilon > 0$.

In the situation above one has $(\frac{2}{\sqrt{\frac{d}{2}}}H)^2 = 8$, so we see that, under these numerical conditions, the class $\zeta + \pi^* H$ is pseudoeffective for an ample $\mathbb{R}$-divisor class H of degree at least eight. In view of this observation we make the following

0.9. **Conjecture.** *Fix an even natural number $2n$. Then there exists only finitely many deformation families of polarised Hyperkähler manifolds (X, H) such that $\dim X = 2n$ and H is ample Cartier divisor on X such that $\zeta + \pi^* H$ is not pseudoeffective.*

This conjecture should be seen as an analogue of the situation for uniruled manifolds: in this case Ω_X is not even generically nef in the sense of Miyaoka, but $\Omega_X \otimes H$ is generically nef unless X is very special ([**Hör14**, Thm.1.1], see [**AD17**, Cor.1.3] for a stronger version).

In this paper we give a sufficient condition for the pseudoeffectivity of twisted cotangent bundles for Hyperkähler manifolds. Since deformations to non-projective Hyperkähler manifolds are crucial for the proof we state the result in the analytic setting:

0.10. **Theorem.** *Let X be a (not necessarily projective) Hyperkähler manifold of dimension $2n$, and denote by $q(\cdot)$ its Beauville-Bogomolov form. Denote by $\pi : \mathbb{P}(\Omega_X) \to X$ the projectivisation of the cotangent bundle, and by $\zeta \to \mathbb{P}(\Omega_X)$ the tautological class. There exists a constant $C \geq 0$ depending only on the deformation family of X such that the following holds:*

- *Let ω_X be a nef and big $(1,1)$-class on X such that $q(\omega_X) \geq C$. Then $\zeta + \pi^*\omega_X$ is pseudoeffective.*
- *Suppose that X is very general in its deformation space, and let ω_X be a nef and big $(1,1)$-class on X. Then $q(\omega_X) \geq C$ if and only if $\zeta + \pi^*\omega_X$ is nef.*

The proof of the second statement is a combination of Demailly-Păun's criterion for nef cohomology classes with classical results on the cohomology ring of very general Hyperkähler manifolds: we show in Lemma 19.1 that all the relevant intersection numbers are in fact polynomials in one variable, the variable being the Beauville-Bogomolov form $q(\omega_X)$. The largest real roots of these polynomials turn out to be bounded from above, this yields the existence of the constant C. The first statement then follows by a folklore degeneration argument that was explained to me by my advisor S. Diverio. A detailed explanation can be founded in Section 9.

As an immediate consequence we obtain some good evidence for Conjecture 0.9:

0.11. **Corollary.** *Let X_0 be a differentiable manifold of real dimension $4n$. Then there exist at most finitely many deformation families of polarised Hyperkähler manifolds (X, H) such that $X_0 \overset{diff.}{\simeq} X$ and H is an ample Cartier divisor on X such that $\zeta + \pi^*H$ is not pseudoeffective.*

While Theorem 0.10 is quite satisfactory from a theoretical point of view, it it is not clear how to compute the constant C in practice. We therefore prove a more explicit version under a technical assumption:

0.12. **Theorem.** *Let X be a (not necessarily projective) Hyperkähler manifold of dimension $2n$. Suppose that a very general deformation of X does not contain any proper subvarieties. Let ω_X be a Kähler class on X.*

- *Suppose that*
$$(\zeta + \lambda\pi^*\omega_X)^{4n-1} > 0 \qquad \forall\ \lambda > 1.$$

Then $\zeta + \pi^* \omega_X$ *is pseudoeffective.*

- *Suppose that* X *is very general in its deformation space. Then* $\zeta + \pi^* \omega_X$ *is nef if and only if*
$$(\zeta + \lambda \pi^* \omega_X)^{4n-1} > 0 \qquad \forall \ \lambda > 1.$$

We also prove in Proposition 20.2 that for very general X, the class $\zeta + \pi^* \omega_X$ is pseudoeffective if and only if it is nef. Thus Theorem 0.12 is optimal at least for very general X. Since $(\zeta + \lambda \pi^* \omega_X)^{4n-1}$ can be expressed as a polynomial depending only on the Segre classes of X (see equation (5)), the sufficient condition can be written down explicitly.

If ω_X is the class of an ample divisor, the condition in Theorem 0.12 essentially says that the leading term of the Hilbert polynomial
$$\chi(\mathbb{P}(\Omega_X), \mathcal{O}_{\mathbb{P}(\Omega_X)}(l(\zeta + \pi^* \omega_X)))$$
is positive. It is however possible that the higher cohomology of $\mathcal{O}_{\mathbb{P}(\Omega_X)}(l(\zeta + \pi^* \omega_X))$ grows with order $4n-1$, so it is not obvious that $\zeta + \pi^* \omega_X$ is pseudoeffective.

Let S be a K3 surface, and denote by $X := S^{[n]}$ the Hilbert scheme parametrizing 0-dimensional subschemes of length n. Then X is Hyperkähler [**Bea96**], and by a theorem of Verbitsky [**Ver98**, Thm.1.1] a very general deformation does not contain any proper subvarieties. Thus the technical condition in Theorem 0.12 is satisfied for a Hyperkähler manifold of deformation type $K3^{[n]}$. We compute the constant C for Hilbert schemes of low dimension. In particular we obtain

0.13. **Corollary.** *Let* S *be a (not necessarily projective) K3 surface. Let* ω_S *be a nef and big* $(1,1)$*-class on* S *such that* $\omega_S^2 \geq 8$*. Then* $\zeta + \pi^* \omega_S$ *is pseudoeffective.*

The theorem of Gounelas and Ottem shows that this result is optimal for infinitely many 19-dimensional families of projective K3 surfaces. Their results also show that for certain families, e.g. general smooth quartics in $\mathbb{P}^3$, our estimate is not optimal [**GO18**, Cor.4.2]. In these cases the obstruction comes from the projective geometry of X [**GO18**, Sect.4.2].

In higher dimension the situation becomes much more complicated. We show in Corollary 21.3 that for a nef and big class ω_X on a Hilbert square $X := S^{[2]}$ such that
$$q(\omega_X) \geq 3 + \sqrt{\frac{21}{5}},$$
the class $\zeta + \pi^* \omega_X$ is pseudoeffective. This bound is optimal for a very general deformation of X. However a Hilbert square $S^{[2]}$ deforms as a complex manifold in a 21-dimensional space, while its deformations as a Hilbert square only form a 20-dimensional family. In Section 21 we

study in detail very general elements of the family of Hilbert squares: since the Hilbert square always contains an exceptional divisor, it is obvious that the nef cone and the pseudoeffective cone of $\mathbb{P}(\Omega_{S^{[2]}})$ do not coincide. It is much more difficult to decide if $\zeta + \pi^*\omega_X$ is nef if it is pseudoeffective. For this purpose we construct in Subsection 21.D a "universal" subvariety $Z \subset \mathbb{P}(\Omega_{S^{[2]}})$ that surjects onto $S^{[2]}$ and is an obstruction to the nefness of $\zeta + \pi^*\omega_X$ (cf. Proposition 21.10).

Structure of the thesis.

Part one. This part is mainly devoted to introduce some facts useful for the subsequent parts. In order to make the second part more readable, most of the theory needed to study Calabi–Yau fiber spaces is developed in this part.

In Section 1, I fix some notations about the singularities that will appear in Part 2 and prove an easy lemma to detect the singularities of a relatively minimal pair.

In Section 2, I recall some well-known facts about reflexive sheaves. Then I focus on some properties of the cotangent sheaf and recall some facts on the tautological line bundle.

In Section 3, I introduce some useful terminology, I recall two lemmas needed for the study of degenerate divisors and then I focus on orbibundles. Some of the key lemmas of the results in Part 2 are in this section.

In Section 4, I recall a result of Kawamata and two results of Hacon and McKernan useful to control the exceptional locus of a morphism. A minor generalization to rational map is introduced.

In Section 5, I give two proofs of Fischer–Grauert Theorem in the étale topology under some assumption on the fibers.

In Section 6, I recall the definition of augmented irregularity and prove some properties of this invariant.

In Section 7, I explain how some facts about Chern classes that are well-known for smooth varieties can be extended to varieties with mild singularities.

In Section 8, I proved some facts of the cones inside the Néron–Severi group needed in Section 15.

In Section 9, I recall some notions on the positivity of real cohomology classes of Hodge type $(1,1)$ for compact Kähler manifolds. I explain why the relative pseudoeffective cone of a proper submersion is locally closed.

Part two. This part is devoted to find some conditions such that a Calabi–Yau fiber space contains rational curves. The first five sections of this part are about the case of genus one fibrations, the subsequent two sections are about, respectively, fibrations over curves and over a base of arbitrary dimension. The last section is a collection of examples.

In Section 10, I prove Theorem 0.2 and explain the consequences in the case $K_X \equiv_{\text{num}} 0$.

In Section 11, I focus on the case of relatively minimal genus one fibrations. In particular there is the proof of Theorem 0.4.

In Section 12, I study the case of non relatively minimal fibrations. I prove, among the others, Theorem 0.6.

In Section 13, I talk about some particular cases and some application for smooth varieties.

In Section 14, I recall some necessary and sufficient numerical conditions to have a relatively minimal genus one fibration.

In Section 15, I study the case the base of the fibration has dimension one.

In Section 16, I explain which are the problems in the general case of Calabi–Yau fiber spaces, and prove Theorem 0.7.

In Section 17, I collect all the examples of the first two parts.

Part three. This part is devoted to the study of the Kähler and pseudoeffective cones of the projectivised cotangent bundle of a Hyperkähler manifold. All the results in this part are proved in collaboration with Andreas Höring.

In Section 18, I fix the notation for the third part. After that, I recall some facts about the geometry of Hyperkähler manifolds, their moduli spaces and some of their invariants that are needed in the subsequent sections.

In Section 19, I study some properties of particular cohomology classes of $\mathbb{P}(\Omega_X)$, prove Theorem 0.10 and Corollary 0.11. I conclude this section studying some subvarieties of $\mathbb{P}(\Omega_X)$.

In Section 20, I prove Theorem 0.12 and compute the pseudoeffective threshold for K3 surfaces. As an application I explicitly describe the Kähler cone of $\mathbb{P}(\Omega_X)$ is some cases.

In Section 21, I describe in details the situation if $X \sim_{\text{def}} K3^{[2]}$. If X is the Hilbert scheme of length two points on a K3 surface I construct an obstruction for a pseudoeffective class to be Kähler.

In Section 22, I write down the computations for the pseudoeffective threshold in the case $X \sim_{\text{def}} K3^{[3]}$.

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Contents

Part 1

Preliminary results

We assume the reader knows the basic properties of algebraic geometry. The general notations we use can be found in [Har77]. We assume that the reader is familiar with the basic notions on the singularities that appear in birational geometry. The reader can find all the unreferenced notations about the singularities that we will use in [KM98].

1. Singularities

Let Y be a normal projective variety. A *boundary* on Y is an effective Weil $\mathbb{Q}$-divisor on Y. We say that Y is *potentially klt*, *i.e.* potentially kawamata log-terminal, if there exists some boundary on Y such that the pair (Y, Δ) is klt.

The *pushforward* of a divisor is intended to be as Weil divisor, *i.e.* the direct image as cycle. The pushforward as Cartier divisor, and hence as a locally free sheaf is different.

Let $f : Y' \to Y$ be a finite cover of normal projective varieties. Under the finiteness assumption one can pullback arbitrary Weil divisors in the following way. The preimage of the singular locus $f^{-1}(Y_{\text{sing}})$ has codimension at least two. Hence to a Weil divisor D on Y one can associate the Cartier divisor $D_{Y_{\text{reg}}}$ on Y_{reg}. Then one consider the pullback as a Cartier divisor to $f^{-1}(Y_{\text{reg}})$ that, by the condition on the codimension, has a well defined extension f^*D to a Weil divisor in Y'. This construction is clearly equivalent to the standard one if D is a $\mathbb{Q}$-Cartier divisor. With this preliminary explanation we recall the following proposition.

1.1. **Proposition.** *Let $f : Y' \to Y$ be a finite cover of normal projective varieties. Let Δ and Δ' be $\mathbb{Q}$-Weil divisors respectively on Y and Y' such that $K_{Y'} + \Delta' = f^*(K_Y + \Delta)$. Then*

- *The divisor $K_Y + \Delta$ is $\mathbb{Q}$-Cartier if and only if $K_{Y'} + \Delta'$ is.*
- *The pair (K_Y, Δ) is klt if and only if $(K_{Y'}, \Delta')$ is.*

PROOF. See Proposition [KM98, Proposition 5.20]. □

As an application of the results in the minimal model program we have the following fact that that can be found in [BCHM06, Corollary 1.4.3].

1.2. **Proposition.** *Let (Y, Δ) be a klt pair. Let $\mathcal{E}$ be a finite collection of exceptional divisors over Y with non positive discrepancies. Then there is birational morphism f from a $\mathbb{Q}$-Cartier klt variety Y' nef over Y such that $K_{Y'} + \Gamma = f^*(K_Y + \Delta)$ and the contracted divisors are exactly the divisors in $\mathcal{E}$.*

If $\mathcal{E} = \emptyset$ then the map f is small and Y' will be denoted by $Y_{\mathbb{Q}}$ and it is called $\mathbb{Q}$-factorialization. The opposite extremal case, where $\mathcal{E}$ is given by all the divisors with negative discrepancies, will be denoted by Y_{term} and it is called terminalization.

Suppose that the pair (Y, Δ) is klt and that Y' is isomorphic in codimension one to Y. Unfortunately we cannot say that (Y', Δ') is klt for some boundary Δ' because there is no reasons to expect $K_{Y'} + \Delta'$ to be $\mathbb{Q}$-Cartier. This phenomenon happens for example taking a small contraction of an extremal ray with non zero intersection with the canonical bundle. However we can prove the following lemma that will be useful.

1.3. **Lemma.** *Let X_1, X_2 be normal projective varieties which are isomorphic in codimension one via a rational map f over a base B*

$$\begin{tikzcd} X_1 \arrow[rr, dashed, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & B. & \end{tikzcd}$$

Suppose that there exists a boundary Δ_1 such that (X_1, Δ_1) is klt and $f_1^*L \sim_{\mathbb{Q}} K_{X_1} + \Delta_1$ for some $\mathbb{Q}$-Cartier divisor on B. Then, denoting with $\Delta_2 := g_*\Delta_1$, also (X_2, Δ_2) is klt and $f_2^*L \sim_{\mathbb{Q}} K_{X_2} + \Delta_2$.

Proof. Let Z be any common log-resolution of X_i.

$$\begin{tikzcd} & Z \arrow[dl, "\nu_1"'] \arrow[dr, "\nu_2"] & \\ X_1 \arrow[rr, dashed, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & B. & \end{tikzcd}$$

Since X_1 and X_2 are isomorphic in codimension one the exceptional divisors for the two varieties are the same. In order to write the canonical bundle of Z respect the two resolutions to compute the discrepancies, we should firstly check that the $K_{X_2} + \Delta_2$ is $\mathbb{Q}$-Cartier. Let m be an integer such that mL is Cartier. Then

$$m(K_Z + (\nu_1)_*^{-1}\Delta_1 - \sum a_j E_j) \sim m\nu_1^*(K_{X_1} + \Delta_1) \sim$$

$$\sim \nu_1^*(f_1^*(mL)) \sim \nu_2^*(f_2^*(mL))$$

Let us spend few words explaining in which sense the $\mathbb{Q}$-Cartier divisor f_2^*L is equal to $(\nu_2)_*(\nu_2^*(f_2^*L))$. The Cartier divisor $f_2^*(mL)$ can be pulled back as line bundle and then one choose a non necessarily effective Weil divisor as rational section of $\nu_2^*(f_2^*(mL))$, and then push it forward as cycle to get a Weil divisor on X_2. We obtain a Weil divisor $(\nu_2)_*(\nu_2^*(f_2^*(mL)))$ that is a rational section of $f_2^*(mL)$ and hence is also Cartier. In conclusion we can write

$$f_2^*(mL) \sim (\nu_2)_*\nu_2^*(f_2^*(mL)) \sim (\nu_2)_*(m(K_Z + (\nu_1^{-1})_*\Delta_1 - \sum a_j E_j))$$

that is also equal to $m(K_{X_2} + \Delta_2)$.

The discrepancies of (X_1, Δ_1) and (X_2, Δ_2) are the same by a direct computation of the canonical bundle of Z. □

With the notations that we will introduce later the conditions on (X_i, Δ_i) mean that they are relatively minimal log Calabi–Yau fiber spaces. In other words we proved that a variety which is isomorphic in codimension one to a relatively minimal log Calabi–Yau fiber space is itself a relatively minimal log Calabi–Yau fiber space.

2. Reflexive sheaves

For the missing proofs of this section the interested reader can see [**Har80**].

Let Y be a quasi-projective variety and $\mathcal{F}$ a coherent sheaf on it. One can associate to $\mathcal{F}$ its dual and we denote it by $\mathcal{F}^* := \hom(\mathcal{F}, \mathcal{O}_Y)$. Iterating this process we get a natural map from $\mathcal{F}$ to $\mathcal{F}^{**}$.

2.1. **Definition.** *With the above notations the sheaf $\mathcal{F}$ is* reflexive *if the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism of sheaves on Y. The sheaf $\mathcal{F}^{**}$ is called* reflexive hull *of $\mathcal{F}$.*

There are many observations that we can do. The kernel of the map $\mathcal{F} \to \mathcal{F}^{**}$ is exactly the torsion subsheaf of $\mathcal{F}$. In particular the reflexive sheaves are torsion free. A locally free sheaf is a reflexive sheaf. So the reflexive sheaves form a wider class than the locally free sheaves, but are not as general as all torsion free sheaves. The reflexive hull of a coherent sheaf is reflexive. More generally one can prove that the dual of a coherent sheaf is reflexive. If we suppose Y to be smooth then any reflexive sheaf is locally free in codimension two. In particular a reflexive sheaf on a smooth surface is nothing but a locally free sheaf. Since we will work with singular varieties, the previous observation is not enough. However we have the following explicative result.

2.2. **Proposition.** *Assume Y is a normal variety and $\mathcal{F}$ a coherent sheaf on Y. Then the following conditions are equivalent:*

- *$\mathcal{F}$ is reflexive.*
- *$\mathcal{F}$ is torsion free, and for each open $U \subset Y$ and each closed subset $Z \subset U$ of codimension greater or equals than two, denoting by $j : U - Z \to U$ the inclusion map, it induces an isomorphism of sheaves $\mathcal{F}_U \simeq j^*\mathcal{F}_{U-Z}$.*

This result tells us that the sections of a reflexive sheaf doesn't see the geometry of the variety in codimension at least two. This is particularly useful if we know that $\mathcal{F}$ is locally free over an open set whose complementary has codimension at least two, *e.g.* the cotangent sheaf on the smooth locus of Y.

2.A. The cotangent sheaf. Suppose for simplicity that Y is a normal variety. The cotangent sheaf of a Y can be defined in several equivalent ways. One can consider Y embedded via the diagonal morphism Δ in the product $Y \times Y$. The closed subvariety $\Delta(Y)$ is defined by a sheaf of ideals $\mathcal{J}$.

2.3. **Definition.** *With the above notation the* cotangent sheaf *of Y is the coherent sheaf $\Omega^1_Y := \Delta^*(\mathcal{J}/\mathcal{J}^2)$. The* reflexive cotangent sheaf *of Y is $\Omega^{[1]}_Y := (\Omega^1_Y)^{**}$. The* p-reflexive exterior power *of the cotangent sheaf is $\Omega^{[p]}_Y := (\Lambda^p \Omega^1_Y)^{**}$.*

The cotangent sheaf is locally free is and only if Y is smooth. More precisely it is locally free exactly over the smooth locus of Y. By this observation and by Proposition 2.2 the sections of the p-reflexive exterior powers of the cotangent sheaf on Y are exactly the sections of the cotangent sheaf on the regular part of Y. This implies that denoting by $i : Y_{\mathrm{reg}} \to Y$ the inclusion we have $\Omega^{[p]}_Y = i_* \Omega^p_{Y_{\mathrm{reg}}}$.

If Y is smooth then we denote $\Omega_Y := \Omega^1_Y$ that is a locally free sheaf of rank $n = \dim(Y)$. One can consider the *projectivised cotangent bundle* $\mathbb{P}(\Omega_Y)$ that comes with a natural projection $\mathbb{P}(\Omega_Y) \xrightarrow{\pi} Y$. This object is a projective bundle over Y that encodes a lot of informations on Y. To understand some of these informations we recall some definitions that we will use for an arbitrary vector bundle and not only Ω_Y.

For this part see also [**Har77**, Section II.7] and [**Laz04b**, Chapter 6]. Let $\mathcal{F}$ be a locally free sheaf on Y. The projectivisation of $\mathcal{F}$ is denoted by $\mathbb{P}(\mathcal{F}) \xrightarrow{\pi} Y$. It comes with the *tautological line bundle* that is $z := \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ and its first Chern class is the *tautological class* that will be denoted with $\zeta := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$. Let L be a line bundle on Y. The global sections of the sheaf $z^{\otimes k} \otimes \pi^* L$ are in correspondence with the global sections of $\mathrm{Sym}^k \mathcal{F} \otimes L$. This remark is useful for the following characterization of pseudoeffective vector bundle: a vector bundle $\mathcal{F}$, *i.e.* a locally free sheaf, is *pseudoeffective* if the tautological class ζ is. Equivalently it is pseudoeffective if for any ample line bundle on X and any small rational $\varepsilon > 0$ the locally free sheaf $\mathrm{Sym}^k(\mathcal{F} \otimes A^{\otimes \varepsilon})$ has global sections for $k >> 0$ enough divisible.

3. Fibrations

In this section we collect some definitions and some results from the literature as Lemma 3.6, Lemma 3.7 and Theorem 3.18. The rest of the section is devoted to some preliminary results that are needed in the subsequent sections. In particular we do a careful analysis of orbibundles. We denote by n the dimension of Y.

3.1. **Definition.** *A* fibration *is a proper surjective morphism $Y \to B$ between normal quasi-projective varieties with connected fibers.*

An interesting geometrical object associated to a fibration is the locus of singular values. Let us recall its definition.

3.2. **Definition.** *Let $f : X \to Y$ be a surjective projective morphism of normal variety. The subset of* singular values *of f is the following subset of Y*

$\mathrm{Sv}(f) = \{y \in Y |\ \dim(f^{-1}(y)) > \dim(X) - \dim(Y) \vee f^{-1}(y) \textit{ is singular}\}.$

3.3. **Remark.** The singular values of f is the image of the singular locus of f. For the interested reader the definition of singular locus of a morphism can be found at the following link: http://stacks.math.columbia.edu/tag/01V5. We do not give the definition of singular locus of a morphism because we just need the given characterization of the image of the singular locus.

3.4. **Definition.** *A fibration is* generically isotrivial *if any two general fibers are isomorphic. A fibration is* of maximal variation *if a fixed general fiber is isomorphic only to finitely many other fibers.*

3.A. Degenerate divisors.

To understand the geometry of the fibration in the case of higher dimensional algebraic varieties, the first thing one should do is to understand some properties of $\mathbb{Q}$-divisors. In order to do that we introduce the following:

3.5. **Definition.** *Let $f : Y \to B$ be a fibration between normal projective varieties and let D be a prime Weil divisor on Y. We say that D is* exceptional *if $\mathrm{cod}_B(f(D)) \geq 2$. We say that D is of* insufficient fiber type *if $\mathrm{cod}_B(f(D)) = 1$ and there exists another prime Weil divisor $D' \neq D$ such that $f(D') = f(D)$. In either of the above cases, we say that D is* degenerate.

These divisors are very useful because of the following versions of the negativity lemma:

3.6. **Lemma.** *Let $Y \xrightarrow{f} B$ be a fibration between projective varieties. Let D be an effective, irreducible and exceptional $\mathbb{Q}$-Cartier divisor on Y. Then the support of D is covered by curves contracted by f and* *intersecting D negatively. More precisely these curves are obtained as general element in $D \cap H^{n-2}$ for some very ample divisor H on Y.*

Proof. Let H be a very ample line bundle on Y and $S \subset X$ be a surface obtained as a complete intersection of $n-2$ general elements in $|H|$. Since there is an exceptional divisor the dimension of B is at least two. Hence the map $f|_S$ is generically finite over its image and $D \cap S$ is contracted to a point. By the Hodge index Theorem $D^2 \cdot H^{n-2} < 0$. This means that the curves obtained as a complete intersection of $n-2$ general elements in $|H|$ and D are curves contained in D, that have negative intersection with D. $\square$

With minor changes this lemma can be found in [**Kol15**, Lemma 18] and [**Lai10**, Lemma 2.9].

3.7. **Lemma.** *[**Lai10**, Lemma 2.9] Let $Y \xrightarrow{f} B$ be a fibration between projective varieties. Suppose that Y is $\mathbb{Q}$-factorial. For an effective insufficient type Weil divisor D on Y we can always find a component $F \subseteq \mathrm{Supp}(D)$ which is covered by curves contracted by f and intersecting D negatively. More precisely these curves are obtained as general element in $F \cap H^{n-2}$ for some ample divisor H on Y.*

3.8. **Remark.** If we suppose that there exists a small $\mathbb{Q}$-factorialization of Y, *e.g.* Y is potentially klt, then the assumption on the $\mathbb{Q}$-factoriality can be ignored working on a small $\mathbb{Q}$-factorialization.

The following lemma will be useful.

3.9. **Lemma.** *Let Y and Y' be projective varieties with two fibrations f and f' to a normal variety B of dimension $n-1$. Suppose both fibrations have no degenerate divisors and they are birational via a map g. Then g contracts some divisors if and only if g^{-1} does.*

Proof. Suppose $D \subset Y$ is a prime divisor contracted in Y' by g and that f has no degenerate divisors. Let us observe that a rational map between normal varieties is always defined in codimension one, so we can decide if a divisor is contracted also for rational map. If $f(D) = B$ then by dimensional reasons $\dim(g(D)) = n-1$. The divisor D is not f-exceptional, so $f(D)$ is a divisor in B. By dimensional reasons its preimage f'^{-1} in X' contains a Weil divisor W. This divisor is not the image of a Weil divisor of Y because we are supposing the only divisor in $f^{-1}(f(D))$ is D that is contracted by g. With this in mind it follows easily that g^{-1} contracts W. $\square$

3.10. **Remark.** The previous lemma is false without the assumption on the dimension of the base because it may exist a divisor contracted by g that dominates the base. However this hypothesis can be replaced in the following way.

3.11. **Lemma.** *Let Y and Y' be projective varieties with two fibrations f and f' onto a normal variety B of dimension d. Suppose both fibrations have no degenerate divisors and they are birational via a map g that is an isomorphism over a non-empty open subset of B. Then g contracts some divisors if and only if g^{-1} does.*

Proof. Let $D \subset Y$ be a prime divisor contracted in Y', *i.e.* $\dim(g(D)) < n-1$. Since the map g is an isomorphism over an open subset of B, if a divisor dominates the base it cannot be contracted in Y', *i.e.* $f(D) \subsetneq B$. Since Y has no f-exceptional divisor $\dim(f(D)) = d-1$, and hence $\dim((g(D)) = n-2$. The scheme $f'^{-1}(f(D))$ contains some variety of dimension $n-1$ that cannot be in the image of g, and hence is contracted by g^{-1}. $\square$

3.B. Calabi–Yau fiber spaces. Now we fix some notations about the fibrations.

