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Identifiability of small rank tensors and related problems

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IDENTIFIABILITY OF SMALL RANK TENSORS AND RELATED PROBLEMS

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Introduction

Over the last 60 years multilinear algebra made its way in the applied sciences. As a consequence, tensors acquired an increasingly central role in the applications and the problem of tensor rank decomposition has started to be studied by several non-mathematical communities. One of the main advantages of working with tensors instead of matrices is that tensors *very often* admit a unique rank decomposition. Under this perspective, after translating applied problems of different fields in the language of tensors, the uniqueness of the tensor rank decomposition represents a unique way of interpreting the initial datas of the corresponding application.

In the following we will always work over an algebraically closed field of characteristic zero. The central problem studied in this thesis is the so-called *identifiability problem for tensors* and it amounts to understand whether a tensor admits a unique tensor rank decomposition. Fix vector spaces $V_1, \ldots, V_k$ of dimensions $n_1+1, \ldots, n_k+1$ respectively. A tensor $Q \in V_1 \otimes \cdots \otimes V_k$ is called an *elementary tensor* if $Q = v_1 \otimes \cdots \otimes v_k$ for some $v_i \in V_i$ with $i = 1, \ldots, k$. Elementary tensors are the building block of the tensor rank decomposition and the rank of a tensor $Q \in V_1 \otimes \cdots \otimes V_k$ is the minimum integer r such that we can write Q as a combination of r elementary tensors

$$Q = \sum_{i=1}^{r} v_{1,i} \otimes \cdots \otimes v_{k,i}, \text{ where all } v_{j,i} \in V_j \text{ for } j = 1, \ldots, k.$$

A rank-r tensor Q is *identifiable* if admits a unique rank decomposition up to reordering the elementary tensors and up to scalar multiplication.

Keeping this parallelism with applied sciences, we want to remark that the word *identifiability* find its scientific roots in the field of statistic: indeed, a statistical model is identifiable if it is possible to understand the values of the underlying parameters of the model itself. Perhaps it is not a coincidence that the first modern contribution on identifiability of tensors has been given by J. B. Kruskal [Kru77], a mathematician by training that worked at the *Bell Labs* for over 30 years. Kruskal criterion represents a milestone of the literature related to this problem, it has been reproved different times (see e.g. [Rho10], [Lan09], [SS07]) and it has been generalized in many ways (cf. [SB00], [COV17], [Chi19]). The result relies on the so-called *Kruskal rank* of a set S of vectors, which is the maximum number k such that any subset of k vectors of S is indeed a subset of linearly independent vectors. Kruskal theorem involves 3-ways tensors Q of the form

$$Q = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i.$$

Starting with a decomposition of Q given by r elements as above, if $2(r-1)$ is less or equal than the sum of the Kruskal ranks of $\{a_1, \ldots, a_r\}, \{b_1, \ldots, b_r\}, \{c_1, \ldots, c_r\}$, then the rank of Q is r and the decomposition of Q we started with is actually unique.

The standpoint from which we study the identifiability problem for tensors is a geometrical point of view and we propose to tackle the identifiability problem with tools coming from classical algebraic geometry, such as Segre varieties and secant varieties of Segre varieties. Let $N = \prod_{i=1}^{k}(n_i + 1) - 1$. The map going from a multiprojective space $Y_{n_1,\dots,n_k} := \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k$ to $\mathbb{P}(V_1 \otimes \cdots \otimes V_k) \cong \mathbb{P}^N$ sending classes of vectors into the class of their tensor product is the *Segre embedding*. The *Segre variety* $X_{n_1,\dots,n_k}$ is the image of the Segre embedding of $Y_{n_1,\dots,n_k}$ and it is the projective algebraic variety parametrizing classes of elementary tensors. The variety needed to study higher rank tensors is the secant variety of the Segre variety. If we fix a positive integer r, the r-th *secant variety* $\sigma_r(X_{n_1,\dots,n_k})$ of a Segre variety $X_{n_1,\dots,n_k}$ is the Zariski closure of the union of all possible points spanned by projective $(r-1)$-planes given by r independent points of the Segre variety. Another auxiliary variety needed to study the identifiability problem is the so-called r-th *abstract secant variety* of a Segre variety $X_{n_1,\dots,n_k} \subset \mathbb{P}^N$, namely

$$AbSec_r(X_{n_1,\dots,n_k}) := \overline{\{(q,(p_1,\dots,p_r)) \in \mathbb{P}^N \times (X_{n_1,\dots,n_k})^r \mid q \in \langle p_1,\dots,p_r\rangle \cong \mathbb{P}^{r-1}\}}.$$

The first projection T_r of the open part $AbSec^0_r(X_{n_1,\dots,n_k}) = \{(q,(p_1,\dots,p_r)) \in \mathbb{P}^N \times (X_{n_1,\dots,n_k})^r \mid q \in \langle p_1,\dots,p_r\rangle \cong \mathbb{P}^{r-1}\}$ onto the ambient space $\mathbb{P}^N$ that contains the r-th secant variety is the *Terracini map*. Under this perspective it is possible to rephrase the identifiability problem as follows:

a rank-r tensor $q \in \mathbb{P}^N$ is identifiable if the fiber $T_r(q)^{-1}$ is a singleton.

Starting from Kruskal result, over the years there have been a lot of contributions on the identifiability problem coming from the field of pure mathematics as well as applied sciences. Most of the mathematical literature on the subject is related to the identifiability problem for generic tensors of fixed rank. Knowing if a generic tensor of a certain rank is identifiable can be useful in the applications because it gives an indication on the identifiabilty of specific tensors of the same rank. Indeed, looking at the Terracini map, we recall that the dimension of the fiber of an element of the r-th secant variety of a Segre variety is greater or equal than the difference between the dimension of the r-th abstact secant variety and the dimension of the r-th secant variety itself and equality holds for the generic element. Therefore, the dimension of the space containing all rank-1 tensors computing a rank decomposition of a specific tensor is at least the dimension of the space of rank-1 tensors computing the rank decomposition of a generic element of the same rank.

A whole line of research on the identifiability of generic tensors begun after the results of L. Chiantini and C. Ciliberto collected in [CC02], where the authors introduced in a modern language the concept of weak defectivity. Before stating this concept it is useful to recall the notion of defectivity. A Segre variety $X_{n_1,\dots,n_k}$ is *r-defective* if the dimension of the r-th secant variety of $X_{n_1,\dots,n_k}$ is strictly smaller than the minimum between the dimension of the ambient space in which $X_{n_1,\dots,n_k}$ lives and the value $r(\dim X_{n_1,\dots,n_k}+1)-1$. A Segre variety $X_{n_1,\dots,n_k} \subset \mathbb{P}^N$ is *r-weakly defective* if the general r-tangent hyperplane has a contact variety of positive dimension, where by general r-tangent hyperplane we mean the general hyperplane $H \subset \mathbb{P}^N$ containing the span of the r tangent spaces to $X_{n_1,\dots,n_k}$ at generic r points of $X_{n_1,\dots,n_k}$. In a second work [CC06], the authors linked the notion of weak defectivity with the notion of identifiability. They proved that if the secant variety of a Segre variety is non defective, the fiber with respect to the corresponding Terracini map can be positve dimensional only if $X_{n_1,\dots,n_k}$ is weakly defective. Few years later, the second author together with G. Ottaviani introduced in [CO12] the notion of r-tangentially weak defectiveness, which is a useful concept related to the identifiability of generic tensors. A Segre variety $X_{n_1,\dots,n_k}$ is *r-tangentially weakly defective* if the span of the tangent spaces at r general points of $X_{n_1,\dots,n_k}$ is tangent also in some other point. In [CO12], the authors introduced also an inductive method for the study of the identifiability of generic 3-way tensors based on the notion of weak defectivity but we refer to Section 2.1 for a more detailed literature review on the identifiability of generic tensors.

Working in the applied fields, one may also be interested in the identifiability of specific tensors. Indeed, when translating an applied problem in the language of tensors one may be forced to deal with a very specific tensor that has a precise structure by reasons related to the nature of the applied problem itself. In these cases the knowledge of the identifiability of the generic tensor of the same rank may not be useful because it may happen that the tensor we are dealing with is not identifiable even if the generic element of the same rank is identifiable. If we address the identifiability problem to specific tensors, we can no longer use tools of weak defectivity and of tangential weak defectivity introduced in the generic context because the existence of a particular r-uple of points that have the behaviour described by the notion of (tangentially) weak defectivity does not imply the existence of a whole contact subvariety having the same behaviour. Hence these concepts cannot be adapted to a non-generic framework.

In this thesis we focus on the identifiability of specific tensors of fixed rank. Working with specific tensors, the literature review became more scattered and the few results can be considered extensions and/or generalizations of Kruskal's result (cf. [DDL14], [SDL15], [DDL13] and [LP21]). As a consequence, a complete classification on the identifiability of all tensors of small ranks was still missing.

The first project presented in this thesis is devoted to classify all identifiable tensors of rank either 2 or 3.

One of the building blocks on which our classification is based on is the classical concision Lemma (cf. [Lan12, Prop. 3.1.3.1]). The lemma states that for any tensor $Q \in V_1 \otimes \cdots \otimes V_k$ there exists a unique minimal tensor space included in $V_1 \otimes \cdots \otimes V_k$ that contains both the tensor and all its possible rank decompositions. In the first part of this thesis, we deal with the identifiability of all tensors of rank $r \leq 3$ working with concision assumptions. In particular, for $r = 2$ we prove that the only non-identifiable rank-2 tensors are 2×2 matrices. A more interesting situation occurs with rank-3 tensors. All possible cases of non-identifiable rank-3 tensors are collected in Theorem 2.6.1, which is the main theorem of the second chapter. In the theorem we present the following 6 different families of non-identifiable rank-3 tensors.

$\boxed{\text{1) Matrix case}}$

The first trivial example of non-identifiable rank-3 tensors are 3×3 matrices, which is a very classical case.

$\boxed{\text{2) Tangential case}}$

The *tangential variety* of a variety is the tangent developable of the variety itself. A point q lying on the tangential variety of the Segre image $X_{1,1,1}$ of three copies of the projective line is actually a point of the tangent space $T_pX_{1,1,1}$ for some $p = u \otimes v \otimes w$. Therefore there exists some $a, b, c \in \mathbb{C}^2$ such that q can be written as

$$q = a \otimes v \otimes w + u \otimes b \otimes w + u \otimes v \otimes c$$

and hence q is actually non-identifiable.

$\boxed{\text{3) Defective case}}$

Working with concision assumptions, the only defective case of a third secant variety of a Segre variety occurs for the third secant variety of the Segre variety $X_{1,1,1,1}$ of 4 copies of the projective line (cf. [AOP09, Theorem 4.5]). By defectivity, the dimension of $\sigma_3(X_{1,1,1,1})$ is strictly smaller than the expected dimension and this proves that the generic element of $\sigma_3(X_{1,1,1,1})$ has an infinite number of rank-3 decompositions and therefore all the rank-3 tensor of this variety have an infinite number of decompositions.

$\boxed{\text{4) − 5) Conic cases}}$

In this case we consider the Segre variety $X_{2,1,1}$ given by the image of a projective plane and two projective lines.

Consider the Segre embedding of the two projective lines in $\mathbb{P}^3$ and take a hyperplane section which intersects the 2-dimensional Segre variety in a conic $\mathcal{C}$. Let $L_{\mathcal{C}}$ be the Segre image of the product given by a projective plane and the conic $\mathcal{C}$, therefore $L_{\mathcal{C}} \subset X_{2,1,1}$. The family of non-identifiable rank-3 tensors are points lying in the span of $L_{\mathcal{C}}$. In this case, the non-identifiability comes from the fact that the points on $\langle \mathcal{C} \rangle$ are not identifiable and the distinction between the two cases reflects the fact that the conic $\mathcal{C}$ can be irreducible or not.

$\boxed{\text{6) General case}}$

The last family of non-identifiable rank-3 tensors relates the Segre variety $X_{n_1,n_2,1^{k-2}}$ that is the image of the multiprojective space $Y_{n_1,n_2,1^{k-2}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{(k-2)}$, for some $k \geq 3$ and $n_1, n_2 \in \{1, 2\}$. The non-identifiable rank-3 tensors of this case are as follows. Let $Y' := \mathbb{P}^1 \times \mathbb{P}^1 \times \{u_3\} \times \cdots \times \{u_k\}$ be a proper subset of $Y_{n_1,n_2,1^{k-2}}$, take q' in the span of the Segre image of Y' with the constrain that q' is not an elementary tensor. Therefore q' is a non-identifiable tensor of rank-2 since it can be seen as a 2×2 matrix of rank-2. Let $p \in X_{n_1,n_2,1^{k-2}}$ be a rank-1 tensor taken outside the Segre image of Y'. Now any point $q \in \langle \{q', p\} \rangle \setminus \{q', p\}$ is actually by construction a rank-3 tensor and it is not identifiable since q' has an infinite number of decompositions and each of these decompositions can be taken by considering p together with a decomposition of q'.

This is a first step towards a complete classification of the identifiability of all tensors of small rank. A natural question that arises after the short analysis we just provided, is to understand what happens in the case of higher rank tensors. In the present thesis we will not go further in this direction but it is reasonable to think that as the rank grows it also grows the number of families of non-identifiable tensors.

Changing perspective on this problem, as mentioned above, given the projective class $q \in \langle X_{n_1,\ldots,n_k} \rangle \subset \mathbb{P}^N$ of a tensor, to prove that q is identifiable we have to look at the fiber $T_r(q)^{-1}$ and verify that it is a singleton. To do so, we can start by verifying that the fiber $T_r(q)^{-1}$ is 0-dimensional. Indeed, given a non-defective variety $X \subset \mathbb{P}^N$ and taken points $p_1, \ldots, p_r \in X_{n_1,\ldots,n_k}$, if the p_i's are contained in a family of decompositions of positive dimension, then the dimension of the span of the r tangent spaces of the variety $X_{n_1,\dots,n_k}$ at points $p_1,\dots,p_r$ is strictly smaller than the dimension of the r-th secant variety of $X_{n_1,\dots,n_k}$ (cf. [COV17, Lemma 37]). Therefore, starting with a rank decomposition, the first preliminary test to understand if the corresponding tensor is identifiable, is to compute the dimension of the span of the corresponding tangent spaces.

This idea brings us to the second part of this thesis.

We recall that an extremely powerful tool to compute dimensions of secant varieties is the Terracini's Lemma (cf. [Ter11]). This lemma says that for an irreducible non-degenerate projective variety $X_{n_1,\dots,n_k} \subset \mathbb{P}^N$, taken a generic point $q \in \sigma_r(X_{n_1,\dots,n_k})$ such that $q \in \langle p_1,\dots,p_r\rangle$, for generic $p_1,\dots,p_r \in X_{n_1,\dots,n_k}$, then the tangent space of $\sigma_r(X_{n_1,\dots,n_k})$ at q is equal to the span of the r tangent spaces of $X_{n_1,\dots,n_k}$ at points $p_1,\dots,p_r$. Even when $X_{n_1,\dots,n_k}$ is not r-defective, there may exist special points $q_1,\dots,q_r \in X_{n_1,\dots,n_k}$ for which

$$\dim\langle T_{q_1}X_{n_1,\dots,n_k},\dots,T_{q_r}X_{n_1,\dots,n_k}\rangle < \min\{N, r(\dim X_{n_1,\dots,n_k}+1)-1\}.$$

We define the r-th *Terracini locus* of the Segre variety $X_{n_1,\dots,n_k}$ as the space containing all r-uples of points that have this behaviour.

In the third chapter of the present thesis we introduce the notion of r-th Terracini locus of a Segre variety and we completely characterize it for $r = 2,3$ working with minimality assumptions, i.e. working with the smallest multiprojective space containing the set of particular points whose differential of the Terrcini map drops rank.

For $r = 2$ the minimal multiprojective space containing a set of two distinct points is a multiprojective space given by the product of just projective lines, i.e. $Y_{1^k} = (\mathbb{P}^1)^{\times k}$ for some $k \geq 1$. In this case we prove that the Terracini locus is always empty.

Let now $k = 3$. The minimal multiprojective space is given by products of projective lines and planes, i.e. $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where all $n_i \in \{1,2\}$. We prove that the third Terracini locus is empty if and only if either $k = 1,2$ or $Y_{2^k} = (\mathbb{P}^2)^k$, for all $k \geq 3$. Moreover the non-empty sets S of three points lying in a third Terracini locus can only be as follows:

- For $Y_{m,1^k} = \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$, with $k \geq 4$ the points of the corresponding third Terracini locus are all set $S = \{a,b,c\}$ of three points such that a and b share all the last $k-1$ components and the projection of a and b in the first factor are linearly independent. The projection on the last $k-1$ components of the point c is different from the values of a and b and if $m = 2$ then we request the projection of c on the first factor to be linearly independent with respect to the projections of a and b.