3.12. **Definition.** *A normal projective variety Y (resp. a pair (Y, Δ)) together with a fibration $Y \xrightarrow{f} B$ is a* Calabi–Yau fiber space *(resp. a* log Calabi–Yau fiber space*) if the generic fiber Y_t of f (resp. the pair (Y_t, Δ_t)) has numerically trivial canonical bundle (resp. $K_{Y_t} + \Delta_t \equiv_{num} 0$).*

We are mainly interested in the case where the boundary Δ does not intersect the general fiber. Indeed if this intersection is non-trivial then Y is uniruled. Let us also fix the following convention.

3.13. **Definition.** *A variety with at most log-terminal singularities Y (resp. a klt pair (Y, Δ)) is a* Calabi–Yau *variety (resp. a* log Calabi–Yau*) if $K_Y \equiv_{num} 0$ (resp. $K_Y + \Delta \equiv_{num} 0$) and $\tilde{q}(Y) = 0$.*

In our Definition 3.13 we include also products of Calabi–Yau and irreducible holomorphic symplectic varieties in the sense of [**GKP16c**]. However we do not allow quotients of abelian varieties. In particular a Calabi–Yau fiber space over a point is not necessarily a Calabi–Yau itself.

3.14. **Definition.** *A* genus one fibration *is a fibration such that the general fiber is a smooth genus one curve. An* elliptic fibration *is a genus one fibration with a fixed section.*

A particular case of Calabi–Yau fiber space is the following.

3.15. **Definition.** *A Calabi–Yau fiber space $Y \xrightarrow{f} B$ (resp. a log Calabi–Yau fiber space $(Y, \Delta) \xrightarrow{f} B$) is called a* relatively minimal *(resp.* log*)* Calabi–Yau fiber space *if Y is log-terminal (resp. the pair (Y, Δ)*

is klt) and $K_Y \sim_{\mathbb{Q}} f^*L$ $(K_Y + \Delta \sim_{\mathbb{Q}} f^*L)$ *for some* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* B*. A Calabi–Yau fiber space is* linearly relatively minimal *if* $K_Y \sim f^*L$ *for some line bundle on* B*.*

In particular when we say that genus one fibration is relatively minimal we are saying that X has at most log-terminal singularities.

3.16. **Remark.** Suppose that the general fiber is contained in the smooth locus of Y. In this context we claim that the conditions $K_Y \equiv_{\mathrm{num}} f^*L$ and $K_Y \sim_{\mathbb{Q}} f^*L$ are equivalent. Indeed suppose that $K_Y \equiv_{\mathrm{num}} f^*L$. By adjunction formula the canonical bundle of a general fiber is numerically trivial and by [**Amb05**, Theorem 0.1] also $\mathbb{Q}$-linearly trivial. The exponential exact sequence on X gives us

$$H^1(Y, \mathcal{O}_Y) \to \mathrm{Pic}(Y) \xrightarrow{c_1} H^2(Y, \mathbb{Z}).$$

Hence under our assumption $c_1(K_{Y_t} - f^*L) \in H^2(X, \mathbb{Z})_{\mathrm{tors}}$ there exists a positive integer m such that $m \cdot c_1(K_{Y_t} - f^*L) = 0$. By exactness this means that $K_{Y_t} - f^*L|_{Y_t}$ lies in $\mathrm{Pic}^0(Y)$. By [**Lan19**, Chapter VIII Theorem 13] there is an exact sequence of abelian varieties

$$0 \to J(B) \xrightarrow{f^*} \mathrm{Pic}^0(Y) \xrightarrow{\alpha} P \to 0$$

where P is a part of the Jacobian of the general fiber. Essentially P is composed by those line bundles on the generic fiber that extend to the variety Y. By our assumption $m \cdot \alpha(K_{Y_t} - f^*L|_{Y_t}) = 0$. This means that $m \cdot c_1(K_{Y_t} - f^*L)$ is in the image of f^* that proves our claim.

3.C. Orbibundles. An important class of examples of Calabi–Yau fiber spaces is given by orbibundles. We recall the construction of orbibundles because they are very useful example af Calabi–Yau fiber spaces. Let $\tilde{B}$ be a normal variety, F a variety with at most log-terminal singularities, $K_F \equiv_{\mathrm{num}} 0$ and $\tilde{Y} := \tilde{B} \times F$ their product. Let G be a finite group and $\rho_B : G \to \mathrm{Aut}(\tilde{B})$, $\rho_F : G \to \mathrm{Aut}(F)$, two faithful representations.

3.17. **Definition.** *An* orbibundle *is the fibration*

$$(Y \to B) := \tilde{Y}/G \to \tilde{B}/G$$

obtained as the quotient with respect to the diagonal representation of G*.*

Since both the representations of G are faithful, the quotient map is quasi-étale, hence in the case of genus one curve the augmented irregularity of an orbibundle is always at least one. Moreover they provide many examples of complete, generically isotrivial families of varieties with numerically trivial canonical bundle. One of the key point is that every generically isotrivial family of such varieties is birational to an orbibundle:

3.18. **Theorem.** *[**Kol15**, Theorem 44] Let $Y \xrightarrow{f} B$ be a projective, generically isotrivial Calabi–Yau fiber space. There is a unique orbibundle $Y_{orb} \to B$ birational to Y.*

More precisely it follows from the proof of this theorem that the birational map from the orbibundle to Y is an isomorphism over the points where the fibers of f are isotrivial.

3.19. **Remark.** In [**Kol15**, Theorem 44] Kollár claims that under some further assumptions Y is isomorphic to Y_{orb}. More precisely under the assumptions that Y and B are $\mathbb{Q}$-factorial, f is relatively minimal and has no exceptional divisors, then he claims that Y is an orbibundle over B. Unfortunately it is not clear to us how he excludes the case of some divisorial contractions K_Y-trivial. Also with the further assumption that Y, B are smooth and f is equidimensional there are some counterexamples to his statement. Such examples are isotrivial relatively minimal Calaby–Yau fiber spaces which have a divisorial contraction of insufficient fiber type divisors onto an orbibundle. For such examples see below.

3.20. **Example.** Let E be an elliptic curve. The quotient $(E \times E)/\pm$ under the diagonal action with the projection onto a factor is an example of orbibundle

$$Y := (E \times E)/\pm \xrightarrow{f} E/\pm = \mathbb{P}^1.$$

A minimal resolution S of Y is a smooth K3 surface with a generically isotrivial genus one fibration over $\mathbb{P}^1$. The generically isotrivial genus one fibration $S \to \mathbb{P}^1$ is birational to the orbibundle $Y \to \mathbb{P}^1$ but they are not isomorphic.

This construction has a natural generalization for Hyperkähler manifolds of dimension $n > 2$. Let S be a K3 surface with a generically isotrivial fibration. The Hilbert scheme of lenght $d = n/2$, 0-dimensional subschemes of S is a Hyperkäler manifold that comes with a generically isotrivial morphism to $\mathbb{P}^d$. Moreover f is equidimensional and relatively minimal. The fibration $S^{[n]} \to \mathbb{P}^n$ is birational to the orbibundle $S^{(n)} \to \mathbb{P}^n$ but they are not isomorphic.

For more details see Example 17.4.

For our purpose it will be useful to understand when an isotrivial genus one fibration is isomorphic in codimension one to an orbibundle. For convenience we introduce the following terminology.

3.21. **Definition.** *A Calabi–Yau fiber space $Y \to B$ is a* quasi-product *if Y is isomorphic in codimension one over B to an orbibundle.*

Some criteria to prove that a fixed variety is isomorphic in codimension one to an orbibundle are proven in the previous section (see Lemma 3.9). In order to apply such criteria we need also the following useful lemma.

3.22. **Lemma.** *The orbibundle X_{orb} has no degenerate divisors.*

PROOF. By definition of the fiber of an orbibundle

$$\begin{tikzcd} \tilde{B} \times F \arrow[r, "\pi"] \arrow[d] & X_{\mathrm{orb}} \arrow[d, "f"] \\ \tilde{B} \arrow[r] & B \end{tikzcd}$$

the fiber of f over a point where the orbit for the action to B is maximal is isomorphic to F. Over a point of ramification of the map $\tilde{B} \to B$ the fiber is a quotient of F. Let us be more precise: a point in B is a class $\bar{b}$ for $b \in \tilde{B}$. Its fiber in X_{orb} is by definition $\{\overline{(b,t)}\}$, and in the orbibundle the equivalence relation is

$$\overline{(b,t)} = \overline{(b',t')} \Leftrightarrow \exists g \in G \ s.t. \ g(b) = b' \text{ and } g(t) = t'.$$

Let G' be the subgroup of G of elements that fix b. By definition the fiber over $\bar{b}$ is F/G' with the induced action. In particular all the fibers are irreducible, hence there are no insufficient type divisors. Since the acting group is finite then there are no exceptional divisors. By definition this means that there are no degenerate divisors in X_{orb}. □

One cannot control a priori the singularities of an orbibundle. Indeed if the singularities of B or the singularities of the fibers are bad, then also the singularities of X_{orb} are bad. For an example of this situation see Example 17.5. The previous lemma is useful, among the others, to control the singularities of orbibundles. In particular it is useful for the proof of the following lemma.

3.23. **Lemma.** *Suppose that an orbibundle Y_{orb} is birational to a relatively minimal log Calabi–Yau fiber space $(Y, \Delta) \xrightarrow{f} B$. Suppose moreover the general fiber F of f is smooth. Then there exists a boundary such that the pair (Y_{orb}, Δ') is klt.*

PROOF. Since (Y, Δ) is klt there exists a boundary D on B such that (B, D) is klt. More precisely $f^*(K_B + D) \sim_{\mathbb{Q}} K_Y + \Delta$. By Proposition 1.1 also $(\tilde{B}, \tilde{D})$ is klt and then $(\tilde{B} \times F, \tilde{D} \times F)$ is also klt. Then applying again Proposition 1.1 we conclude the proof. □

The hypothesis of this lemma can be relaxed. What we actually need is only that there exists a boundary Δ on $\tilde{B} \times F$ such that the pair $(\tilde{B} \times F, \Delta)$ is klt.

Now we provide a characterization of those orbibundles which contain a divisor covered by rational curves.

3.24. **Lemma.** *Let $Y_{orb} := (E \times \tilde{B})/G$ be an orbibundle of relative dimension one, that is a genus one fibration. Then Y_{orb} contains a divisor covered by vertical rational curves if and only if the quotient map $E \times \tilde{B} \xrightarrow{\pi} (E \times \tilde{B})/G$ is not étale in codimension two.*

Proof. By definition of the orbibundle the quotient map π fails to be étale in codimension two if and only if there exists an element $e \neq g \in G$ that fixes a divisor D_B in $\tilde{B}$ and a divisor D_E in E. Hence if π is not étale in codimension one then for any point over the image of D_B in B there is the image of a genus one curve under a finite map which ramifies, that is a rational curve by Hurwitz formula.

For the converse, if π is étale in codimension two then there are two possibilities: if it does not exist an element $e \neq g \in G$ which fixes a divisor, this means that the map $\tilde{B} \to B$ is quasi-étale. Where this map is étale the fibers are isomorphic to E. The other case is when the action of G on E is free, in this case all the fibers of π are genus one curve. This concludes the converse. In the last case the fibers over the locus of ramification of the map $\tilde{B} \to B$ are multiple genus one curves. □

This lemma will be very useful for the study of rational curves in genus one fibrations. The following is an example where this situation appears.

3.25. **Example.** Let E be an elliptic curve and denote by $X := (E \times E)/\pm$ the quotient of the product of two copies of E by the involution. This surface comes with a natural genus one fibration $X \to \mathbb{P}^1$ with four singular fibers sitting above the branch points of $E \to \mathbb{P}^1$ and each singular fibre consists of a rational curves with multiplicity two. This is an example of an orbibundle such that its quotient map is not étale in codimension two, and hence has some uniruled divisors. For further details see Example 17.3.

Another example of this situation in higher dimension is Example 17.4.

3.D. The j-invariant. One can associate to any elliptic curve a complex number called its j-invariant. This association is modular, which means that an elliptic fibration $f : Y \to B$ comes with a rational map $j : B \dashrightarrow \mathbb{P}^1$ called j-map that is at least defined over the smooth values of f. For some standard facts about the j-map of an elliptic fibration the references can be found in [**Kod63**] or [**Har10**].

3.26. **Remark.** The j-invariant is well-defined also for genus one curve, *i.e.* without fixing the origin. Indeed different choices of the origin does not change the j-invariant because the translation of the origin is an isomorphism of elliptic curves. In particular, as for elliptic curves, two genus one curves are isomorphic if and only if they have the same j-invariant.

A genus one fibration $X \xrightarrow{f} B$ gives a rational map that we call the j-map as in the case of elliptic fibrations. To show this consider an open subset U of B contained in the smooth values of f and in the regular part of B. Let Σ be a general element in a very ample linear series on X restricted over U. The pullback $\Sigma \times_B X \to \Sigma$ is a smooth elliptic fibration. Up to shrink U out of the ramification of Σ over U, we can suppose that $\Sigma \to U$ is a finite étale cover. If necessary we can consider a further finite étale cover Σ' of Σ such that the composition $\Sigma' \to U$ is Galois. Since $X \times_U \Sigma' \to \Sigma'$ is an elliptic fibration it is well-defined the j-function $\Sigma' \to \mathbb{C}$. The regular functions on U are the regular functions on Σ' that are invariant under the Galois group. All the fibers in an orbit of the Galois group are isomorphic, so the j-map on Σ' descends to a regular function on U, that is a rational map on B.

3.27. **Remark.** Consider the following two different definitions of generic isotriviality for a flat family. One can ask that two general fibers are isomorphic, or that the smooth fibers are isomorphic. In the general setting the first definition is strictly more general than the second one. An example of this situation is given by a degeneration of an Hirzebruch surface F_n into an F_m with $m > n$, [**Ser06**, See Example 1.2.11(iii)]. For elliptic fibrations these two definitions coincide. Indeed a smooth degeneration of an elliptic curve is again elliptic by Kodaira's table [**BHPVdV04**]. Since the j-invariant is constant on a dense subset of the base it is constant. We can conclude that every smooth fiber is a smooth elliptic curve with the same j-invariant, so the smooth fibers are isomorphic. Since a smooth projective morphism étale-locally admits a section, the same statement holds for a smooth genus one fibration.

Also if all the fibers of a fibration are genus one curves, this does not mean that all of them are isomorphic. Indeed it is easy to construct some (non reduced) fibers isogenus to the general fiber.

The following lemma is essentially stated in [**DFM19**].

3.28. **Lemma.** *Let $X \xrightarrow{\pi} B$ be a genus one fibration. If the subvariety of singular values $Z := \mathrm{Sv}(\pi)$ has codimension at least two then the family π is generically isotrivial.*

Proof. Since B is normal it is smooth in codimension one and also the subvariety $Z \cup B_{\text{sing}}$ has codimension at least two. We denote $B_0 := Z^c \cap B_{\text{reg}}$. The j-map $B \dashrightarrow \mathbb{P}^1$ is well-defined on B_0. Moreover the image of B_0 under this map is contained in $\mathbb{A}^1_{\mathbb{C}}$. Since B is normal and $(B_0)^c$ has codimension at least two, this map extends to a holomorphic function $j : B \to \mathbb{C}$. This function must be constant because B is projective and this means that π is generically isotrivial. $\square$

4. Rational curves on exceptional locus

In this section we collect some useful tools to find rational curves on the exceptional locus of a morphism.

A classical result in the literature that is useful for our purpose is the following result of Kawamata:

4.1. **Theorem.** *[**Kaw91**, Theorem 2] Let $Y \xrightarrow{f} B$ be a fibration between normal projective varieties. Suppose there exists a boundary on Y such that (Y, Δ) is klt and $-(K_Y + \Delta)$ is f-nef. Then any irreducible component of the f-exceptional locus is covered by rational curves contracted in B.*

In the statement of Kawamata it is not explicitly said that the rational curves are contracted, but it is clear from his proof.

In the case the morphism is birational there are several other useful results. In particular we need the following result due to Hacon and McKernan.

4.2. **Theorem.** *[**HM+07**, Corollary 1.6] Let (Y, Δ) be a divisorially log-terminal pair. If $Z \xrightarrow{g} Y$ is any birational morphism, then the fibers of g are rationally chain connected.*

A version of this result holds also if we assume the map to be rational.

4.3. **Corollary.** *Let Y, Y' be normal projective varieties of dimension n. Suppose there is a birational map $g : Y \dashrightarrow Y'$. Assume moreover that Y' is potentially divisorially log-terminal. Then the divisors contracted by g are covered by rational curves contracted by g.*

Proof. Since we are working with normal varieties any rational map can be extended to a domain whose complementary has codimension at least two. So it makes sense to ask if a divisor is contracted or not.

Consider a resolution of g

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "h"'] \arrow[d, "h'"] \\ Y \arrow[r, dashed, "g"] & Y'. \end{tikzcd}$$

Suppose by contradiction that there is a divisor $D \subset Y$ contracted in Y', *i. e.* $\dim(g(D)) < n-1$. By Theorem 4.2 the strict transform $(h^{-1})_*D$ in $\tilde{Y}$ is covered by rational curves contracted in Y'. Since the diagram commutes and the image of rational curves is again a rational curve or a point, the divisor D is covered by rational curves contracted by f. $\square$

As an application of this theorem one find rational curves in the indeterminacy locus of a rational morphism.

4.4. **Definition.** *Let $f : Y \dashrightarrow W$ be a rational morphism between normal proper varieties. Let $Z \subset Y \times W$ be the closure of the graph of f and denote by p and q the two projections from Z respectively to Y and W. The* indeterminacy locus of y *is $q(p^{-1}(y)) \subset W$.*

4.5. **Theorem.** *[**HM⁺07**, Corollary 1.7] Let $f : Y \dashrightarrow W$ be a rational morphism of normal proper varieties such that (Y, Δ) is a divisorially log terminal pair for some effective divisor Δ. Then, for each closed point $y \in Y$, the indeterminacy locus of y is covered by rational curves.*

In particular a rational map from a potentially divisorially log-terminal pair to a variety with no rational curves can be extended globally.

5. Fischer–Grauert Theorem

A well-known theorem proved by Fischer and Grauert [**FG65**] tells us that a proper holomorphic submersion with isomorphic fibers is locally a product in the complex topology. This means that given a proper holomorphic submersion $f : X \to B$ between complex manifolds such that for any $t, s \in B$ the fibers X_t and X_s are isomorphic, then for any $p \in B$ there exists a neighborhood $U_p \subset B$ open in complex topology such that the family $X_{U_p} \simeq X_p \times U_p$ splits in a product over the base. The same statement does not hold in the Zariski topology as we can see in the following example.

5.1. **Example.** Let $f : X' \to X$ be any finite unramified (hence étale) morphism between varieties of degree $d > 1$. For example f can be a finite unramified morphism of degree d from a smooth curve of genus $d(g-1)$ to a smooth curve of genus g. For any $p \in X$, the fiber over p is a scheme given by d distinct reduced points. In particular any two fibers are isomorphic. However for any $U \subseteq X$ open in the Zariski topology, the preimage $U' := f^{-1}(U)$ is a non-empty Zariski-open subset of X'. In particular since U' is connected it is not isomorphic to the product between d points and U that has d connected components.

For the general philosophy about the relation between complex topology and étale topology one can expect that the same statement of Fischer–Grauert Theorem holds for the étale topology. A strong solution in the case $K_F \equiv_{\text{num}} 0$ is Theorem 3.18. Since we were unable to find a neat reference in the case of curves, for the reader's convenience we prove some statements that will be useful for what follows.

5.2. **Proposition.** *Let $Y \to B$ a smooth proper morphism between normal quasi-projective varieties such that for any $t \in B$ the variety Y_t is a smooth curve of genus $g \geq 1$. Suppose moreover that for any $s, t \in B$ the curves Y_t and Y_s are isomorphic. Then there exists a finite étale morphism $\tilde{B} \to B$ such that the pullback $Y_{\tilde{B}} \simeq Y_t \times \tilde{B}$ is a product.*

Proof. Fix a point $0 \in B$. By GAGA's principle we can consider B and Y_0 as complex manifolds, in this way we can study the monodromy around zero as follows. Fix an integer number n greater than three and consider the action of the fundamental group of the base on the first cohomology group of the central fiber with coefficient in $\mathbb{Z}_n := \mathbb{Z}/(n)$

$$\phi : \pi_1(B, 0) \to \operatorname{Aut}(H^1(Y_0, \mathbb{Z}_n)).$$

Since Y_0 is a complete curve of genus g the group $H^1(Y_0, \mathbb{Z}_n) \simeq (\mathbb{Z}_n)^{2g}$ is finite. This implies that $\operatorname{Aut}(H^1(Y_0, \mathbb{Z}_n))$ is finite and hence the kernel $\operatorname{Ker}(\phi) \trianglelefteq \pi_1(B, 0)$ is a normal subgroup of finite index of the fundamental group of the base. By the standard correspondence between subgroup of index d of $\pi_1(B, 0)$ and étale cover of B of degree d, the subgroup $\operatorname{Ker}(\phi)$ corresponds to a finite étale cover $\tilde{B}$ of B. Moreover the action of $\pi_1(\tilde{B}, \tilde{0})$ is trivial on the first cohomology group with coefficients in $\mathbb{Z}_n$ of the pullback family $Y \times_B \tilde{B}$. This construction, called J_n-rigidification, is useful because for $n \geq 3$ there are no automorphisms of a curve with positive genus acting in a trivial way on $H^1(C, \mathbb{Z}_n)$. In particular there exists a fine moduli space with a universal family $\mathcal{U}_{g,n} \to \mathcal{M}_{g,n}$ (see for example [**Bea96**]). The classifying morphism $\tilde{B} \to \mathcal{M}_{g,n}$ is constant because the morphism $B \to \mathcal{M}_g$ is constant (this morphism is constant since all the fibers are pairwise isomorphic). It follows that there is a pullback diagram as follows

$$\begin{tikzcd}
Y \times_B \tilde{B} = \tilde{B} \times_{\mathcal{M}_{g,n}} \mathcal{U}_{g,n} \arrow[r] \arrow[d] & \mathcal{U}_{g,n} \arrow[d] \\
\tilde{B} \arrow[r] & \mathcal{M}_{g,n}
\end{tikzcd}$$

and since the classifying morphism $\tilde{B} \to \mathcal{M}_{g,n}$ is constant, the variety $Y_{\tilde{B}}$ is isomorphic to the product $\tilde{B} \times Y_0$. $\square$

We need the previous result only for genus one fibrations. Since the previous proof uses many topological tools, we give another proof more algebraic in spirit of the following statement, that is essentially Proposition 5.2 for curves with genus one.

5.3. **Proposition.** *Let $Y \to B$ be a smooth projective morphism between normal varieties such that for any $t \in B$ the variety Y_t is isomorphic to a fixed curve of genus $g = 1$, i.e. a smooth isotrivial genus one fibration. Suppose moreover B is smooth. Then there exists a finite étale morphism $\tilde{B} \to B$ such that the pullback $Y_{\tilde{B}} \simeq Y_t \times \tilde{B}$ is a product.*

To prove this proposition we need two results.

5.4. **Lemma.** *If $f : Y \to B$ is a smooth isotrivial elliptic fibration, then there exists a finite étale map $\tilde{B} \to B$ such that the pullback family $Y \times_B \tilde{B}$ is isomorphic to the trivial family.*

This result is [**Har10**, Corollary 26.5]. The difference between Proposition 5.3 and Lemma 5.4 is that in the lemma the family of genus one curves has a section. So we have to combine this result with the following.

5.5. **Lemma.** *Let $f : Y \to B$ be a projective morphism between normal varieties. Assume that B is smooth and f is étale locally trivial and the generic fiber F has numerically trivial canonical bundle. Then there is a finite étale cover $B' \to B$ such that the pull back $Y_{B'} \simeq F \times B'$ is globally trivial.*

This lemma is stated and proved in [**KL09**, Lemma 17]. Finally we can give an algebraic proof of Proposition 5.3.

Proof of 5.3. We have to prove that f is étale locally trivial, *i.e.* for any $p \in B$ there exists an étale neighborhood U of p such that $Y_U \simeq U \times Y_p$. Choose a point $p \in B$. The morphism is smooth and projective so locally around p there exists a multi-section Σ of f that is étale at p. Indeed the local structure of smooth morphism can be described in the following way: for any point $y \in Y$ and $t = f(y)$ there exist open neighborhood V_t and U_y with $U_y \subset f^{-1}(V_t)$ such that $f|_{U_t}$ factorizes as an étale morphism $g : U_y \to \mathbb{A}^d_{V_t}$ followed by the canonical projection $\mathbb{A}^d_{V_t} \to V_t$. Consider a section s of $\mathbb{A}^d_{V_t} \to V_t$ and the associated fiber product $U \times_{\mathbb{A}^d_V} s(V_t)$. The image of this fiber product in U_y is the desired étale multi-section. Shrinking Σ we can suppose that the fiber product $Y_\Sigma \to \Sigma$ is a family of smooth elliptic curves and the fibers are pairwise isomorphic, so by Lemma 5.4 $Y_\Sigma \simeq \Sigma \times Y_p$. This proves that f is étale locally trivial. We can apply Lemma 5.5 and the proof is completed. □

6. Augmented irregularity

In this section we introduce the *augmented irregularity.* This invariant has been recently introduced by many authors because it carries some informations on the one forms on the covers of a fixed variety. In particular it has been studied in order to characterize the universal cover of varieties with trivial canonical bundle, see for example [**Dru18**], [**GGK19**], [**HP19**] and related papers.

6.1. **Definition.** *A morphism $f : Z \to Y$ between normal quasi-projective varieties is called* quasi-étale *if f is quasi-finite and étale in codimension one. If f is finite we call it a cover.*

6.2. **Remark.** The above definition of quasi-étale morphism is not the same of [**Cat07**].

6.3. **Remark.** A quasi-étale morphism to a smooth variety is globally étale by standard arguments on purity of the branch locus.

6.4. **Definition.** *The* irregularity *of a normal projective variety Y is the non negative integer $q(Y) := h^1(Y, \mathcal{O}_Y)$. The* augmented irregularity *of Y is the following, not necessarily finite, positive integer*

$$\tilde{q}(Y) := \sup\{q(Z) \mid Z \to Y \textit{ is a finite quasi-étale cover}\}.$$

For any variety the inequality $\tilde{q}(Y) \geq q(Y)$ holds. In general the equality does not hold. Moreover it may happen that this supremum is not achieved; this is exactly the case when the augmented irregularity is infinite. This can happen also in dimension one, as we see in the following example.

6.5. **Example.** The augmented irregularity of a genus zero curve is zero. Any finite étale cover of a genus one curve has irregularity one, so if $g(C) = 1$ then $\tilde{q}(C) = 1$. A curve C with $g(C) \geq 2$ has étale covers of arbitrary large irregularity, hence $\tilde{q}(C) = \infty$. For a more accurate explanation see Example 17.1.

It may also happen that a smooth projective variety has zero irregularity but its augmented irregularity does not vanish. For an example of this situation see Example 17.3.

It is natural to ask whether there exists some manageable conditions for the vanishing of the augmented irregularity of a variety. The first basic observation is the case Y has finite fundamental group.

6.6. **Remark.** The augmented irregularity of a smooth projective variety Y with finite fundamental group is trivial.

PROOF OF THE REMARK. A quasi-étale cover $\tilde{Y}$ of Y is an étale cover for purity of branch locus. The fundamental group of $\tilde{Y}$ is a subgroup of the fundamental group of Y, so it is finite. The first Betti number of a variety with finite fundamental group is zero, so by Hodge theory also $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0$, and hence $\tilde{q}(Y) = 0$. $\square$

6.7. **Remark.** It is not difficult to show that the augmented irregularity is a birational invariant for smooth projective varieties. To show this fact we take two smooth birational projective varieties X and X'. We can suppose, considering a resolution of the birational map, that there is a well-defined morphism $X' \to X$. Any quasi-étale cover Z of X (hence globally étale) can be pulled back to an étale cover Z' of X'. These two covers are smooth and birational, so $q(Z) = q(Z')$. Since this argument works for any quasi-étale cover taking the sup we get $\tilde{q}(X) \geq \tilde{q}(X')$. For the converse let $Z' \xrightarrow{h'} X'$ any quasi-étale cover. By purity of the branch locus h' is globally étale. Consider the Stein factorization of the composition $Z' \xrightarrow{g} Z \xrightarrow{h} X$. Since h and h' are étale Z and Z' are smooth and moreover they are birational, hence $q(Z) = q(Z')$. As before this implies that $\tilde{q}(X) \leq \tilde{q}(X')$ that proves our remark.

6.8. **Remark.** Since log-terminal singularities are always rational, passing throught a resolution one can show that the irregularity is a birational invariant for projective varieties with log-terminal singularities.

However the augmented irregularity is not a birational invariant for projective varieties with canonical singularities. Indeed the standard construction of a Kummer surface is a counterexample:

6.9. **Example.** Let E be an elliptic curve and denote by $X := (E \times E)/\pm$ the quotient of the product of two copies of E by the involution. One can easily check that $\tilde{q}(X) = 2$. However a minimal resolution $\tilde{X}$ of X is a K3 surface. In particular $\tilde{q}(\tilde{X}) = 0$. The variety X is thus also an example of a variety with zero irregularity but non trivial augmented irregularity. For more details on this example see Example 17.3.

6.10. **Remark.** Let $X' \to X$ be a birational morphism of log-terminal projective varieties. As we have seen in the previous example the irregularity X and X' may be different. However we always have the inequality $\tilde{q}(X') \leq \tilde{q}(X)$. Indeed let Z' be a quasi-étale cover of X'. Let Z be the Stein factorization of $Z' \to X$. By construction the map $Z \to X$ is quasi-étale. The varieties Z and Z' are birational and log-terminal by

Proposition 1.1. Hence we have $\tilde{q}(X') \geq q(Z') = q(Z) \leq \tilde{q}(X)$ where we use Remark 6.8. If the augmented irregularity of X' is finite we can conclude choosing a quasi-étale cover Z' such that $q(Z') = \tilde{q}(X')$. If $\tilde{q}(X') = \infty$ then the above argument shows that there is an sequence of quasi-étale cover of X with unbounded irregularity, *i.e.* $\tilde{q}(X) = \infty$. This remark should be compared to [**Dru18**, Lemma 4.4].

In the other direction we can prove this lemma that will be useful in the following.

6.11. **Lemma.** *Let Y, Y' be projective varieties with at most log-terminal singularities. Suppose they are isomorphic in codimension one, then $\tilde{q}(Y) = \tilde{q}(Y')$.*

Proof. By symmetry it is sufficient to show $\tilde{q}(Y) \geq \tilde{q}(Y')$. We claim that any quasi-étale cover of Y is isomorphic in codimension one to a quasi-étale cover of Y'. Let $U \subseteq Y$ and $U' \subseteq Y'$ be the subsets where there exists an isomorphism $f : U \to U'$. Let $Z \to Y$ be any quasi-étale cover of Y. The restriction over U gives a quasi-étale morphism $Z_U \xrightarrow{f_U} Y$ whose image is U'. By Zariski's Main Theorem [**Gro67**] the quasi-finite morphism f_U is always the composition of an open immersion $Z_U \xrightarrow{i} W$ and a finite morphism $W \xrightarrow{f} Y'$. It turns out that the compactification f of f_U is a quasi-étale cover of Y. Indeed f is finite and étale outside the subset of U' where $f \circ f_U$ ramifies. By construction W and Z are isomorphic in codimension one, that proves the claim.

By Proposition 1.1 a quasi-étale cover of a variety with log-terminal singularities has the same kind of singularities. To conclude we need to prove that the irregularity is an invariant for projective varieties with log-terminal singularities isomorphic in codimension one. By [**GKP16c**, Proposition 6.9] the irregularity of a projective log-terminal variety X is the dimension of the space of global sections of $\Omega_X^{[1]}$. By definition $\Omega_X^{[1]}$ is a reflexive sheaf, so its sections are not affected by any codimension two subset. Applying this argument to Z and W we conclude the proof. $\square$

6.12. **Remark.** In the previous lemma one can be interested in the case one of the two varieties is not log-terminal. This may happens if one of the two is not $\mathbb{Q}$-Gorenstein. In this case one can say the following. Let Y, Y' be projective varieties isomorphic in codimension one. Suppose Y has most log-terminal singularities. We claim that $\tilde{q}(Y) \geq H^0(\tilde{Y}, \Omega_{\tilde{Y}}^{[1]})$ for any quasi-étale cover $\tilde{Y}$ of Y'. Indeed by the argument of the above lemma, any quasi-étale cover $\tilde{Y}$ of Y' is isomorphic in codimension one to a quasi-étale cover Z of Y. As above the global sections of $\Omega_{\tilde{Y}}^{[1]}$ correspond to the global sections of $\Omega_Z^{[1]}$, that since Z has klt singularities we can apply duality and obtain our statement.

For varieties with numerically trivial canonical divisor an interesting characterization is given in [**GGK19**, Theorem 11.1], where the authors proved that, in this setting, $\tilde{q}(X) = 0$ if and only if for any $k > 0$ there are no non-trivial symmetric reflexive forms, *i.e.* $H^0(X, \mathrm{Sym}^{[k]}\,\Omega^1_X) = 0$ $\forall k > 0$. We prove that an implication still holds without the assumption on the canonical bundle: the following is a sufficient condition for the vanishing of the augmented irregularity which does not rely on computations of invariants on quasi-étale covers, but only on invariants of the variety under investigation.

6.13. **Proposition.** *Let X be a projective variety with at most log terminal singularities. If $H^0(X, \mathrm{Sym}^{[k]}\,\Omega^1_X) = 0$ for every $k > 0$, then $\tilde{q}(X) = 0$.*

Proof. Suppose by contradiction that there is a quasi-étale cover $\tilde{X} \to X$ with $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \neq 0$. The variety $\tilde{X}$ is log-terminal Proposition 1.1 and by [**GGK19**, Proposition 6.9] there is a non-zero reflexive form $\omega \in H^0(\tilde{X}, \Omega^{[1]}_{\tilde{X}})$. By definition the sections of a reflexive sheaf are exactly the sections on the regular part of X. So to construct a non-zero global section of $\mathrm{Sym}^{[k]}\,\Omega^1_X$ we construct an element in $H^0(X_{\mathrm{reg}}, \mathrm{Sym}^k\,\Omega^1_{X_{\mathrm{reg}}})$. Now we consider just the restriction to the regular locus of X:

$$Y := \tilde{X} \times_X X_{\mathrm{reg}} \to X_{\mathrm{reg}}.$$

This is a finite étale cover, so we can find a further finite étale cover over the regular part $\tilde{Y} \to X_{\mathrm{reg}}$, that is Galois. Let G be the group of deck transformations of $\tilde{Y}$ over X_{reg}. By abuse of notations we call again ω the pullback to $\tilde{Y}$ of ω. Now consider the section

$$\tilde{\alpha} := \sum_{\tau \in G} \bigotimes_{\rho \in G} \tau^* \rho^* \omega \in H^0(\tilde{Y}, (\Omega^1_{\tilde{Y}})^{\otimes N})$$

where $N = |G|$. This section is invariant under the action of the deck trasformations, so it descends to a section α of $H^0(X_{\mathrm{reg}}, (\Omega^1_{X_{\mathrm{reg}}})^{\otimes N})$. By construction it is easy to check that this section is symmetric, *i.e.* ω belongs to $H^0(X_{\mathrm{reg}}, \mathrm{Sym}^N(\Omega^1_{X_{\mathrm{reg}}}))$. It is less trivial to prove that $\tilde{\alpha}$, and hence α, is non-zero.

For any non-zero element $\gamma \in H^0(\tilde{Y}, \Omega^1_{\tilde{Y}})$ and a generic point $p \in \tilde{Y}$ the space $\mathrm{Ker}(\gamma) \subset T_{\tilde{Y},p}$ is a proper subspace. Since $\omega \neq 0$ and the elements $\rho \in G$ are automorphisms (and we are working over $\mathbb{C}$), also

$\rho^*\omega$ are non-zero elements in $H^0(\tilde{Y}, \Omega^1_{\tilde{Y}})$. So for generic $p \in \tilde{Y}$ we can choose a tangent vector

$$0 \neq v \in T_{\tilde{Y},p} \setminus \bigcup_{\rho \in G} \operatorname{Ker}((\rho^*\omega)_p).$$

Now we can evaluate our section $\tilde{\alpha}$ at the vector $v^{\otimes N}$. The computations are the following:

$$\tilde{\alpha}(v^{\otimes N}) = \sum_{\tau \in G} \bigotimes_{\rho \in G} \tau^*\rho^*\omega(v) = \sum_{\tau \in G} \prod_{\rho \in G} \tau^*\rho^*\omega(v) = N \prod_{\rho \in G} \rho^*\omega(v) \neq 0.$$

So we have constructed a non-zero section of $H^0(X_{\mathrm{reg}}, \operatorname{Sym}^N \Omega^1_{X_{\mathrm{reg}}}))$ that corresponds to a non-zero section of $H^0(X, \operatorname{Sym}^{[N]} \Omega^1_X)$. $\square$

6.14. **Remark.** The sections of the symmetric powers of the cotangent bundle correspond to the sections of the powers of the tautological class $\zeta := c_1(\mathcal{O}_{\mathbb{P}(\Omega^1_X)})$. In particular the vanishing of $H^0(X, \operatorname{Sym}^{[k]} \Omega^1_X)$ for any positive k is equivalent to say that no multiple of ζ is effective.

7. Chern classes of singular varieties

One of the difficulties of singular varieties respect the smooth ones is that it is in general more difficult to use Riemann–Roch formula. Restricting the kind of singularities that we allow, we can work without problems in the following parts.

We start spending some words about the Chern classes for singular varieties. The Todd and Chern classes of an arbitrary algebraic scheme are defined in [Ful84, Section 18.3].

7.1. **Remark.** Let $\pi : \tilde{X} \to X$ be a proper birational morphism that is an isomorphism outside $Z \subset X$. Then

$$\operatorname{Td}(X) = \pi_* \operatorname{Td}(\tilde{X}) + \alpha \in A_*(X)_{\mathbb{Q}}$$

where α is a class supported in Z. In particular this tells us that if X is a variety smooth in codimension two, the definition $c_2(X) := \pi_* c_2(\tilde{X})$ for some resolution $\pi : \tilde{X} \to X$ agrees with the definition in [Ful84]. We want to prove that this two definitions agrees also for varieties with rational singularities.

7.2. **Remark.** Using the definitions in [Ful84] the Hirzebruch-Riemann-Roch Theorem [Ful84, Corollary 18.3.1] holds for any complete scheme. Let X be a projective variety with rational singularities, *e.g.* with at most dlt singularities, and $\pi : \tilde{X} \to X$ a resolution of singularities. By definition of rational singularities for any line bundle L

on X we have $\chi(X, L)) = \chi(\tilde{X}, \pi^* L)$. Applying Hirzebruch-Riemann-Roch to X and to $\tilde{X}$ we get

$$\int_X \mathrm{ch}(L) \cdot \mathrm{Td}(X) = \int_{\tilde{X}} \mathrm{ch}(\pi^* L) \cdot \mathrm{Td}(\tilde{X}) = \int_X \mathrm{ch}(L) \cdot \pi_* \mathrm{Td}(\tilde{X})$$

where the last equality follows from the projection formula [**Ful84**, Proposition 2.5 (c)]. Since this equality holds for any line bundle L, this tells us that $c_2(X) = \pi_* c_2(\tilde{X})$ as elements in $N^2(X)$. In particular for a variety Y with at most klt singularities we can take as definition $c_2(Y) = \pi_* c_2(\tilde{Y})$ for some resolution of the singularities of Y.

The pseudo-effectiveness of the second Chern class proved by Miyaoka holds also in our setting.

7.3. **Lemma.** *Let X be a normal projective variety with canonical singularities and $K_X \equiv_{num} 0$. For any $D_1, \ldots, D_{n-2}$ nef divisors on X we have $c_2(X) \cdot D_1 \cdots D_{n-2} \geq 0$.*

Proof. Let $\nu : \tilde{X} \to X$ be a terminalization of X. The canonical bundle of $\tilde{X}$ is still numerically trivial and $\tilde{X}$ is smooth in codimension two. The divisors $\nu^* D_1, \ldots, \nu^* D_{n-2}$ are nef, so applying [**Miy87**, Theorem 6.6] and the projection formula we get $c_2(\tilde{X}) \cdot D_1 \cdots D_{n-2} \geq 0$. The conclusion follows applying another time the projection formula to ν. □

In our setting to prove that the second Chern class of X is trivial it is sufficient to show that $c_2(X) \cdot H^{n-2} = 0$ for some ample divisor H.

7.4. **Lemma.** *Let X be a normal projective variety with canonical singularities and $K_X \equiv_{num} 0$. Then $c_2(X) = 0$ in $N^2(X)$ if and only if there exist $H_1, \ldots, H_{n-2}$ ample line bundles on X such that $c_2(X) \cdot H_1 \cdots H_{n-2} = 0$. In particular if $c_2(X) \neq 0$ in $N^2(X)$ then for any ample line bundle $c_2(X) \cdot H^{n-2} > 0$.*

Proof. The proof of this lemma is an adaptation of the proof of [**GKP16a**, Proposition 4.8]. We start proving that if there exist ample line bundle $H_1, \ldots, H_{n-2}$ on X such that $c_2(X) \cdot H_1 \cdots H_{n-2} = 0$ then for any line bundle $L_1, \ldots, L_{n-2}$ we have $c_2(X) \cdot L_1 \cdots L_{n-2} = 0$, *i.e.* $c_2(X) = 0$ in $N^2(X)$. Since the ample cone is open in $N^1(X)$ and the intersection product is multilinear it is enough to prove that the intersection is trivial for L_i ample line bundle.

Up to taking large multiples of the divisors H_i we can also assume that $H_i \pm L_i$ are ample divisors in X. We prove by induction on k that

$$c_2(X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot H_{k+1} \cdots H_{n-2} = 0.$$

For $k = 0$ this is the hypothesis. Suppose it is true for k, we have

$$0 = c_2(X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot H_{k+1} \cdots H_{n-2} =$$

$$= c_2(X) \cdot (H_1 + L_1) \cdots (H_k + L_k) \cdot (H_{k+1} \pm L_{k+1}) \cdots H_{n-2}.$$

Both the summands are non negative by Lemma 7.3, so they must be zero. For $k = n-2$ and expanding the product we get

$$0 = \sum c_2(X) \cdot A_1 \cdots A_{n-2}$$

where $A_i \in \{H_i, L_i\}$. Since A_i is nef, this is a zero sum of non-negative numbers whose sum is zero, so every summand must be zero. In particular we get $c_2(X) \cdot L_1 \cdots L_{n-2} = 0$.

To conclude we have to prove that if $c_2(X)$ is non-zero then for any ample divisor H the number $c_2(X) \cdot H^{n-2}$ is positive, but this is immediate since it is a non zero number by the above arguments, and it is non-negative by Lemma 7.3. □

7.5. **Remark.** The second Chern class of a Calabi–Yau variety X is non-zero. This is well-known under the further assumption that X is smooth in codimension two. This is proved under the further assumption that X is canonical and $\mathbb{Q}$-factorial in [**GKP16b**, Theorem 1.4]. In a very recent paper it is proved that a normal projective variety with at most log-terminal singularities, trivial canonical bundle and trivial second Chern class is a quasi-étale quotient of an abelian variety [**LT18**, Theorem 1.2 and Remark 1.5], so the augmented irregularity is equal to the dimension. Note that the converse does not hold because $c_2((E \times E)/\pm) = 24$. Using only the results in [**GKP16b**] we cannot prove that the second Chern class of a Calabi–Yau variety is non-zero because X is not $\mathbb{Q}$-factorial, and a priori the second Chern class of a $\mathbb{Q}$-factorialization can be contracted in X.

8. Properties of cones

In this section there are some results that are well-known to the experts. For the reader convenience let me recall the following.

8.1. **Definition.** *Let Y be a normal variety. The* numerical dimension *of a nef class $x \in N^1(Y)$ is the maximum integer k such that $x^k \neq 0$ as element in $N^k(Y)$.*

8.2. **Remark.** By definition $N^k(Y)$ is the set of codimension k cycles on Y modulo numerical equivalence. An element in $N^k(Y)$ is 0 if and only if its intersection with any k dimensional cycle is 0. This definition is not the same of [**Ful84**, Definition 19.1] but in the smooth cases they are equivalent.

The well-known statement for $\mathbb{Q}$-divisors that a nef divisor is big if and only if it has positive top self-intersection [**Laz04a**, Theorem 2.2.13] holds also for $\mathbb{R}$-divisors. This fact is certainly well-known to the experts and an exercise for others.

8.3. **Lemma.** *Let Y be a normal projective of dimension n and $D \in N^1(Y)$ a nef $\mathbb{R}$-divisor. Then D is big if and only if $D^n > 0$.*

This lemma can be easily deduced by [**DP04**, Theorem 0.5], but invoke this result to prove this statement is certainly unnecessary.

A detailed direct proof of this fact can be found on the web page of professor A. Lopez at http://ricerca.mat.uniroma3.it/users/lopez/Note.html.

The following is an interesting consequence for varieties with no rational curves and numerically trivial canonical bundle.

8.4. **Proposition.** *[**DF14**, Proposition 2.1] Let X be a projective variety with at most log-terminal singularities, numerically trivial canonical bundle and no rational curves. Then the ample cone and the big cone coincide.*

Proof. Let D be any effective $\mathbb{Q}$-divisor. For small positive and rational ε the pair $(X, \varepsilon D)$ is klt. Since there are no rational curves in X, the cone theorem [**KM98**, Theorem 3.7] tells us that εD is also nef. It follows that the effective cone is contained in the nef cone. Passing to the interior of such cones we get the thesis. □

Now following [**Laz04a**] we define two cones that help us to study nef divisors that are not ample.

8.5. **Definition.** *The* null cone $\mathcal{N}_X \subset N^1(X)$ *is the set of classes of divisors D such that $D^n = 0$. The* boundary cone $\mathcal{B}_X \subset N^1(X)$ *is the boundary of the nef cone.*

Note that these cones are not convex cones. The following corollary that is already known by the experts explains the relation between these cones in our context.

8.6. **Proposition.** *Let X be a variety with log-terminal singularities, numerically trivial canonical bundle and no rational curves. The boundary of the ample cone is contained in the null cone, i.e. $\mathcal{B}_X \subset \mathcal{N}_X$.*

Proof. In the boundary of the ample cone there are nef $\mathbb{R}$-divisors that by Proposition 8.4 are not big $\mathbb{R}$-divisors. These $\mathbb{R}$-divisors has trivial top self-intersection and so they are in the null cone. □

9. Positivity properties of real (1,1)-classes

In this section we recall some basic facts about the positivity of $(1, 1)$ real cohomology classes that generalize the corresponding notions for classes of divisors. A complete treatment on this subject would be to burdensome in this thesis and we claim no originality on this part. For a detailed explanation we refer to [**Dem12**].

On a smooth projective variety X the ampleness of the class of a line bundle can be checked, by Kleiman's criterion [**Deb01**, Theorem 1.27], computing its intersection with the curves (and the limits of classes represented by sums of effective one cycles) contained in X. In the more general context of compact Kähler manifold this is no longer sufficient. This happens because a non projective manifold may contain no curves at all. The situation gets even worst for the definition of pseudoeffective line bundles. For projective varieties a line bundle is pseudoeffective if it is limit of effective line bundles, *i.e.* which have some sections. If there are no divisors in X, what does it mean for a line bundle to be pseudoeffective?

For a line bundle L on a smooth projective manifold, the standard definitions of numerically effective, ample, pseudoeffective and big, can be characterized in terms of positivity conditions on the metrics that one can put on L. More precisely in the following way:

9.1. **Proposition.** *[**Dem12**, Chapter 6] Let L be a line bundle on a projective manifold X, on which a hermitian metric ω is given. Then*

- *The line bundle L is ample if and only if there is a smooth metric h on L such that $i\Theta_{L,h} > 0$.*
- *The line bundle L is nef if and only if for every $\varepsilon > 0$, there is a smooth metric h_ε on L such that $i\Theta_{L,h_\varepsilon} > -\varepsilon\omega$.*
- *The line bundle L is pseudoeffective if and only if can be equipped with a singular hermitian metric h with $T = \frac{i}{2\pi}\Theta_{L,h} \geq 0$ as a current.*

These characterizations for line bundles on projective varieties make sense in the general setting of real $(1,1)$ classes on a compact Kähler manifold, and this can be taken as definition.

9.2. **Definition.** *Let X be a compact Kähler manifold and $\alpha \in H^{1,1}(X,\mathbb{R})$. The class α is a Kähler class if it can be represented by a smooth real form of type $(1,1)$ that is positive definite at every point. The class α is pseudoeffective if it can be represented by a closed real positive $(1,1)$-current.*

The cone generated by the Kähler forms is the open convex cone $\mathcal{K}(X)$ in $H^{1,1}(X,\mathbb{R})$ called Kähler cone. The cone generated by closed positive real $(1,1)$-currents is a closed convex cone denoted by $\mathcal{E}(X)$ called pseudoeffective cone. The closure of the Kähler cone is the nef cone and the interior of the pseudoeffective cone is the big cone. Clearly the pseudoeffective cone contains the Kähler cone, and hence its closure that is the nef cone.

Suppose now that X is a projective manifold. Inside the real vector space $H^{1,1}(X,\mathbb{R})$ there is the group of real divisors modulo numerical equivalence or real Néron–Severi space

$$\mathrm{NS}_{\mathbb{R}}(X) = \big(H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})_{\mathrm{tf}}\big) \otimes_{\mathbb{Z}} \mathbb{R}$$

Then we have

$$\mathcal{K}(X) \cap \mathrm{NS}_{\mathbb{R}}(X) = \mathrm{Nef}(X), \qquad \mathcal{E}(X) \cap \mathrm{NS}_{\mathbb{R}}(X) = \overline{\mathrm{Eff}(X)}$$

where $\mathrm{Nef}(X)$ (resp. $\overline{\mathrm{Eff}(X)}$) is the nef cone (resp. pseudoeffective cone) well-known to algebraic-geometers (cf. [**BDPP13**] for more details).

As we noted before in the analytic context it is difficult to characterise the positivity of a $(1,1)$-class via intersection numbers, however we have the following easy consequence of the Demailly-Păun criterion [**DP04**, Theorem 0.1]:

9.3. **Lemma.** *Let X be a compact Kähler manifold, and let V be a vector bundle over X. Denote by $\pi : \mathbb{P}(V) \to X$ the natural morphism, and by ζ the tautological class on $\mathbb{P}(V)$. Let ω_X be a Kähler class on X such that for all $\lambda \geq 1$ we have*

$$(\zeta + \lambda\pi^*\omega_X)^{\dim Z} \cdot Z > 0 \qquad \forall\ Z \subset \mathbb{P}(V) \textit{ irreducible.}$$

Then $\zeta + \pi^*\omega_X$ is a Kähler class.

Proof. By assumption the class $\zeta + \lambda\pi^*\omega_X$ is an element of the positive cone $\mathcal{P} \subset H^{1,1}(\mathbb{P}(V))$ of classes having positive intersection with all subvarieties. By the Demailly-Păun criterion [**DP04**, Theorem 0.1] the Kähler cone $\mathcal{K}$ is a connected component of $\mathcal{P}$. Since ζ is a relative Kähler class, we know that $(\zeta + \lambda\pi^*\omega_X) \in \mathcal{K}$ for $\lambda \gg 0$ [**Voi02**, Proof of Prop.3.18]. Conclude by connectedness. □

9.A. The limit of pseudoeffective classes. It is not difficult to show, using the relative Hilbert scheme, that a line bundle that is effective on the general fiber of a projective family is still effective on the central fiber. It seems to be well-known to the experts that the same statements holds in the case of real $(1,1)$ pseudoeffective classes on a family of compact Kähler manifolds. We didn't found a neat reference in the literature, but a detailed proof of this fact was given to us by my advisor, Simone Diverio. For the sake of completeness we write his proof that is the following.

Let $\pi \colon \mathfrak{X} \to \Delta$ be a proper holomorphic submersion onto the complex unit disc of relative complex dimension n, and call $X_t = \pi^{-1}(t)$ the compact complex manifold over the point $t \in \Delta$.

Suppose also that π is a weakly Kähler fibration, *i.e.* there exists a real 2-form ω on $\mathfrak{X}$ such that its restriction $\omega_t = \omega|_{X_t}$ is a Kähler form on X_t, for each $t \in \Delta$.

By Ehresmann's fibration theorem, π it is a locally trivial fibration in the smooth category. Thus, after possibly shrinking Δ, we may suppose that we are given a smooth compact real manifold F of real dimension $2n$ and a smooth diffeomorphism $\theta\colon \mathfrak{X} \to F \times \Delta$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "\theta"] \arrow[dr, "\pi"'] & & F \times \Delta \arrow[dl, "\mathrm{pr}_2"] \\ & \Delta. & \end{tikzcd}$$

Next, call $\theta_t := \theta|_{X_t}\colon X_t \underset{\simeq_{C^\infty}}{\longrightarrow} F$. For any $t \in \Delta$, given a real $(1,1)$-cohomology class $\alpha_t \in H^{1,1}(X_t, \mathbb{R})$, we can then think of it as an element β_t of $H^2(F, \mathbb{R})$, by pulling-back *via* θ_t^{-1}, that is $\beta_t := \left(\theta_t^{-1}\right)^* \alpha_t$.

Now, suppose that we are given a class $\alpha_0 \in H^{1,1}(X_0, \mathbb{R})$ with the following property: there is a sequence of points $\{t_k\} \subset \Delta$ converging to 0, for each k it is given a $(1,1)$-class $\alpha_{t_k} \in H^{1,1}(X_{t_k}, \mathbb{R})$ which is pseudoeffective and the corresponding classes β_{t_k} converge to β_0 in the finite dimensional vector space $H^2(F, \mathbb{R})$. Then we have the following statement.