- For $Y_{n_1,n_2,1^{k-2}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, with $k \geq 3$ the points of the corresponding third Terracini locus are given by all sets $S = \{u,v,o\}$ of three points such that the projections of u and v coincide on all the last $k-2$ factors and they are different on the first two factors. For the other point o, its projection on each of the last $k-2$ factors differs from the projection of u and if $n_j = 2$ for $j = 1,2$ then the projection of S on the j-th spans a two dimensional projective plane. If $k \geq 4$ then these are the set of three points in the corresponding third Terracini locus. If $k = 3$ then we will need to add more restrictive conditions to the points to get them in the corresponding third Terracini locus.

- For $Y_{1^4} = (\mathbb{P}^1)^4$ all set of three points that have Y_{1^4} as minimal multiprojective space lie in the corresponding third Terracini locus.

Given a multiprojective space $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, we look at the problem of finding all finite sets $S \subset Y$ whose differential of the Terracini map drops rank from an algebraic point of view by means of 0-dimensional scheme of r double fat points. Given a point $p \in Y_{n_1,\dots,n_k}$, denote by $(2p, Y_{n_1,\dots,n_k})$ the first infinitesimal neighbourhood of p in $Y_{n_1,\dots,n_k}$, which is the closed subscheme of $Y_{n_1,\dots,n_k}$ with $(\mathcal{I}_{p,Y_{n_1,\dots,n_k}})^2$ as its ideal sheaf. For any finite set $S \subset Y_{n_1,\dots,n_k}$ let $(2S, Y_{n_1,\dots,n_k}) := \cup_{p\in S}(2p, Y_{n_1,\dots,n_k})$. Consider the exact sequence of the inclusion of $(2S, Y_{n_1,\dots,n_k})$ in $Y_{n_1,\dots,n_k}$ with respect the Segre embedding. If we consider the corresponding cohomology exact sequence and we look at its dimensions we get

$$h^0(O_{Y_{n_1,\dots,n_k}}(1,\dots,1)) - h^0(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)) = \deg(2S, Y_{n_1,\dots,n_k}) - h^1(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)), \qquad (*)$$

where we recall that the codimension of the span of the tangent spaces to $X_{n_1,\dots,n_k}$ at all points of S is actually the dimension of the global section of the ideal sheaf of $(2S, Y_{n_1,\dots,n_k})$ embedded via Segre. Fixing $Y_{n_1,\dots,n_k}$ and r, the only integers that change in $(*)$, as S varies, are $h^0(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1))$ and $h^1(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1))$. These values represent the *defect* of that particular S. In order to get a proper defect, we require at the same time that

$$h^0(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)) > 0 \text{ and } h^1(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)) > 0. \qquad (\diamond)$$

The r-th Terracini locus is the set containing all r-uples $S \subset Y_{n_1,\dots,n_k}$ that show this behaviour.

Remark that this idea can be developed for any pair $(Y, \mathcal{L})$ given by an irreducible non-degenerate projective variety Y embedded via a line bundle $\mathcal{L}$ for which $h^1(Y, \mathcal{L}) = 0$. Moreover, if we are dealing with a 0-dimensional scheme of generic r double fat points satisfying conditions $(\diamond)$, then the corresponding r-th secant variety is defective and does not fill the ambient space. There are a lot of results working with generic points because of its link with defectivity and complete classifications have been made in the following cases

$(Y, \mathcal{L})$	reference
$(\mathbb{P}^n, \mathcal{O}(d))$	[AH95]
$((\mathbb{P}^1)^k, \mathcal{O}(d_1,\dots,d_k))$	[LP13]
$(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(d_1, d_2))$	[GO21]

As mentioned, these classifications are related to generic 0-dimensional schemes of double fat points, while, to the best of our knowledge, nobody tackled the same problem when the 0-dimensional scheme is not necessarily supported on generic points.

Working with non-generic points we lose the equivalence between the codimension of the secant variety and the dimension of the global sections of the ideal sheaf related to the scheme of double fat points. Nevertheless, this remains an interesting problem from the pure mathematical point of view and it has turned out to be an interesting problem also for the numerical community working on tensor rank decomposition (cf. [BV18], [BV20], also the introduction of Chapter 2 for a more elaborate discussion).

The thesis is organized as follows.

Chapter 1 contains the basic notions of tensors and secant varieties of Segre varieties that will be used in the sequel.

Chapter 2 is devoted to the identifiability problem for any tensor of rank at most 3. After a literature review on the identifiability of tensors, we worked on the identifiability of rank-2 tensors. The rest of the chapter is devoted to the study of rank-3 tensors. We first presented all 6 families of non-identifiable tensors and then we proved that any non-identifiable rank-3 tensor belongs to one of these 6 families. We concluded the chapter by presenting an algorithm that is able to recognize if a tensor belongs to one of these families.

Chapter 3 is devoted to first introduce the notion of r-th Terracini locus of a Segre variety and then to completely characterize it for $r = 2, 3$.

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Chapter 1

Preliminary notions

The first part of this chapter is devoted to review basic notions on tensors from a geometrical perspective and to fix the notation used in the sequel. The second part contains some standard cohomological tools that are used in the second and third chapter of the present thesis respectively.

Unless specified we always work over $\mathbb{C}$.

1.1 Geometry of tensors

For this part we refer to [Lan12], [Chi04] and [Har95].

Let $V_1, \ldots, V_k$ be vector spaces of dimensions $n_1 + 1, \ldots, n_k + 1$ respectively and consider the tensor space $V_1 \otimes \cdots \otimes V_k$. Any element $Q \in V_1 \otimes \cdots \otimes V_k$ of the form

$$Q = v_1 \otimes \cdots \otimes v_k$$

where $v_i \in V_i$ for all $i = 1, \ldots, k$ is an *elementary tensor* (also called *simple tensor* or *indecomposable tensor*). Given a tensor $Q \in V_1 \otimes \cdots \otimes V_k$, one may wonder which is the minimum number of elementary tensors needed in order to write Q as a linear combination of these elementary tensors, that is the rank of the tensor.

Definition 1.1.1. The *rank* of a tensor $Q \in V_1 \otimes \cdots \otimes V_k$ is

$$r(Q) := \min\{r \in \mathbb{N} \mid T = \sum_{i=1}^{r} v_{1,i} \otimes \cdots \otimes v_{k,i}, \text{ where all } v_{j,i} \in V_j\}.$$

Therefore elementary tensors are by definition rank-1 tensors. For all $i = 1, \ldots, k$ let $\{e_{i,0}, \ldots, e_{i,n_i}\}$ be a basis of V_i and denote by $(\alpha_{i,j})_{j=0,\ldots,n_i} = v_i$ the coordinates of the vector $v_i \in V_i$ with respect to the above bases. A rank-1 tensor $Q = v_1 \otimes \cdots \otimes v_k$ can be written in coordinates as

$$Q = \sum_{\substack{i_1,\ldots,i_k \\ i_j=0,\ldots,n_j, j=0,\ldots,k}} \underbrace{\alpha_{1,i_1} \cdots \alpha_{k,i_k}}_{t_{i_1\ldots i_k}} e_{1,i_1} \otimes \cdots \otimes e_{k,i_k} = (t_{i_1,\ldots,i_k}). \tag{1.1.1}$$

The coordinate description of Q completely characterizes Q and the scalars $t_{i_1\ldots i_k}$ can be stored in a multidimensional array of size $(n_1 + 1) \times \cdots \times (n_k + 1)$.

Remark 1.1.2. The characterization of rank-1 tensors as multidimensional arrays can be extended by linearity to any element $Q \in V_1 \otimes \cdots \otimes V_k$.

Moreover, since the rank of a tensor is invariant under scalar multiplication, it becomes natural to look at tensors in the projective space.

Multidimensional arrays describing tensors can be parametrized by points of $\mathbb{P}^N$ where $N = \prod_{i=1}^{k}(n_i + 1) - 1$. If we consider the Segre embedding

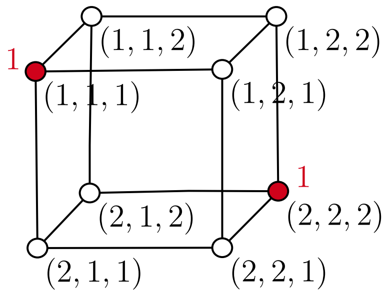
$$\begin{aligned} \nu : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k &\longrightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_k) = \mathbb{P}^N \\ ([v_1], \ldots, [v_k]) &\mapsto [v_1 \otimes \cdots \otimes v_k] \end{aligned}$$

it becomes immediately clear that Segre varieties parametrize rank-1 tensors. Note that if we use the coordinate expression of $v_1 \otimes \cdots \otimes v_k$ as in (1.1.1), then $[v_1 \otimes \cdots \otimes v_k] = [t_{0\ldots 0} : t_{0\ldots 01} : \cdots : t_{i_1 \ldots i_k} : \cdots : t_{n_1 \ldots n_k}]$.

Example 1.1.3. Fix the tensor space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Let $\{e_1, e_2\} \subset \mathbb{C}^2$ be the canonical basis of $\mathbb{C}^2$ and let $Q \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ be

$$Q = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2.$$

The point $p_1 = ([e_1], [e_1], [e_1])$ is sent to $\nu(p_1) = [1 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$ via the Segre map and similarly $p_2 = ([e_2], [e_2], [e_2])$ is sent to $\nu(p_2) = [0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1]$. Their sum is therefore $\nu(p_1) + \nu(p_2) = [Q] = [e_1 \otimes e_1 \otimes e_1] + [e_2 \otimes e_2 \otimes e_2]$. Since we are considering a three factors tensor its representation as multidimensional array can be seen as a cube in which each direction corresponds to a factor of the tensor space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Each vertex of the cube represents a coordinate of Q and we can mark the coordinates of Q. Therefore a picture of the coordinate description of Q is the following.



We recall that Segre varieties are irreducible non-degenerate smooth projective varieties (cf. [Har95, Example 2.11]). For the sake of completeness, we recall the action of the general linear group on tensor spaces.

Remark 1.1.4. For $i = 1, \ldots, k$, let V_i be a vector space of dimension $n_i + 1$ and consider $V_1 \otimes \cdots \otimes V_k$. The group $GL(V_1) \times \cdots \times GL(V_k)$ acts on $V_1 \otimes \cdots \otimes V_k$ and the action of $g = (g_1, \ldots, g_k) \in GL(V_1) \times \cdots \times GL(V_k)$ on a rank-1 element $v_1 \otimes \cdots \otimes v_k$ is defined as

$$g \cdot v_1 \otimes \cdots \otimes v_k = (g_1 v_1) \otimes \cdots \otimes (g_k v_k).$$

This action is extended by linearity on any element $Q \in V_1 \otimes \cdots \otimes V_k$ and preserves the rank of Q.

Notation 1.1.5. We will use use lower case letters p, q to denote the projective classes of tensors, while we will use capital letters T, Q to denote tensors in their natural vectorial ambient space.

1.1.1 Notation and concision of tensors

We will keep the following notation for Segre varieties.

Notation 1.1.6. We denote by $Y_{n_1,\dots,n_k}$ the multiprojective space

$$Y_{n_1,\dots,n_k} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

and by $X_{n_1,\dots,n_k}$ the image of $Y_{n_1,\dots,n_k}$ via Segre embedding, i.e. $X_{n_1,\dots,n_k} = \nu(Y_{n_1,\dots,n_k})$. If $n_1 = \cdots = n_k = n$ we will simply write Y_{n^k}.
We denote the projection on the i-th factor as

$$\pi_i : Y_{n_1,\dots,n_k} \longrightarrow \mathbb{P}^{n_i}.$$

The space given by all factors of $Y_{n_1,\dots,n_k}$ but the i-th one is denoted by $Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i}$, where, as usual, the hat symbol represents eliminating the corresponding element:

$$Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i} := \mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_k}.$$

Let $N_i' = \prod_{j\neq i}(n_j+1) - 1$. With $\nu_i : Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i} \longrightarrow \mathbb{P}^{N_i'}$ we denote the corresponding Segre embedding, in particular $X_i := \nu(Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i})$.
The projection on all the factors of $Y_{n_1,\dots,n_k}$ but the i-th one is denoted with η_i:

$$\eta_i : Y_{n_1,\dots,n_k} \longrightarrow Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i}.$$

Obviously all fibers of η_i are isomorphic to $\mathbb{P}^{n_i}$.

A very basic property of Segre vareities is the following.

Remark 1.1.7. Since the Segre variety $X_{n_1,\dots,n_k} \subset \mathbb{P}^N$ is cut out by quadrics (cf. [Gro77]), any projective line $L \subset \mathbb{P}^N = \langle X_{n_1,\dots,n_k}\rangle$ that intersect the Segre variety $X_{n_1,\dots,n_k}$ in more than two points is actually all contained in $X_{n_1,\dots,n_k}$.

1.1.1.1 Concision Lemma

It is also useful to recall the so-called Concision Lemma (cf. [Lan12, Prop. 3.1.3.1]).

Lemma 1.1.8 (Concision/Autarky)**.** *Let $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. For any $q \in \langle \nu(Y_{n_1,\dots,n_k})\rangle$, there is a unique minimal multiprojective space $Y' \simeq \mathbb{P}^{n_1'} \times \cdots \times \mathbb{P}^{n_k'} \subseteq Y_{n_1,\dots,n_k}$ with $n_i' \leq n_i$, $i = 1,\dots,k$ such that*

$$\{A \subset Y' \mid \#A = r(q) \,\text{and}\, q \in \langle\nu(A)\rangle\} = \{A \subset Y_{n_1,\dots,n_k} \mid \#A = r(q) \,\text{and}\, q \in \langle\nu(A)\rangle\}.$$

One may look at a tensor in the smallest tensor space containing all its possible rank decompositions. Let us review more in details the concision process working in coordinates.

Concise tensor space of a tensor

Fix a tensor $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$, where $k \geq 2$ and $n_1,\dots,n_k \geq 1$.
For all $\ell = 1,\dots,k$, denote by $\mathcal{B}_\ell = \{e_1^\ell,\dots,e_{n_\ell}^\ell\}$ an ordered basis of $\mathbb{C}^{n_\ell}$ and by $\mathcal{B}_\ell^* = \{\eta_1^\ell,\dots,\eta_{n_\ell}^\ell\}$ the corresponding dual basis. Let $T = (t_{i_1,i_2,\cdots,i_k})$ be the coordinates of T with respect to those bases.

A useful operation that allows to store the elements of a tensor as a matrix is the flattening, also called matrix-unfolding of a tensor in [DLDMV00, Definition 1], which is the oldest reference we found for a formal definition of this operation.

Definition 1.1.9. The ℓ-th *flattening* of a tensor $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ is the linear map

$$\begin{aligned}\varphi_\ell \colon (\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_{\ell-1}} \otimes \mathbb{C}^{n_{\ell+1}} \otimes \cdots \otimes \mathbb{C}^{n_k})^* &\to \mathbb{C}^{n_\ell} \\ f &\mapsto \sum_{i_1,\dots,i_k} t_{i_1\dots i_k} f(e^1_{i_1} \otimes \cdots \otimes e^{\ell-1}_{i_{\ell-1}} \otimes e^{\ell+1}_{i_{\ell+1}} \cdots \otimes e^k_{i_k}) e^\ell_{i_\ell}.\end{aligned}$$

We denote by A_ℓ the $n_\ell \times (\prod_{i\neq \ell} n_i)$ associated matrix with respect to bases $\mathcal{B}_\ell$ and $\{\eta^1_1 \otimes \cdots \otimes \eta^{\ell-1}_1 \otimes \eta^{\ell+1}_1 \otimes \cdots \otimes \eta^k_1, \eta^1_1 \otimes \cdots \otimes \eta^{\ell-1}_1 \otimes \eta^{\ell+1}_1 \otimes \cdots \otimes \eta^k_2, \dots, \eta^1_{n_1} \otimes \cdots \otimes \eta^{\ell-1}_{n_{\ell-1}} \otimes \eta^{\ell+1}_{n_{\ell+1}} \otimes \cdots \otimes \eta^k_{n_k}\}$.

Definition 1.1.10 ([Hit28])**.** Let $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$. For all $\ell = 1, \dots, k$ let A_ℓ be the ℓ-th flattening of T as in Definition 1.1.9 and denote by $r_\ell := r(A_\ell)$. The *multilinear rank* of T is the k-uple

$$mr(T) := (r_1, \dots, r_k)$$

containing all the ranks of the flattenings.

We remark that (cf. [CK11, Theorem 7]) for all $\ell = 1, \dots, k$

$$r_\ell \leq r(T) \leq \prod_{i\neq \ell} r_i \tag{1.1.2}$$

and moreover it is classically known that

$$r(T) = 1 \iff \textit{the multilinear rank of } T \textit{ is } (1, \dots, 1).$$

We are ready to describe a procedure that gives the concise tensor space $\mathcal{T}_{n'_1,\dots,n'_{k'}}$ of a given tensor $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ (cf. [Lan12, Subsection 3.1.3]).