9.4. **Theorem.** *The class α_0 is also pseudoeffective.*

Proof. To start with, we select for each k a closed, positive $(1,1)$-current $T_k \in \mathfrak{D}'^{+}_{n-1,n-1}(X_{t_k})$ representing the cohomology class α_{t_k}. Each of these, being a positive current, is indeed a real current of order zero.

Now, set $\Theta_k := \left(\theta_{t_k}\right)_* T_k$. This is a closed, real 2-current of order zero on the compact real smooth manifold F.

The first step is to produce a weak limit Θ of the sequence Θ_k on F. In order to to this, by the standard Banach–Alaoglu theorem, it suffices to show that for every fixed test form $g \in \mathfrak{D}^{2n-2}(F)$ we have that the sequence $\langle \Theta_k, g\rangle$ is bounded. By definition, we have

$$\langle \Theta_k, g\rangle = \left\langle \left(\theta_{t_k}\right)_* T_k, g\right\rangle = \left\langle T_k, \left(\theta_{t_k}\right)^* g\right\rangle,$$

and of course $\left\langle T_k, \left(\theta_{t_k}\right)^* g\right\rangle = \langle T_k, f_k\rangle$, where f_k is the $(n-1, n-1)$ component of $\left(\theta_{t_k}\right)^* g$ on the complex manifold X_{t_k}. The $(n-1, n-1)$-forms f_k are real, since $\left(\theta_{t_k}\right)^* g$ is so.

9.5. **Lemma.** *Let (X, ω) be a compact Kähler manifold, T be a closed positive current of X, and f be a real smooth $(n-1, n-1)$-form. Then, there exists a constant $C > 0$ depending continuously on f and ω such that we have*

$$|\langle T, f\rangle| \leq C\,[T] \cdot [\omega]^{n-1},$$

where the right hand side is intended to be the intersection product in cohomology.

PROOF. Since f is real, we are enabled to define the following (possibly indefinite) hermitian form on T^*_X:

$$(\xi,\eta)_f \mapsto \frac{f\wedge i\xi\wedge\bar\eta}{\omega^n}.$$

We also have the positive definite hermitian form given by

$$(\xi,\eta)_\omega \mapsto \frac{\omega^{n-1}\wedge i\xi\wedge\bar\eta}{\omega^n}.$$

It is positive because $(\xi,\xi)_\omega = \frac{1}{n}\operatorname{tr}_\omega(i\,\xi\wedge\bar\xi)$. By compactness of the bundle of $(\cdot,\cdot)_\omega$-unitary $(1,0)$-forms on X, we can define

$$C' := -\min_{(\xi,\xi)_\omega=1}\{(\xi,\xi)_f\},$$

and we have that $(\xi,\eta)'\mapsto(\xi,\eta)_f + C'\,(\xi,\eta)_\omega$ is positive semidefinite. This constant C' depends manifestly continuously on f and ω. We can do the same job with $-f$ in the place of f thus obtaining another constant C'', still depending continuously on f and ω such that

$$(\xi,\eta)''\mapsto -(\xi,\eta)_f + C''\,(\xi,\eta)_\omega$$

is positive semidefinite. Now set $C:=\max\{C',C''\}\geq 0$, which again depends continuously on f and ω. This means exactly that both $f + C\,\omega^{n-1}$ and $-f + C\,\omega^{n-1}$ are positive $(n-1,n-1)$-forms.

But then, begin T positive on positive forms,

$$\begin{aligned}\langle T,f\rangle &= \langle T, f + C\,\omega^{n-1} - C\,\omega^{n-1}\rangle\\ &\geq -C\,\langle T,\omega^{n-1}\rangle = -C\,[T]\cdot[\omega]^{n-1},\end{aligned}$$

and

$$\begin{aligned}\langle T,f\rangle &= \langle T, f - C\,\omega^{n-1} + C\,\omega^{n-1}\rangle\\ &= -\langle T, -f + C\,\omega^{n-1}\rangle + \langle T, C\,\omega^{n-1}\rangle\\ &\leq C\,\langle T,\omega^{n-1}\rangle = C\,[T]\cdot[\omega]^{n-1}.\end{aligned}$$

□

Now, we apply the above lemma with $(X,\omega)=(X_{t_k},\omega_{t_k})$, $T=T_k$ and $f=f_k$. We therefore obtain positive constants C_k such that

$$|\langle T_k,f_k\rangle| \leq C_k\,[T_k]\cdot[\omega_{t_k}]^{n-1}.$$

The right hand side is equal to $C_k\,\beta_{t_k}\cdot\Omega_{t_k}^{n-1}$, where $\Omega_t\in H^2(F,\mathbb{R})$ is the cohomology class of $\big(\theta_t^{-1}\big)^*\omega_t$. It converges to the quantity $C_0\,\alpha_0\cdot\Omega_0$, where C_0 is the constant obtained if one applies the above lemma with $(X,\omega)=(X_0,\omega_0)$, and f the $(n-1,n-1)$ component of $\big(\theta_0\big)^*g$. Thus, the left hand side is uniformly bounded independently of k.

We finally come up with a real 2-current Θ on F which is a weak limit of the Θ_k's. By continuity of the differential with respect to the weak topology we find also that Θ is closed and of course its cohomology class is β_0. Being Θ trivially with compact support since it lives on the compact manifold F, by [**dR84**, Corollary on p. 43], it is of finite order, say of order p.

9.6. **Remark.** We can then look at the whole sequence $\{\Theta_k\}$ together with its weak limit Θ as a set of currents of order p. In particular, this is a set of continuous linear functionals on the Banach space ${}^p\mathfrak{D}^{2n-2}(F)$ which are pointwise bounded. By the Banach–Steinhaus theorem this set is uniformly bounded in operator norm, *i.e.* there exists a constant $A > 0$ such that for each positive integer k and each $g \in {}^p\mathfrak{D}^{2n-2}(F)$ we have

$$|\Theta_k(g)| \leq A\,||g||_{{}^p\mathfrak{D}^{2n-2}(F)}.$$

This remark will be crucial in what follows.

Next, set $T := \left(\theta_0^{-1}\right)_*\Theta$. It is a real current of degree 2 on X_0. We are left to show that T is indeed a $(1,1)$-current which is moreover positive.

9.7. **Proposition.** *The current T is of pure bidegree* $(1,1)$.

Proof. If not, there exists a $(n, n-2)$-form h on X_0 such that $\langle T, h\rangle \neq 0$. Fix a finite open covering of X_0 by coordinate charts and a partition of unity $\{\varphi_j\}$ relative to this covering. Since

$$0 \neq \langle T, h\rangle = \big\langle T, \sum_j \varphi_j h\big\rangle = \sum_j \langle T, \varphi_j h\rangle,$$

there exists a j_0 such that $\langle T, \varphi_{j_0} h\rangle \neq 0$. Thus, we may assume that h is compactly supported in a coordinate chart (U, z). Without loss of generality, we can also suppose that such a coordinate chart is adapted to the fibration π, *i.e.* $U = \mathcal{U} \cap X_0$, where $\mathcal{U}$ is a coordinate chart for $\mathfrak{X}$ with coordinates (t, z) such that $\pi(t, z) = t$.

In this way, we can extend h "constantly" on the nearby fibres of π: call this extension $\tilde{h}$ and write $\tilde{h}_t$ for $\tilde{h}|_{\mathcal{U}\cap X_t}$. If we set $u_t := \left(\theta_t^{-1}\right)^*\tilde{h}_t$ we obtain a family of test form on F such that, for k sufficiently large, we have

$$\langle T_k, \tilde{h}_{t_k}\rangle = \langle \Theta_k, u_{t_k}\rangle.$$

By Remark 9.6, we have

$$\begin{aligned}
|\langle T_k, \tilde{h}_{t_k}\rangle - \langle T, \tilde{h}\rangle| &= |\langle \Theta_k, u_{t_k}\rangle - \langle \Theta, u_0\rangle| \\
&\leq |\langle \Theta_k, u_{t_k} - u_0\rangle| + |\langle \Theta_k, u_0\rangle - \langle \Theta, u_0\rangle| \\
&\leq A \underbrace{||u_{t_k} - u_0||_{{}^p\mathfrak{D}^{2n-2}(F)}}_{\to 0,\text{ by construction}} + \underbrace{|\langle \Theta_k, u_0\rangle - \langle \Theta, u_0\rangle|}_{\to 0,\text{ by weak convergence}}.
\end{aligned}$$

Being the T_k's of bidegree $(1,1)$ and $\tilde{h}_{t_k}$ of bidegree $(n, n-2)$, we have that $\langle T_k, \tilde{h}_{t_k}\rangle \equiv 0$ and we deduce then that $\langle T, h\rangle = 0$, contradiction. □

9.8. **Proposition.** *The current T is positive.*

PROOF. The proof is almost identical to that of the above proposition. We want to show that for any positive $(n-1,n-1)$-form h on X_0 we have that $\langle T, h\rangle \geq 0$. As before the question is local, so we can suppose that h is compactly supported in U as above. Now the "constant" extensions $\tilde{h}_t$ are again positive $(n-1,n-1)$-forms on X_t, so that $\langle T_k, \tilde{h}_{t_k}\rangle \geq 0$ and we still have convergence to $\langle T, h\rangle$. But then $\langle T, h\rangle \geq 0$. □

This concludes the proof of the theorem, since we have represented α_0 by a closed positive $(1,1)$-current, *i.e.* α_0 is a pseudoeffective class. □

Part 2

Rational curves in Calaby–Yau fiber spaces

In this part we denote by X a normal projective variety of dimension $n \geq 2$. In the following three sections we study the case X admits a genus one fibration onto a normal base B. Section 10 is taken from [**Ane18a**], the sections 11 and 12 has some ideas of [**Ane18b**] but with much extra work. Then in Section 13 we discuss some consequences of the results we achieved and in Section 14 we present some results of Kollár that gives necessary and sufficient numerical conditions on a line bundle on X to have a genus one fibration. Then in Section 15 we study the case the canonical bundle of X is trivial and the base of the fibration is a curve. This section is taken from [**Ane18a**]. Then I conclude in Section 16 with some results in the general case.

10. Linearly relatively minimal genus one fibration

The following theorem is one of the main results in [**Ane18a**]. It will be generalized in the following but we want to give the proof in this easier case to make the comparison between the proofs.

10.1. **Theorem.** *Let X be a projective variety with at most log-terminal singularities that admits a surjective morphism : $X \to B$ to a variety of dimension $n-1$. Suppose moreover that $\tilde{q}(X) = 0$ and there exists a Cartier divisor L on B such that $f^*L \sim K_X$, then there exists a subvariety of codimension one in X that is covered by rational curves contracted by f.*

Proof. The proof is divided in several steps some of which might be already known to the experts. In particular Step 1, Step 3 and Step 5 adapt arguments used in [**DFM19**].

Step 1 : the morphism $f : X \to B$ is a genus one fibration. We can suppose, by taking the normalization of B and passing to Stein factorization, that the morphism f has connected fibers and the base B is normal. For dimensional reasons the generic fiber is a curve. Since X is normal $X_{\text{sing}} \subset X$ has codimension at least two, so $f(X_{\text{sing}}) \subset B$ has positive codimension. The restriction on the regular part of X is a morphism from a smooth variety, so there is a non-empty open subset $U \subset B$ where the morphism $f : X \cap f^{-1}(U) \to U$ is smooth [**Har77**, III.10.7]. Let $Z \subset B$ be the union of the singular locus of B and the singular values of f, *i.e.* $Z := B_{\text{sing}} \cup \text{Sv}(f)$. Now $B_0 := Z^c$ and $B_0 \cap f(X_{\text{sing}})^c$ are non-empty open sets and the morphism $f_0 : X_0 := f^{-1}(B_0) \to B_0$ is a smooth proper surjective morphism. Taking the determinant of the relative cotangent bundle sequence restricted to a fiber that is in the regular part of X, we get the isomorphism $K_E \sim K_{X_0}|_E$. It follows that $K_E \sim K_{X_0}|_E \sim f^*L|_E \sim \mathcal{O}_E$. A smooth curve with trivial canonical bundle is a genus one curve and a smooth degeneration of a genus one curve has again genus one [**BHPVdV04**, See Section V.7], so every fiber of $f_0 : X_0 \to B_0$ is a smooth genus one curve.

Step 2: we reduce to the case where the subvariety Z has codimension one in B. Suppose every irreducible component of Z has codimension at least two. By Lemma 3.28 the family f is isotrivial, so by Proposition 5.2 or 5.3 there exists a variety C_0 and a finite étale cover $C_0 \xrightarrow{\tau} B_0$ such that the pullback $C_0 \times_{B_0} X_0$ is globally trivial, *i.e.* $C_0 \times_{B_0} X_0 \overset{\psi}{\cong} C_0 \times E$. The morphism induced by the following diagram

$$\begin{tikzcd} C_0 \times E \arrow[r, "\psi^{-1}"] & X_0 \times_{B_0} C_0 \arrow[r, "\tau'"] \arrow[d] & X_0 \arrow[d] \\ & C_0 \arrow[r, "\tau"] & B_0 \end{tikzcd}$$

given by the composition $\alpha_0 := \psi^{-1} \circ \tau' : C_0 \times E \to X_0$ is finite étale because τ is the pullback of a finite étale morphism. In particular the composition of the morphisms $C_0 \times E \xrightarrow{\alpha_0} X_0 \xrightarrow{i} X$ is quasi-finite and étale. By Zariski's Main Theorem [**Gro67**] a quasi-finite morphism is always the composition of an open immersion and a finite morphism, so there is a commutative diagram

$$\begin{tikzcd} C_0 \times E \arrow[r, "\alpha_0"] \arrow[d, "i'"] & X_0 \arrow[d, "i"] \\ Y \arrow[r, "\alpha"] & X \end{tikzcd}$$

where i' is an open immersion and α is a finite morphism. The exceptional locus of f is by definition $\operatorname{Exc}(f) = \{x \in X \mid \dim(f^{-1}(f(x))) > 1\}$. The variety $X_0^c = X_Z$ can be splitted as a disjoint union

$$X_Z = (X_Z \cap \operatorname{Exc}(f)) \sqcup (X_Z \cap \operatorname{Exc}(f)^c).$$

Since the subvariety $X_Z \cap \operatorname{Exc}(f)^c$ has dimension at most $\dim(Z)+1$ and we are assuming that $\operatorname{cod}_B(Z) \geq 2$, the dimension of X_Z is bounded by $\dim(X_Z \cap \operatorname{Exc}(f)^c) \leq n-2$. Since $K_X \sim f^*(L)$ the anticanonical bundle $-K_X$ is f-nef, hence by Theorem 4.1 the f-exceptional locus is covered by rational curves contracted by f (in the cited reference Kawamata didn't say explicitly that the rational curves are contracted by f, however this is clear from his proof). This implies that if the exceptional locus of f has codimension one in X, it is a uniruled subvariety of codimension one of X. This allows us to assume that $\operatorname{cod}_X(X_Z) \geq 2$.

Since α is finite, also $i'(C_0 \times E)^c$ has codimension at least two in Y. In particular since α is étale in $i'(C_0 \times E) \subset Y$, this argument proves that α is a finite quasi-étale cover of X, so by hypothesis $H^1(Y, \mathcal{O}_Y) = 0$. By [**KM98**, Proposition 5.20] Y has log-terminal singularities. As proved in [**GKP16c**, Proposition 6.9] there is an isomorphism $\overline{H^0(Y, \Omega_Y^{[1]})} \simeq$

$H^1(Y, \mathcal{O}_Y)$. By definition $\Omega_Y^{[1]}$ is a reflexive sheaf, so it is isomorphic to the sheaf of one forms on the regular part. The variety C_0 is smooth because it is a finite étale cover of B_0, so $\Omega_Y^{[1]} = i'_*\Omega^1_{C_0\times E}$. Since

$$0 = \overline{H^1(Y, \mathcal{O}_Y)} \simeq H^0(Y, \Omega_Y^{[1]}) \simeq H^0(C_0 \times E, \Omega^1_{C_0\times E}) =$$
$$= H^0(C_0, \Omega^1_{C_0}) \oplus H^0(E, \Omega^1_E) \neq 0$$

we reach a contradiction, so if there are no uniruled divisors on X then Z has codimension one in B. This last part can be also done applying Lemma 6.11.

Step 3: restriction to a fibration onto a curve with some singular values. Let H be a very ample divisor on B such that $(n-2)H + L$ is globally generated. The pullback f^*H is a globally generated Cartier divisor. Moreover there is an isomorphism

$$H^0(X, f^*H) \simeq H^0(B, f_*(f^*H)) \simeq H^0(B, H)$$

because f has connected fibers. This isomorphism implies that general elements in $|H|$ are general also in $|f^*(H)|$. So we can choose $n-2$ general divisors $D_1, \ldots, D_{n-2} \in |H|$ such that $C := D_1 \cap \ldots \cap D_{n-2}$ is a smooth irreducible curve in B_{reg} not contained in Z and $S := f^{-1}(D_1) \cap \ldots \cap f^{-1}(D_{n-2})$ is a normal surface. We call again f the morphism $f|_S$. Since Z has codimension one, it must intersect C. Indeed $Z \cdot C = Z \cdot D_1 \cdot \ldots \cdot D_{n-2} = Z \cdot H^{n-2} > 0$ because H is ample in B. This means that f must have some singular fiber. To prove the existence of a uniruled divisor in X it is sufficient to find a rational curve in the general S.

Step 4: the case where $f^{-1}(p_i) \cap \mathrm{sing(S)} \neq \emptyset$. Let $\overline{S}$ be a minimal resolution of S

$$\begin{tikzcd}
\overline{S} \arrow[d, "\nu"'] \arrow[dr, "\beta"] & \\
S \arrow[r, "f"] & C.
\end{tikzcd}$$

We can assume β is relatively minimal. Indeed if there is a (-1)-curve on $\overline{S}$ contracted by β, the image of such a curve is again a rational curve in S because it cannot be contracted to a point by minimality of the resolution. If there are (-1)-curves in the general surface S constructed above, then the union of such rational curves cover a divisor of X. Let $p_1, \ldots, p_k$ be the points of $C \cap Z$. The singular curves $f^{-1}(p_i) \subset S$ are exactly $f^{-1}(p_i) = \nu(\beta^{-1}(p_i))$. Since β is a minimal genus one fibration, by Kodaira's table [**BHPVdV04**, Section V.7] a fiber of β can be a smooth genus one curve, a sum of (possibly non reduced) rational curves or a non reduced genus one curve. If $f^{-1}(p_i)$ contains some singular point of S, then $\beta^{-1}(p_i) = \nu^{-1}(f^{-1}(p_i))$ contains an exceptional divisor of ν, in particular $\beta^{-1}(p_i)$ must be a sum of rational curves. Since not every rational curve of $\beta^{-1}(p_i)$ can be contracted in

S, the curve $f^{-1}(p_i) = \nu(\beta^{-1}(p_i))$ must be a sum of rational curves in S.

Step 5: the case where $f^{-1}(p_i) \subset S_{\text{reg}}$. The curve $f^{-1}(p_i)$ is the central fiber of a family $S_0 \xrightarrow{f} \Delta$ of genus one curves. Since $f^{-1}(p_i)$ is not smooth, by Kodaira's table it is a rational curve or a non reduced irreducible genus one curve. We need to exclude the last possibility.

By adjunction formula the canonical bundle of S_{reg} is base point free. Indeed

$$K_{S_{\text{reg}}} \sim (K_X + (n-2)f^*H)|_{S_{\text{reg}}} \sim f^*(L + (n-2)H)|_{S_{\text{reg}}}$$

the canonical bundle is the restriction of the pullback of a base point free divisor.

By [**BHPVdV04**, V.12.3] the canonical bundle of S_{reg} can be computed using the formula

$$K_{S_{\text{reg}}} \sim f^*D + \sum (m_i - 1)F_i$$

for some divisor D on the base and the sum runs over all the multiple fiber F_i with multiplicity m_i. The restriction of the canonical bundle of S_{reg} to F_i is base point free because $K_{S_{\text{reg}}}$ is base point free. By the above formula for any i the canonical bundle restricted to F_i is

$$K_{S_{\text{reg}}}|_{F_i} \sim (f^*D + \sum (m_i - 1)F_i)|_{F_i} = \mathcal{O}_{F_i}((m_i - 1)F_i).$$

that in particular has some sections because it is the restriction of a base point free line bundle. The line bundle $\mathcal{O}_{Fi}(F_i)$ is torsion of order m_i by [**BHPVdV04**, Lemma III.8.3]. A non-trivial torsion line bundle has no sections, so for any i the line bundle $\mathcal{O}_{F_i}((m_i - 1)F_i)$ must be trivial, hence the multiplicity of the fiber m_i is one for any i and this is a contradiction. □

This Theorem is inspired to [**DFM19**] where they proved a similar result in the case X is a smooth projective manifold with finite fundamental group. Remark 6.6 implies that Theorem 10.1 is stronger than [**DFM19**, Theorem 1.1] also for smooth varieties.

The proof is somehow constructive: if the family of genus one curves varies in moduli then there is a vertical divisor over $j^{-1}(\infty)$ that is covered by rational curves; if the family is generically isotrivial then there is an exceptional divisor covered by rational curves. We used the linear condition $K_X \sim f^*L$ instead $K_X \sim_{\mathbb{Q}} f^*L$ only to exclude the case of multiple fibers. We will see in the following that we can unroll these fibers relaxing this hypothesis.

It is not sufficient the assumption $q(X) = 0$ instead of the vanishing $\tilde{q}(X) = 0$. Indeed the following is a counterexample.

10.2. **Example.** Let $V \subset \mathbb{P}^N$ be a smooth variety of dimension $k \geq 3$ with cyclic fundamental group of order $2q$. Consider in this projective space $\mathbb{P}^N$ a generic hypersurface Z of degree $d' \geq 2N-1$. By [**Cle86**, Theorem 1.1] there are no rational curves in Z. We denote by $B := Z \cap V$ their intersection. Let $\tilde{B}$ be the universal cover of B.

Consider an elliptic curve E and an automorphism η of order $2q$ that does not fix ω_E the generator of Ω^1_E and the diagonal action of $G := \mathbb{Z}/(2q)$ induced on the product $Y := \tilde{B} \times E$. The quotient of Y under this action gives a genus one fibration $X := Y/G \xrightarrow{\pi} B$. The variety X has zero irregularity but does not contain rational curves at all. A detailed explanation of this example can be found in Example 17.6.

An interesting application of Theorem 10.1 is the following corollary, already observed in their context in [**DFM19**].

10.3. **Corollary.** *A variety X with at most log-terminal singularities, $\tilde{q}(X) = 0$, $\kappa(X) = n-1$ and whose canonical bundle is nef of exponent one does contain a uniruled divisor.*

The condition on the exponent is needed because we need to assume in Theorem 10.1 that the canonical bundle of X is linearly equivalent to the pullback of a line bundle on B. This hypothesis can be relaxed as we will see in Corollary 13.2. In particular for the proof of this corollary we refer to Corollary 13.2.

10.A. Trivial canonical bundle. For Calabi–Yau varieties we can improve Theorem 10.1 proving the following result.

10.4. **Theorem.** *Let X be Calabi–Yau variety. Suppose that there exists a morphism $f : X \to B$ whose general fiber is a curve. Then there exists a uniruled subvariety of codimension one in X that is covered by rational curves contracted by f.*

Proof. By global index one theorem [**GGK19**, Proposition 2.18] there is a variety X' with canonical singularities and a finite quasi-étale morphism $\alpha : X' \to X$ such that $K_{X'} \sim 0$. A finite quasi-étale cover $Y \to X'$ is also (after composition with α) a finite quasi-étale cover of X. This proves that $\tilde{q}(X') \leq \tilde{q}(X)$ and so $\tilde{q}(X') = 0$. If there is a subvariety $V \subset X'$ of dimension $n-1$ that is covered by rational curves, then also the variety $\alpha(V) \subset X$ is covered by rational curves. Since the canonical bundle of X' is linearly equivalent to the trivial line bundle it is automatically the pullback of the trivial line bundle. The hypotheses of Theorem 10.1 are satisfied, so the theorem is proved. □

To preserve the dichotomy given by Beauville–Bogomolov decomposition between irreducible symplectic varieties and Calabi–Yau varieties in the singular setting, a useful definition is given for example in [**GGK19**], [**Dru18**], [**HP19**] and related papers. In particular, in [**HP19**], they prove that there exists a version of the Beauville–Bogomolov decomposition for varieties with canonical singularities and smooth in codimension two. In these definitions of Calabi–Yau varieties and irreducible symplectic varieties there are some conditions on the reflexive exterior algebra of forms, that in particular imply that such varieties must have vanishing augmented irregularity.

Being a Calabi–Yau variety as in Definition 3.12 means that in the Beauville–Bogomolov decomposition [**HP19**, Theorem 1.5] the abelian factor is trivial. Without the assumption on the smoothness in codimension two an analogous statement is [**GGK19**, Theorem B]. In particular Theorem 10.4 can be applied to any product of Calabi–Yau's and irreducible symplectic varieties with such a definition. But let us be more precise.

Let X be a variety with at most log-terminal singularities. Suppose moreover $K_X \equiv_{\mathrm{num}} 0$ and that there is a surjective morphism $f : X \to B$ to a variety of dimension $n-1$. By [**GGK19**, Theorem B] there is a quasi-étale map $g : A \times Y \to X$ with A an abelian variety of dimension $\tilde{q}(X)$, and $\tilde{q}(Y) = 0$. Passing through the Stein factorization of $f \circ g$ we get a genus one fibration $\alpha : A \times Y \to \tilde{B}$. If the restriction of α to $\{t\} \times Y$ for generic t is a family of curves, then we can apply Theorem 10.4 and find a uniruled divisor on $\{t\} \times Y$. This implies that there is also a uniruled divisor on $A \times Y$ and hence its image under g is again a uniruled subvariety of codimension one in X.

10.5. **Remark.** We can't expect that we can always apply Theorem 10.4 to the restriction of the fibration to $\{t\} \times \tilde{X}$ because it may happen that α is a projection, *i.e.* $X = E \times Y \to Y$ for some genus one curve E.

Also for smooth varieties, Theorem 10.4 seems more general than [**DFM19**, Corollary 1.2] because in their context a Calabi–Yau variety must have finite fundamental group. Such finiteness condition is a priori stronger than the vanishing of the augmented irregularity (see Remark 6.6). However one can see as consequences of Beauville–Bogomolov decomposition for smooth varieties that this two conditions are equivalent: a smooth projective variety with numerically trivial canonical bundle and vanishing augmented irregularity has finite fundamental group. It is conjectured that the same implication holds also in the singular case, at least for varieties with mild singularities.

11. Relatively minimal genus one fibrations

In this section we generalize the results of the previous one. The key ingredient for this generalization is Theorem 3.18 of Kollár that leads us to relax the condition $K_X \sim f^*L$. However some of the results we obtain in this section can be obtained directly with a semistable reduction and the results of Section 5.

11.A. Singular fibers of genus one fibrations. Now we study the general fibers over the singular values of an genus one fibration.