Let $T = (t_{i_1,\dots,i_k}) \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$, where all $n_i \geq 1$ and $k \geq 2$. For all $\ell = 1, \dots, k$ consider the ℓ^{th} flattening A_ℓ of T as in Definition 1.1.9. For the sake of simplicity take $\ell = 1$. The first column of A_1 is

$$(t_{1,1,\dots,1}, t_{2,1,\dots,1}, \dots, t_{n_1,1,\dots,1})^T = \sum_{i=1}^{n_1} t_{i,1,\dots,1} u^1_i = \sum_{i,j=1}^{n_1} t_{i,1,\dots,1} \alpha^1_j(u^1_i),$$

which is referred to $u^2_1 \otimes \cdots \otimes u^k_1$. The same holds for the other columns of A_1. Once we have computed $n'_1 := r(A_1)$ we can extract n'_1 linearly independent columns from A_1, say $u^1_1, \dots, u^1_{n'_1}$. Since $\mathrm{Im}(\varphi_1) = \langle u^1_1, \dots, u^1_{n'_1}\rangle \cong \mathbb{C}^{n'_1} \subseteq \mathbb{C}^{n_1}$, we rewrite the other columns as a linear combination of the independent ones. The resulting tensor T' will therefore live in a smaller space $\mathbb{C}^{n'_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_k}$. By continuing this process for each flattening we arrive to a concise tensor space

$$\mathcal{T}_{n'_1,\dots,n'_{k'}} = \mathbb{C}^{n'_1} \otimes \cdots \otimes \mathbb{C}^{n'_{k'}}$$

where we may assume $n'_i > 1$ for all $i = 1, \dots, k'$ and $k' \leq k$ since $\mathbb{C}^{n'_1} \otimes \cdots \otimes \mathbb{C}^{n'_{k'}} \otimes \{u_1\} \otimes \cdots \otimes \{u_{k-k'}\} \cong \mathbb{C}^{n'_1} \otimes \cdots \otimes \mathbb{C}^{n'_{k'}}$.

Minimal multiprojective space containing a set of points

Let us see now how to recognize the minimal multiprojective space containing a set of points $S \subset Y_{n_1,\dots,n_k}$.

Remark 1.1.11. Let $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space and consider a set of r distinct points $S \subset Y_{n_1,\dots,n_k}$. For all $i = 1,\dots,k$ we can look at

$$\#(\pi_i(S)).$$

If $\#(\pi_i(S)) > 1$ then we can actually reduce ourselves to consider on the i-th factor the projective space $\langle \pi_i(S) \rangle \cong \mathbb{P}^{n'_i}$. If there is a factor $j \in \{1,\dots,k\}$ for which $\#(\pi_j(S)) = 1$ then we can ignore it. Indeed, since $\mathbb{P}^n \times \{o\} \cong \mathbb{P}^n$, if we denote by $\{o\} = \pi_j(S)$, then each point $p \in S$ is such that $\pi_j(p) = o$, which means that the contribution of the j-th factor of $Y_{n_1,\dots,n_k}$ to each point of $\nu(S)$ is always the same and therefore it can be ignored. Therefore, rearranging the factors of $Y_{n_1,\dots,n_k}$ if necessary, we may reduce to work with the following multiprojective space

$$Y' := \mathbb{P}^{n'_1} \times \cdots \times \mathbb{P}^{n'_{k'}} \subseteq Y_{n_1,\dots,n_k},$$

where the integer $k' \leq k$ is the maximum integer such that $\#(\pi_i(S)) > 1$ for all $i = 1,\dots,k$. By construction Y' is the minimal multiprojective space containing S.

Notation 1.1.12. If S is a set of multiprojective points, when we write $\dim\langle S \rangle$ we will mean the projective dimesnion of the projective space spanned by the points of S

Dealing with the problem of tensor rank decomposition, it is useful to consider the object containing all $(r-1)$-projective linear spaces spaned by r elementary projective classes of tensors. This brings us to the notion of secant variety.

1.1.2 Secant varieties

Secant varieties are very classical objects an their interest dates back to the beginning of the 20th century when the italian school started a systematic study of dimensions of such varieties with the works of F. Palatini ([Pal06], [Pal09]), G. Scorza ([Sco08],[Sco09]) and A. Terracini ([Ter11], [Ter21])). The interest on these varieties has then been renewed due to the work of F. Zak ([Zak93]) and since then they have been continuously studied (cf. e.g. [AH95], [CC02], [CGG02], [AOP09], [LO13], [BL13]).

To introduce secant varieties we start with the notion of join of two varieties.

Definition 1.1.13. Let $X, Z \subset \mathbb{P}^N$ be two irreducible non-degenerate projective varieties. The *join of* X and Z is the Zariski closure of the union of all lines in $\mathbb{P}^N$ spanned by a point of X and a point of Z, namely

$$\mathrm{Join}(X,Z) := \overline{\{q \in \langle x, z \rangle \mid x \in X \text{ and } z \in Z, \text{ with } x \neq z\}}$$

The definition of join of two varieties can be naturally extended to more than two varieties. Indeed, if we consider r varieties $X_1,\dots,X_r \subset \mathbb{P}^N$ we define the *join of* r varieties as

$$\mathrm{Join}(X_1,\dots,X_r) := \overline{\{q \in \langle x_1,\dots,x_r \rangle \mid x_i \in X_i \text{ and if } i \neq j \text{ then } x_i \neq x_j\}}.$$

Considering the join of a variety $X \subset \mathbb{P}^N$ with itself r times corresponds to the definition of the r-th secant variety of X.

Definition 1.1.14. Let $X \subset \mathbb{P}^N$ be and irreducible non-degenerate projective variety and let r be a positive integer. The *r-th secant variety* of X is

$$\sigma_r(X) := \overline{\bigcup_{p_1,\dots,p_r \in X} \langle p_1, \dots, p_r \rangle},$$

where the closure is the the Zariski closure. If $r = 1$ then $\sigma_1(X) = X$. Moreover, the set of points of X-rank equal to r is denoted as $\sigma_r^0(X)$.

Secant varieties form a chain of subvarieties in which the previous variety is contained in the subsequential variety until we reach the ambient space:

$$X \subset \sigma_2(X) \subset \cdots \subset \sigma_{s-1}(X) \subset \sigma_s(X) = \mathbb{P}^N,$$

where the smallest integer s such that $\sigma_s(X) = \mathbb{P}^N$ is called the *generic rank*.

Remark 1.1.15. Any of the above inclusions is proper until we reach the ambient space. Indeed assume by contradiction that $\sigma_t(X) = \sigma_{t+1}(X)$ for some positive integer $t < s$. Consider a point $q \in \sigma_{t+2}(X)$, i.e. $q \in \langle p_1, \dots, p_{t+2} \rangle$ where all $p_j \in X$. Since by assumption $\sigma_t(X) = \sigma_{t+1}(X)$, then there exist t points $q_1, \dots, q_t \in X$ such that $q \in \langle q_1, \dots, q_t, p_{t+2} \rangle$, therefore $q \in \sigma_{t+1}(X) = \sigma_t(X)$. We can continue this procedure for all t until we reach $\sigma_s(X) = \mathbb{P}^N$ but this is in contradiction with the fact that $t < s$.

A first step to better understand these objects would be computing their projective dimensions.

Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety of dimension n and let $r > 1$. Let us count the number of parameters needed to define a point in $\sigma_r(X)$.

We recall that, since $\dim X = n$, to define a point of X we need n parameters. To define a generic point of $\sigma_r(X)$ we need r points of X (which require rn parameters) and then we need to specify a point on the linear space of projective dimension $r - 1$ spanned by those r points. Therefore we expect that the r-th secant variety $\sigma_r(X)$ has dimension $\min\{rn + r - 1, N\}$.

The integer arising from the parameter count is called the *expected dimension* of the r-th secant variety and we talk about an expected dimension because in general

$$\dim \sigma_r(X) \leq \min\{rn + r - 1, \dim\langle X \rangle\}$$

and sometimes the inequality is strict.

Definition 1.1.16. An irreducible non-degenerate variety $X \subset \mathbb{P}^N$ is *r-defective* if

$$\dim \sigma_r(X) < \min\{rn + r - 1, N\}.$$

If X is r-defective, the difference $\delta_r = \min\{rn + r - 1, N\} - \dim \sigma_r(X)$ is called the *r-th secant defect of X*.

In general, studying secant varieties is very challenging and the following lemma is a powerful tool to compute dimensions of secant varieties.

Lemma 1.1.17 (Terracini's Lemma [Ådl87, Corollary 1.11], [Ter11])**.** *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Let $q \in \sigma_r(X)$ be a generic point such that $q \in \langle p_1, \dots, p_r \rangle$, for generic $p_1, \dots, p_r \in X$. The tangent space of $\sigma_r(X)$ at q is*

$$T_q\sigma_r(X) = \langle T_{p_1}X, \dots, T_{p_r}X \rangle.$$

Example 1.1.18 ([Lan12, Example 5.1.2.2])**.** Let $X_{m,n}$ be the Segre variety of $Y_{m,n} = \mathbb{P}^m \times \mathbb{P}^n$. The variety $X_{m,n}$ parametrizes projective classes of rank-1 matrices of size $(m+1) \times (n+1)$ and the generic element of $\sigma_2(X_{m,n})$ parametrizes projective classes of rank-2 matrices of the same size. Let us bound its dimension with the expected dimension

$$\dim \sigma_2(X_{m,n}) \leq \min\{2(m+n)+1, (m+1)(n+1)-1\}.$$

It is easy to see that actually $\dim \sigma_2(X_{m,n}) = 2(m+1)+2(n-1)$. Indeed let $v, w \in \mathbb{C}^{m+1}$ with $v \neq \alpha w$ for all $\alpha \in \mathbb{C}$ with $\alpha \neq 0$. For $i = 1, \dots, n-1$ let $\alpha, \beta \in \mathbb{C}$ be non-zero scalars. A general rank-2 matrix A of size $(m+1) \times (n+1)$ can be written as

$$A = \begin{bmatrix} v & w & \alpha v + \beta w & \alpha^2 v + \beta^2 w & \cdots & \alpha^{n-1}v + \beta^{n-1}w \end{bmatrix}.$$

Therefore it is sufficient to use $2(m+1)+2(n+1-2)$ parameters to describe a general rank-2 matrix. Thus $X_{m,n}$ is 2-defective with 2-nd secant defect $\delta_2 = 1$.

From the following more general result we see that the above example is the only case of a 2-defective Segre variety.

Proposition 1.1.19 ([Lan12, Proposition 5.3.1.6])**.** *Let $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space of at least 3 factors and let $r \leq \min\{n_i + 1 \mid i = 1, \dots, k\}$. Then $\sigma_r(X_{n_1,\dots,n_k})$ is of the expected dimension, i.e.*

$$\dim(\sigma_r(X_{n_1,\dots,n_k})) = r(n_1 + \cdots + n_k) + r - 1.$$

The second secant variety of a Segre variety $X_{n_1,\dots n_k}$ is a well understood object. C. Raicu proved in [Rai12] that the defining ideal of $\sigma_2(X_{n_1,\dots,n_k})$ is generated by the 3×3 minors of all generalized flattenings (cf. also [LM04]). Moreover, even though a general tensor of $\sigma_2(X_{n_1,\dots,n_k})$ is a rank-2 tensor, in this variety we can find all ranks $r \leq k$. To better explain this behaviour we can to introduce another auxiliary variety that is contained in the second secant variety of an irreducible non-degenerate projective variety $X \subset \mathbb{P}^N$.

Definition 1.1.20. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. The *tangential variety* of X is the tangent developable of X that we denote by $\tau(X)$. In other words $\tau(X)$ is defined by the union of all tangent spaces to X.

We recall that the tangent space to a Segre variety $X_{n_1,\dots,n_k} = \nu(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k) \subset \mathbb{P}^N$ at a point $p = [v_1 \otimes \cdots \otimes v_k] \in X_{n_1,\dots,n_k}$ is defined by its tangent affine cone as

$$\widehat{T_pX} = V_1 \otimes v_2 \otimes \cdots \otimes v_k + v_1 \otimes V_2 \otimes v_3 \otimes \cdots \otimes v_k + \cdots + v_1 \otimes \cdots \otimes v_{k-1} \otimes V_k.$$

Indeed, taken $w_i \in V_i$ for $i = 1, \dots, k$, it is sufficient to compute

$$\lim_{t \longrightarrow 0} \frac{\mathrm{d}}{\mathrm{dt}}\big((v_1 + tw_1) \otimes \cdots \otimes (v_k + tw_k)\big) =$$
$$w_1 \otimes v_2 \otimes \cdots \otimes v_k + v_1 \otimes w_2 \otimes v_3 \otimes \cdots \otimes v_k + \cdots + v_1 \otimes \cdots \otimes v_{k-1} \otimes w_k.$$

Let $p \in X$ and consider the tangent space T_pX of X at p . Given a point $q \in T_pX$, then there exists a curve $\mathcal{C} \subset X$ such that $q \in T_p\mathcal{C}$ (cf. [Har95, Example 15.7]). This makes possible to express $\tau(X)$ as the Zariski closure of the collection of all points that lie on all the tangent lines. Tangential varieties are contained in the closure of the second secant varieties. Moreover by [Zak93, Theorem 1.4] when $\sigma_2(X)$ is not defective then $\tau(X)$ is always a hypersurface, otherwise $\tau(X) = \sigma_2(X)$. In general we have

$$\sigma_2(X) = \sigma_2^0(X) \cup (\tau(X) \setminus X) \cup X.$$

Example 1.1.21. If we consider a Segre variety $X_{m,n}$ of two factors, by Example 1.1.18 we know that $X_{m,n}$ is always 2-defective. Therefore in this case

$$\tau(X_{m,n}) = \sigma_2(X_{m,n}).$$

Another way to look at this equality is to observe that the limit of a rank-2 matrix cannot have rank greater than 2.

Set-theoretic defining equations of tangential varieties of Segre varieties have been found in [Oed] by L. Oeding. Moreover, it was found independently by Ballico-Bernardi and Buczyński-Landsberg all possible tensor ranks that appear in a tangential variety of a Segre variety (cf. [BB13b, Theorem 1], [BL14, Proposition 1.1]).

Theorem 1.1.22. *Let $X_{n_1,\dots,n_k}$ be the Segre image of the multiprojective space $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and let $\tau(X_{n_1,\dots,n_k})$ be its tangential variety. The rank of a point $p \in \tau(X_{n_1,\dots,n_k})$ is such that*

$$1 \leq r(p) \leq k$$

and all such integers appear.

In [BB13b] the authors proved that the concise space of a rank-r tensor $T \in \tau(X_{n_1,\dots,n_k})$ is actually $\mathbb{P}((\mathbb{C}^2)^{\otimes r})$, which means that the minimal multiprojective space Y' such that $T \in \langle \nu(Y') \rangle$ is given by $Y' = (\mathbb{P}^1)^r$.

Remark 1.1.23. An element $q \in \tau(X_{n_1,\dots,n_k}) \setminus X_{n_1,\dots,n_k}$ has rank equal to 2 if and only if the minimal multiprojective space Y' such that $q \in \langle \nu(Y') \rangle$ is $Y' = \mathbb{P}^1 \times \mathbb{P}^1$.

Before proceeding, it is worthwhile to mention that classifying dimensions of secant varieties of Segre varieties is still an open problem and we refer to [BCC+18, Sections 3 and 4] where there are collected several results on this problem.

Since it will be useful in the sequel, we recall the classification of any defective third secant variety of a Segre variety $X_{n_1,\dots,n_k}$.

Theorem 1.1.24 ([AOP09, Theorem 4.5])**.** *The third secant variety of a Segre variety $X_{n_1,\dots,n_k}$ is never defective unless either $X_{1,1,1,1} = \nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ or $X_{1,1,a} = \nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^a)$, with $a \geq 3$.*

Another extremely useful object to study secant varieties is the so-called abstract secant variety of a given $X \subset \mathbb{P}^N$.

Definition 1.1.25. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety and denote by X_{reg} the set of non-singular points of X. The *r-th abstract secant variety of* X is

$$AbSec_r(X) := \overline{\{(q, (p_1, \dots, p_r)) \in \mathbb{P}^N \times X_{\mathrm{reg}}^r \,|\, q \in \langle p_1, \dots, p_r \rangle \cong \mathbb{P}^{r-1}\}} \subset \mathbb{P}^N \times X^r.$$

Denote by

$$AbSec_r^0(X) = \{(q, (p_1, \ldots, p_r)) \in \mathbb{P}^N \times X_{\text{reg}}^r \,|\, q \in \langle p_1, \ldots, p_r \rangle \cong \mathbb{P}^{r-1}\}.$$

Consider the projection T_r of $AbSec_r^0(X)$ onto $\mathbb{P}^N$

$$T_r : AbSec_r^0(X) \longrightarrow \sigma_r^0(X).$$

The projection T_r is called the r^{th} *Terracini map*.

Definition 1.1.26. A rank-r point $q \in \mathbb{P}^N$ is *identifiable* if $T_r^{-1}(q)$ is a singleton.

Identifiable tensors are very important for the applications and we will introduce the identifiability problem more in detail together with some applications in Section 2.1. Before going further, let us just recall that matrices are non-identifiable tensors.