11.1. **Lemma.** *Let $f : X \to B$ be a genus one fibration and $Z := \mathrm{Sv}(f)$. Suppose that $cod_B(Z) = 1$, then a general fiber over Z is $mE + \sum m_i R_i$ where E is a genus one curve and R_i are rational curves.*

PROOF. We can study the restriction of f to a surface as follows. Let H be a very ample divisor on B such that $(n-2)H + L$ is globally generated. The pullback f^*H is a globally generated Cartier divisor. Moreover there is an isomorphism

$$H^0(X, f^*H) \simeq H^0(B, f_*(f^*H)) \simeq H^0(B, H)$$

because f has connected fibers. This implies that general elements in $|H|$ are general also in $|f^*(H)|$. So we choose $n-2$ general divisors $D_1, \ldots, D_{n-2} \in |H|$ such that $C := D_1 \cap \ldots \cap D_{n-2}$ is a smooth irreducible curve in B_{reg} not contained in the locus of singular values of f and $S := f^{-1}(D_1) \cap \ldots \cap f^{-1}(D_{n-2})$ is a normal surface. Looking at the Kodaira's table [**BHPVdV04**, Section V.7] it is easy to check that the singular fibers of $f|_S$ are $mE + \sum m_i R_i$ where E is an elliptic curve and R_i are rational curves. The condition on the dimension of Z insures that a general point in Z lies on a curve obtained as general intersection of hyperplane sections. □

11.2. **Remark.** If X contains no uniruled codimension one subvarieties but $\mathrm{Sv}(f)$ has codimension one in B, then the fibers over any general point of $\mathrm{Sv}(f)$ of dimension $n-2$ are multiple genus one curves.

11.3. **Remark.** It follows from Lemma 11.1 and from Theorem 4.1 of Kawamata that a relatively minimal genus one fibration with no uniruled codimension one subvarieties has no degenerate divisors.

Lemma 11.1 can be seen as a soft version of Kodaira's table in higher dimension. With the same strategy of the proof of this lemma one can certainly do a better classification of singular fibers. Using the techniques of Lemma 11.1 we can control only the general singular fiber in codimension one.

Now we can merge together these lemmas and prove the following result.

11.4. **Proposition.** *Let $X \xrightarrow{f} B$ be a genus one fibration between normal projective varieties such that X does not contain codimension one subvarieties that are uniruled. Then the family $X \xrightarrow{f} B$ is isotrivial.*

This proposition should be compared with [**LP18**, Proposition 6.5] and [**V+03**, Proof of Corollary 3.34].

Proof of Proposition 11.4. We can assume by Lemma 3.28 that $Z := \mathrm{Sv}(f))$ has codimension one in B. The general fiber over Z is classified by Lemma 11.1. Since there are no uniruled codimension one subvarieties, in the general fiber over Z there are only multiple genus one curves.

Now we can proceed cutting with hyperplane sections as in the proof of Lemma 11.1. In this way we get many curves C in B with only genus one fibers (possibly multiple) over them. Up to consider a finite possibly ramified base change we can assume this map has a section. The j-invariant for multiple elliptic curves is well-defined as one can easily check with a semistable reduction. Since the curve C is complete this implies that the j-map is constant, *i.e.* the family π restricted over C is isotrivial.

For each curve C obtained in this way we get an isotrivial family. From this fact it follows that the all the family over B is isotrivial. Let us prove this fact by induction on the dimension of B.

There is nothing to prove if the dimension is one. By induction we can suppose that the family is isotrivial when restricted to an ample subvariety $H \subset B$. The fibers over curves that are general complete intersections are pairwise isomorphic and these curves must intersect H. The union of these curves dominates B. This implies that the family $X \xrightarrow{f} B$ is isotrivial. □

Another way to prove this lemma is to consider the j-map and the semistable reduction directly from B.

11.B. Relatively minimal genus one fibrations. Here we present the first generalization of Theorem 10.1 based on a result of Kollár which allows us to face the case the singular locus of the fibration has codimension one, but there are only multiple genus one curves over it.

11.5. **Theorem.** *Let $X \xrightarrow{f} B$ be a relatively minimal genus one fibration.*

- *The variety X does not contain a divisor covered by vertical rational curves if and only if X is isomorphic in codimension one to a variety which has a finite cover, étale in codimension two, isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*
- *The variety X does not contain vertical rational curves if and only if there is a finite globally étale cover of X isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*

Proof. It follows from standard arguments that a general fiber of f is a smooth curve contained in the smooth locus of X, so by adjunction formula $K_{X_t} \sim K_X|_{X_t}$. This implies that $K_{X_t} \sim 0$ and hence the general fiber has genus one.

If the fibration is not generically isotrivial then there exists a uniruled divisor in X by Proposition 11.4. In particular all the implications are satisfied. So we can assume that f is generically isotrivial. Under these assumption by Theorem 3.18 X is birational over B via a rational map g to a unique orbibundle $X_{\text{orb}} \xrightarrow{f'} B$. By uniqueness the orbibundle is the only candidate to be a quotient of a product.

If there are some degenerate divisors on X then by Remark 11.3 they are covered by rational curves contracted by f, and by Lemma 3.22 these divisors must be contracted in X_{orb}. In particular also in this case all the implications are satisfied.

We can also assume by Lemma 3.9 that g is an isomorphism in codimension one between X and X_{orb}.

Let us focus on the first equivalence. Since we can assume that X and X_{orb} are isomorphic in codimension one over B, and one contains a divisor covered by vertical rational curves if and only if the other does, we can replace in the statement X with X_{orb}. Then the conclusion follows from the characterization in Lemma 3.24.

Now let us focus on the second characterization. By Lemma 1.3 also X_{orb} is log-terminal. Hence we can apply Theorem 4.5 to g^{-1}. The indeterminacy locus of g^{-1} is covered by rational curves. We claim that such curves must be vertical. Indeed consider a resolution of the morphism

$$\begin{tikzcd} & Z \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X_{\text{orb}} \arrow[rr, dashed, "g^{-1}"] \arrow[dr] & & X \arrow[dl] \\ & B. & \end{tikzcd}$$

If the β-exceptional locus contains the α-exceptional locus then g^{-1} is globally defined by [**Deb01**, Lemma 1.15]. Suppose this inclusion does not hold and denote by $p \in X_{\text{orb}}$ a point in the image of $\text{Exc}(\alpha) \setminus \text{Exc}(\beta)$. By Theorem 4.2 $\alpha^{-1}(p)$ is rationally connected. Choose two points in $\alpha^{-1}(p)$ that maps into different points of X and take a rational curves R passing through these points. Its image $\beta(R)$ is a rational curve in X and, since the diagram above commutes, it must be contracted in B.

Assume that X does not contain vertical rational curves. Then by the above argument there is a diagram

$$\begin{tikzcd} X_{\mathrm{orb}} \arrow[rr, "g^{-1}"] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

and since all the fibers of the orbibundle are irreducible curves, by dimensional reasons g^{-1} cannot contract anything, and hence X is isomorphic to X_{orb}. If the finite map $\tilde{B} \times E \xrightarrow{h} X$ ramifies at some point (b, z) then the map restricted to the fiber $b \times E \to X$ is a finite ramified map from a genus one curve onto its image, that by Hurwitz formula is a genus zero curve that is contracted in B. Hence h is globally étale.

Assume that there is a finite globally étale cover of X isomorphic to $\tilde{B} \times E$ over B, then all the fibers are the image of the genus one curve E under a finite unramified map, that by Hurwitz formula is again a genus one curve. $\square$

In particular an immediate consequence of this theorem is the following.

11.6. **Corollary.** *In the setting of the previous theorem, if B does contain no rational curves, then X contains rational curves if and only if does not exists a finite étale cover of the form $\tilde{B} \times E \to X$ over B.*

The proof of Theorem 11.5 essentially is in the same spirit of the proof of Theorem 10.1. The key points for this generalization are Theorem 3.18, the uniruledness of the divisors of X contracted in the orbibundle and a careful analysis of orbibundles.

12. Non relatively minimal genus one fibrations

It is natural to ask what happens if we do not assume the fibration is relatively minimal. There are several obstruction to be minimal. The first one is that the canonical bundle is not a $\mathbb{Q}$-Cartier divisor. The second one is that the canonical bundle has some exceptional component, *i.e.* $K_X \sim_{\mathbb{Q}} f^*L + \sum a_i E_i$ for some exceptional divisors E_i.

12.A. Log relatively minimal fibration. One way to face off the first problem is to add an effective boundary Δ to make $K_X + \Delta$ a $\mathbb{Q}$-Cartier divisor. This case can be treated exactly as the case of Theorem 12.2:

12.1. **Theorem.** *Let $(X, \Delta) \xrightarrow{f} B$ be a relatively minimal log Calabi–Yau fiber space of relative dimension one. Then*

- *The variety X does not contains a divisor covered by vertical rational curves if and only if X is isomorphic in codimension one to a variety which has a finite cover, étale in codimension two, isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*
- *The variety X does not contain vertical rational curves if and only if there is a finite globally étale cover of X isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*

Proof. Since $K_X + \Delta \sim_{\mathbb{Q}} f^*L$, the restriction of $(K_X + \Delta)$ to a general fiber of f, *i.e.* $(K_X + \Delta)|_{X_t}$ is numerically trivial. A general fiber of f is a smooth curve contained in the smooth locus of X, so by adjunction formula $K_{X_t} \sim K_X|_{X_t} \sim_{\mathbb{Q}} -\Delta|_{X_t}$ and hence the general fiber has genus at most one. If the genus is zero the variety X is uniruled, so we can suppose f is a genus one fibration. Note that even if Δ and K_X are not $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors their restrictions on a neighborhood of a general fiber X_t are $\mathbb{Q}$-Cartier. Then follows verbatim the proof of Theorem 12.2. ☐

12.B. Genus one fibrations with exceptional locus. In the general case it is not easy to control the exceptional locus. As in Example 17.5 there are cases where one cannot find rational curves on the exceptional locus also if we assume X to be smooth. The main tool we can use is the the result of Kawamata Theorem 4.1.

12.2. **Proposition.** *Let (X, Δ) be a klt pair such that there exists a surjective morphism $f : X \to B$ to a variety of dimension $n-1$ that is not a quasi-product over B. Suppose moreover $K_X + \Delta \sim_{\mathbb{Q}} f^*L + \sum a_i E_i$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on B, some f-exceptional divisor E_i and whose coefficients are not all strictly negative. Then X does contain rational curves.*

Proof. Let us start assuming that X is $\mathbb{Q}$-factorial. We can write $K_X + \Delta \sim_{\mathbb{Q}} f^*L + D - D'$ with D and D' f-exceptional effective divisors with no common components. If there are no exceptional divisors this is Theorem 11.5. Otherwise it follows from Lemma 3.7 that there exists a component E of the divisor D that is covered by curves that intersect negatively $K_X + \Delta$. By Cone Theorem this implies that there are rational curves in X.

If X is not $\mathbb{Q}$-factorial, in order to apply Lemma 3.7, we consider a $\mathbb{Q}$-factorialization $X_{\mathbb{Q}} \to X$ and with the same argument we get that there exists a divisor in $X_{\mathbb{Q}}$ covered by curves that intersect negatively $K_{X_{\mathbb{Q}}} + \Delta_{\mathbb{Q}}$. Since a $\mathbb{Q}$-factorialization is small, not all these curves can be contracted in X. By projection formula we get curves in X with negative intersection with $K_X + \Delta$. The conclusion follows again by the Cone Theorem. ☐

12.3. **Remark.** Up to consider the new pair $K_X + \Delta + \varepsilon E_i$ one can prove that there exists rational curves, provided that not every $-a_i$ is bigger than the log-canonical threshold of E_i.

With the same strategy we can find a uniruled divisor with a further assumption on the singularities of X.

12.4. **Corollary.** *In the setting of Theorem 12.2 with the further condition that X is smooth in codimension two, e.g. X has terminal singularities. Then X contains an uniruled divisor.*

Proof. Since we are looking for a uniruled divisor and a $\mathbb{Q}$-factorialization is small, we can assume that X is $\mathbb{Q}$-factorial. By the proof of Theorem 12.2 there exist some very ample line bundles H_i on X and an irreducible component E of D such that $E \cdot H_1 \cdot \ldots H_{n-2} \cdot (K_X + \Delta) < 0$. Since X is smooth in codimension two, the general complete intersection of elements in $|H_i|$ is contained in X_{reg}, so the general curve obtained as intersection $E \cap H_1 \cap \cdots \cap H_{n-2}$ is contained in the regular part of X and intersects negatively $K_X + \Delta$. Moreover, by the construction of H_i, these curves cover E. To conclude it is sufficient to apply Bend and Break Theorem [**Deb01**, Theorem 3.6]. □

With a different strategy we can find a uniruled divisor also in the case X is singular in codimension two.

12.5. **Proposition.** *Let $X \xrightarrow{f} B$ be a genus one fibration that is not a quasi-product and $K_X \sim_{\mathbb{Q}} f^*L + \sum a_i E_i$. If we suppose that some a_i is non-negative, then X does contain a uniruled divisor.*

Proof. If there are no exceptional divisors we can apply Theorem 12.2 to conclude. Since we are looking for a uniruled divisor and a $\mathbb{Q}$-factorialization is small we can assume that X is $\mathbb{Q}$-factorial. We can write $K_X \sim_{\mathbb{Q}} f^*L + \sum a_i E_i - \sum b_i F_i$ with $E_i \neq F_j$ the exceptional divisors and with all the coefficients non-negative. The divisor $\sum a_i E_i$ has a component E_1 covered by curves C_t contracted in B such that $\sum a_i E_i \cdot C_t < 0$ by Lemma 3.7. Moreover the curves C_t are complete intersections of the form $E_1 \cap H_1 \cap \cdots \cap H_{n-2}$ for some very ample divisors H_i in X.

Take a terminalization $\tilde{X} \xrightarrow{\nu} X$. The canonical bundle of this partial resolution is $K_{\tilde{X}} \sim_{\mathbb{Q}} \nu^* K_X - \sum c_i G_i$ for some non negative number c_i. A component of the divisor $\nu^* E_1$, that is $(\nu^{-1})_* E_1$, is covered by the strict transforms $\tilde{C}_t$ and satisfies $\sum a_i \nu^* E_i \cdot \tilde{C}_t < 0$. The family of curves $\tilde{C}_t$ is not contained in the other components of the support of $K_{\tilde{X}}$, so we have $K_{\tilde{X}} \cdot \tilde{C}_t < 0$. Since $\tilde{X}$ is terminal, it is smooth in codimension 2. The curves $\tilde{C}_t$ are contained in the intersection $\nu^* H_1 \cap \cdots \cap \nu^* H_{n-2}$.

The divisors ν^*H_i are base point free, so the general element of this family does not intersect the singular points of $\tilde{X}$. This means that for a general point in $(\nu^{-1})_*E_1$ there is a curve contained in the regular part of X that intersects negatively the canonical bundle $K_{\tilde{X}}$, hence we can apply [**Deb01**, Theorem 3.6] and get a family of rational curves that covers $(\nu^{-1})_*E_1$. Since the image of a rational curve is again rational, this implies that also E_1 is covered by rational curves. □

Another strong way to control the exceptional locus is Theorem 4.2. Using much of the theory we developed so far we can prove the following:

12.6. **Theorem.** *Let $X \xrightarrow{f} B$ be a genus one fibration. Assume the base B is potentially klt. Then*

- *If X is not a quasi-product, then X contains a uniruled divisor.*
- *If X is not an orbibundle, then X contains a vertical rational curve.*

More precisely:

- *The variety X does not contains a divisor covered by vertical rational curves if and only if X is isomorphic in codimension one to a variety which has a finite cover, étale in codimension two, isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*
- *The variety X does not contain vertical rational curves if and only if there is a finite globally étale cover of X isomorphic to $\tilde{B} \times E$ over B, for some cover $\tilde{B}$ of B.*

Proof. By Proposition 11.4 we can assume that f is a generically isotrivial fibration. By Lemma 11.1 we can assume that there are no degenerate divisors. Since we have no hypothesis on the canonical bundle of X we cannot apply Theorem 4.1.

Let E be a general fiber of f. By Theorem 3.18, X is birational via a rational map g to a unique orbibundle $X_{\mathrm{orb}} \xrightarrow{f'} B$. Since B is potentially klt, by Proposition 1.1 also the cover $\tilde{B}$ is potentially klt. Since E is a smooth curve also the product $E \times \tilde{B}$ is potentially klt and applying again Proposition 1.1 X_{orb} has the same singularities.

We can apply Corollary 4.3 to all the divisors contracted by g. This implies that we can assume there are no divisors contracted by g. In particular by Lemma 3.22 all the divisors f-exceptional are contracted by g and hence are uniruled. We can apply Lemma 3.9 and obtain that X and X_{orb} are isomorphic in codimension one. This proves that if there are no uniruled divisor in X then it is isomorphic in codimension one to an orbibundle. At this point the first equivalence is given by our characterization of those orbibundles with divisors covered by vertical rational curves given in Lemma 3.24.

For the second point we apply Lemma 1.3 that tells us that f is a log relatively minimal genus one fibration for some boundary. Then we get the conclusion following verbatim the last part of the proof of Theorem 11.5. $\square$

This result is more general than Theorem 12.1 because the base of a relatively minimal Calabi–Yau fiber space is potentially klt by [**Amb05**, Theorem 0.2]. The only assumption that we make on the singularities of B is needed: if B has only log-canonical singularities that are non log-terminal, the thesis fails. The following is an example of this situation:

12.7. **Example.** Let $B := \{x_0^3 + x_1^3 + x_2^3\} \subset \mathbb{P}^3$ be a cone over a genus one curve and E an elliptic curve. Consider the product $X := B \times E$ and its minimal resolution $\tilde{X}$. The variety $\tilde{X}$ has a generically isotrivial genus one fibration with an exceptional divisor. There are no vertical rational curves in $\tilde{X}$ because the exceptional locus is isomorphic to the product of two elliptic curve, and the other fibers are genus one curves. This is because B has log-canonical but non divisorially log-terminal singularities. For further details see Example 17.5.

13. Particular cases and consequences

We mention some particular cases and prove some interesting consequences.

13.1. **Corollary.** *Let X be a projective variety of dimension n with at most canonical singularities, $\kappa(X) = n - 1$ and $\tilde{q}(X) = 0$. Then X does contain rational curves.*

Proof. If the canonical bundle of X is not nef, then there are rational curves in X by the Cone Theorem. The numerical dimension of K_X is greater than the Kodaira dimension of X that is $n-1$. If $\nu(K_X) = n$ then K_X is a nef $\mathbb{Q}$-Cartier divisor with positive self-intersection, hence by Riemann–Roch Theorem $k(X) = n$, that is a contradiction. So $\nu(K_X) = n - 1 = \kappa(K_X)$ and following the definition of Kawamata this means that X is a good minimal model. By [**Kaw85**, Theorem 1.1] the canonical divisor is semi-ample, hence the Iitaka fibration of the canonical bundle gives a genus one fibration $\varphi_{mK_X} : X \to B$ with $K_X \sim_{\mathbb{Q}} \varphi^*_{mK_X}(H)$ for an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on B. In particular we can apply Theorem 11.5 and get the thesis. $\square$

For smooth varieties we can do slightly better. The key point for this improvement is a very nice work on varieties covered by elliptic curves by Lazic and Peternell [**LP18**]. For smooth varieties we can prove, using their results, the following corollary.

13.2. **Corollary.** *Let X be a smooth projective variety of dimension $n \geq 2$ that is not a quasi-product, e.g. $\tilde{q}(X) = 0$. If X is covered by genus one curves, then it contains a rational curve.*

Proof. Suppose by contradiction that X does not contain rational curves. We can apply [**LP18**, Theorem 6.12] and find an equidimensional fiber space $X \to W$. This fibration is relatively minimal and then by Theorem 12.2 we can conclude. $\square$

More precisely we can also apply the last part of Theorem 12.2 proving the following.

13.3. **Corollary.** *Let X be a smooth projective variety covered by elliptic curves but with no rational curves. Then X is a finite étale quotient of $E \times \tilde{B}$. In particular the fundamental group of X is infinite.*

Proof. As before by [**LP18**, Theorem 6.12] under our assumption there exists a relatively minimal genus one fibration $X \to B$. Then we can apply Theorem 12.2 to f to get the conclusion. $\square$

14. Conditions to have minimal genus one fibration

Here we state an useful criterion to find genus one fibration due to Kollár.

14.1. **Theorem.** *Let X be a projective variety with at most log-terminal singularities of dimension n, nef canonical bundle and L a Cartier divisor on X. Assume moreover*

1) $L^{n-2} \cdot \mathrm{Td}_2(X) > 0$.
2) L *is nef.*
3) $L - \varepsilon K_X$ *is nef for* $0 \leq \varepsilon << 1$.
4) $L^n = 0$.
5) $L^{n-1} \neq 0$ *in* $H^{2n-2}(X, \mathbb{Q})$.

Then X with the Iitaka fibration associated to L is a relatively minimal genus one fibration.

This result is [**Kol15**, Theorem 10]. In the same article there is also a log version of this theorem.

14.2. **Remark.** In [**Kol15**, Theorem 10] there is the further hypothesis $L^{n-1} \cdot K_X = 0$, but this condition actually follows from the others. Indeed if $L - \varepsilon K_X$ is nef then

$$0 \leq (L - \varepsilon K_X)^n = L^n - n\varepsilon L^{n-1} \cdot K_X + ... = -n\varepsilon L^{n-1} \cdot K_X + ...$$

The divisors L and K_X are nef, hence $L^{n-1} \cdot K_X \geq 0$. It follows that $L^{n-1} \cdot K_X = 0$.

The conditions 2), 3) and 4) mean that L is a line bundle whose class is contained in an extremal face of the nef cone, and this face contains the class of the canonical bundle of X.

It follows from Theorem 14.1 the following result.

14.3. **Corollary.** *Let X be a variety with log terminal singularities and $\tilde{q}(X) = 0$. If there exists a line bundle L on X such that the conditions from 1) to 5) of Theorem 14.1 are satisfied, then X does contain rational curves.*

In [**Kol15**] Kollár provides also a log version of Theorem 14.1.

15. Fibration over curves

In this section we study the dual case of a genus one fibration: the case of a surjective morphism $\pi : X \to C$ to a curve. Passing through the Stein factorization we can assume π has connected fibers and since X is normal we can assume that C is smooth. So it is sufficient to study the geometry of a morphism with connected fibers onto a smooth curve. A fiber of a morphism onto a curve is a semiample divisor with numerical dimension one. So it is a priori more general to work only with a nef divisor with numerical dimension one than with a fibration onto a curve.

In this section we prove the following statement.

15.1. **Theorem.** *Let X be a Calabi–Yau variety. Suppose there exists a nef $\mathbb{Q}$-divisor D with numerical dimension one such that $c_2(X) \cdot D = 0$ in $N^3(X)$. Then X does contain rational curves.*

15.2. **Remark.** Theorem 15.1 is a generalization of [**DFM19**, Theorem 1.6] also for smooth varieties. Indeed for smooth varieties with trivial canonical bundle with a fibration onto a curve with general fiber an abelian variety F, the class of F in $N^1(X)$ has numerical dimension one and intersect in zero the second Chern class of X, *i.e.* $F \cdot c_2(X) = 0$ [**DFM19**, Section 3]. Moreover a divisor with numerical dimension one which intersects in zero the second Chern class of X is just conjecturally semiample.

The geometric meaning of Theorem 15.1 is clear if the divisor is also semiample. In this case the Itaka fibration associated to D is a fibration onto a curve. A general fiber F of such a morphism intersects trivially $c_2(X)$, *i.e.* $c_2(F) = F \cdot c_2(X) = 0$. If F is contained in the regular part of X, then by adjunction formula F has automatically trivial canonical bundle. This in particular implies that there is an abelian variety with a finite quasi-étale cover to F.

15.3. **Proposition.** *Let X be a variety with log terminal singularities, numerically trivial canonical bundle and without rational curves. Let H and D be two divisors on X that are respectively ample and nef of numerical dimension one. There is a (unique) rational number t_0 such that the $\mathbb{Q}$-divisor $\overline{N}(D,H) = H - t_0 \cdot D$ is nef and has numerical dimension $n-1$.*

Proof. The line in $N^1(X)$ for $t \in \mathbb{R}$

$$N_t = H + t \cdot D$$

gives us an interesting divisor in the intersection with the null cone. This line is parallel to the extremal ray of the nef cone generated by $[D]$. The divisor D is nef so the line N_t is contained in the nef cone for $t \geq \frac{-H^n}{nH^{n-1} \cdot D}$ and intersect the null cone when there is the equality. The divisor in the intersection $\overline{N} = H - \frac{H^n}{nH^{n-1} \cdot D} D$ is a $\mathbb{Q}$-divisor because H and D are $\mathbb{Q}$-divisors and $\frac{H^n}{nH^{n-1}} \in \mathbb{Q}$. The divisor $\overline{N}$ has numerical dimension $n-1$ because $\overline{N}^{n-1} \cdot D = H^{n-1} \cdot D \neq 0$ and it is not big. □

In particular this proposition implies the following corollary.

15.4. **Corollary.** *Let X be a variety with canonical singularities, numerically trivial canonical bundle and with no rational curves. If $c_2(X) \neq 0$ as element in $N^2(X)$ but $c_2(X) \cdot D = 0$ in $N^3(X)$ for some nef $\mathbb{Q}$-divisor D with $\nu(D) = 1$, then there exists an ample $\mathbb{Q}$-divisor H such that the $\mathbb{Q}$-divisor $\overline{N}(D,H)$ constructed in Proposition 15.3 satisfies $c_2(X) \cdot \overline{N}(D,H)^{n-2} > 0$.*

Proof. By Proposition 15.3, X contains a $\mathbb{Q}$-divisor N of numerical dimension $n-1$. By Lemma 7.3 we know that the intersection of $c_2(X)$ with $n-2$ nef divisors is non negative. By Lemma 7.4 for any ample divisor H we have $H^{n-2} \cdot c_2(X) > 0$. By hypothesis $c_2(X) \cdot D = 0$, so $c_2(X) \cdot (\overline{N}(D,H))^{n-2} = c_2(X) \cdot (H - \frac{H^n}{nH^{n-1} \cdot D} D)^{n-2} = c_2(X) \cdot H^{n-2} > 0$. □

15.A. Proof of Theorem 15.1. Now the proof of Theorem 15.1 follows from the results obtained in this section and by Theorem 10.4.

Proof of Theorem 15.1. If the singularities of X are not canonical then X is uniruled by [**KL09**, Theorem 11], so we can suppose that X has canonical singularities. Moreover by Remark 7.5 we now that $c_2(X) \neq 0$. Suppose by contradiction that there are no rational curves in X. Thanks to Corollary 15.4 we can find a nef $\mathbb{Q}$-divisor $\overline{N}$ such that $0 < c_2(X) \cdot \overline{N}^{n-2} = 12 \operatorname{Td}_2(X) \cdot \overline{N}^{n-2}$. So applying [**Kol15**, Theorem 10] the divisor $\overline{N}$ induces an genus one fibration

$X \to B$. Thus we can apply Theorem 10.4 to find rational curves in X, which gives a contradiction. □

The idea of Theorem 15.1 is to find a nef divisor D in X with Itaka dimension $n-1$. In the proof of Theorem 15.1 we explained that in our setting it is sufficient to find a nef $\mathbb{Q}$-divisor D with numerical dimension $n-1$ such that $D^{n-2} \cdot c_2(X) > 0$. A careful analysis in dimension three can be found in [**DF14**]. They work with smooth varieties but their proofs works verbatim also for Calabi–Yau varieties as in Definition 3.12.