Example 1.1.27. Let $A \in \mathbb{C}^m \otimes \mathbb{C}^n$ be a rank-r matrix, i.e. there exist column vectors $u_i \in \mathbb{C}^m$, $v_i \in \mathbb{C}^n$ such that

$$A = \sum_{i=1}^{r} u_i \otimes v_i = \sum_{i=1}^{r} u_i v_i^T.$$

Let us denote the matrix whose columns are the vectors u_i by $U = [u_1 \cdots u_r]$ and the matrix whose columns are the vectors v_i by $V = [v_1 \cdots v_r]$ for all $i = 1, \ldots, r$. Then

$$A = \sum_{i=1}^{r} u_i v_i^T = UV^T = (UX^{-1})(VX^T)^T$$

for any $X \in GL_r(\mathbb{C})$. If we denote by $U_X = UX^{-1}$ and by $V_X = VX^T$, then for any $X \in GL_r(\mathbb{C})$ we have a different decomposition of A given by $A = U_X V_X^T$.

1.2 Cohomology for tensors

In this section we introduce all the cohomological tools needed in the sequel. For standard notion related to this part we refer to [Har77].
Let $Y_{n_1,\ldots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We recall that for all $i = 1, \ldots, k$ we denoted by $\pi_i : Y_{n_1,\ldots,n_k} \to \mathbb{P}^{n_i}$ the projection of $Y_{n_1,\ldots,n_k}$ onto the i-th factor (cf. Notation 1.1.6). The Segre embedding is the map associated to the linear system $|\mathcal{O}_{Y_{n_1,\ldots,n_k}}(1, \ldots, 1)| = |\pi_1^*(\mathcal{O}_{\mathbb{P}^{n_1}}(1)) \otimes \cdots \otimes \pi_k^*(\mathcal{O}_{\mathbb{P}^{n_k}}(1))|$ and we recall that

$$h^0\left(\mathcal{O}_{Y_{n_1,\ldots,n_k}}(1, \ldots, 1)\right) = \prod_{i=1}^{k} (n_i + 1).$$

Notation 1.2.1. for $1 \le i \le k$, $\varepsilon_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, where the only 1 is in the i-th place and $\widehat{\varepsilon_i}$ which is a k-uple with all one's but the i-th place, which is filled by 0, i.e. $\widehat{\varepsilon_i} := (1, \ldots, 1, 0, 1, \ldots, 1)$. If we denote by $\varepsilon_I := \sum_{i \in I} \varepsilon_i$, where $I \subset \{1, \ldots, k\}$, then $\widehat{\varepsilon_I}$ is a k-uple with 0's in position of the indices appearing in ε_I and 1's everywhere else.

We saw that problems related to rank-r tensors can be translated into problems related to r independent points of a Segre variety. Therefore, working with tensors, we are implicitly working with 0-dimensional schemes of reduced points, namely simple points. Simple points are not the only type of 0-dimensional schemes that one can use when dealing with tensors. Indeed, by means of the Terracini's Lemma, there is a very powerful connection between secant varieties and schemes of double fat points. Before stating this connection, let us introduce the notion of double fat point adapted for multiprojective spaces.

Definition 1.2.2. . For any $p \in Y_{n_1,\dots,n_k}$, denote by $(2p, Y_{n_1,\dots,n_k})$ the first infinitesimal neighbourhood of p in $Y_{n_1,\dots,n_k}$, i.e. the closed subscheme of $Y_{n_1,\dots,n_k}$ with $(\mathcal{I}_{p,Y_{n_1,\dots,n_k}})^2$ as its ideal sheaf. For any finite set $S \subset Y_{n_1,\dots,n_k}$ let

$$(2S, Y_{n_1,\dots,n_k}) := \bigcup_{p\in S}(2p, Y_{n_1,\dots,n_k}).$$

Remark 1.2.3 ([CGG02, Corollary 1.2])**.** Let $p_1, \dots, p_r \in X_{n_1,\dots,n_k}$ be generic points and consider a generic point $q \in \langle p_1, \dots, p_r\rangle$. By Terracini's Lemma

$$\dim \sigma_r(X_{n_1,\dots,n_k}) = \dim\langle T_{p_1}X_{n_1,\dots,n_k}, \dots, T_{p_r}X_{n_1,\dots,n_k}\rangle.$$

The Segre variety $X_{n_1,\dots,n_k}$ is embedded in $\mathbb{P}^N = \mathbb{P}H^0\left(\mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)\right)^*$ and we can look at elements of $H^0\left(\mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)\right)$ as hyperplanes in $\mathbb{P}^N$. A hyperplane containing $T_{p_i}X_{n_1,\dots,n_k}$ corresponds to an element of $H^0\left(\mathcal{I}_{(2p_i,Y_{n_1,\dots,n_k})}(1,\dots,1)\right)$. Therefore, if we denote by $(2S, Y_{n_1,\dots,n_k})$ the 0-dimensional scheme of double fat points such that $S = \{p_1, \dots, p_r\}$, we have that hyperplanes containing $\langle T_{p_1}X_{n_1,\dots,n_k}, \dots, T_{p_r}X_{n_1,\dots,n_k}\rangle$ are the sections of $H^0\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k}}(1,\dots,1)\right)$. Thus

$$\dim(\sigma_r(X_{n_1,\dots,n_k})) = N - h^0\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right).$$

Therefore, another way to compute the dimension of the r-th secant variety of a given projective variety X is to compute the dimension of the global sections of the ideal sheaf associated to a scheme of r generic double fat points on X. A very useful tool to compute such dimensions is the so-called Horace method.

1.2.1 Horace method and differential Horace method

The *postulation* of a 0-dimensional scheme $Z \subset \mathbb{P}^n$ is the sequence of values

$$(h^0(\mathcal{I}_Z(d))_{d\geq 1}.$$

The Horace method was introduced by A. Hirschowitz in [Hir85] and it was aimed to compute the *postulation* of a 0-dimensional scheme Z in a projective space $\mathbb{P}^n$.

Given a 0-dimensional scheme $Z \subset \mathbb{P}^n$, in order to prove that

$$h^0\left(\mathcal{I}_Z(d)\right) = 0$$

for some $d \geq 0$, we can use two other ideals sheaves that are related to $\mathcal{I}_Z(d)$ through an exact sequence and we can prove that the dimensions of their global sections are equal to zero. More precisely, given a hyperplane $H \subset \mathbb{P}^n$,

- the *trace* of Z with respect to H is the scheme-theoretic intersection $H \cap Z$;
- the *residue* $\mathrm{Res}_H(Z)$ of Z with respect to H is the 0-dimensional scheme defined by the ideal sheaf $\mathcal{I}_Z : \mathcal{I}_H$, which is the ideal sheaf corresponding to the colon ideal $I_Z : I_H$ that contains all polynomials p such that $pI_H \subseteq I_Z$.

This leads to consider the exact sequence of sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{H\cap Z,H}(d) \to 0,$$

from which follows that

$$h^0(\mathcal{I}_Z(d)) \leq h^0\big(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)\big) + h^0\big(\mathcal{I}_{H\cap Z,H}(d)\big).$$

Remark that on the left hand side of the exact sequence we deal with the ideal sheaf of a zero-dimensional scheme in degree $d-1$, while on the right hand side we deal with a zero-dimensional scheme that belongs to a hyperplane of $\mathbb{P}^n$ and therefore we are working in dimension $n-1$. Therefore, proving that both $h^0\big(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)\big) = h^0\big(\mathcal{I}_{H\cap Z,H}(d)\big) = 0$ is easier than proving that $h^0\big(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)\big) = 0$. The idea behind this method is to kill one member at a time in order to solve our final problem, just like, in the ancient roman legend, Publius of the Horatius brothers killed one at a time all the three Curiatus brothers, which is where the name of the method comes from.

The differential Horace method ([AH92, AH00]) is a degeneration of the Horace method and it has been introduced to overcome situations in which the Horace method does not succeed. We refer to [BCC$^+$18, Subsections 2.2.1 and 2.2.2] for a more detailed introduction of both methods and we present here a particular application of the differential horace method that will be used in the following.

Lemma 1.2.4. *Let X be an integral projective variety, D an integral effective Cartier divisor of X and $\mathcal{L}$ a line bundle on X such that $h^i(\mathcal{L}) = 0$ for all $i > 0$ and $h^1(\mathcal{L}(-D)) = 0$. Let $Z \subset X$ be a closed subscheme. Suppose*

$$h^1\left(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D)\right) = 0 \text{ and } h^1\left(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}\right) = 0.$$

Fix $i \in \{0,1\}$. To prove that a general union A of Z and one double point satisfies $h^i(\mathcal{I}_A \otimes \mathcal{L}) = 0$ it is sufficient to prove that

$$h^i\left(\mathcal{I}_{\mathrm{Res}_D(Z)\cup(2o,D)} \otimes \mathcal{L}(-D)\right) = 0$$

and

$$h^i\left(D, \mathcal{I}_{(Z\cap D)\cup\{o\}} \otimes \mathcal{L}_{|D}\right) = 0,$$

where o is a general point of D.
Since o is general in D, $h^1\left(D, \mathcal{I}_{(Z\cap D)\cup\{o\}} \otimes \mathcal{L}_{|D}\right) = 0$ if and only if

$$h^1\left(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}\right) = 0 \text{ and } h^0\left(D, \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D}\right) > 0.$$

Remark 1.2.5. The Horace method together with its differential version was used by J. Alexander and A. Hirschowitz to completely characterize the postulation of a 0-dimensional scheme of generic double fat points in the projective space $\mathbb{P}^n$ (cf. [AH95]). Therefore, knowing the postulation of a 0-dimensional scheme of generic double fat points in the projective space $\mathbb{P}^n$ corresponds to find all dimensions of secant varieties of any Veronese variety. We refer to [BO08] for a modern and simplified proof of the classification theorem and in particular to [BO08, Section 7] for some interesting historical remarks on the problem.

At the end of the section we provide an example of the use of this method.

When dealing with multiprojective spaces $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ (not necessarily embedded via $\mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)$) one can use another method to compute the dimension of the global sections of the ideal sheaf of a 0-dimensional scheme.

The Multiprojective-Affine-Projective Method [CGG05]

For $i = 1,\dots,k$ let $d_i \geq 1$ and let V_i be a vector space of dimension $\dim V_i = n_i + 1$. The Segre-Veronese embedding of $Y_{n_1,\dots,n_k} = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k$ is

$$\begin{aligned} \nu_{(d_1,\dots,d_k)} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k &\to \mathbb{P}(Sym^{d_1}V_1 \otimes \cdots \otimes Sym^{d_k}V_k) \\ ([v_1],\dots,[v_k]) &\mapsto [v_1^{d_1} \otimes \cdots \otimes v_k^{d_k}]. \end{aligned}$$

Segre-Veronese varieties parametrize rank-1 partially symmetric tensors and the Segre-Veronese embedding corresponds to the linear system

$$|\mathcal{O}_{Y_{n_1,\dots,n_k}}(d_1,\dots,d_k)| = |\pi_1^*(\mathcal{O}_{\mathbb{P}^{n_1}}(d_1)) \otimes \cdots \otimes \pi_k^*(\mathcal{O}_{Y_{n_1,\dots,n_k}}(d_k))|.$$

The multiprojective-affine-projective method was introduced by Catalisano-Geramita-Gimigliano in [CGG05] to compute dimensions of some Segre-Veronese varieties. The method allows to avoid the multigraded structure of the problem by passing to a standard projective space. Indeed, given a zero-dimensional scheme $Z \subset Y_{n_1,\dots,n_k}$, the idea is to construct a scheme $W \subset \mathbb{P}^{n_1+\cdots+n_k}$ such that $h^0(\mathcal{I}_W(d_1+\cdots+d_k)) = h^0(\mathcal{I}_Z(d_1,\dots,d_k))$.

Let $n = n_1+\cdots+n_k$, consider the map $g : Y_{n_1,\dots,n_k} \to \mathbb{A}^n$ and then take the embedding $\mathbb{A}^n \to \mathbb{P}^n$. Composing the two maps one gets $f \colon Y_{n_1,\dots,n_k} \to \mathbb{P}^n$. Let $Z \subset Y_{n_1,\dots,n_k}$ be a zero-dimensional scheme contained in the affine chart on which g is defined and let

$$Z' = f(Z).$$

We recall that given an integer $k \geq 0$ and a linear space $L \subset \mathbb{P}^n$, then kL represents the scheme given by $\mathcal{I}_L^k$ as its ideal sheaf. For $i = 1,\dots,k$ let $\Lambda_i \cong \mathbb{P}^{n_i-1} \subset \mathbb{P}^n$ and set

$$W_i = \left(\sum_{j\neq i} n_j\right)\Lambda_i.$$

We have now all the necessary tools to state the result.

Theorem 1.2.6 ([CGG05, Theorem 1.1])**.** *Let $Z, Z', W_1,\dots,W_k$ be as above and consider $W = Z' + W_1 + \cdots + W_k \subset \mathbb{P}^n$, which is the smallest projective space containig $Z', W_1,\dots,W_k$. Then we have*

$$h^0(\mathcal{I}_W(d_1+\cdots+d_k)) = h^0(\mathcal{I}_Z(d_1,\dots,d_k)).$$

We conclude this subsection with the following example in which we compute the dimension of the global sections of the ideal sheaf of a zero-dimensional scheme with both methods.

Example 1.2.7. Let $Z \subset Y_{1,1,1} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional scheme given by the union of 1 double fat point and 4 simple points in general position. We will show that

$$h^0(\mathcal{I}_Z(1,1,1)) = 0$$

by using both the Horace method in the multiprojective space and the multiprojective-affine-projective method.

Horace method in the multiprojective space
Let $H \in |\mathcal{O}_{Y_{1,1,1}}(0,1,1)|$ containing the double point and one of the generic simple points. The residual exact sequence of Z with respect to H is

$$0 \to \mathcal{I}_{\mathrm{Res}_{H(Z)}}(1,0,0) \to \mathcal{I}_Z(1,1,1) \to \mathcal{I}_{H\cap Z,H}(1,1,1) \to 0.$$

In this case $\mathrm{Res}_H(Z)$ is given by 3 simple points, while $H \cap Z$ is given by the double point and the simple point we selected. Now $h^0(\mathcal{I}_{Z\cap H,H}(1,1,1)) = h^0(\mathcal{I}_{\mathbb{P}^1}(1,1))$ and since there are no hyperplanes of $\mathbb{P}^3$ containing both a double point and a generic point then $h^0(\mathcal{I}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)) = 0$. On the other side $h^0\big(\mathcal{I}_{\mathrm{Res}_{H(Z)}}(1,0,0)\big) = 0$ because a hyperplane of $\mathbb{P}^1$ cannot contain 3 simple points. Therefore we conclude that $h^0(\mathcal{I}_Z(1,1,1)) = 0$. Now let us prove the same result with the other method.

Multiprojective-affine-projective method
Keeping the notation of Theorem 1.2.6, in this case we have $n = 3$, $d_1 = d_2 = d_3 = 1$. Therefore

$$h^0(\mathcal{I}_Z(1,1,1)) = h^0(\mathcal{I}_W(3))$$

where W is a zero-dimensional scheme given by 4 double fat points and 4 simple points in $\mathbb{P}^n$.

Let $H \subset \mathbb{P}^3$ be a plane containing 3 double points of W. We want to specialize another simple point on H. We can do this specialization since we recall that if there exists a hypersurface of degree k passing through ℓ generic points then there exists a hypersurface of the same degree passing through ℓ specific points. In other words, if we denote by $\overline{W}$ the specialization of W in which one of the simple points lie on H, then we have that

$$h^0(\mathcal{I}_W(k)) \leq h^0(\mathcal{I}_{\overline{W}}(k))$$

for all $k > 0$. Therefore, since we want to show that $h^0(\mathcal{I}_W(3)) = 0$ then it is enough to show that $h^0(\mathcal{I}_{\overline{W}}(3)) = 0$. Thus let us work with $\overline{W}$ and consider the following exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{H(\overline{W})}}(2) \to \mathcal{I}_{\overline{W}}(3) \to \mathcal{I}_{H\cap\overline{W},H}(3) \to 0.$$

In this case $\mathrm{Res}_H(\overline{W})$ is the scheme of what is left from $\overline{W}$ outside the plane, i.e. it is given by 6 simple points and a double point, while $H\cap\overline{W}$ is the zero-dimensional scheme containing 3 double fat points and a simple point.

Let us focus for the moment on $\mathcal{I}_{H\cap\overline{W},H}(3)$. The intersection $H \cap \overline{W}$ is the scheme of 3 double fat points and a simple point on a plane. We recall that a zero-dimensional scheme S has a good postulation if for all $k \in \mathbb{Z}$ we have either that $h^0(\mathcal{I}_S(k)) = 0$ or that $h^1(\mathcal{I}_S(k)) = 0$. If $\ell \neq 2, 5$ then ℓ double fat points on a plane have a good postulation. Therefore, in this case it is sufficient to make a parameter count to verify that $h^0\left(\mathcal{I}_{H\cap\overline{W},H}(3)\right) = 0$. Having 6 degrees of freedom, we see that the double point gives us 3 conditions and we have to add another 3 conditions given by the remaining 3 simple points. Therefore $h^0\left(\mathcal{I}_{H\cap\overline{W},H}(3)\right) = 0$.