16. Calabi–Yau fiber spaces, the general case

The case where the dimension of the fibers is intermediate is much more difficult. In the case of non-isotrivial fibrations the main new problem is the existence of complete families of abelian varieties. In the case of isotrivial fibrations we can say something, however many cases are completely open, *e. g.* $F \times B \to B$ where F is a Calabi–Yau of dimension at least 3. Many of the results that we are going to present in this section can be adapted to the case X has mild singularities, but we will state everything in the smooth case.

We start proving an analogue of Lemma 11.1 in the case the dimension of the fibers is at least two.

16.1. **Lemma.** *Let $X \xrightarrow{h} B$ be a smooth relatively minimal Calabi–Yau fiber space. Then all the degenerate divisors are uniruled.*

Proof. Since the canonical bundle of X is relatively trivial we can directly apply Theorem 4.1 to all the h-exceptional divisors of X. Let D be an insufficient fiber type divisor. Cutting with general hyperplane sections of the base we can assume the base to be a smooth curve and X is still smooth. Over this locus we can apply [**Tak08**] to conclude. □

16.2. **Remark.** In this lemma we do not use that X is smooth but only that it has canonical singularities. A more careful analysis can also be done in the case the singularities of X are not canonical. We decide to work in the smooth case to avoid technicalities.

16.A. Isotrivial relatively minimal fibrations.

In the case of isotrivial fibrations using the orbibundles we can prove the following.

16.3. **Theorem.** *Let $X \xrightarrow{f} B$ be a relatively minimal Calabi–Yau fiber space which is generically isotrivial. Denote by F the generic fiber. Suppose that X is smooth and does not contain rational curves. Then X is isomorphic to an orbibundle.*

Proof. Under these hypothesis we know by Theorem 3.18 that there exists a birational map

$$\begin{tikzcd} X_{\mathrm{orb}} \arrow[rr, dashed, "g"] \arrow[dr] & & X \arrow[dl, "f"] \\ & B. & \end{tikzcd}$$

Since X is smooth, by generic smoothness of f [**Har77**, Corollary 10.7] the generic fiber of f is smooth, hence we can apply Lemma 3.23 that tells that X_{orb} is potentially klt. Hence we can apply Theorem 4.5 to obtain that h extends to a morphism $X_{\mathrm{orb}} \xrightarrow{g} X$ over B. By Lemma 3.22 X_{orb} has no degenerate divisors, so we can apply Lemma 3.11 to get that g is an isomorphism in codimension one. This implies that g is a small globally defined morphism. Suppose that g is not an isomorphism, and hence contracts a curve $C \subset X_{\mathrm{orb}}$. Chose an ample line bundle $\tilde{H}$ on X_{orb} and define $H := g_*\tilde{H}$. If H would be $\mathbb{Q}$-Cartier then $\tilde{H} \equiv_{\mathrm{num}} g^*H$ but this is impossible because $\tilde{H}$ is ample but $C \cdot g^*H = 0$. This is a contradiction because X is smooth and hence the $\mathbb{Q}$-Weil divisor H is also a $\mathbb{Q}$-Cartier. As conclusion g is an isomorphism. $\square$

16.4. **Remark.** By purity of branch locus, being the map $\tilde{B} \times F \xrightarrow{g} X$ quasi étale, it is globally étale.

16.5. **Remark.** In this situation we have $\tilde{q}(X) = \tilde{q}(X_{\mathrm{orb}}) \geq H^0(\tilde{B} \times F, \Omega^{[1]}_{\tilde{B}\times F}) \geq q(F)$. In particular a Hyperkähler or a Calabi–Yau manifold with a generically isotrivial fibration contains rational curves.

16.6. **Corollary.** *Let X be a projective Hyperkähler manifold with admits an isotrivial fibration. Then X does contain rational curves.*

One can construct many explicit examples of Hyperkähler manifolds with an isotrivial lagrangian fibration, see Example 17.7 and 17.8

For more details on these examples and in isotrivial lagrangian fibrations the interested reader can see the paper of Sawon [**Saw14**].

16.B. Non-isotrivial relatively minimal fibrations. Lets consider as before the case X is a smooth projective variety. Suppose that there exists a fibration $f : X \to B$ relatively minimal, *i.e.* $K_X \sim_{\mathbb{Q}} f^*L$. By generic smoothness [**Har77**, Corollary 10.7] and adjunction formula the generic fiber X_t is a smooth variety with trivial canonical bundle. Suppose that f is non-isotrivial, *i.e.* two generic fibers X_t and X_s are not isomorphic. Since we already studied the case of genus one fibration we can suppose in this section $\dim(X_t) = d \geq 2$.

In the smooth fibers it can be very difficult to find rational curves: if $\tilde{q}(X_t) = d$ it never contains rational curves. Indeed by Beauville–Bogomolov decomposition there exists a finite étale cover $A \to X_t$ that is an abelian variety, hence the universal cover of X_t is isomorphic to $\mathbb{A}^d$. Since $\mathbb{P}^1$ is simply connected, a rational curve $\mathbb{P}^1 \to X_t$ lifts to a map $\mathbb{P}^1 \to \mathbb{A}^d$ that must be constant. If $\tilde{q}(X_t) < d$ it is conjectured that the answer is positive, but certainly non trivial. Indeed using the same trick one can proves that this is equivalent to the existence of rational curves in Calabi–Yau or Hyperkähler projective manifolds.

16.7. **Remark.** By the construction of the Satake compactification of the moduli space of abelian varieties, one can find complete non-isotrivial families of projective abelian varieties. This phenomenon does not appear for $d = 1$ because to compactify the moduli space of elliptic curves $\mathbb{A}^1$ one need to add a codimension one object to get $\mathbb{P}^1$. In higher dimension one can compactify $\mathcal{A}_d$ with subsets of greater codimension, so one can find subvarieties of $\overline{\mathcal{A}_d}$ contained in $\mathcal{A}_d$ that gives smooth non-isotrivial families of abelian varieties. In this case one can use the hyperbolicity of the base for families with maximal variation to study the problem. In particular this gives a lot of examples with no rational curves at all.

However in some sense one can find conditions on X such that the fibration f must have some singular fibers. For example if the general fiber is an abelian variety and f is smooth in codimension one then the fundamental group of X cannot be finite.

16.8. **Lemma.** *Let $X \xrightarrow{f} B$ be an equidimensional relatively minimal Calabi–Yau fiber space. Assume the generic fiber is an abelian variety. Assume moreover that X is smooth and its fundamental group of X is finite. Then there exists some codimension one component of the singular values* $\mathrm{Sv}(f)$ *in B.*

PROOF. Suppose by contradiction $\mathrm{cod}_B(\mathrm{Sv}(f)) \geq 2$. Then an easy adaptation of the proof of [**DFM19**, Lemma 2.3] works also in this context. $\square$

With this technique one can focus on the singular fibers that appear in codimension one. The main tool to control the central fiber is the following result of Takayama.

16.9. **Theorem.** *[**Tak08**] Let $X \xrightarrow{f} C$ be a projective surjective morphism with connected fibers, from a normal variety X with canonical singularities to a smooth curve with a fixed point $0 \in C$. Let $X_0 = \sum m_i F_i$ be the irreducible decomposition of the central fiber as a Weil divisor. Assume that a general fiber X_t has numerically trivial canonical divisor. Then*

- *Every component F_i is either uniruled or the Kodaira dimension $k(F_i)$ is zero. In addition there exists at most one component F_i with $k(F_i) = 0$.*
- *Assume the central fiber F is irreducible (possibly non reduced). If F is non-normal or contains a codimension 2 singular locus of X, then it is uniruled.*
- *Assume the central fiber F is irreducible (possibly non reduced), but it is normal and does not contain any codimension 2 singular locus of X. In this case its canonical bundle is torsion $K_F \sim_{\mathbb{Q}} 0$ and in particular it is uniruled if and only if F has non canonical singularities.*

This is not the complete original statement of Takayama but only the part that is useful for our aims. This is a beautiful theorem but it is not sufficient because if all the singular fibers have mild singularities we can expect that they contain no rational curves.

In the case the general fiber is a Calabi–Yau the situation is not clear, but there is an interesting example in the case of abelian fibrations: Pirola proved in [**P⁺89**, Theorem 2] that the Kummer associated to a very general principally polarized abelian variety of dimension $q \geq 3$ does not contain curves of geometric genus at most $q-3$. This leads us to construct examples of abelian fibration with some singular fibers but no rational curves, see Example 17.9. Unfortunately we are able only to construct generically isotrivial examples.

17. Examples

17.1. **Example.** The behaviour of the augmented irregularity for smooth curves is easy to describe using Riemann–Hurwitz formula. The augmented irregularity of a genus zero curve is zero. Indeed $\mathbb{P}^1$ is simply connected. Any finite étale cover of a genus one curve is again a genus one curve by Riemann–Hurwitz formula, so $\tilde{q}(C) = 1$. A curve C with $g(C) \geq 2$ has a cover of degree d from a curve C' of genus $g(C') = d \cdot (g(C) - 1) + 1$. Indeed its fundamental group is the free group generated by $2g$ elements, that has subgroups of index d arbitrary large. This subgroup corresponds to an étale cover $\tilde{C}$ of degree d, whose genus is given by Riemann–Hurwitz formula and equals $g(\tilde{C}) = d(g-1)+1$. So we can find an étale cover of C with arbitrary large irregularity. Hence $\tilde{q}(C) = \infty$.

17.2. **Example.** Fix two integer numbers $r \geq 1$ and $d \geq 2$. Consider a smooth hypersurface $X_{3,r} \subset \mathbb{P}^2 \times \mathbb{P}^d$ given by the zero locus of a bihomogeneous polynomial of bedegree $(3, r)$. Consider the natural projection $\pi : X_{3,r} \to \mathbb{P}^d$. The augmented irregularity of $X_{3,r}$ is zero because it is simply connected by Lefschetz hyperplane theorem. By

Grothendieck–Lefschetz Theorem the Picard group of $X_{3,r}$ is isomorphic to $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^d)$ and by adjunction formula the canonical bundle is $K_{X_{3,r}} \sim \mathcal{O}_{X_{3,r}}(0, r-d-1)$. In particular $K_{X_{3,r}} \sim \pi^*\mathcal{O}_{\mathbb{P}^d}(r-d-1)$. So we can apply Theorem 10.1: it follows that this kind of family of genus one curves can't be everywhere smooth but it degenerates over a divisor of the base in (singular) rational curves.

17.3. **Example.** Let E be an elliptic curve and denote by $X := (E \times E)/\pm$ the quotient of the product of two copies of E by the involution. This surface comes with a natural genus one fibration $X \to \mathbb{P}^1$ with four singular fibers sitting above the branch points of $E \to \mathbb{P}^1$ and each singular fibre consists of a rational curves with multiplicity two. This is an example of an orbibundle such that its quotient map is not étale in codimension two, and indeed has some rational curves, that are uniruled divisors (see Lemma 3.24).

The quotient map $E \times E \to X$ is a quasi-étale cover so $\tilde{q}(X) \geq 2$, moreover by [**Dru18**, Remark 4.3] it holds the equality. However a minimal resolution $\tilde{X}$ of X is a K3 surface. In particular $\tilde{X}$ is simply connected, hence $\tilde{q}(\tilde{X}) = 0$. The variety X is thus also an example of a variety with zero irregularity but non trivial augmented irregularity.

17.4. **Example.** Let E be an elliptic curve. The quotient $(E \times E)/\pm$ under the diagonal action with a projection to a factor gives the easiest example of orbibundle

$$Y := (E \times E)/\pm \xrightarrow{f} E/\pm = \mathbb{P}^1.$$

A minimal resolution S of Y is a smooth K3 surface with an induced generically isotrivial genus one fibration over $\mathbb{P}^1$. It is obtained by blowing up the 16 singular points of Y, and in particular it is simply connected and it is not an orbibundle. This construction has a natural generalization for Hyperkähler manifolds of dimension $n > 2$. Let S be a K3 surface with a generically isotrivial fibration as before. The Hilbert scheme of lenght $d = n/2$, 0-dimensional subschemes of S is an Hyperkäler manifold that comes with a generically isotrivial morphism to $\mathbb{P}^d$. Indeed a natural way to construct $\mathrm{Hilb}^d(S)$ is as resolution of $\mathrm{Sym}^d S$:

$$Y := \mathrm{Hilb}^d S \xrightarrow{HC} \mathrm{Sym}^d S \to \mathrm{Sym}^d \mathbb{P}^1 \simeq \mathbb{P}^d$$

where HC is the Hilbert-Chow morphism that is essentially the blow-up of the diagonals. Outside the diagonal the fibers of the composition $Y \xrightarrow{f} \mathbb{P}^d$ is by construction $E \times \cdots \times E$ d times. By the results of Matsushita [**Mat99**], [**Mat00**, Corollary 2] f is an equidimensional lagrangian fibration. In particular where f is non-isotrivial there are some insufficient fiber type divisors obtained by blowing-up the singular locus. These divisors are not exceptional for f but are exceptional for the birational map to the orbibundle. A divisor contracted in the orbibundle must be of insufficient fiber type and this is one way to fix the second statement of [**Kol15**, Theorem 44].

17.5. **Example.** Let B be a cone over a genus one curve and E any elliptic curve. For an explicit example take $B := \{x_0^3 + x_1^3 + x_2^3\} \subset \mathbb{P}^3$. Consider the product $X := B \times E$ and its minimal resolution $\tilde{X}$. The variety X has a trivial genus one fibration onto B that gives a trivial orbibundle structure and, composing with the resolution of singularities, also $\tilde{X}$ has a generically isotrivial genus one fibration with an exceptional divisor. There are no vertical rational curves in $\tilde{X}$ because the exceptional locus is isomorphic to the product of two elliptic curve. This is because X has log-canonical but non divisorially log-terminal singularities. Indeed the exceptional locus of a minimal resolution of B is a genus one curve. In this case $\tilde{X}$ is not isomorphic in codimension one to an orbibundle over B, but it is itself a trivial orbibundle over a minimal resolution $\tilde{B}$ of B.

17.6. **Example.** Let V be a smooth projective variety of dimension $k \geq 3$ with finite, but non trivial, fundamental group. An easy way to obtain such variety is considering the quotient V of the Fermat $Y = \{y_0^d + \cdots + y_{k+1}^d\} \subset \mathbb{P}^{k+1}$ by a cyclic group of order $2q$ for some divisor of d acting by multiplication on the coordinates for the $2q$-roots of the unity. In order to make this action free we need to assume $k < 2q$. In this way V is smooth, its fundamental group is cyclic of order $2q$ and can be embedded in $\mathbb{P}^N$ for some N. Consider in this projective space $\mathbb{P}^N$ a generic hypersurface Z of degree $d' \geq 2N - 1$. By [**Cle86**, Theorem 1.1] there are no rational curves in Z. We denote by $B := Z \cap V$ their intersection. By Lefschetz Hyperplane Theorem [**Laz04a**, Theorem 3.1.19] the fundamental group of B is isomorphic to the fundamental group of V. Let $\tilde{B}$ be the universal cover of B.

Consider an elliptic curve E and an automorphism η of order $2q$ that does not fix ω_E the generator of Ω^1_E. For example one can consider $E = \mathbb{C}/(\mathbb{Z} \oplus \tau \cdot \mathbb{Z})$ and the automorphism $\eta(z) = -z + \tau/q$. Consider the diagonal action of $G := \mathbb{Z}/(2q)$ induced on the product $Y := \tilde{B} \times E$. The quotient of Y under this action gives a genus one fibration $X := Y/G \xrightarrow{\pi} B$. The quotient map $Y \to X$ is globally étale because the map is globally étale on the first component. Indeed the action of G is free on Y and hence it is free on the product. The holomorphic one forms on X are exactly the holomorphic one forms on Y that are invariant under the action of G. Since $\tilde{B}$ is simply connected a one form ω on Y is exactly a one form on E. By the choice of the automorphism on E we see that a generator of G acts as minus one on the holomorphic one forms of Y, hence $q(X) = 0$. Since all the vertical curves have genus one and an horizontal rational curve gives a rational curve on B, there are no rational curves on X. So this is an example of a variety with admits a genus one fibration but has zero irregularity and contains no rational curves. The augmented irregularity is one, so this does not contradict Theorem 10.1.

17.7. **Example.** If $S \to \mathbb{P}^1$ is an isotrivial elliptic K3 surface whose general fiber is isomorphic to E, than $S^{[n]}$ has a natural lagrangian fibration onto $\operatorname{Sym}^n \mathbb{P}^1 \simeq \mathbb{P}^n$ whose general fiber is isomorfic to E^n. In this case there are clearly many rational curve in $S^{[n]}$ beacause of the exceptional divisor of the Hilbert–Chow morphism. With our language the exceptional divisors are of insufficient fiber type for the lagrangian fibration $S^{[n]} \to \mathbb{P}^n$.

17.8. **Example.** Let E and F two complex tori and let $A = E \times F$. Then the generalized Kummer variety $K_n(A)$ admits an isotrivial lagrangian fibration [**Saw14**, Lemma 10] and hence contains rational curves by Theorem 16.3.

17.9. **Example.** Let A be an abelian variety such that $A/\pm$ has no rational curves. By [**P$^+$89**, Theorem 2] a very general polarized abelian variety of dimension at least 3 satisfies this condition. Consider a genus two curve $\tilde{B}$ with an automorphism of order two which is free outside two points. Consider the associated orbibundle $(A \times \tilde{B})/\pm \to B = E$. Outside the points of ramification the fibers are isomorphic to A and the two singular fibers are $A/\pm$ that has no rational curves.

In particular consider the quotient $X := (A \times E)/G$ where $G = \mathbb{Z}/(2)$ and the action induced on E is free. It has a generically isotrivial fibration over E/G whose generic fiber is an abelian variety and has some singular fiber. However X has no rational curves at all. Indeed no rational curves are contained in any fiber, and the base is a genus one curve.

Part 3

Twisted cotangent bundles of Hyperkähler manifolds (joint work with A. Höring)

18. Notation and basic facts

We work over $\mathbb{C}$, for general definitions we refer to [**Har77**, **Dem12**]. Manifolds and normal complex spaces will always be supposed to be irreducible. We will not distinguish between an effective divisor and its first Chern class.

A (not necessarily projective) Hyperkähler manifold is a simply connected compact Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by a symplectic form σ, *i.e.* an everywhere non-degenerate holomorphic two form. The existence of the symplectic form σ implies that $\dim X$ is even, so we will write $\dim(X) = 2n$. The symplectic form defines an isomorphism $T_X \to \Omega_X$, so the odd Chern classes of X vanish.

The second cohomology group with integer coefficients $H^2(X, \mathbb{Z})$ is a lattice for the Beauville-Bogomolov quadratic form $q = q_X$ [**Bea96**, Sect.8]. Somewhat abusively we denote by $q(\cdot, \cdot)$ the associated bilinear form. If X is projective it follows from the Bochner principle that all the symmetric powers $S^l\Omega_X$ are slope stable with respect to any polarization H on X [**Kob80**, Thm.6].

We will frequently use basic facts about the deformation theory of Hyperkähler manifolds, as explained in [**Bea96**, Sect.8] [**Huy99**, Sect.1]. In particular we use that a very general point of the deformation space corresponds to a non-projective manifold, but the projective manifolds form a countable union of codimension one subvarieties that are dense in the deformation space. A very general deformation of X is a manifold X_t which corresponds to a very general point t in the Kuranishi space of X.

The Picard group $\mathrm{Pic}(X)$ is by definition the group of isomorphism classes of line bundles on X. Since $H^1(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathbb{Z})$ is torsion-free, the Lefschetz $(1,1)$-theorem [**Huy05**, Prop.3.3.2] gives an isomorphism

$$H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R}) \simeq \mathrm{Pic}(X).$$

18.1. **Remark.** By Hodge theory a class $\alpha \in H^2(X, \mathbb{Z})$ is of type $(1,1)$ if an only if it is orthogonal to the symplectic form σ_X. If σ_X is not orthogonal to any non zero element of the lattice $H^2(X, \mathbb{Z})$ then there are no integral cohomology classes of type $(1,1)$ in X. For any $0 \neq \alpha \in H^2(X, \mathbb{Z})$ the orthogonal $\alpha^\perp \subset H^2(X, \mathbb{C})$ is a proper hyperplane because the Beauville form q is non degenerate. By local Torelli Theorem [**Bea96**, Théorème 5] the moduli space of the deformations of X is locally an open inside the quadric $\{q(\beta) = 0\} \subset \mathbb{P}(H^2(X, \mathbb{C}))$. So a very general Hyperkähler manifold can be taken outside all the hyperplanes $\alpha^\perp$ such that $0 \neq \alpha \in H^2(X, \mathbb{Z})$, hence has trivial Picard group.

18.2. **Remark.** For any very general Hyperkähler X we have by [**Huy03a**, Cor.1]

$$\mathcal{E}^0(X) = \mathcal{K}(X) = \mathcal{C}(X)$$

where $\mathcal{C}(X)$ is the connected component of $\{\alpha \in H^{1,1}(X,\mathbb{R}) \mid q(\alpha) > 0\}$ that contains $\mathcal{K}(X)$. In particular the classes in the boundary of the Kähler cone cannot be in the interior of the pseudoeffective cone because they have trivial top self intersection. Thus a big class, being in the interior of $\mathcal{E}(X)$, is in fact Kähler.

Finally let us recall that for a vector bundle of rank r over a compact Kähler manifold M, the k-th Segre class is defined as $\pi_*\zeta^{r+k} = (-1)^k s_k(V)$, where $\pi : \mathbb{P}(V) \to M$ is the projectivisation and ζ the tautological class.

19. The projectivised cotangent bundle

Let X be a compact Kähler manifold, and let $V \to X$ be a vector bundle over X. Denote by $\zeta := c_1(\mathcal{O}_V(1))$ the tautological class on $\mathbb{P}(V)$ and by $\pi : \mathbb{P}(V) \to X$ the projection. By [**Kob87**, Chapter 2] the cohomology ring with integral coefficients is

$$H^\bullet(\mathbb{P}(V),\mathbb{Z}) = H^\bullet(X,\mathbb{Z})[\zeta]/p(\zeta)$$

where $p(\zeta) = \zeta^n + \zeta^{n-1}\pi^*c_1(V) + \ldots + \pi^*c_n(V)$.

Passing to complex coefficients we get that any class $\alpha \in H^{2k}(\mathbb{P}(V),\mathbb{C})$ can be uniquely written as

$$\alpha = \sum_{p=0}^{k} \zeta^p \cdot \pi^*\beta_{2k-2p}$$

where $\beta_{2k-2p} \in H^{2k-2p}(X,\mathbb{C})$.

Since X is Kähler we can consider the Hodge decomposition of $H^{2k}(\mathbb{P}(V),\mathbb{C})$ and obtain a decomposition

$$H^{k,k}(\mathbb{P}(V)) = \bigoplus_{i=0}^{k} \mathbb{C}\zeta^{k-i} \otimes \pi^* H^{i,i}(X).$$

Using the canonical inclusion $H^{k,k}(\mathbb{P}(V)) \subset H^{2k}(\mathbb{P}(V),\mathbb{C})$ we can compare the two decompositions and obtain

$$(1) \quad H^{k,k}(\mathbb{P}(V)) \cap H^{2k}(\mathbb{P}(V),\mathbb{Z}) = \bigoplus_{i=0}^{k} \mathbb{Z}\zeta^{k-i} \otimes \pi^*(H^{i,i}(X) \cap H^{2i}(X,\mathbb{Z})).$$

In particular the cohomology class of a codimension k subvariety Z of $\mathbb{P}(V)$ can be uniquely written as

$$(2) \qquad [Z] = \beta_0\zeta^k + \zeta^{k-1} \cdot \pi^*\beta_1 + \zeta^{k-2} \cdot \pi^*\beta_2 + \ldots + \pi^*\beta_k$$

where $\beta_i \in H^{i,i}(X) \cap H^{2i}(X,\mathbb{Z})$ and $\beta_0 \in \mathbb{Z}$.

In this section we will first use this decomposition to establish Theorem 0.10, see Subsection 19.A. Then we will prove an additional restriction on the component β_1 that allows us to describe the varieties $Z \subset \mathbb{P}(\Omega_X)$ in some cases, see Subsection 19.B.

19.A. Proof of the main result. It is well-known that the cohomology ring of a very general Hyperkähler manifold X is governed by its Beauville-Bogomolov form. We start by showing a similar property for the cohomology ring of $\mathbb{P}(\Omega_X)$:

19.1. **Lemma.** *Let X be a Hyperkähler manifold of dimension $2n$, and denote by $q(\cdot)$ its Beauville-Bogomolov form. Let*

$$\Theta \in H^{k,k}(\mathbb{P}(\Omega_X)) \cap H^{2k}(\mathbb{P}(\Omega_X), \mathbb{Z})$$

be an integral class of type (k,k). Suppose that the class Θ is of type (k,k) for every small deformation of X. Then there exists a polynomial $p_\Theta(t) \in \mathbb{Q}[t]$ such that for any $(1,1)$-class ω on X, one has

$$(\zeta + \pi^*\omega)^{4n-1-k} \cdot \Theta = p_\Theta(q(\omega)).$$

Proof. Observe first that both sides of the equation are polynomial functions on $H^{1,1}(X)$. In particular they are determined by their values on an open set and we can assume without loss of generality that ω is Kähler. Let

$$\Theta = \sum_{i=0}^{k} \zeta^{k-i} \pi^* \beta_i \tag{3}$$

be the decomposition of Θ according to (1) where $\beta_i \in H^{i,i}(X) \cap H^{2i}(X, \mathbb{Z})$. By our assumption, for any small deformation $\mathfrak{X} \to \Delta$, the class Θ deforms as an integral class Θ_t of type (k,k). Thus we can write

$$\Theta_t = \sum_{i=0}^{k} \zeta_t^{k-i} \pi^* \beta_{i,t}$$

with $\beta_{i,t} \in H^{i,i}(X_t) \cap H^{2i}(X_t, \mathbb{Z})$. Since the family $\mathbb{P}(\mathfrak{X}) \to \Delta$ is locally trivial in the differentiable category, we can consider the classes β_i as elements of $H^{2i}(X_t, \mathbb{Z})$ for $t \neq 0$. The integral cohomology class $\Theta_t \in H^{2k}(\mathbb{P}(\Omega_{\mathfrak{X}_t}, \mathbb{Z})$ does not depend on t, so (3) induces a decomposition

$$\Theta_t = \sum_{i=0}^{k} \zeta_t^{k-i} \pi^* \beta_i.$$

By uniqueness of the decomposition we have $\beta_i = \beta_{i,t}$, in particular the classes β_i are of type (i,i) in $\mathfrak{X}_t$.

We have

$$(\zeta + \pi^*\omega)^{4n-1-k} = \sum_{j=0}^{4n-1-k} \binom{4n-1-k}{j} \zeta^{4n-1-k-j} \pi^* \omega^j,$$

so

$$(\zeta+\pi^*\omega)^{4n-1-k}\cdot\Theta=\sum_{j=0}^{4n-1-k}\binom{4n-1-k}{j}\sum_{i=0}^{k}\zeta^{4n-1-j-i}\pi^*(\beta_i\cdot\omega^j).$$

By the projection formula and the definition of Segre classes one has for $i+j\leq 2n$

$$\zeta^{4n-1-j-i}\pi^*(\beta_i\cdot\omega^j)=(-1)^{i+j}s_{2n-j-i}\cdot\beta_i\cdot\omega^j.$$

Since the odd Segre classes of a Hyperkähler manifold vanish, we can implicitly assume that $i+j$ is even. In particular $(-1)^{i+j}=1$. We claim that we can also assume that j is even.