To conclude, we have to show that $h^0\big(\mathcal{I}_{\mathrm{Res}_{H(\overline{W})}}(2)\big) = 0$. Let us call $T = \mathrm{Res}_H(\overline{W})$ and denote by $\overline{T}$ its specialization obtained by specifying the double fat point on a hyperplane $H' \subset \mathbb{P}^3$. Therefore $H' \subset \mathbb{P}^3$ contains 3 simple points and a double fat point of $\overline{T}$ and we can consider the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{H'(\overline{T})}}(1) \to \mathcal{I}_{\overline{T}}(2) \to \mathcal{I}_{H'\cap\overline{T},H'}(2) \to 0.$$

In this case $h^0\big(\mathcal{I}_{\mathrm{Res}_{H'}(\overline{T})}(1)\big) = 0$ because $\mathrm{Res}_{H'}(\overline{T})$ is the union of 4 simple points and there is no plane of $\mathbb{P}^3$ passing through 4 general points. One can easily prove that also $h^0\left(\mathcal{I}_{H'\cap\overline{T},H'}(2)\right) = 0$ since one double point has a good postulation in the plane. Therefore, going back to the beginning, we get that

$$h^0(\mathcal{I}_T(2)) = h^0\big(\mathcal{I}_{\mathrm{Res}_H(\overline{W})}(2)\big) = 0.$$

Thus $h^0(\mathcal{I}_{\overline{W}}(3)) = 0$, from which follows that

$$h^0(\mathcal{I}_W(3)) = 0.$$

1.2.2 A very useful lemma for zero-dimensional schemes

An extremely useful tool that will turn out to be crucial in many proofs of Chapter 2 is [BB13a, Lemma 5.1]. We first recall the analogous statement given in [BBCG19, Lemma 2.4] in the setting of zero-dimensional schemes and then we explain how to use the forthcoming lemma in our context.

Lemma 1.2.8 (Ballico–Bernardi–Christandl–Gesmundo)**.** *Let $X \subseteq \mathbb{P}^n$ be an irreducible variety embedded by the complete linear system associated with $\mathcal{L} = \mathcal{O}_X(1)$. Let $p \in \mathbb{P}^n$ and let A, B be zero-dimensional schemes in X such that $p \in \langle A\rangle$, $p \in \langle B\rangle$ and there are no $A' \subsetneq A$ and $B' \subsetneq B$ with $p \in \langle A'\rangle$ or $p \in \langle B'\rangle$. Suppose $h^1(\mathcal{I}_B(1)) = 0$. Let $C \subseteq \mathbb{P}^n$ be an effective Cartier divisor such that $\mathrm{Res}_C(A) \cap \mathrm{Res}_C(B) = \emptyset$. If $h^1\big(X, \mathcal{I}_{\mathrm{Res_C(A\cup B)}}(1)(-C)\big) = 0$ then $A \cup B \subseteq C$.*

Let us rephrase it in terms of sets of points of multiprojective spaces embedded via $|\mathcal{O}(1,\ldots,1)|$.
Let $k \geq 2$, let $Y_{n_1,\ldots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $q \in \mathbb{P}^N = \langle X_{n_1,\ldots,n_k}\rangle$ be a point of rank r and let $A, B \subset Y_{n_1,\ldots,n_k}$ be sets of points evincing the rank of q, i.e. such that $\#A = \#B = r$ and $q \in \langle \nu(A)\rangle \cap \langle \nu(B)\rangle$. Write $S := A \cup B$.

In this setting, the irreducible variety X considered in Lemma 1.2.8 is the Segre variety $X_{n_1,\ldots,n_k}$. The residual scheme $\mathrm{Res}_C(S)$ is therefore $S \setminus (S \cap C)$. The assumption $h^1(\mathcal{I}_B(1)) = 0$ of [BBCG19, Lemma 2.4], in the setting of Segre varieties becomes $h^1(\mathcal{I}_B(1,\ldots,1)) = 0$, which means that the points of $\nu(B)$ are linearly independent and this assumption is satisfied since both A and B are sets evincing the rank of q.

With all this said we can state the specific version of [BBCG19, Lemma 2.4] and [BB13a, Lemma 5.1] which is needed in the following chapter.

Lemma 1.2.9. *Let $k \geq 2$ and consider $Y_{n_1,\ldots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where all $n_i \geq 1$. Let $q \in \mathbb{P}^N$, $A, B \subset Y_{n_1,\ldots,n_k}$ be two different subsets evincing the rank of q and write $S = A \cup B$. Let $D \in |\mathcal{O}_{Y_{n_1,\ldots,n_k}}(\varepsilon)|$ be a divisor such that $A \cap B \subset D$, where $\varepsilon = \sum_{i\in I} \varepsilon_i$ for some $I \subset \{1,\ldots,k\}$ as introduced in Notation 1.2.1. If*

$$h^1\big(\mathcal{I}_{S\setminus S\cap D}(\hat{\varepsilon})\big) = 0$$

then $S \subset D$.

The above lemma gives a sufficient condition so that the whole $S = A \cup B$ is contained in a given divisor D of the variety $X_{n_1,\ldots,n_k}$. As already mentioned, in the next chapter we will use Lemma 1.2.9 by taking as D a divisor which is still a multiprojective space, i.e. we will take $D \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_i)|$ for some i, which means that

$$D \cong \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{i-1}} \times \mathbb{P}^{n_i - 1} \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n}.$$

Working in the Autarky assumption (cf. Lemma 1.1.8), by using this type of divisor in Lemma 1.2.9 one can easily get a contradiction because in this way we would have that our two distinct decompositions lie in a smaller multiprojective space.

Remark 1.2.10. If A, B are two disjoint distinct sets evincing the rank of a tensor q then the assumption $A \cap B \subset D$ of Lemma 1.2.9 is always satisfied.

Since in the next chapter we will study the identifiability of rank-3 tensors, we will deal with sets of three points in some Segre variety. Therefore it is convenient to understand the structure of the dependent sets of at most 3 points of a Segre variety (i.e. 3 rank-1 tensors).

Lemma 1.2.11. *A set of points $E \subset Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ of cardinality at most 3 does not impose independent conditions to multilinear forms over $Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i} := \mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_k}$ for some $i = 1, \dots, k$, (i.e. $h^1(\mathcal{I}_E(\hat{\varepsilon}_i)) > 0$) if and only if one of the following cases occurs:*

1. *$\#(E) = 3$ and there is $j \in \{1, \dots, k\} \setminus \{i\}$ such that $\#(\pi_h(E)) = 1$ for all $h \notin \{i, j\}$;*
2. *there are $u, v \in E$ such that $u \neq v$ and $\eta_i(u) = \eta_i(v)$.*

Proof. The fact that both items 1. and 2. imply that $h^1(\mathcal{I}_E(\hat{\varepsilon}_i)) > 0$ is obvious. Let us describe the other implication.

By definition $H^0\big(\mathcal{O}_{Y_{n_1,\dots,n_k}}(\hat{\varepsilon}_i)\big) \cong H^0\big(\mathcal{O}_{Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i}}(1, \dots, 1)\big)$, and $\mathcal{O}_{Y_{n_1,\dots,n_k}}(\hat{\varepsilon}_i)$ is not a very ample line bundle. So we cannot be sure about the injectivity of the restriction $\eta_{i|E}$ of η_i to the finite set E.

If $\eta_{i|E}$ is not injective one immediately gets that $h^1(\mathcal{I}_E(\hat{\varepsilon}_i)) > 0$. Moreover if $\eta_{i|E}$ is not injective it means that there are 2 distinct points of E, say u and v which are mapped by η_i onto the same point, i.e. we are in item 2. of this lemma.

Now assume that $\eta_{i|E}$ is injective (i.e. we are not in item 2.). This implies that $\#(E) = \#(\eta_i(E))$. We have by hypothesis that $h^1(\mathcal{I}_E(\hat{\varepsilon}_i)) > 0$. Since by definition $h^1(\mathcal{I}_E(\hat{\varepsilon}_i)) = h^1\big(Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i}, \mathcal{I}_{\eta_i(E)}(1, \dots, 1)\big)$ we have that $\eta_i(E)$ does not impose independent conditions to the multilinear forms over $Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i}$, therefore $\#(\eta_i(E)) \geq 3$ which clearly implies that $\#(\eta_i(E)) = 3$ since by hypothesis the cardinality of E is at most 3. Now $\eta_i(E)$ is a set of 3 distinct points on $Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i}$ which does not impose independent conditions to the multilinear forms over $Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i}$, and $\mathcal{O}_{Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i}}(1, \dots, 1)$ is very ample, therefore the 3 points of $\eta_i(E)$ must be mapped to collinear points by the Segre embedding ν_i of Y_i. Hence, by the structure of the Segre variety $\nu_i(Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i})$, we get that $\langle \nu_i(\eta_i(E)) \rangle \subseteq \nu_i(Y_{n_1,\dots,\hat{n}_i,\dots,n_k;i})$ and there is $j \in \{1, \dots, k\} \setminus \{i\}$ such that $\#(\pi_h(\eta_i(E))) = 1$ for all $h \notin \{i, j\}$. Since $h \neq i$, we have $\pi_h(\eta_i(E)) = \pi_h(E)$. □

1.2.3 Tools for a scheme of double fat points

As we said at the beginning of the section, zero-dimensional schemes of double fat points are strictly connected to secant varieties (cf. Remark 1.2.3). In the following we prove some useful results on the dimension $h^1\big(\mathcal{I}_{(2S, Y_{n_1,\dots,n_k})}(1, \dots, 1)\big)$ for some finite $S \subset Y_{n_1,\dots,n_k}$.

Notation 1.2.12. for a zero-dimensional scheme $A \subset Y_{n_1,\dots,n_k}$ we denote by $\delta(A, Y_{n_1,\dots,n_k}) = h^1(\mathcal{I}_A(1,\dots,1))$. In particular, given a set of points $S \subset Y_{n_1,\dots,n_k}$ we will denote the dimension of the first cohomology group of the ideal sheaf of $(2S, Y_{n_1,\dots,n_k})$ embedded via Segre as

$$\delta(2S, Y) := h^1(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)).$$

Let us first observe how two zero-dimensional schemes $A \subset B \subset Y_{n_1,\dots,n_k}$ are related to each other.

Remark 1.2.13. If $A \subset B \subset Y_{n_1,\dots,n_k}$ are zero-dimensional schemes, then

$$\delta\left(A, Y_{n_1,\dots,n_k}\right) \le \delta(B, Y_{n_1,\dots,n_k}) \le \delta(A, Y_{n_1,\dots,n_k}) + \deg(B) - \deg(A). \tag{1.2.3}$$

Indeed the first inequality is clear since $A \subset B$. Moreover we remark that if $A \subset B$ then $h^0(\mathcal{I}_B(1,\dots,1)) \le h^0(\mathcal{I}_A(1,\dots,1))$. So by the restriction exact sequences of both A and B with respect to $Y_{n_1,\dots,n_k}$, we get the second inequality. In particular for all $S' \subset S \subset Y_{n_1,\dots,n_k}$ we have

$$\delta(2S', Y_{n_1,\dots,n_k}) \le \delta(2S, Y_{n_1,\dots,n_k}) \le \delta(2S', Y_{n_1,\dots,n_k}) + (\#S - \#S')(\dim Y_{n_1,\dots,n_k} + 1). \tag{1.2.4}$$

The following key lemma proves a sort of concision for the value $\delta(2S, Y_{n_1,\dots,n_k})$ of a finite set $S \subset Y_{n_1,\dots,n_k}$. By the previous remark, if $S' \subset S$ is a scheme of r double points, $\delta(2S', Y_{n_1,\dots,n_k})$ is smaller than $\delta(2S, Y_{n_1,\dots,n_k})$.
In the following lemma we fix the finite set $S \subset W$ of cardinality r and we compare the behaviour of the two values $\delta(2S, W)$ and $\delta(2S, Y_{n_1,\dots,n_k})$, where $W \subsetneq Y_{n_1,\dots,n_k}$ is a smaller multiprojective space. Since $Y_{n_1,\dots,n_k}$ is no longer the minimal multiprojective space containing $S \subset W \subsetneq Y_{n_1,\dots,n_k}$, the value $\delta(2S, Y_{n_1,\dots,n_k})$ may be bigger than $\delta(2S, W)$. In case (a) of Lemma 1.2.14 we give an upper bound for $\delta(2S, Y_{n_1,\dots,n_k})$ via $\delta(2S, W)$. Case (b) can be considered as a strong version of concision because the achievement of equality $\delta(2S, W) = \delta(2S, Y_{n_1,\dots,n_k})$ is telling that the defect of $2S$ is independent from the number of factors of the multiprojective space where S is embedded.

Lemma 1.2.14. *Let $W \subsetneq Y_{n_1,\dots,n_k}$ be multiprojective spaces. Let $S \subset W$ be a finite set. Then:*

(a) $\delta(2S, W) \le \delta(2S, Y_{n_1,\dots,n_k}) \le \delta(2S, W) + (\#S - 1)(\dim Y_{n_1,\dots,n_k} - \dim W)$.

(b) If W is isomorphic to a factor of $Y_{n_1,\dots,n_k}$, i.e. $Y_{n_1,\dots,n_k} = W \times Y'$, with Y' a multiprojective space of positive dimension and $\nu(S)$ is linearly independent, then $\delta(2S, W) = \delta(2S, Y_{n_1,\dots,n_k})$.

Proof. Since the restriction map $H^0\left(Y_{n_1,\dots,n_k}, \mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)\right) \to H^0\left(W, \mathcal{O}_W(1,\dots,1)\right)$ is surjective and $(2S, W) \subseteq (2S, Y_{n_1,\dots,n_k})$, the first inequality of part (a) is the first inequality of (1.2.3). Therefore, we just need to prove the second inequality of (a) and we will do it by induction on the integer $\dim Y_{n_1,\dots,n_k} - \dim W$.

First assume $\dim Y_{n_1,\dots,n_k} = \dim W + 1$. Thus there is $i \in \{1,\dots,k\}$ such that $W \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_i)|$. Note that $W \cap (2S, Y_{n_1,\dots,n_k}) = (2S, W)$ and that $\mathrm{Res}_W(2S, Y_{n_1,\dots,n_k}) = S$. Thus the residual exact sequence of W gives the following exact sequence

$$0 \longrightarrow \mathcal{I}_S(\hat{\varepsilon}_i) \longrightarrow \mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1) \longrightarrow \mathcal{I}_{(2S,W)}(1,\dots,1) \longrightarrow 0. \tag{1.2.5}$$

Since the restriction map $H^0\left(Y_{n_1,\dots,n_k},\mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)\right) \to H^0\left(W,\mathcal{O}_W(1,\dots,1)\right)$ is surjective, $h^1\left(Y_{n_1,\dots,n_k},\mathcal{I}_{(2S,W)}(1,\dots,1)\right) = h^1\left(W,\mathcal{I}_{(2S,W)}(1,\dots,1)\right)$. Since S is a finite set, $h^i(\mathcal{L}) = 0$ for all $i > 0$ and all line bundles $\mathcal{L}$ on S. The long cohomology exact sequence of the exact sequence

$$0 \longrightarrow \mathcal{I}_S(\hat{\varepsilon}_i) \longrightarrow \mathcal{O}_{Y_{n_1,\dots,n_k}}(\hat{\varepsilon}_i) \longrightarrow \mathcal{O}_S(\hat{\varepsilon}_i) \longrightarrow 0$$

gives $h^2\left(\mathcal{I}_S(\hat{\varepsilon}_i)\right) = h^2\left(\mathcal{O}_{Y_{n_1,\dots,n_k}}(\hat{\varepsilon}_i)\right) = 0$. Since $h^1\left(\mathcal{O}_{Y_{n_1,\dots,n_k}}(\hat{\varepsilon}_i)\right) = 0$ and $\mathcal{O}_{Y_{n_1,\dots,n_k}}(\hat{\varepsilon}_i)$ is globally generated, $h^1\left(\mathcal{I}_S(\hat{\varepsilon}_i))\right) \le \#S - 1$. Thus (1.2.5) gives part (a). Note that we have $h^1\left(W,\mathcal{I}_{(2S,W)}(1,\dots,1)\right) = h^1\left(Y_{n_1,\dots,n_k},\mathcal{I}_{(2S,Y)}(1,\dots,1)\right)$ if $h^1\left(\mathcal{I}_S(\hat{\varepsilon}_i)\right) = 0$.

Now assume $\dim Y_{n_1,\dots,n_k} \ge \dim W + 2$. We can always find a multiprojective space M such that $W \subsetneq M \subseteq Y_{n_1,\dots,n_k}$ and in particular we take $M \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_i)|$ for some i. The inductive step follows by applying the codimension one case to the inclusion $M \subset Y_{n_1,\dots,n_k}$ and we conclude by applying the inductive assumption on the inclusion $W \subset M$.