Proof of the claim. Note that $f(\omega):=s_{2n-j-i}\cdot\beta_i\cdot\omega^j$ defines a polynomial on $H^{1,1}(X)$. Thus, up to replacing ω by a general Kähler class, we can assume that $s_{2n-j-i}\cdot\beta_i\cdot\omega^j=0$ if and only if $s_{2n-j-i}\cdot\beta_i\cdot(\omega')^j=0$ for every $(1,1)$-class ω'. As we have already observed at the start of the proof, we can make this generality assumption without loss of generality. If $s_{2n-j-i}\cdot\beta_i\cdot\omega^j=0$, the term is irrelevant for our computation. If $s_{2n-j-i}\cdot\beta_i\cdot\omega^j\neq 0$, then by [**Ver96**, Thm.2.1] the degree of the cohomology class $s_{2n-j-i}\cdot\beta_i$ is divisible by 4 (here we use that ω is a Kähler class). Since $s_{2n-j-i}\cdot\beta_i\in H^{4n-2j}(X,\mathbb{R})$, the claim follows.

Thus we obtain

$$(\zeta+\pi^*\omega)^{4n-1-k}\cdot\Theta=\sum_{j=0}^{4n-1-k}\binom{4n-1-k}{j}\sum_{i=0}^{k}s_{2n-j-i}\cdot\beta_i\cdot\omega^j.$$

We have shown above that the classes $s_{2n-j-i}\cdot\beta_i$ are of type $(2n-j,2n-j)$ on all small deformations of X. Since j is even, we know by [**Huy97**, Theorem 5.12] that there exist constants $d_{i,j}\in\mathbb{Q}$ such that for any $\delta\in H^{1,1}(X,\mathbb{R})$ we have

$$s_{2n-j-i}\cdot\beta_i\cdot\delta^j=d_{i,j}q(\delta)^{j/2}$$

The polynomial

$$p_\Theta(t):=\sum_{j=0}^{4n-1-k}\binom{4n-1-k}{j}\sum_{i=0}^{k}d_{i,j}t^{j/2}$$

has the claimed property. □

Proof of Theorem 0.10. Suppose first that X is very general in its deformation space. Let $Z\subset\mathbb{P}(\Omega_X)$ be a subvariety. Since X is very general, we know that for any small deformation $\mathfrak{X}\to\Delta$, the variety Z deforms to a variety $Z_t\subset\mathbb{P}(\Omega_{\mathfrak{X}_t})$ (by countability of the parameter space). In particular its cohomology class $[Z]$ is of type (k,k) for every small deformation. Thus Lemma 19.1 applies and there exists a polynomial $p_Z(t)=p_{[Z]}(t)$ such that

$$(\zeta+\pi^*\omega)^{4n-1-k}\cdot[Z]=p_Z(q(\omega))$$

for any $(1,1)$-class ω on X. Since intersection numbers are invariant under deformation and the cycle space has only countably irreducible components, we obtain a countable number of polynomials $(p_m(t))_{m\in\mathbb{N}}$ such that for every subvariety $Z \subset \mathbb{P}(\Omega_X)$ there exists a polynomial p_m such that

$$(\zeta + \pi^*\omega)^{4n-1-k} \cdot [Z] = p_m(q(\omega)).$$

Denote by c_m the largest real root of the polynomial p_m. We claim that

$$\sup_{m\in\mathbb{N}}\{c_m\} < \infty.$$

Indeed fix a Kähler class η on X such that $\zeta + \pi^*\eta$ is a Kähler class on $\mathbb{P}(\Omega_X)$. Then $\zeta + \lambda\pi^*\eta$ is a Kähler class for all $\lambda \geq 1$, so

$$p_m(\lambda^2 q(\eta)) = (\zeta + \lambda\pi^*\eta)^{4n-1-k} \cdot [Z] > 0$$

for all $\lambda \geq 1$. In particular $c_m \leq q(\eta)$, and hence $\sup_{m\in\mathbb{N}}\{c_m\} \leq q(\eta)$. This shows the claim and we denote the real number $\sup_{m\in\mathbb{N}}\{c_m\}$ by C.

Proof of the second statement. Since X is very general, we know by Remark 18.2 that the nef and big class ω_X is Kähler. If $q(\omega_X) > C$ then by construction of the constant C one has

$$(\zeta + \lambda\pi^*\omega_X)^{4n-1-k} \cdot [Z] = p_m(\lambda^2 q(\omega_X)) > 0$$

for every subvariety Z. By Lemma 9.3 this implies that $\zeta + \pi^*\omega_X$ is Kähler. If $q(\omega_X) \geq C$ then $q((1+\varepsilon)\omega_X) > C$, so $\zeta + (1+\varepsilon)\pi^*\omega_X$ is Kähler. Thus $\zeta + \pi^*\omega_X$ is nef.

Vice versa suppose that $\zeta + \pi^*\omega_X$ is nef. Then $\zeta + \lambda\pi^*\omega_X$ is nef for all $\lambda \geq 1$. Thus

$$p_m(\lambda^2 q(\omega_X)) = (\zeta + \lambda\pi^*\omega_X)^{4n-1-k} \cdot [Z] \geq 0$$

for all $\lambda \geq 1$. Since $\lim_{\lambda\to\infty} \lambda^2 q(\omega_X) = \infty$, this implies $c_m \leq q(\omega_X)$ for all $m \in \mathbb{N}$. Hence we obtain $q(\omega_X) \geq C$.

Proof of the first statement. We claim that we can assume that ω_X is a Kähler class with $q(\omega_X) > C$. Indeed let δ be any Kähler class on X, then $\omega_X + \delta$ is Kähler. Moreover one has

$$q(\omega_X + \delta) = q(\omega_X) + q(\delta) + 2q(\delta, \omega) > q(\omega_X) \geq C$$

Thus if $\zeta + \pi^*(\omega_X + \delta)$ is pseudoeffective for every δ, then the closedness of the pseudoeffective cone implies the statement by taking the limit $\delta \to 0$. This shows the claim.

We denote by $0 \in \mathrm{Def}(X)$ the point corresponding to X in its Kuranishi family. By [**Huy16**, Proposition 5.6] we can assume that in a neighborhood U of $0 \in \mathrm{Def}(X)$ the Kähler class ω_X deforms as a Kähler class $(\omega_{X_t})_{t\in U}$. In order to simplify the notation we replace U with a very general disc Δ centered at 0 and consider the family $\mathcal{X} \to \Delta$. Since the Beauville–Bogomolov form is continuous we have, up to replacing Δ by a smaller disc, that $q(\omega_{X_t}) > C$ for every $t \in \Delta$. By the second statement this implies that for $t \in \Delta$ very general the class $\zeta_t + \pi_t^* \omega_{X_t}$ is nef, in particular it is pseudoeffective. Now we apply Theorem 9.4 to the family $\mathbb{P}(\Omega_{\mathfrak{X}}) \to \Delta$ and the classes $\zeta_t + \pi^* \omega_{X_t}$: this shows that $\zeta + \pi^* \omega_X$ is pseudoeffective. □

Proof of Corollary 0.11. By [**Huy03b**, Theorem 2.1] there exist at most finitely many different deformation families of irreducible holomorphic symplectic complex structures on X_0. For any such deformation type, Theorem 0.10 gives a constant C_k such that $\zeta + \pi^* \omega_X$ is pseudoeffective for every Kähler class ω_X such that $q(\omega_X) > C_k$. Let C be the maximum among the constants C_k. Since the differentiable structure on X is fixed, the constant of proportionality between the Beauville–Fujiki form $q(\omega_X)$ and the top intersection ω_X^{2n} is fixed. Thus the polarised Hyperkähler manifolds (X, H) such that $X_0 \overset{\text{diff.}}{\simeq} X$ and $\zeta + \pi^* H$ is not pseudoeffective satisfy $H^{2n} \leq b$ for some constant b. By a theorem of Matsusaka-Mumford [**MM64**] there are for any fixed $0 < i \leq b$ only a finite number of deformation families of polarised Hyperkähler manifolds (X, H) such that $H^{2n} = i$. Thus the cases where $\zeta + \pi^* H$ is not pseudoeffective belong to one of these finitely many families. □

19.B. Subvarieties of the projectivised cotangent bundle. We start with a technical observation:

19.2. **Lemma.** *Let X be a projective Hyperkähler manifold of dimension $2n$. Let Z be an effective cycle on $\mathbb{P}(\Omega_X)$ of codimension $k > 0$ such that $\pi(\operatorname{Supp} Z) = X$. Denote by*

$$[Z] = \beta_0 \zeta^k + \zeta^{k-1} \cdot \pi^* \beta_1 + \zeta^{k-2} \cdot \pi^* \beta_2 + \ldots + \pi^* \beta_k$$

the decomposition (2) of its cohomology class. Then we have $\beta_1 \neq 0$.

Proof. We argue by contradiction and suppose that $\beta_1 = 0$. Let $C \subset X$ be a general complete intersection of sufficiently ample divisors $D_i \in |H|$ so that the Mehta–Ramanathan theorem [**MR84**, Thm.4.3] applies for Ω_X. Then the restriction $\Omega_X|_C$ is stable, and by a result of Balaji and Kollár [**BK08**, Prop.10] its algebraic holonomy group is $\mathrm{Sp}_{2n}(\mathbb{C})$. Thus not only $\Omega_X|_C$, but also all its symmetric powers $S^l \Omega_X|_C$ are stable. Denote by Z_C the restriction of the effective cycle Z to $\mathbb{P}(\Omega_X|_C)$. Since $\pi(\operatorname{Supp} Z) = X$ the effective cycle Z_C is not zero. Then its cohomology class is

$$[Z_C] = (\beta_0 \zeta^k + \zeta^{k-2} \cdot \pi^* \beta_2 + \ldots + \pi^* \beta_k) \cdot \pi^* H^{2n-1} = \beta_0 \zeta_C^k$$

where ζ_C is the restriction of the tautological class. In particular, since $c_1(\Omega_X|_C) = 0$, we have $\zeta_C^{2n-k} \cdot [Z_C] = \beta_0 \zeta_C^{2n} = 0$. Yet this is a contradiction to [**HP19**, Prop.1.3]. □

19.3. **Remark.** Lemma 19.2 also holds if X is a Calabi-Yau manifold (in the sense of [**Bea96**]): the cotangent bundle Ω_X is also stable and the algebraic holonomy is $\mathrm{SL}_{\dim X}(\mathbb{C})$ [**BK08**, Prop.10]. Thus the proof above applies without changes.

In [**COP10**, Cor.2.6] it is shown that a very general Hyperkähler manifold is not *covered* by proper subvarieties. We show an analogue for the projectivised cotangent bundle Ω_X:

19.4. **Lemma.** *Let X be a Hyperkähler manifold of dimension $2n$. Suppose that X is very general in the following sense: we have*

1.) $\mathrm{Pic}(X) = 0$;
2.) *if $\mathfrak{X} \to \Delta$ is a deformation of $X = \mathfrak{X}_0$, then every irreducible component of the cycle space $\mathcal{C}(\mathbb{P}(\Omega_{\mathfrak{X}_0}))$ deforms to $\mathcal{C}(\mathbb{P}(\Omega_{\mathfrak{X}_t}))$ for $t \neq 0$.*

Let $Z \subsetneq \mathbb{P}(\Omega_X)$ be a compact analytic subvariety. Then $\pi(Z) \subsetneq X$.

By countability of the irreducible components of the relative cycle space [**Fuj79**, Thm.] and by Remark 18.1 we know that for a very general choice of X the hypothesis of the lemma are satisfied.

Proof. We argue by contradiction, and suppose that Z is a subvariety of $\mathbb{P}(\Omega_X)$ of codimension $k > 0$ such that $\pi(Z) = X$. Denote by

$$[Z] = \beta_0\zeta^k + \zeta^{k-1} \cdot \pi^*\beta_1 + \zeta^{k-2} \cdot \pi^*\beta_2 + \ldots + \pi^*\beta_k$$

the decomposition (2) of its cohomology class. Since $\mathrm{Pic}(X) = 0$ we know that $\beta_1 = 0$.

Projective Hyperkähler manifolds are dense in the deformation space of any Hyperkähler manifold [**Bea96**, Sect.9] [**Buc08**, Prop.5], so we can consider a small deformation of X

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ 0 \arrow[r] & \Delta \end{tikzcd}$$

such that $\mathfrak{X}_{t_0}$ is projective for some point $t_0 \in \Delta$. This deformation comes naturally with a deformation of the cotangent bundle, so we have a diagram

$$\begin{tikzcd} \mathbb{P}(\Omega_X) \arrow[r] \arrow[d, "\pi"] & \mathbb{P}(\Omega_{\mathfrak{X}/\Delta}) \arrow[d] \\ X \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ 0 \arrow[r] & \Delta \end{tikzcd}$$

By the second assumption the subvariety $Z \subset \mathbb{P}(\Omega_X)$ deforms in a family of subvarieties $Z_t \subset \mathbb{P}(\Omega_{X_t})$ having cohomology class

$$[Z_t] = \beta_0 \zeta^k + \zeta^{k-2} \cdot \pi^* \beta_2 + \ldots + \pi^* \beta_k.$$

Since the cycle space is proper over the base Δ [**Bar75**, Théorème 1] we obtain in particular that the class $\beta_0 \zeta^k + \zeta^{k-2} \cdot \pi^* \beta_2 + \ldots + \pi^* \beta_k$ is effectively represented on $\mathbb{P}(\Omega_{X_{t_0}})$. This contradicts Lemma 19.2. $\square$

19.5. **Corollary.** *Let X be a Hyperkähler manifold of dimension $2n$. Suppose that X is very general in the sense of Lemma 19.4. Suppose also that X contains no proper compact subvarieties. Let $Z \subsetneq \mathbb{P}(\Omega_X)$ be a compact analytic subvariety. Then $\pi(Z)$ is a point.*

Proof. By Lemma 19.4 we have $\pi(Z) \subsetneq X$ for every subvariety $Z \subsetneq \mathbb{P}(\Omega_X)$. By our assumption this implies that $\pi(Z)$ is a point. $\square$

19.6. **Remark.** A very general deformation of Kummer type does *not* satisfy the assumptions of the corollary ([**KV98**, Sect.6.1])

20. The positivity threshold

In view of the results from Subsection 19.B, we will deduce Theorem 0.12 from the main result:

20.1. **Proposition.** *Let X be a Hyperkähler manifold of dimension $2n$. Suppose that a very general deformation of X contains no proper compact subvarieties. Let $p_X(t)$ be the polynomial defined by applying Lemma 19.1 to $[\mathbb{P}(\Omega_X)]$. Then the constant C appearing in Theorem 0.10 is the largest real root of $p_X(t)$.*

Proof. Since C only depends on the deformation family we can assume that X is very general in its deformation space. In the proof of Theorem 0.10 we defined the constant C as $\sup_{m \in \mathbb{N}} \{c_m\}$ where c_m is the largest real root of the polynomials $p_m(t)$, and the family of polynomials $(p_m(t))_{m \in \mathbb{N}}$ is obtained by applying Lemma 19.1 to the classes of *all* the subvarieties $Z \subset \mathbb{P}(\Omega_X)$.

By our assumption and Corollary 19.5 we know that a proper subvariety $Z \subsetneq \mathbb{P}(\Omega_X)$ is contained in a fibre. Thus for any Kähler class ω_X the restriction

$$(\zeta + \pi^* \omega_X)_Z = \zeta|_Z = c_1(\mathcal{O}_{\mathbb{P}^{2n-1}}(1))|_Z$$

is ample. Hence the corresponding polynomial $p_m(t)$ is constant and positive. In particular there is no real root to take into account for the supremum. $\square$

Proof of Theorem 0.12. By Proposition 20.1 the constant C in Theorem 0.10 is the largest real root of the polynomial $p_X(t)$ defined by

$$p_X(q(\omega)) = (\zeta + \pi^*\omega)^{4n-1}.$$

Thus the condition $q(\omega_X) \geq C$ is equivalent to

$$(\zeta + \lambda\pi^*\omega)^{4n-1} > 0$$

for all $\lambda > 1$. Conclude with Theorem 0.10. □

We have already observed that for a very general Hyperkähler manifold the pseudoeffective cone and the nef cone coincide. This also holds for the projectivised cotangent bundle:

20.2. **Proposition.** *Let X be a Hyperkähler manifold of dimension $2n$. Suppose that X is very general in the sense of Lemma 19.4. Suppose also that X contains no proper compact subvarieties.*

Let $C \geq 0$ be the constant from Theorem 0.10. Then we have

$$\mathcal{E}(\mathbb{P}(\Omega^1_X)) = \{a\zeta + \pi^*\delta |\ a \geq 0, \delta \in \overline{\mathcal{K}(X)},\ q(\delta) \geq a^2 C\} \tag{4}$$

and

$$\mathcal{E}(\mathbb{P}(\Omega^1_X)) = \overline{\mathcal{K}(\mathbb{P}(\Omega^1_X))}.$$

Proof. We start proving the last statement. We recall the definition of the *Null cone* of $\mathbb{P}(\Omega^1_X)$ that is the following set

$$\mathcal{N} := \{x \in H^{1,1}(\mathbb{P}(\Omega^1_X), \mathbb{R}) \mid \int_{\mathbb{P}(\Omega^1_X)} x^{2n-1} = 0\}.$$

For any class $\gamma \in \partial\mathcal{K}(\mathbb{P}(\Omega^1_X))$ there exists a subvariety V of $\mathbb{P}(\Omega^1_X)$ such that $\int_V \gamma^{\dim(V)} = 0$. Since we are assuming that there are no proper subvarieties in X, by Lemma 19.4 we know that the proper subvarieties of $\mathbb{P}(\Omega^1_X)$ are contracted to points in X. Since $\mathbb{P}(\Omega^1_X)$ is a projective bundle the integral along a contracted subvariety V has the following property

$$\int_V (a\zeta + \pi^*\delta)^{\dim(V)} = 0 \Leftrightarrow a = 0.$$

This implies using [**DP04**, Theorem 0.1] that

$$\partial\mathcal{K}(\mathbb{P}(\Omega^1_X)) \subseteq \mathcal{N} \cup \{a = 0\}.$$

A $(1,1)$ form in the hyperplane $\{a = 0\}$ is in the null cone. This tells that the Kähler cone is one of the connected component of $H^{1,1}(\mathbb{P}(\Omega^1_X), \mathbb{R}) \setminus \mathcal{N}$. Hence the classes in the boundary of the Kähler cone are nef classes with trivial self intersection, so they are also in the boundary of the pseudoeffective cone [**DP04**, Thm.0.5]. This proves that the closure of the Kähler cone is the pseudoeffective cone.

For notation's convenience we call $\mathcal{A} := \{a\zeta + \pi^*\delta |\ a \geq 0, \delta \in \overline{\mathcal{K}(X)},\ q(\delta) \geq a^2 C\}$. The inclusion $\mathcal{E}(\mathbb{P}(\Omega^1_X)) \supseteq \mathcal{A}$ follows from the first statement of Theorem 0.10. To prove the other inclusion we argue as follows. The points of $\partial\mathcal{A}$ are contained in the set $\{a = 0 \vee q(\delta) = a^2 C\}$. By definition of the constant C the self intersection of the classes $a\zeta + \pi^*\delta$ vanishes. We also have $(\pi^*\delta)^{2n-1} = 0$, hence

$$\partial\mathcal{A} \subset \mathcal{N}.$$

Moreover there are no points in the interior of $\mathcal{A}$ contained in the null cone, so $\mathcal{A}^\circ$ must be a connected component of $H^{1,1}(\mathbb{P}(\Omega^1_X), \mathbb{R}) \setminus \mathcal{N}$. Since the intersection of $\mathcal{E}(\mathbb{P}(\Omega^1_X))$ and $\mathcal{A}$ is non-empty and both are closed convex cones the conclusion follows. □

20.3. **Remark.** The rest of the paper is devoted to giving more explicit expressions of the conditions in Theorem 0.10 and Theorem 0.12, so for clarity's sake let us write down the polynomial $p_X(t)$ from Proposition 20.1: let X be a Hyperkähler manifold of dimension $2n$, and denote by ζ the tautological class of $\pi : \mathbb{P}(\Omega_X) \to X$. Recall that by definition of the Segre classes we have $\pi_*\zeta^{2n+i} = (-1)^i s_i(X)$. Since the odd Chern classes of a Hyperkähler manifold are trivial, the odd Segre classes vanish. Note also that $(\pi^*\omega_X)^i = 0$ if $i > 2n$. The top self-intersection is thus

$$\begin{aligned} p_X(\lambda q(\omega_X)) = (\zeta + \lambda\pi^*\omega_X)^{4n-1} &= \sum_{i=0}^{2n} \binom{4n-1}{i} \zeta^{4n-1-i} \cdot \pi^*\omega_X^i \lambda^i \\ &= \zeta^{2n-1} \sum_{i=0}^{n} \binom{4n-1}{2i} \zeta^{2n-2i} \cdot \pi^*\omega_X^{2i} \lambda^{2i} \\ &= \sum_{i=0}^{n} \binom{4n-1}{2i} s_{2n-2i}(X) \cdot \omega_X^{2i} \lambda^{2i}. \end{aligned} \tag{5}$$

Recall also that by [**Fuj87**, Remark 4.12] there exist constants $d_{2i} \in \mathbb{R}$ that depend only on the family such that

$$s_{2n-2i}(X) \cdot \omega_X^{2i} = d_{2i} q(\omega_X)^i \tag{6}$$

for any $(1,1)$-class ω_X. Note that $s_0(X) \cdot \omega_X^{2n} = \omega_X^{2n} = d_{2n} q(\omega_X)^n$, so $d_{2n} > 0$.

20.4. **Example.** For $n = 1$ we obtain

$$(\zeta + \lambda\pi^*\omega_X)^3 = -c_2(X) + 3\omega_X^2\lambda^2.$$

For $n = 2$ we obtain

$$(\zeta + \lambda\pi^*\omega_X)^7 = (c_2(X)^2 - c_4(X)) - 21c_2(X) \cdot \omega_X^2\lambda^2 + 35\omega_X^4\lambda^4.$$

Proof of Corollary 0.13. By Proposition 20.1 we only have to compute the largest real root of $p_X(t)$. By Formula (5) and Example 20.4 the constant C is the largest root of $-c_2(X) + 3t = 0$. Since $c_2(X) = 24$ the result follows. □

20.5. **Definition.** *Let X be a Hyperkähler manifold of dimension $2n$, and let ω_X be a nef and big class on X. The* positivity threshold *of (X, ω_X) is defined as*

$$\gamma_p(\omega_X) := \inf\{\lambda_0 \in \mathbb{R} \mid (\zeta + \lambda\pi^*\omega_X)^{4n-1} > 0 \qquad \forall \lambda > \lambda_0\}.$$

20.6. **Remark.** Since $(\zeta + \lambda\pi^*\omega_X)^{4n-1} \sim \lambda^{2n}\omega_X^{2n}$ for $t \gg 0$ we have $\gamma_p(\omega_X) < +\infty$. It seems unlikely that $(\zeta+\lambda\pi^*\omega_X)^{4n-1} > 0$ for all $\lambda \in \mathbb{R}$. If (a very general deformation of) X contains no proper subvarieties, this can be seen as follows: since X has no subvarieties, the nef and big class ω_X is Kähler. By Corollary 19.5, the class $\zeta + \lambda\pi^*\omega_X$ satisfies the condition of Lemma 9.3 for any $\lambda \in \mathbb{R}$, so $\zeta + \lambda\pi^*\omega_X$ is Kähler for any $\lambda \in \mathbb{R}$. But $\mathcal{K}(\mathbb{P}(\Omega_X))$ does not contain any lines.

Let X be a Hyperkähler manifold, and let ω_X be a Kähler class on X. We define the *pseudoeffective threshold*

$$\gamma_e(\omega_X) := \inf\{t \in \mathbb{R} \mid \zeta + t\pi^*\omega_X \text{ is big/pseudoeffective}\}$$

and the *nef threshold*

$$\gamma_n(\omega_X) := \inf\{t \in \mathbb{R} \mid \zeta + t\pi^*\omega_X \text{ is Kähler/nef}\}.$$

Since $\zeta + t\pi^*\omega_X$ is Kähler for $t \gg 0$, both thresholds are real numbers.

20.7. **Proposition.** *Let X be a (not necessarily projective) Hyperkähler manifold of dimension $2n$. Suppose that a very general deformation of X does not contain any proper subvarieties. Let ω_X be a Kähler class on X. Then we have*

$$\gamma_e(\omega_X) \leq \gamma_p(\omega_X) \leq \gamma_n(\omega_X).$$

For a very general deformation of X these inequalities are equalities for any Kähler class ω_X.

Proof. The top self-intersection of a Kähler class is certainly positive, so the inequality $\gamma_p(\omega_X) \leq \gamma_n(\omega_X)$ is trivial. The inequality $\gamma_e(\omega_X) \leq \gamma_p(\omega_X)$ follows from Theorem 0.12. For a very general deformation of X we can apply Proposition 20.2, so the nef cone and the pseudoeffective cone coincide. Thus we have $\gamma_e(\omega_X) = \gamma_n(\omega_X)$. □

We will show in Section 21 that for the Hilbert square of a K3 surface the second inequality is strict.

21. Hilbert square of a K3 surface

21.A. Setup. We recall the basic geometry of the Hilbert square, using the notation and results of [**Bea96**, Sect.6]: let S be a (not necessarily algebraic) K3 surface, and let $\rho : \widetilde{S \times S} \to S \times S$ be the blow-up along the diagonal $\Delta \subset S \times S$. We denote the exceptional divisor of this blowup by E. The natural involution on the product $S \times S$ lifts to an involution

$$i_{\widetilde{S\times S}} : \widetilde{S \times S} \to \widetilde{S \times S},$$

and we denote by $\eta : \widetilde{S \times S} \to X$ the ramified two-to-one covering defined by taking the quotient with respect to this involution. It is well-known that X is smooth and Hyperkähler. Finally we denote by $\pi : \mathbb{P}(\Omega_X) \to X$ the natural projection, and by $\zeta \to \mathbb{P}(\Omega_X)$ the tautological divisor.

Recall that X is isomorphic to the Hilbert scheme of length two zero dimensional subschemes $S^{[2]}$, and denote by

$$\varepsilon : S^{[2]} \to S^{(2)}$$

the natural map to the symmetric product. We denote by $E_X \subset X$ the exceptional divisor of this contraction, and observe that $\eta|_E$ induces an isomorphism $E \simeq E_X$. Since ρ is the blowup of the diagonal one has $E \simeq \mathbb{P}(\Omega_S)$, and we denote by

$$\pi_S := \rho|_E \simeq \eta|_{E_X} : \mathbb{P}(\Omega_S) \to S$$

the natural map. Denote by $\zeta_S \to \mathbb{P}(\Omega_S)$ the tautological divisor.