Assume that W is isomorphic to a factor of $Y_{n_1,\dots,n_k}$, say $Y_{n_1,\dots,nk} \cong W \times Y'$. We will show (b) by induction on the number of factors of Y'. Assume $Y_{n_1,\dots,n_k} = W \times \mathbb{P}^m$ for some $m > 0$, where $W \cong W' \times \{o\}$ for some $o \in \mathbb{P}^m$ and some positive dimensional projective space W'. We will work by induction on $m \ge 1$.
First assume $m = 1$, so $W \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_2)|$ and in particular $W = \pi_2^{-1}(o)$ where $o \in \mathbb{P}^1$. We remark that the Segre embedding ν_2 of W can be seen as the restriction to W of the Segre embedding of $Y_{n_1,\dots,n_k}$. Thus $\nu(S)$ is linearly independent if and only if $\nu_2(S)$ is linearly independent. Note that the linear independence of $\nu_2(S)$ is equivalent to $h^1\left(\mathcal{I}_S(1,0)\right) = 0$ because $\pi_2(S) = \{o\}$. Since we already proved part (a) and $h^1\left(\mathcal{I}_S(1,0)\right) = 0$ we get the result.

Assume now $m \ge 2$ and fix $H \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_2)|$ containing W. By induction we get $\delta(2S,W) = \delta(2S,H)$. Since H is a divisor of $Y_{n_1,\dots,n_k}$ and $h^1(\mathcal{I}_S(0,1)) = 0$ we get the result by applying the base case of (a).

Assume now $Y_{n_1,\dots,n_k}$ has $k \ge 3$ factors, i.e. $Y_{n_1,\dots,n_k} \cong W \times Y'$ where Y' is a multiprojective space with at least two factors. Let $\mathbb{P}^{n_k}$ be the last factor of $Y_{n_1,\dots,n_k}$, again we will show the result by induction on $n_k \ge 1$. If $n_k = 1$, one can always find $M \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_k)|$ containing W and by induction we get $\delta(2S,W) = \delta(2S,M)$. We remark as before that the Segre embedding of $\nu(S)$ is linearly independent if and only if $\nu_k(S)$ is linearly independent and this is equivalent to say that $h^1(\mathcal{I}_S(\hat{\varepsilon}_k)) = 0$. Since $M = \pi_k^{-1}(o)$, for some $o \in \mathbb{P}^1$ we get the result by applying (a).

Assume now $n_k \ge 2$, and take some $M \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_k)|$ containing W. By induction we get $\delta(2S,W) = \delta(2S,M)$, since $h^1(\mathcal{I}_S(\hat{\varepsilon}_k)) = 0$ and M is a divisor of $Y_{n_1,\dots,n_k}$ we get $\delta(2S,M) = \delta(2S,Y_{n_1,\dots,n_k})$ by (a). □

Let us prove now that, working with two factors, to compute $\delta(2S,Y_{n_1,n_2})$ for some particular $S \subset Y_{n_1,n_2}$, it is enough to compute the dimension of the first cohomology group of the ideal sheaf of $2S$ seen in the smallest multiprojective space containing the set of points.

Lemma 1.2.15. *Let $Y_{n_1,n_2} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and $Y' \subseteq Y_{n_1,n_2}$ with $Y' := \mathbb{P}^{m_1} \times \mathbb{P}^{m_2}$ for some $m_1, m_2 > 0$. Let $S \subset Y'$ be a finite subset such that Y' is the minimal multiprojective space containing S and suppose that both $\pi_{1|S}$ and $\pi_{2|S}$ are injective and both $\pi_1(S)$ and $\pi_2(S)$ are linearly independent. Then*

$$m_1 = m_2 = \#S - 1 \text{ and } h^1\left(Y',\mathcal{I}_{(2S,Y')}(1,1)\right) = h^1\left(Y_{n_1,n_2},\mathcal{I}_{(2S,Y_{n_1,n_2})}(1,1)\right).$$

Proof. Since $\pi_i(S)$ is linearly independent and Y' is the minimal multiprojective space containing S, then $m_1 = m_2 = \#S - 1$. Moreover since $h^0(\mathcal{I}_S(1,0)) = h^0(\mathcal{I}_S(0,1)) = 0$, then $h^1(\mathcal{I}_S(1,0)) = h^1(\mathcal{I}_S(0,1)) = 0$. To conclude it is sufficient to use the proof of part (a) of Lemma 1.2.14. □

Proposition 1.2.16. *Write $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times Y_{n_2,\dots,n_k;1}$. Fix $o \in \mathbb{P}^{n_1}$ and take a closed subscheme $Z_1 \subset Y_{n_2,\dots,n_k;1}$ such that $\{o\} \times Z_1 \subset Y_{n_1,\dots,n_k}$. Then*

$$\dim\langle \nu(Z_1)\rangle = (n_1+1)(\dim\langle \nu_1(Z_1)\rangle + 1) - 1.$$

Proof. By assumption

$$h^0\big(Y_{n_2,\dots,n_k;1}, \mathcal{I}_{Z_1, Y_{n_2,\dots,n_k;1}}(1,\dots,1)\big) = h^0\big(\mathcal{O}_{Y_{n_2,\dots,n_k;1}}(1,\dots,1)\big) - \dim\langle \nu_1(Z_1)\rangle - 1.$$

The Künneth formula gives

$$h^0(\mathcal{I}_{Z_1}(1,\dots,1)) = (n_1+1)\big(h^0(\mathcal{O}_{Y_{n_2,\dots,n_k;1}}(1,\dots,1)) - \dim\langle \nu_1(Z_1)\rangle - 1\big) - 1.$$

Since $h^0\left(\mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)\right) = (n_1+1)h^0\big(\mathcal{O}_{Y_{n_2,\dots,n_k;1}}(1,\dots,1)\big)$, we get the lemma. □

We conclude the section by proving that if there exists an index $i = 1,\dots,k$ such that $\eta_{i|S}$ is injective for some $S \subset Y_{n_1,\dots,n_k}$ then to prove that $\delta(2S, Y_{n_1,\dots,n_k}) = 0$ it is enough to prove the same result working with $2\eta_i(S)$.

Proposition 1.2.17. *Fix a finite set $S \subset Y_{n_1,\dots,n_k}$. Assume that there exist and index $i \in \{1,\dots,k\}$ for which the projection $\eta_{i|S} : S \longrightarrow Y_{n_1,\dots,n_k;i}$ is injective. If $\delta(2\eta_i(S), Y_{n_1,\dots,n_k;i}) = 0$, then also $\delta(2S, Y_{n_1,\dots,n_k}) = 0$.*

Proof. With no loss of generality we may assume $i = 1$. Set $S' := \eta_1(S)$, $s := \#S$ and $m := n_2 + \dots + n_k = \dim Y_{n_2,\dots,n_k;1}$. The submersion $\eta_1 : Y_{n_1,\dots,n_k} \longrightarrow Y_{n_2,\dots,n_k;1}$ has the property that $\eta_1(i^*(\mathcal{O}_{Y_{n_2,\dots,n_k;1}}(1,\dots,1))) \cong \mathcal{O}_{Y_{n_1,\dots,n_k}}(\hat{\varepsilon}_1)$ and this isomorphism induces an isomorphism of global section. By assumption $2S'$ imposes $s(n-n_1+1)$ independent conditions to $H^0\big(Y_{n_2,\dots,n_k;1}, \mathcal{O}_{Y_{n_2,\dots,n_k;1}}(1,\dots,1)\big)$. Thus the scheme $\eta_1^{-1}(2S')$ imposes $s(n-n_1+1)$ independent conditions to $H^0\big(\mathcal{O}_{Y_{n_1,\dots,n_k}}(\hat{\varepsilon}_1)\big)$. The scheme $\eta_1^{-1}(S')$ is the union of s disjoint varieties isomorphic to $\mathbb{P}^{n_1}$ and embedded by ν as linear spaces and $\eta_1^{-1}(2S')$ is the union of the first infinitesimal neighborhoods of $\mathbb{P}^{n_1}$ in $Y_{n_1,\dots,n_k}$. By Proposition 1.2.16 the scheme $(2\eta_1^{-1}(S'), Y_{n_1,\dots,n_k})$ imposes $s(n_1+1)(m+1)$ independent conditions to $H^0\big(\mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)\big)$, i.e. the s connected components of $\eta_1^{-1}(2S')$ spans linearly independent linear spaces. For each $o \in S$ the scheme $\eta_i^{-1}(2o') = 2\eta_i^{-1}(o')$, $o' := \eta_1(o)$, contains the double point $(2o, Y_{n_1,\dots,n_k})$. In the Segre embedding the scheme $\nu((2o, Y_{n_1,\dots,n_k}))$ gives $\dim Y_{n_1,\dots,n_k} + 1$ independent conditions. Since the s subspaces spanned by the connected components of $\nu(\eta_1^{-1}(2S'))$ are linearly independent, $\nu(2S, Y_{n_1,\dots,n_k})$ is linearly independent, i.e. $\delta(2S, Y_{n_1,\dots,n_k}) = 0$. □

Chapter 2

Identifiability of tensors

The present chapter is devoted to the description of the identifiability problem for tensor rank decomposition, namely, understanding if a given tensor admits a unique rank decomposition. Section 2.1 is an introductory section to the problem and it is structured as follows. First, we present the concept of identifiability of tensors, which is a very useful notion in numerous applications, and we provide a classical example to explain the importance of studying the identifiability problem in applied fields. Then, we make a brief literature review of the identifiability of both generic tensors and specific tensors, by recalling interesting and useful results. After that, we start working on the identifiability of rank-3 tensors, which is the main core of the present chapter.

We provide a complete classification of the identifiability of any tensor up to rank 3. In particular, in Section 2.2 we recall the notion of concision for a tensor and we prove the identifiability result in the case of rank-2 tensors. In Section 2.3 we present all the families of rank-3 tensors that are not identifiable. We prove that these families are the only non-identifiable ones in both Sections 2.4 and 2.5. Section 2.6 is devoted to actually present the main theorem of the present chapter, which recollect the results proved in the previous sections. We conclude the chapter with an algorithm aimed to recognize if a given tensor is a non-identifiable rank-3 tensor.

2.1 Literature review of the identifiability of tensors

A very interesting question related to tensors is to understand if a given tensor $T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_1+1}$ can be decomposed in a unique way as a sum of elementary tensors. Clearly, the uniqueness of decomposition is understood up to both permutation of the summands and scalar multiplication. Therefore, it is natural to extend the definition of uniqueness of a decomposition in the projective setting.

Definition 2.1.1 (A)**.** Let $T \in \mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1}$ be a rank r tensor. The tensor T is *identifiable* if it can be decomposed in a unique way as

$$T = \sum_{i=1}^{r} v_{1,i} \otimes \cdots \otimes v_{k,i},$$

where all $v_{i,j} \in \mathbb{C}^{n_i+1}$ for all $i = 1, \ldots, k$.

Another way to look at the identifiability property is by looking a the fiber of the r-th Terracini map

$$T_r : Abs_r^0(X_{n_1,\ldots,n_k}) \longrightarrow \sigma_r^0(X_{n_1,\ldots,n_k}),$$

where we denoted by $X_{n_1,\dots,n_k} = \nu(\mathbb{P}\mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{P}\mathbb{C}^{n_k+1})$ the Segre variety.

Definition 2.1.2 (B)**.** A point $q \in \mathbb{P}(\mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1})$ of rank r is identifiable if the fiber $T_r^{-1}(q)$ is just a singleton.

From a pure mathematical point of view, the identifiability problem is a very interesting problem on its own. Nevertheless, knowing if a tensor admits a unique decomposition can be meaningful when working with the applications. A straightforward result is that matrices are highly not identifiable (cf. Example 1.1.27). Contrary to this, a rank decomposition of a tensor is often unique and therefore, it can be useful to model an applied problem with tensors instead of matrices. To better explain the importance of the identifiability problem in the applications, we present the following example (cf. [HLB+18], [AGH+14]).

Example: identifiability of tensors for topic models

A topic model is a statistical model used to recover abstract topics that occur in a collection of documents. If we denote by W a finite set of words, then every document D_i can be described by a finite number of elements of W, for $i = 1, \dots, N$, with $N > 0$. Any document D_i is described by r distinct topics coming from W and we denote by $T \subset W$ the set of those r elements. We assume that the order of appearance of the topics in each D_i is irrelevant and that the topics in a document are independently and identically distributed, conditional on the topic. Moreover, the probability of each topic to appear in a document is conditional to the topic of the document itself. With these assumptions, we would like to recover the probability distributions p_i for each topic $t_i \in T$, where the total probability distribution is

$$P(W) = \sum_{i=1}^{r} P(T = t_i) P(W|T = t_i) = \sum_{i=1}^{r} \alpha_i p_i,$$

where all p_i are vectors of size N containing the probability distributions of each topic conditioned to each document. It is clear that one cannot recover the conditional probabilities since there does not exist a unique choice of $p_1, \dots, p_r$ whose linear combination gives $P(W)$.
To overcome this issue, one may think of looking at the total joint probability distribution, which is $\sum_{i=1}^{r} \alpha_i p_i p_i^T$. This is done by counting word pairs and it leads to working with stochastic matrices. However, also in this case one cannot recover the initial parameters since also matrices are not identifiable.
Working with triplets of words we are able to solve the problem. More precisely, we consider

$$P(W = w_i, W = w_j, W = w_k) = \sum_{\ell=1}^{r} \alpha_i P(W = w_i|T = t_\ell) P(W = w_j|T = t_\ell) P(W = w_k|T = t_\ell).$$

We are now working with a (symmetric) tensor and if the above 3-way tensor is identifiable, then computing its unique rank-r decomposition allows us to recover all the conditional probability distributions $p_1, \dots, p_r$.

The problem of identifiability of tensors has widely been treated from both mathematicians and other areas. From an applied point of view, the problem has been tackled in a non generic scenario, i.e. working with specific tensors, while the geometrical perspective focused more on the identifiability of generic tensors. We will give an overview of the state of the art in both cases by recalling interesting and useful results proved either for generic tensors (of fixed rank) or for specific tensors.

Remark 2.1.3. The literature review that we are going to present focuses on tensors without any kind of symmetries, therefore we will only report results on identifiability of tensors lying in the ambient space of some Segre variety.

2.1.1 Identifiability of generic tensors of fixed rank

Working with the applications, knowing if a generic tensor of a certain rank is identifiable can give an indication regarding the behaviour of specific tensors of the same rank. Recall the following result.

Proposition 2.1.4 ([Har77, Cap II, Ex 3.22, part (b)]). *Let L, M be two irreducible and reduced projective varieties and let $f: L \to M$ be a dominant morphism, i.e. $f(L)$ is dense in M. For any point $v \in f(L)$ every irreducible component of the fiber $f^{-1}(v)$ has dimension*

$$\dim \left(f^{-1}(v)\right) \geq \dim L - \dim M.$$

We want to apply the above proposition in the following framework, where we denote by $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ a multiprojective space of $k \geq 1$ factors and we call $X_{n_1,\dots,n_k} = \nu(Y_{n_1,\dots,n_k})$ the corresponding Segre variety. Applying the above result to the Terracini map, i.e. taking $L = Abs_r(X_{n_1,\dots,n_k})$ and $M = \sigma_r(X_{n_1,\dots,n_k})$ for some $r > 0$, it tells us that the dimension of the space $\mathcal{S}(Y_{n_1,\dots,n_k}, [T])$ of rank-1 tensors computing the rank of a specific rank-r tensor $[T]$ (cf. Definition 2.2.2) can only be bigger or equal than the dimension of $\mathcal{S}(Y_{n_1,\dots,n_k}, q)$, where q is a generic tensor of rank equal to the rank of $[T]$. Therefore the knowledge of the behaviour on the generic element of some fixed rank can be useful to understand the behaviour of specific tensors of the same rank.

Moreover, if the generic tensor of rank r is identifiable, then there exist a Zariski set that can be seen as an euclidean set of measure zero in which the elements of $\sigma_r(X_{n_1,\dots,n_k})$ outside this set are r-identifiable.

From the pure mathematical point of view, the identifiability problem remains a very interesting problem related to the behaviour of the fiber of the r-th Terracini map. Moreover, the knowledge of the generic non-identifiability is a necessary condition for r-defectivity, a problem which is still far from being completely understood. Indeed if X is r-defective (cf. Definition 1.1.16) then by Proposition 2.1.4 the fiber of the general element of the r-th Terracini map is positive dimensional. Therefore the general element of $\sigma_r(X)$ is not identifiable and so any other specific tensor of the same border rank is not identifiable.

We recall also that when the r-th secant variety $\sigma_r(X) \subset \mathbb{P}^N$ fills the ambient space such that $\dim \sigma_r(X) > N$ the general fiber of the r-th Terracini map is trivially positive dimensional. For the purpose of this subsection, it is useful to recall the following definitions that hold in the more general setting of $X \subset \mathbb{P}^N$ being any irreducible non degenerate projective variety and that we adapt to the case of X being a Segre variety.

Definition 2.1.5. Let $X \subset \mathbb{P}^N$ be the Segre image of a multiprojective space of $k > 0$ factors. We say that

1 X is *r-identifiable* if the general element of the r-th secant variety $\sigma_r(X)$ is identifiable;

2 X is *generically identifiable* if the general element of $\mathbb{P}^N$ is identifiable.

There is a vast literature on the identifiability of generic tensors and a lot of results are scattered through the literature. We recall some of them in the following discussion, by focusing on identifiability of tensors.