By [**Bea96**, Sect.6, Prop.6] we have a canonical inclusion of groups $i : H^2(S, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{Z})$ inducing a morphism of Hodge structures

$$H^2(X, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta$$

where δ is a primitive class such that $2\delta = E_X$. This decomposition is orthogonal with respect to the Beauville–Bogomolov quadratic form q [**Bea96**, Sect.9, Lemma 1] and one has $q(\delta) = -2$ [**Bea96**, Sect.1, Rque.1]. By construction of the inclusion i [**Bea96**, Sect.6, Prop.6] we have

$$\alpha_X|_{E_X} = 2\pi_S^*\alpha_S, \tag{7}$$

and by [**Bea96**, Sect.9, Rque. 1] one has $q(\alpha_X) = \alpha_S^2$.

Since E is the ramification divisor of the two-to-one cover η, we have $\eta^* E_X = 2E$. Since $E|_E = -\zeta_S$ and $2\delta = E$, we obtain

$$\delta|_{E_X} = -\zeta_S. \tag{8}$$

By [**Bea96**, Sect.9, Lemma 1] we have

$$\alpha^4 = 3q(\alpha)^2 \tag{9}$$

for any $\alpha \in H^{1,1}(X)$. If α_S is any $(1,1)$-class on S, we set $\alpha_X := (i \otimes \mathrm{id}_{\mathbb{C}})(\alpha_S)$.

The second Chern class $c_2(X)$ is a multiple of the Beauville–Bogomolov form. More precisely we have

$$c_2(X) \cdot \alpha^2 = 30q(\alpha) \tag{10}$$

for any $\alpha \in H^{1,1}(X)$ [**Ott15**, Section 3.1].

21.B. Intersection computation on X. Denote by $p_i : S \times S \to S$ the projection on the i-th factor. The composition of p_i with the blow-up ρ defines a submersion

$$p_i \circ \rho : \widetilde{S \times S} \to S,$$

the fibre over a point $x \in S$ being isomorphic to the blow-up of S in x. We denote by F_i a $p_i \circ \rho$-fibre and by $\bar{S} = \eta(F_i)$ its image[1] in X. We will denote by $\bar{S}_x$ the image of the fibre $p_i \circ \rho^{-1}(x) \subset \widetilde{S \times S}$ in X.

The tangent sequence for ρ

$$0 \to \rho^*\Omega_{S\times S} \to \Omega_{\widetilde{S\times S}} \to \mathcal{O}_E(2E) \to 0$$

immediately yields

(11)

$$c_1(\Omega_{\widetilde{S\times S}}) = E, \qquad c_3(\Omega_{\widetilde{S\times S}}) = E^3 + 24(F_1 + F_2) \cdot E,$$
$$c_2(\Omega_{\widetilde{S\times S}}) = 24(F_1 + F_2) - E^2 \quad c_4(\Omega_{\widetilde{S\times S}}) = -E^4 - 24(F_1 + F_2) \cdot E^2 + 576.$$

From tangent sequence for η

$$0 \to \eta^*\Omega_X \to \Omega_{\widetilde{S\times S}} \to \mathcal{O}_E(-E) \to 0$$

one deduces

$$\begin{aligned} &c_1(\eta^*\Omega_X) = 0, && c_3(\eta^*\Omega_X) = 0, \\ &c_2(\eta^*\Omega_X) = 24(F_1 + F_2) - 3E^2, && c_4(\eta^*\Omega_X) = 648. \end{aligned} \tag{12}$$

We can then deduce the Segre and Chern classes of X :

$$\begin{aligned} &s_1(X) = 0 = c_1(X), && s_3(X) = 0 = c_3(X) \\ &s_2(X) = -24\bar{S} + 3\delta^2 = -c_2(X), && s_2(X)^2 = 828 = c_2(X)^2 \\ &s_4(X) = 504, && c_4(X) = 324. \end{aligned} \tag{13}$$

More precisely these formulas follow from (12), the projection formula and the following lemmas.

21.1. **Lemma.** *In the setup of subsection 21.A, one has*

$$\bar{S} \cdot \delta = l$$

where l is the class of a fibre of $\varepsilon|_{E_X} : E_X \to S$. Moreover one has

$$\bar{S} \cdot \delta \cdot \alpha_X = 0, \qquad \bar{S} \cdot \delta^2 = -1, \qquad \bar{S}^2 = 1, \qquad \bar{S} \cdot \alpha_X^2 = \alpha_S^2.$$

Proof. The first statement is equivalent to $\bar{S} \cdot E_X = 2l$. Since $\bar{S} = \eta_* F_1$ and $\eta^* E_X = 2E$ we know by the projection formula that

$$\bar{S} \cdot E_X = \eta_* F_1 \cdot E_X = F_1 \cdot \eta^* E_X = 2F_1 \cdot E.$$

Now recall that F_i is the blow-up of $p \times S$ in the point (p, p). Thus the intersection $F_1 \cdot E$ is the exceptional divisor of the blowup $F_i \to p \times S$.

[1] Note that the involution $i_{\widetilde{S\times S}}$ maps F_1 onto F_2, so $\overline{S}$ is well-defined.

This exceptional $\mathbb{P}^1$ maps isomorphically onto a fibre of $\varepsilon|_{E_X}$. This shows the first statement.

The equalities $\bar{S}\cdot\delta\cdot\alpha_X = 0$, $\bar{S}\cdot\delta^2 = -1$ now follow from (7) and (8). Since $\eta^*\bar{S} = F_1 + F_2$ the projection formula implies

$$\bar{S}^2 = \frac{1}{2}(\eta^*\bar{S})^2 = \frac{1}{2}(F_1+F_2)^2 = F_1\cdot F_2 = 1,$$

where the last equality is due to the fact that the strict transform of $p\times S$ and $S\times q$ intersect exactly in (p,q) if $p\neq q$.

Finally the equality $\bar{S}\cdot\alpha_X^2 = \alpha_S^2$ follows from the construction of α_X [**Bea96**, Sect.6, Prop.6] and observing that if $F_{1,x}$ is the fibre of $p_1\circ\rho$ over $x\in S$, then $\alpha_X|_{\mu(F_{1,x})} = \rho_x^*\alpha_S$ where $\rho_x : F_1\times S$ is the blow-up in x. □

21.2. **Lemma.** *In the setup of subsection 21.A, one has*

$$\alpha_X^4 = 3(\alpha_S^2)^2,\quad \alpha_X^3\cdot\delta = 0,\quad \alpha_X^2\cdot\delta^2 = -2\alpha_S^2,\quad \alpha_X\cdot\delta^3 = 0,\quad \delta^4 = 12$$

Proof. A standard intersection computation based on (9), (7), (8) and $q(\delta) = -2$. □

21.C. Positive threshold. Using the preceding section we can easily compute the positive threshold:

21.3. **Corollary.** *Let X be a four-dimensional Hyperkähler manifold of deformation type $K3^{[2]}$. Let ω_X be a nef and big $(1,1)$-class on X such that*

$$q(\omega_X) \geq 3 + \sqrt{\frac{21}{5}} \sim 5.0493.$$

Then $\zeta + \pi^*\omega_X$ is pseudoeffective. This bound is optimal for a very general deformation of X.

Proof. By Proposition 20.1 we only have to compute the largest real root of $p_X(t)$. By Formula (5) and Example 20.4 we have to compute the largest solution of

$$d_0 + 21d_2t + 35d_4t^2 = 0,$$

where the constants d_{2i} are defined by (6). By (9) and (10) we have

$$c_2(X)\alpha^2 = 30q(\alpha),\quad \alpha^4 = 3q(\alpha)^2$$

for any element $\alpha\in H^{1,1}(X,\mathbb{R})$. By (13) we have $c_4(X) = 324$, $c_2^2 = 828$. Thus we obtain the quadratic equation

$$504 - 630t + 105t^2 = 0.$$

Its largest solution is

$$C = \frac{630 + 42\sqrt{105}}{210} = 3 + \sqrt{\frac{21}{5}}.$$

□

21.4. **Remark.** Let X be a four-dimensional Hyperkähler manifold, not necessarily deformation equivalent to a Hilbert square. In this case the coefficients d_i are not known. However, if a very general deformation of X does not contain any subvarieties, we can use Example 20.4 to show that for a Kähler class ω_X the positivity threshold is

$$\gamma_p(\omega_X) = \sqrt{\frac{21\omega_X^2 c_2 + \sqrt{(21\omega_X^2 c_2)^2 - 140(\omega_X^4)(c_2^2 - c_4)}}{70\omega_X^4}}.$$

21.D. A subvariety of $\mathbb{P}(\Omega_X)$. Denote by $p_i : S \times S \to S$ the projection on the i-th factor. Then $p_i \circ \rho : \widetilde{S \times S} \to S$ is a submersion, the fibre over a point $x \in S$ being isomorphic to the blow-up of S in x. Thus we obtain rank two foliations

$$\operatorname{Ker} T_{p_i \circ \rho} =: \mathcal{F}_i \subset T_{\widetilde{S \times S}}$$

In view of the description of the $\mathcal{F}_i$-leaves it is clear that the natural map $\mathcal{F}_1 \oplus \mathcal{F}_2 \to T_{\widetilde{S \times S}}$ has rank 4 in the complement of the exceptional divisor E, but

$$\mathcal{F}_1|_E \cap T_E = T_{E/S} = \mathcal{F}_2|_E \cap T_E.$$

21.5. **Lemma.** *The composition of the inclusion $\mathcal{F}_i \subset T_{\widetilde{S \times S}}$ with the tangent map $T_{\widetilde{S \times S}} \to \eta^* T_X$ is injective in every point. Thus $\mathcal{F}_i \hookrightarrow \eta^* T_X$ is a rank 2 subbundle.*

Proof. Since T_η is an isomorphism in the complement of E, it is sufficient to study the restriction to E. Note also that

$$T_{\widetilde{S \times S}}|_E \to (\eta^* T_X)|_E$$

has rank three in every point, since $\eta|_E$ induces an isomorphism $E \to E_X$. Arguing by contradiction we assume that there exists a point $x \in E$ such that the map

$$\mathcal{F}_{i,x} \to T_{\widetilde{S \times S},x} \to (\eta^* T_X)_x$$

has rank at most one for some $i \in \{1, 2\}$. Since $\eta \circ i_{\widetilde{S \times S}} = \eta$ this implies that

$$\mathcal{F}_{3-i,x} \to T_{\widetilde{S \times S},x} \to (\eta^* T_X)_x$$

also has rank at most one. Yet $\operatorname{Ker} T_{\eta,x}$ has dimension one, so we obtain

$$\operatorname{Ker} T_{\eta,x} \cap \mathcal{F}_{1,x} = \operatorname{Ker} T_{\eta,x} = \operatorname{Ker} T_{\eta,x} \cap \mathcal{F}_{2,x}.$$

In particular we have

$$\operatorname{Ker} T_{\eta,x} = \mathcal{F}_{1,x} \cap \mathcal{F}_{2,x} = T_{E/S,x}.$$

Yet η induces an isomorphism $E \to E_X$, so $T_{E/S,x} \subset T_{E,x}$ is not in the kernel. $\square$

By Lemma 21.5 we have an injection $\mathcal{F}_i \hookrightarrow \eta^* T_X$. The corresponding quotient $\eta^* T_X \to Q_i$ defines a subvariety $\mathbb{P}(Q_i)$ of $\pi_\eta : \mathbb{P}(\eta^* T_X) \to \widetilde{S \times S}$ that is a $\mathbb{P}^1$-bundle over $\widetilde{S \times S}$. Since $\eta \circ i_{\widetilde{S\times S}} = \eta$ the involution $i^*_{\widetilde{S\times S}}$ acts on $\mathbb{P}(\eta^* T_X)$ and maps $\mathbb{P}(Q_1)$ to $\mathbb{P}(Q_2)$. Thus if we denote by $Z \subset \mathbb{P}(T_X)$ the image of Q_i under the two-to-one cover $\tilde{\eta} : \mathbb{P}(\eta^* T_X) \to \mathbb{P}(T_X)$, we have

$$\tilde{\eta}^*[Z] = [Q_1] + [Q_2].$$

21.6. **Proposition.** *In the situation of Subsection 21.A, denote by $Z \subset \mathbb{P}(T_X) \simeq \mathbb{P}(\Omega_X)$ the subvariety constructed above. Then we have*

$$[Z] = 2\zeta^2 + 2\pi^*\delta \cdot \zeta + \pi^*(24\bar{S} - 6\delta^2). \tag{14}$$

Proof. Consider the exact sequence

$$0 \to \mathcal{F}_i \to T_{\widetilde{S\times S}} \to (p_i \circ \rho)^* T_S \to 0.$$

The Chern classes of $(p_i \circ \rho)^* T_S$ and $T_{\widetilde{S\times S}}$ are known, cf. (11). An elementary computation then yields

$$c_1(\mathcal{F}_i) = -E, \quad c_2(\mathcal{F}_i) = 24F_{3-i} - 3E^2. \tag{15}$$

Denote by ζ_η the tautological bundle on $\mathbb{P}(\eta^* T_X)$. Since $Q_i = \eta^* T_X / \mathcal{F}_i$ we have

$$[Q_i] = \zeta_\eta^2 - \zeta_\eta \cdot \pi_\eta^* c_1(\mathcal{F}_i) + \pi_\eta^* c_2(\mathcal{F}_i) = \zeta_\eta^2 + \zeta_\eta \cdot \pi_\eta^* E + \pi_\eta^*(24F_{3-i} - 3E^2).$$

Since $\tilde{\eta}^*[Z] = [Q_1] + [Q_2]$ and

$$\eta^*\bar{S} = F_1 + F_2, \quad \eta^*\delta = E, \quad \tilde{\eta}^*\zeta = \zeta_\eta$$

the claim follows. $\square$

21.7. **Remark.** The geometry of Z can be understood as follows: on $\widetilde{S \times S}$ we have two distinct families of surfaces $((p_i \circ \rho)^{-1}(x))_{x \in S}$. The images in X of these two families coincide and form a web of surface $(\bar{S}_x)_{x \in S}$. For a point $x \in X$ that is not in E_X there are exactly two members of the web passing through x and they intersect transversally. The projectivisation of their normal bundle defines a projective line in $\mathbb{P}(\Omega_{X,x})$. Since the intersection is transversal, the general fibre of $Z \to X$ is thus a pair of disjoint lines.

For a point $x \in E \subset X$, the involution $i^*_{\widetilde{S\times S}}$ acts on $\mathbb{P}((\eta^* T_X)_x)$ and identifies $\mathbb{P}(Q_{1,x})$ with $\mathbb{P}(Q_{2,x})$. Thus the fibre of $Z \to X$ over a point in $x \in E_X \simeq E$ is a double line. Hence $Z \cap \pi^* E$ is non-reduced with multiplicity two. In fact since $(\eta^* T_X)|_E \simeq T_X|_{E_X}$ we can identify $(Z \cap \pi^* E)_{red}$ to the quotient defined by the inclusion $\mathcal{F}_i|_E \to (\eta^* T_X)|_E$.

21.E. The intersection computation. We will now compute some intersection numbers on $\mathbb{P}(\Omega_X)$.

21.8. **Lemma.** *In the situation of Subsection 21.A, let α_S be a $(1,1)$-class on S and $\alpha_X = (i \otimes \mathrm{id}_{\mathbb{C}})(\alpha_S) \in H^{1,1}(X, \mathbb{R})$. Then one has*

$$\zeta^7 = 504,$$

$$\zeta^5 \cdot \pi^*\delta^2 = 60, \quad \zeta^5 \cdot \pi^*(\delta \cdot \alpha_X) = 0, \quad \zeta^5 \cdot \pi^*\alpha_X^2 = -30\alpha_S^2,$$

$$\zeta^3 \cdot \pi^*\delta^4 = 12, \quad \zeta^3 \cdot \pi^*(\delta^3 \cdot \alpha_X) = 0, \quad \zeta^3 \cdot \pi^*(\delta^2 \cdot \alpha_X^2) = -2\alpha_S^2,$$

$$\zeta^3 \cdot \pi^*(\delta \cdot \alpha_X^3) = 0, \quad \zeta^3 \cdot \pi^*\alpha_X^4 = 3(\alpha_S^2)^2.$$

Proof. Observe first that $\zeta^7 = s_4(X)$, so the first statement is included in (13). Also note that by (13) one has

$$\pi_*\zeta^5 = s_2(X) = -24\bar{S} + 3\delta^2,$$

so the second statement follows from Lemma 21.1 and Lemma 21.2. The intersections with ζ^3 are simply a restatement of Lemma 21.2. □

In order to compute the intersection numbers with $\pi^*\bar{S}$, note that by Lemma 21.1 one has

$$c_1(\Omega_X|_{\bar{S}}) = 0, \quad s_2(\Omega_X|_{\bar{S}}) = s_2(\Omega_X) \cdot \bar{S} = (-24\bar{S} + 3\delta^2) \cdot \bar{S} = -27.$$

Thus we have $\zeta^5 \cdot \pi^*\bar{S} = -27$ and

$$(16) \quad \zeta^3 \cdot \pi^*\bar{S} \cdot \delta^2 = -1, \quad \zeta^3 \cdot \pi^*\bar{S} \cdot \alpha_X \cdot \delta = 0, \quad \zeta^3 \cdot \pi^*\bar{S} \cdot \alpha_X^2 = \alpha_S^2.$$

The intersections with ζ^4 and ζ^6 are all equal to zero: the Segre classes $s_1(X)$ and $s_3(X)$ vanish, so the statement follows from the projection formula.

Let now S be a very general K3 surface such that $\mathrm{Pic}(S) = 0$, in particular S does not contain any curves. The subvarieties of the product $S \times S$ are exactly $S \times x, x \times S$ and the diagonal Δ: the case of curves and divisors is easily excluded. For a surface $Z \subset S \times S$ we first observe that the projection on S is étale, since S does not contain any curve. Since S is simply connected, we obtain that Z is the graph of an automorphism of S. Yet a very general K3 surface has no non-trivial automorphisms [**Ogu08**, Cor.1.6].

21.9. **Lemma.** *In the situation of Subsection 21.A, let S be a very general K3 surface such that Pic$(S) = 0$.*

- *The subvarieties of X are exactly $(\bar{S}_x)_{x \in S}$, the exceptional divisor E_X and the fibres of $E_X \simeq \mathbb{P}(\Omega_X) \to S$.*
- *Let α_S be a Kähler class on S. Then $\alpha_X - \delta$ is a Kähler class if and only if $\alpha_S^2 > 2$.*

Proof. Since η is finite, any subvariety of X corresponds to a subvariety of $\widetilde{S \times S}$. By the discussion above and Corollary 19.5 we know the subvarieties of $S \times S$ and $\mathbb{P}(\Omega_S)$, so the first statement follows. We know that $t\alpha_X - \delta$ is Kähler for $t \gg 0$, so by the Demailly-Păun theorem it is enough to check when $\alpha_X - \delta$ is in the positive cone. By Lemma 21.1 and Lemma 21.2 we have

$$(\alpha_X - \delta)^4 = 3((\alpha_S^2)^2 - 4\alpha_S^2 + 4), \quad (\alpha_X - \delta)^3 \cdot E = 12(\alpha_S^2 - 2)$$
$$(\alpha_X - \delta)^2 \cdot \overline{S} = \alpha_S^2 - 1, \quad (\alpha_X - \delta) \cdot l = 1,$$

which are all positive for $\alpha_S^2 > 2$. ☐

21.10. **Proposition.** *In the situation of Subsection 21.A, let α_S be a Kähler class on S such that $\omega := \alpha_X - \delta$ is a Kähler class. Let $Z \subset \mathbb{P}(T_X) \simeq \mathbb{P}(\Omega_X)$ be the subvariety constructed in Subsection 21.D. Then we have*

$$(\zeta + \pi^*\omega)^5 \cdot [Z] = 15\left((\alpha_S^2)^2 - 8\alpha_S^2 - 56\right).$$

In particular we have

$$(\zeta + \pi^*\omega)^5 \cdot [Z] \geq 0$$

if and only if $\alpha_S^2 \geq \frac{8+\sqrt{288}}{2} \approx 9,6569$.

Proof. The class $[Z]$ is given by (14) and all the intersection numbers are determined in Subsection 21.E. The statement follows from an elementary, but somewhat lengthy computation. ☐

We can summarise our computations on $X = S^{[2]}$ as follows: since $\alpha_S^2 = q(\alpha_X)$ we know that for a very general K3 surface, the class $\alpha_X - \delta$ is Kähler if $q(\alpha_X) > 2$ (Lemma 21.9). The class $\zeta + \pi^*(\alpha_X - \delta)$ is pseudoeffective if $q(\alpha_X) \geq 5 + \sqrt{\frac{21}{5}}$ (Corollary 21.3). If $q(\alpha_X) < \frac{8+\sqrt{288}}{2}$, the class $\zeta + \pi^*(\alpha_X - \delta)$ is not nef (Proposition 21.10). In particular we see that for the Hilbert square of a K3 surface polarised by an ample line bundle L of degree eight, the integral class $\zeta + \pi^*(c_1(L)_X - \delta)$ is big but not nef.

21.F. Remark on subvarieties of X. By [**Ver98**, Thm.1.1] a very general deformation of the Hilbert scheme $S^{[n]}$ does not contain any proper subvarieties. Verbitsky's proof is rather involved, but for the case $n = 2$ general arguments are sufficient: a very general deformation satisfies satisfies $\mathrm{Pic}(X) = 0$, so there are no divisors and by duality there are no curves on X. The vector space

$$H^4(X, \mathbb{Q}) \cap H^{2,2}(X)$$

is one dimensional by [**Zha15**, Table B.1] and thus generated by the non-zero class $c_2(X)$. If X contains a surface S, we obtain that $c_2(X)$ is represented by an effective $\mathbb{Q}$-cycle for X very general. By properness of the relative Barlet space [**Fuj78**, Theorem 4.3] this implies that $c_2(X)$ is effectively represented for every member in the deformation family. Yet this contradicts [**Ott15**, Proposition 2].

22. Hilbert cube of a K3 surface

Now we compute explicitly the positivity threshold for $n = 3$.

22.1. **Corollary.** *Let X be a six-dimensional Hyperkähler manifold of deformation type $K3^{[3]}$. Let ω_X be a nef and big $(1,1)$-class on X such that*

$$q(\omega_X) \geq \frac{2}{21}(18+\sqrt[3]{6(1875-7\sqrt{4233})}+\sqrt[3]{6(1875+7\sqrt{4233})}) \approx 5.9538$$

Then $\zeta + \pi^*\omega_X$ is pseudoeffective. This bound is optimal for a very general deformation of X.

The proof is based on the following proposition, communicated to us by Samuel Boissière :

22.2. **Proposition.** *(S. Boissière) Let X be a six-dimensional Hyperkähler manifold of deformation type $K3^{[3]}$. Then for any $(1,1)$-class α on X one has*

$$\alpha^6 = 15q(\alpha), \qquad c_2\alpha^4 = 108q(\alpha)^2$$
$$c_2^2\alpha^2 = 1848q(\alpha), \qquad c_4\alpha^2 = 2424q(\alpha).$$

Proof. By [**Fuj87**, Remark 4.12] or [**Huy97**, Theorem 5.12] we know that for any element $\gamma \in H^{i,i}(X,\mathbb{R})$ that deforms to a very general deformation of X as element of type (i,i), its intersection with a class in $H^2(X,\mathbb{R})$ satisfies $\gamma \cdot \alpha = C(\gamma)q(\alpha)^{n-i}$ for any $\alpha \in H^2(X,\mathbb{Z})$. We need to compute these constants for varieties that are deformation equivalent to $K3^{[3]}$ and γ in the subalgebra generated by the Chern classes. By abuse of notations we will denote by c_i the Chern classes of X. The constants $C(\gamma)$ are invariant by deformations, so we can assume that X is isomorphic to $S^{[3]}$ for a projective K3 surface S. As we mention before in the case of $S^{[2]}$, there is an isometric inclusion $i : H^2(S,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{Z})$. Geometrically this inclusion is realized sending a line bundle L on S to the line bundle $L_3 := detL^{[3]}$. By Riemann–Roch formula and by [**EGL01**, Lemma 5.1] we have

$$\int_X e^{c_1(L_3)}\,\mathrm{Todd}(X) = \chi_X(L_3) = \binom{\chi_S(L)+2}{2}.$$

From now on we by abuse of notation we will confuse line bundles with their first Chern class. We recall that the Todd class for six dimensional Hyperkähler manifolds is

$$\mathrm{Td}(X) = 1+\frac{1}{12}c_2+\frac{1}{240}c_2^2-\frac{1}{720}c_4+\frac{1}{6048}c_2^3-\frac{1}{6720}c_2c_4+\frac{1}{30240}c_6 \tag{17}$$

and $\chi_S(L) = L^2 + 2 = q(L_3) + 2$. Putting the Todd class and the characteristics in the equation above we get

$$\frac{1}{720}L_3^6+\frac{1}{288}c_2L_3^4+(\frac{1}{480}c_2^2-\frac{1}{1440}c_4)L_3^2+\frac{1}{6048}c_2^3-\frac{1}{6720}c_2c_4+\frac{1}{30240}c_2c_4 =$$
$$= \frac{1}{6}\chi_S(L)(\chi_S(L)+1)(\chi_S(L)+2) = \frac{1}{48}q(L_3)^3+\frac{3}{8}q(L_3)^2+\frac{13}{6}q(L_3)+4$$

that by homogeneity tells us that

$$L_3^6 = 15q(L_3)$$
$$c_2L_3^4 = 108q(L_3)^2.$$

The quadratic term is not sufficient to gives us the other constants but tells only that

$$3c_2^2L_3^2 - c_4L_3^2 = 3120q(L_3). \tag{18}$$

We are going to use a consequence of a formula due to Nieper that can be found in [**Huy03b**, Theorem 4.2]:

$$\int_X \sqrt{\mathrm{Td}(X)}e^x = (1+\lambda(x))^3\int_X\sqrt{\mathrm{Td}(X)} \tag{19}$$

for a quadratic form $\lambda : H^2(X,\mathbb{C}) \to \mathbb{C}$ and any $x \in H^2(X,\mathbb{C})$. One can deduce directly by (17) that

$$\sqrt{\mathrm{Td}(X)} = 1+\frac{1}{24}c_2+\frac{7}{5650}c_2^2-\frac{1}{1440}c_4+\frac{31}{967680}c_2^3-\frac{11}{241920}c_2c_4+\frac{1}{60480}c_6.$$

By the terms of degree 4 and 6 of (19) we deduce that $\lambda(x) = \frac{1}{3}q(x)$. This fact with the degree two component of (19) gives

$$\frac{7}{4}c_2^2x^2 - c_4x^2 = 810q(x). \tag{20}$$

Finally the solution of the system given by (18) and (20) is

$$c_2^2L_3^2 = 1848q(L_3), \qquad c_4L_3^2 = 2424q(L_3).$$

□

Proof of Corollary 22.1. By Proposition 20.1 we only have to compute the largest real root of $p_X(t)$. By Formula (5) we have to compute the largest solution of

$$\binom{11}{6}d_6t^3+\binom{11}{4}d_4t^2+\binom{11}{2}d_2t+d_0 = 0,$$

where the constants d_{2i} are defined by (6). Using Proposition 22.2 we can compute the constant d_{2i} in our setting, one obtains the cubic equation

$$6930t^3 - 35640t^2 - 31680t - 10560. = 0.$$

This polynomial has only one real solution, the one from the statement. The last statement is the second part of Theorem 0.10. □

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