From weak defectivity to identifiability

Numerous contributions on generic identifiability rely on the concept of weak defectivity. Given an irreducible, non-degenerate projective variety $X \subset \mathbb{P}^N$, we say that X is *r-weakly defective* if the general r-tangent hyperplane has a contact variety of positive dimension, where by general r-tangent hyperplane we mean the general hyperplane $H \subset \mathbb{P}^N$ containing $\langle T_{p_1}X, \dots, T_{p_r}X \rangle$, where $p_1, \dots, p_r$ are general points of X. The first time this notion appeared was in the paper [Ter21] of A. Terracini, as stated in [CC02]. The notion of weak defectivity has then been both rediscovered and reformulated in a modern language by L. Chiantini and C. Ciliberto in [CC02]. This notion is strictly connected to the notion of r-defectivity. Indeed, by the Terracini's Lemma if $X \subset \mathbb{P}^N$ is r-defective, then the general hyperplane H containing $T_x\sigma_r(X)$ is tangent to X along a variety $\Sigma(p_1, \dots, p_r)$ of positive dimension containing $p_1, \dots, p_r$. Clearly, a r-defective variety is also r-weakly defective, but the viceversa does not hold. In [CC02], the authors introduced the concept of weak defectivity to better tackle the defectivity problem.

In a second work [CC06], both authors linked the notion of weak defectivity with the notion of identifiability. They proved that, in a non-defective framework, the fiber of the general element with respect to the corresponding Terracini map is zero dimensional unless the variety is weakly defective.

A very useful concept related to the identifiability of generic tensors, is the notion of r-tangentially weak defectiveness, introduced in [CO12] by L. Chiantini and G. Ottaviani. More precisely, an irreducible non-degenerate projective variety $X \subset \mathbb{P}^N$ is *r-tangentially weakly defective* if the span of the tangent spaces at r general points of X, is tangent also in some other point. Since we are interested in recalling the consequences of the introduced tools for the identifiability of generic tensors (of some fixed rank), we will not go further into details of these notions and we refer to [Chi04] for a clear introduction to the topics.

In [CO12], the authors introduced also an inductive method for the study of the identifiability of generic 3-way tensors based on the notion of weak defectivity. The main result of the method is the following bound on the dimensions of the vector spaces forming the tensor space.

Theorem 2.1.6 ([CO12, Theorem 1.1])**.** *Let $a \leq b \leq c$ be dimensions of $\mathbb{C}$-vector spaces A, B, C respectively. Let α, β be maximal such that $2^\alpha \leq a$ and $2^\beta \leq b$. The general tensor $t \in A \otimes B \otimes C$ of rank k has a unique decomposition if $k \leq 2^{\alpha+\beta-2}$.*

They also extended the above bound in the case of an arbitrary number of factors in [CO12, Theorem 6.7].

Remark 2.1.7. Note that the notions of both weak defectivity and tangetially weak defectivity cannot be rearranged in a non-generic scenario since these contact varieties arise from a behaviour of generic points. Indeed, the existence of such contact varieties depends intrinsically on the fact that in the notion of (tangentially) weak defectivity we are considering generic points and the existence of a particular r-uple of points that have the behaviour described by the notion of (tangentially) weak defectivity does not imply the existence of a whole contact subvariety.

Identifiability for generic rank-r tensors in $\langle \nu((\mathbb{P}^n)^k) \rangle$

Let $X_{n^k} = \nu(\mathbb{P}^n \times \cdots \times \mathbb{P}^n)$ be the Segre image of the product of k copies of $\mathbb{P}^n$, with $n \geq 1$. For $n = 1$ (binary case), the identifiability of the generic element holds. This case is relevant in different applications such as for example quantum physics (cf. [BFŻ19]).

A qubit is a state of a two dimensional quantum system, i.e. a vector in the Hilbert space $\mathcal{H} = \mathbb{C}^2$. In the case of a k-qubit system the corresponding Hilbert space is $\mathcal{H} = (\mathbb{C}^2)^{\otimes k}$ and a state $\psi \in \mathcal{H}$ is separable if it can be written as $\psi = \psi_1 \otimes \cdots \otimes \psi_k$, otherwise it is entangled. Thus separable states are rank one tensors and entangled state are tensors of higher rank in $(\mathbb{C}^2)^{\otimes k}$. The rank of a tensor can be considered as a measure of the entanglement and this turns out to be extremely useful in quantum physics.

Remark 2.1.8. The identifiability of rank-2 tensors in the case of binary Segre products is classically known for $k = 3, 4$ factors. The case $k = 4$ has been explicitly treated by C. Segre in [Seg21], where he worked on quadrilinear forms, i.e. elements in $\langle \nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \rangle$. For the case of trilinear forms, i.e. $k = 3$, one can follow the same argument produced by Segre. Indeed, working in the affine setting, we can consider the contraction $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2$. In this case, the kernel of a general tensor is a line in $\mathbb{P}^3$ that meets the quadric $\nu(\mathbb{P}^1 \times \mathbb{P}^1)$ in two points that correspond to the two summands of the decomposition. This was a standard argument for Segre and, even though we cannot find an explicit reference for the 3-factors case, it is reasonable to attribute also this case to Segre.

Recently, in [BC13] the authors proved that working with $k \geq 6$ copies of $\mathbb{P}^1$ embedded via Segre, the general tensor of rank $r+1$ is identifiable if $r + 1 \leq 2^{k-1}/k$, improving the first bound given by [AMR09]. In the follow up paper [BCO14], the two authors jointly with G. Ottaviani improved the bound working with $k \geq 12$ factors (cf. [BCO14, Theorem 4.4]). The bound was then improved by A. Casarotti and M. Mella in [CM22, Theorem 26] in which they also proved generic identifiability for all sub-generic binary tensors.
In the same paper [BCO14], the authors also worked on Segre varieties of many copies of $\mathbb{P}^n$, with $n \geq 2$. They implemented an algorithm based on the notion of weak defectivity that relies on the computation of the Jacobian matrix of the locus C containing points $p \in X$ for which $T_pX \subset \langle T_{p_1}X, \ldots, T_{p_r}X \rangle$, where $p_1, \ldots, p_r$ are random chosen points of a Segre variety X (cf. [BCO14, Section 9]). Their general result on many copies of $\mathbb{P}^n$ is the following theorem.

Theorem 2.1.9 ([BCO14, Theorem 7.1])**.** *Let $k \geq 3$ and consider the Segre image of k copies of $\mathbb{P}^n$. The general rank r tensor is identifiable for*

$$r \leq \frac{(n+1)^k - (3n+1)(n+1)^{k-2}}{nk+1}.$$

The algorithm of [BCO14, Theorem 4.4] was then improved in [COV14, Algorithm 3.1] based on the following sufficient condition for generic identifiability.

Proposition 2.1.10 ([COV14, Proposition 2.3]). *Let X be a non defective Segre variety, let r be a sub-generic rank and assume that X is not generically r-identifiable. Then, for r general points $p_1, \dots, p_r \in X$ the r-th tangential contact locus $\mathcal{C}_r$ contains a curve passing through $p_1, \dots, p_r$.*

The main result they proved on generic identifiability is the following.

Theorem 2.1.11 ([COV14, Theorem 1.1]). *A general tensor in $\langle \nu(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \rangle$ of subgeneric rank r is r-identifiable if $\prod_{i=1}^{k}(n_i + 1) \leq 15000$ unless we have one of the following:*

$(n_1, \dots, n_k)$	r	Type
$(3, 3, 2)$	5	defective [AOP09, Proposition 4.10]
$(3, 3, 3)$	6	sporadic [CO12, Theorem 1.3]
$(5, 5, 2)$	8	sporadic [CMO14]
$(n, n, 1, 1)$	$2n - 1$	defective [AOP09, Proposition 4.7]
$(1, 1, 1, 1, 1)$	5	sporadic [BC13, Proposition 4.1]
$n_1 > \prod_{i=2}^{k} n_i - \sum_{i=2}^{k}(n_i - 1)$	$r \geq \prod_{i=2}^{k} n_i - \sum_{i=2}^{k}(n_i - 1)$	unbalanced [BCO14]

The above statement is still a conjecture for $\prod_{i=1}^{k} n_i > 15000$. We also include the following defective case since it provides an obvious instance of generic non-identifiability.

- $(n_1, \dots, n_k) = (1, 1, 1, 1)$, $r = 3$ (cf. Theorem 1.1.24).

In the same paper the authors gave also a sufficient condition for specific identifiability but we will be more precise about this on the next Subsection 2.1.2 in which we focus on the identifiability of specific tensors.

Generic identifiability

Another interesting case that has been analyzed is the so-called perfect case, that occurs when the biggest r-th secant variety fills the ambient space sharply. In [HOOS19], the authors used homotopy continuation techniques to tackle the problem of classifying identifiability for the general tensor (see 2 of Definition 2.1.5). The homotopy technique works as follows. Let $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ be a rank-r tensor. Starting with a rank decomposition of T, one can consider the equation

$$T = \sum_{i=1}^{r} v_{1,i} \otimes \cdots \otimes v_{k,i},$$

where all $v_{j,i}$ are unknown for $j = 1, \dots, k$. One can fix a closed path and move $T(s)$ along the path such that $T(0) = T(1) = T$, where $s \in [0, 1]$, i.e.

$$T(s) = \sum_{i=1}^{r} v_{1,i}(s) \otimes \cdots \otimes v_{k,i}(s).$$

The key point is that the elements appearing in the decomposition of T may not be the same for $s = 0$ and $s = 1$. This method allows to prove with high probability non-identifiability if the computations provide different decompositions. Otherwise it is not possible to claim identifiability.

Denote by g the generic rank of the corresponding Segre variety. The authors used the homotopy technique in [HOOS19] working with a rank-g tensor T that is given by the sum of g random rank-1 tensors. The computations made were useful for two different reasons. First, with the computations they observed that the cases of $(n_1, \ldots, n_k) = (2,4,5), (2,2,2,3)$ had a different behaviour and indeed in these cases they proved the uniqueness of decomposition. Second, based on their computational evidence, they formulated the following conjecture.

Conjecture 2.1.12 ([HOOS19])**.** *Let $k \geq 3$ and let $n_1 \geq \cdots \geq n_k \geq 1$. Let $X = \nu(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$ be the Segre variety in $\mathbb{P}(\mathbb{C}^{n_1+1} \otimes \cdots \otimes \mathbb{C}^{n_k+1})$ and denote by $\overline{r}_X$ the generic rank. Then X is not generically $\overline{r}_X$- identifiable unless it is one of the following cases:*

1. $X = \nu(\mathbb{P}^4 \times \mathbb{P}^3 \times \mathbb{P}^2)$;
2. $X = \nu(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$;
3. $X = \nu(\mathbb{P}^k \times \mathbb{P}^k \times \mathbb{P}^1)$ *for some k — matrix pencils.*

From an applied point of view, tensors encode very precise models and, in these cases, knowing the behaviour of the generic tensor of the same fixed rank may not be effective.

2.1.2 Identifiability of specific tensors of fixed rank

In the literature of the applied fields, tensor decomposition is also called canonical polyadic decomposition (CPD). It was first described by F. Hitchcock in 1927 ([Hit27]), finding numerous applications in statistics, signal processing, computer vision, computer graphics, psychometrics, linguistics and chemometrics. That is why it is also called CANDECOMP, PARAFAC, or CANDECOMP/PARAFAC (CP). While mathematicians where most interested in the identifiability of generic tensors (of fixed rank), in the applied fields one may also be interested in the identifiability of specific tensors.

Addressing the identifiability problem to specific tensors is far more complicated. It is only right to mention one of the firsts contributions that we found on identifiability of specific tensors, namely the work of R. A. Harshman [Har70], whom turns credit to R. Jennich for the result. His result is related to the identifiability of 3-way arrays and gives a sufficient condition for a 3-way tensor to be identifiable. Few years later, the most celebrated criterion for identifiability has been published in [Kru77]. In order to state properly the result, we need the notion of the so called Kruskal rank for a set of vectors.

Definition 2.1.13. Let V be a $\mathbb{C}$-vector space of finite dimension. Let $S = \{v_1, \ldots, v_r\} \subset V$. The *Kruskal rank* of S is

$$\max\{n \in \mathbb{N} | \text{ every } S' \subseteq S \text{ with } \#S' = n \text{ is a set of linearly independent vectors}\}.$$

Theorem 2.1.14 ([Kru77])**.** *Let $T \in A \otimes B \otimes C$, where $T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$. Denote by k_A, k_B, k_C the Kruskal ranks of $\{a_1, \ldots, a_r\}$, $\{b_1, \ldots, b_r\}$ and $\{c_1, \ldots, c_r\}$ respectively. If*

$$r \leq \frac{1}{2}(k_A + k_B + k_C) - 1$$

then T has rank r and it is identifiable.

A first generalization of the above theorem for an arbitrary number of factors can be found in [SB00, Theorem 3]. The result itself has been analyzed and reproved by different authors coming from the field of pure geometry as well as engineering (see e.g. [Rho10], [Lan09], [SS07]) and we also refer to [DDL13, Section 1.2] for an interesting literature review on identifiability results related to Kruskal criterion. H. Derksen proved in [Der13] that Kruskal's inequality is sharp by finding examples of non-identifiable tensors if $k_A + k_B + k_C = 2r + 1$.

Recently B. Lovitz together with F. Petrov proved a generalization of Kruskal criterion in which they replaced the assumptions on the Kruskal ranks with the standard notion of rank (cf. [LP21]). More precisely, they proved the following statement.

Theorem 2.1.15 ([LP21, Theorem 2]). *Let $r \geq 2, k \geq 3$. Let*

$$T = \sum_{i=1}^{r} x_{i,1} \otimes \cdots \otimes x_{i,k} \in V_1 \otimes \cdots \otimes V_k,$$

where $V_1, \ldots, V_k$ are finite dimensional $\mathbb{C}$-vector spaces. For each subset $S \subset \{1, \ldots, r\}$ and each index $j \in \{1, \ldots, k\}$, let $d_j^S = \dim span\{x_{\ell,j} \colon \ell \in S\}$. If

$$2(\#S) \leq \sum_{i=1}^{k} (d_i^S - 1) + 1 \text{ for every subset } S \subset \{1, \ldots, r\} \text{ with } 2 \leq (\#S) \leq r$$

then the above decomposition of T constitutes a unique tensor rank decomposition.

By using the same examples created by Derksen, they proved that their inequality is sharp (cf. [LP21, Section 6]) and they also compared in [LP21, Section 10] the produced criterion with the previous extensions of Domanov, De Lathauwer, and Sørensen (see [DDL14], [SDL15] and also [DDL13]).

We conclude our literature review with [COV14], in which they adapted their algorithm ([COV14, Algorithm 3.1]) to the case of specific tensors under the assumption that the fixed tensor is not a singular point of the corresponding secant variety (cf. [COV14, Remark 4.6]). They showed with some examples that their method can get uniqueness of decomposition for a specific tensor in cases where neither Kruskal's nor Domanov–De Lathauwer's criterion apply (cf. e.g. [COV14, Table 2])).

However, to the best of our knowledge, nobody tried to give a complete classification on the identifiability of any tensor of a given fixed rank, due to the difficulty of the problem itself. In the following, we start working on the identifiability of all tensors of ranks 2 and 3. We give a complete classification of these first cases describing the structures and the dimensions of all the sets evincing the rank. As already discussed above, in terms of generic tensors of rank either 2 or 3, everything was already well known but, untill now, a complete classification for all the tensors of those ranks was still missing.

In Proposition 2.2.7 we show that a rank-2 tensor T is always identifiable except if T is a 2×2 matrix.

Our **main Theorem 2.6.1** states that a rank-3 tensor T is identifiable except if

1. T is a 3×3 matrix and $\dim(\mathcal{S}(Y_{2,2}, T)) = 6$;
2. there exist $v_1, v_2, v_3 \in \mathbb{C}^2$ s.t. $T \in \mathbb{C}^2 \otimes v_2 \otimes v_3 + v_1 \otimes \mathbb{C}^2 \otimes v_3 + v_1 \otimes v_2 \otimes \mathbb{C}^2$ and $\dim(\mathcal{S}(Y_{1,1,1}, T)) \geq 2$;

3. $T \in (\mathbb{C}^2)^{\otimes 4}$ and $\dim(\mathcal{S}(Y_{1,1,1,1}, T)) \geq 1$;
4. $T \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and it is as in Example 2.3.7. In this case $\dim(\mathcal{S}(Y_{2,1,1}, T)) = 3$;
5. $T \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and it is as in Example 2.3.9. In this case $\mathcal{S}(Y_{2,1,1}, T)$ contains two different 4-dimensional families;
6. $T \in \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes (\mathbb{C}^2)^{k-2}$, where $k \geq 3$ and $m_1, m_2 \in \{2, 3\}$. In this case $\dim(\mathcal{S}(Y_{m_1,m_2,1^{k-2}}, T)) \geq 2$ and T is as in Proposition 2.3.14. If $m_1 + m_2 + k \geq 6$ then $\dim(\mathcal{S}(Y_{m_1,m_2,1^{k-2}}, T)) = 2$.

We will proceed as follows. After introducing the main ingredients needed for the set up, we show the identifiability of rank-2 tensors in Section 2.2. In Section 2.3 we explain in details the examples where the non-identifiability of a rank-3 tensor arises. In Sections 2.4 and 2.5 we show that the examples of the previous section are the only possible exceptions to identifiability of a rank-3 tensor. Section 2.6 is devoted to collect all the information needed (but actually already proved at that stage) to conclude the proof of our main Theorem 2.6.1. This part has alredy been published in a joint work [BBS20a].

2.2 Concise Segre of a tensor and identifiability of rank-2 tensors

After defining the notion of concise Segre of a given tensor, in this section we study and completely determine the identifiability of points on the second secant variety of a Segre variety.

Definition 2.2.1. (Concise Segre) Given a point $q \in \mathbb{P}^N$, we will call *concise Segre* of q the variety $X_{q;n_1,\dots,n_k} := \nu(Y'_{n'_1,\dots,n'_k})$ where $Y'_{n'_1,\dots,n'_k} \subseteq Y_{n_1,\dots,n_k}$ is the minimal multiprojective space $Y'_{n'_1,\dots,n'_k} \subseteq Y_{n_1 \dots n_k}$ such that $q \in \langle \nu(Y'_{n'_1,\dots,n'_k}) \rangle$ as in Concision/Autarky Lemma 1.1.8.

For the rest of this chapter, we will work with the concise Segre $X_{q;n_1,\dots,n_k}$ of a given tensor q, since its span $\langle X_{q;n_1,\dots,n_k} \rangle$ is the projectivization of the smallest tensor space containing q.

Definition 2.2.2. For any $q \in \mathbb{P}^N$, define

$$\mathcal{S}(Y_{n_1,\dots,n_k}, q) := \{A \subset Y_{n_1,\dots,n_k} \mid \#(A) = r_{X_{n_1,\dots,n_k}}(q) \text{ and } q \in \langle \nu(A) \rangle\}.$$

If $A \in \mathcal{S}(Y_{n_1,\dots,n_k}, q)$, we will say either that A *evinces* the rank of q or that A *is a solution* of q.

In the following remark we see how one can build the concise Segre variety of a given tensor $q \in \mathbb{P}^N$, starting from one of its decompositions.

Remark 2.2.3. To obtain the minimal $Y'_{n'_1,\dots,n'_k}$ defining the concise Segre of a point q fix any $A \in \mathcal{S}(Y_{n_1,\dots,n_k}, q)$ and set $A_i := \pi_i(A) \subset \mathbb{P}^{n_i}$, $i = 1, \dots, k$, where the π_i's are the projections on the i-th factor of Notation 1.1.6. Each $\langle A_i \rangle \subseteq \mathbb{P}^{n_i}$ is a well-defined projective subspace of dimension at most $\min\{n_i, r_X(q) - 1\}$ and we will denote by $n'_i = \dim \langle A_i \rangle$ for all $i = 1, \dots, k$. By Concision/Autarky we have $Y'_{n'_1,\dots,n'_k} = \prod_{i=1}^k \langle A_i \rangle$. In particular q does not depend on the i-th factor of $Y_{n_1,\dots,n_k}$ if and only if for one $A \in \mathcal{S}(Y_{n'_1,\dots,n'_k}, q)$ the set $\pi_i(A)$ is a single point.

We remark that, given a solution $A \in \mathcal{S}(Y_{n_1,\dots,n_k}, q)$ of some $q \in \mathbb{P}^N$, any line $L \subset X$ contained in the corresponding Segre variety, can contain at most one point of $\nu(A)$. More precisely, let us state the following.

Remark 2.2.4. Let $q \in \mathbb{P}^N$ and consider $A \in \mathcal{S}(Y_{n_1,\dots,n_k}, q)$. We claim that there is no line $L \subset X_{n_1,\dots,n_k}$ such that $\#(L \cap \nu(A)) \geq 2$. Clearly, if $\#(L \cap \nu(A)) > 2$ we would have at least 3 points that evince the rank of q on a line, which is a contradiction with the linearly independence property that sets in $\mathcal{S}(Y_{n_1,\dots,n_k}, q)$ have.
Assume that there exists a line $L \subset X_{n_1,\dots,n_k}$ such that $\#(L \cap \nu(A)) = 2$ and denote by $u, v \in A$ the preimages of those points, i.e. $u \neq v$ and $\langle \nu(u), \nu(v) \rangle = L$. Then, $r_{X_{n_1,\dots,n_k}}(q) > 2$ because if $r_{X_{n_1,\dots,n_k}}(q) = 2$ then we would have $q \in L \subset X_{n_1,\dots,n_k}$, so the rank of q will be 1. Let $E = A \setminus \{u, v\}$, we can look at q as $q \in \langle \nu(E) \cup L \rangle$, so we can find a point $o \in L$ such that $q \in \langle \nu(E) \cup \{o\} \rangle$, which would imply $r_{X_{n_1,\dots,n_k}}(q) < \#A$.

Now we have all the necessary tools to work on the identifiability of any rank-2 tensor.

2.2.1 Identifiability on the 2-nd secant variety

By Remark 2.2.3, the concise Segre of a border rank-2 tensor q is $X_{q;1^k} = \nu\big((\mathbb{P}^1)^k\big)$. Indeed fix a solution $A \in \mathcal{S}(Y_{n_1,\dots,n_k}, q)$ and note that for all $i = 1, \dots, k$

$$\dim\langle \pi_i(A) \rangle \leq 1.$$

Now

$$\text{if } \#\pi_i(A) = 1 \text{ then clearly } \langle \pi_i(A) \rangle = \mathbb{P}^0, \text{ otherwise } \langle \pi_i(A) \rangle = \mathbb{P}^1.$$

Therefore, for the rest of this section we will focus our attention on Segre varieties of products of $\mathbb{P}^1$'s. We start with the two factors case.

Remark 2.2.5. If the concise Segre $X_{q;1,1}$ of a tensor $q \in \sigma_2(X_{n_1,\dots,n_k})$ is $\nu(\mathbb{P}^1 \times \mathbb{P}^1)$, then $\sigma_2(X_{q;1,1})$ parameterizes the 2×2 matrices for which it is trivial to see that they can be written as sum of two rank-1 matrices in an infinite number of ways.

Recall that, by Remark 1.1.23, a tensor $q \in \tau(X_{1,1,1}) \setminus X_{1,1,1}$ has rank equal to 2 if and only if the concise Segre of q is a two-factors Segre, moreover it is not identifiable for any number of factors.

For the rest of this section we will therefore focus on Segre varieties of $Y_{1^k} = (\mathbb{P}^1)^k$ with $k \geq 3$.

Remark 2.2.6. If $q \in X_{1^k}$ is a non-identifiable rank-2 tensor, then all its decompositions must be disjoint. Indeed let $A, B \in \mathcal{S}(Y_{1^k}, q)$ and assume $\#(A \cap B) = 1$. This means that the line spanned by $\langle \nu(A) \rangle$ meets the line spanned by $\langle \nu(B) \rangle$ in two distinct points, namely q and the image of the intersection point. Therefore $\langle \nu(A) \rangle = \langle \nu(B) \rangle = L$ and L is actually a trisecant line of X_{1^k}, so it is all contained in X_{1^k} (cf. Remark 1.1.7), which is in contradiction with the fact that $r_{X_{1^k}}(q) = 2$.

For $k = 3$, the non-identifiability is classically attributed to Segre (cf. Remark 2.1.8). The following proposition proves our first result on identifiability of non-generic tensors and it focuses on identifiability of any rank-2 tensor.

Proposition 2.2.7. *Let $q \in \sigma_2^0(X_{1^k})$. Then $\#\mathcal{S}(Y_{1^k}, q) > 1$ if and only if the concise Segre of q is $X_{q;1,1} = \nu(\mathbb{P}^1 \times \mathbb{P}^1)$.*

Proof. We only need to check the case of $k \geq 4$ since $k = 2, 3$ are classically known. The case of matrices is obviously not identifiable (cf. Remark 2.2.5), while the identifiabily in the case of $k = 3$ factors is classically attributed to Segre and we refer to Remark 2.1.8. We assume therefore that $k \geq 4$.

Let $A, B \in \mathcal{S}(Y_{1^k}, q)$. By Remark 2.2.6 A and B are two disjoint sets:

$$
\begin{aligned}
A = & \{a, a'\}, \text{ where } a = (a_1, \ldots, a_k), a' = (a'_1, \ldots, a'_k) \in Y_{1^k}, \\
B = & \{b, b'\}, \text{ where } b = (b_1, \ldots, b_k), b' = (b'_1, \ldots, b'_k) \in Y_{1^k}.
\end{aligned}
$$

Since $a \neq a'$, we may assume that at least one of their coordinates is different. Actually we can assume that all the $a_i \neq a'_i$, otherwise, by the concision property, one could consider one factor less. The same considerations hold for B.

In order to proceed further we claim that $\{a_i, a'_i\} = \{b_i, b'_i\}$ for all $i =, 1 \ldots, k$. Indeed, suppose that there exists an index $i \in \{1, \ldots, k\}$ such that $\{a_i, a'_i\} \neq \{b_i, b'_i\}$ and let such an index be $i = 1$, i.e. $\{a_1, a'_1\} \neq \{b_1, b'_1\}$. We proceed by induction on $k \geq 4$. Since a similar argument stands for both the base case $k = 4$ and the inductive step, we will show only the inductive step.
Let η_k, ν_k, and X_k be as in Notation 1.1.6. Let $\tilde{q} = (q_1, \ldots, q_{k-1})$ be the projection $\eta_k(q)$, then $\eta_k(A) \neq \eta_k(B)$ and $\emptyset \neq \langle \nu_k(\eta_k(A)) \rangle \cap \langle \nu_k(\eta_k(B)) \rangle \supset \{\tilde{q}\}$ because $\{q\} \subset \langle \nu(A) \rangle \cap \langle \nu(B) \rangle$. So $r_{X_k}(\tilde{q}) = 2$ and $\#\mathcal{S}(Y_{1^{k-1}}, \tilde{q}) \geq 2$, which is a contradiction because X_k is a concise Segre of $k-1$ factors (where $k \geq 4$) and a point of it cannot have more than a decomposition. Thus for all $i = 1 \ldots, k$ we have that $\{a_i, a'_i\} = \{b_i, b'_i\}$.

Without loss of generality assume that $a_1 = b_1$ and $a'_1 = b'_1$, moreover up to permutation, there exists an index $e \in \{1, \ldots, k-1\}$ such that

$$
\begin{aligned}
b_i = & a_i \text{ and consequently } b'_i = a'_i \text{ for } 1 \leq i \leq e \text{ and} \\
b_i = & a'_i \text{ and consequently } b'_i = a_i \text{ for } e+1 \leq i \leq k.
\end{aligned}
$$

Eventually, by exchanging the role of the first e elements with the others, we have that $k - e \geq 2$ because by assumption $k \geq 4$. Let $H \in |\mathcal{O}_Y(0, \ldots, 0, 1)|$ be the only element containing a', i.e. $H = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times \{a'_k\} \cong (\mathbb{P}^1)^{k-1}$. The residue of $A \cup B$ with respect to H is $\mathrm{Res}_H(A \cup B) = \{a, b'\}$ and since $k - e \geq 2$ we have that $\eta_k(a) \neq \eta_k(b')$, i.e. $h^1\big(\mathcal{I}_{\mathrm{Res}_H(A \cup B)}(1, \ldots, 1, 0)\big) = 0$. Therefore, by applying Lemma 1.2.9, we get $A \cup B \subset H$ which is in contradiction with the concision property. $\square$

The following result is based on the previous Proposition 2.2.7 and explains the geometry of the space of solutions of a non-identifiable rank-2 tensor.

Corollary 2.2.8. *Let q be any rank-2 tensor. If q is not identifiable, then there is a bijection between $\mathcal{S}(Y_{1,1}, q)$ and $\mathbb{P}^2 \setminus L$, where $L \subset \mathbb{P}^2$ is a projective line, $q \in \tau(X_{1,1})$ and L parametrizes the set of all degree 2 connected subschemes V of $Y_{1,1}$ such that $q \in \langle \nu(V) \rangle$.*

Proof. It suffices to work with a Segre variety of 2 factors only because by Proposition 2.2.7 it is the unique not-identifiable case in rank-2. Thus $X_{1,1} \subset \mathbb{P}^3$ is a quadric surface. Denote by $H_q \subset \mathbb{P}^3$ the polar plane of $X_{1,1}$ with respect to q, i.e. $H_q := \{p \in X_{1,1} \,|\, q \in T_p X_{1,1}\}$. Since $q \notin X_{1,1}$ we have that $q \notin H_q$ and the intersection $X_{1,1} \cap H_q$ is a smooth conic. Remark also that by definition a point $o \in X_{1,1}$ is such that $q \in T_o X_{1,1}$ if and only if $o \in X_{1,1} \cap H_q \subset \tau(X_{1,1})$.
Fix $o \in H_q$, then

- if $o \notin X_{1,1}$, the line given by $\langle o, q\rangle$ is not tangent to $X_{1,1}$ and when considering the intersection $\langle o, q\rangle \cap X_{1,1}$, it is given by two points $p_1, p_2 \notin \{o, q\}$ such that $\{p_1, p_2\} \in \mathcal{S}(Y_{1,1}, q)$;

- if $o \in X_{1,1}$, i.e. $o \in X_{1,1} \cap H_q$, then the line $\langle o, q\rangle$ is tangent to $X_{1,1}$.

Consider $\Pi_q = \{\text{lines } L \subset \mathbb{P}^3 \text{ passing through } q\} \cong \mathbb{P}^2$ and consider the following isomorphism $\varphi : H_q \longrightarrow \Pi_q$ defined by $p \mapsto \langle p, q\rangle$. Clearly $\varphi(X_{1,1} \cap H_q)$ is a smooth conic $\mathcal{C}$ of Π_q. Moreover one can notice that $\Pi_q \setminus \varphi(X_{1,1} \cap H_q) \cong \mathbb{P}^2 \setminus \mathcal{C}$ are just the points of the first case. □

2.3 Examples of non-identifiable rank-3 tensors

We start now working with rank-3 tensors. The purpose of this section is to explain in details the phenomena behind the non-identifiable rank-3 tensors. We first recall how to build the concise Segre of a rank-3 tensor, then we review some well known cases of non-identifiable rank-3 tensors, namely the matrix case, points on the tangential variety of $\nu(Y_{1,1,1})$ and points of the defective variety $\sigma_3(\nu(Y_{1,1,1,1}))$. New instances of non-identifiability are contained both in Subsections 2.3.1, with Examples 2.3.7 and 2.3.9, and in Subsection 2.3.2, with Proposition 2.3.14. We will show that essentially, this new cases, together with the well known ones mentioned above, represent the only classes of non-identifiable rank-3 tensors. More precisely, we will see in Theorem 2.6.1 that they will turn out to be the unique cases of non-identifiability for a rank-3 tensor.

From now on we always consider $q \in \mathbb{P}^N$ such that $r_{X_{n_1,\dots,n_k}}(q) = 3$.

Concise Segre of a rank-3 tensor

By Remark 2.2.3, we may assume that q is an order-k tensor with at most 3 entries in each mode. Indeed, if we fix a solution $A \in \mathcal{S}(Y_{n_1,\dots,n_k})$, then for all $i = 1, \dots, k$ we get

$$\dim\langle \pi_i(A)\rangle \leq 2.$$

Clearly,

- if $\#\pi_i(A) = 1$ then $\langle \pi_i(A)\rangle = \mathbb{P}^0$ and in this case we can consider the concise Segre of q without the i-th factor;

- if the three points of $\pi_i(A)$ are linearly dependent in $\mathbb{P}^{n_i}$, then $\langle \pi_i(A)\rangle = \mathbb{P}^1$;

- otherwise $\langle \pi_i(A)\rangle = \mathbb{P}^2$.

Therefore, the concise Segre of q is $X_{q;n_1,\dots,n_k} = \nu(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$, with $n_1, \dots, n_k \in \{1, 2\}$.

First of all, let us remark that the matrix case is highly non-identifiable even for the rank-3 case.

Remark 2.3.1. For the two factors case, i.e. $k = 2$, a rank-3 tensor q is a 3×3 matrix of full rank. The dimension of the concise Segre of q is $\dim(X_{q;2,2}) = 4$ and

$$\dim(\sigma_3(X_{q;2,2})) = \min\{\dim(\mathbb{P}^8), 3\dim(X_{q;2,2}) + 2\} = \min\{8, 14\} = 8.$$

Therefore $\dim \mathcal{S}(Y_{2,2}, q) = 14 - 8 = 6$ for all $q \in \mathbb{P}^8$ of rank 3.

Another well known example of non-identifiable rank-3 tensors is given by points on the tangential variety $\tau(\nu(Y_{1,1,1}))$.

Remark 2.3.2. For the three factors case, the concise Segre of a rank-3 tensor q lying on the tangential variety is $X_{q;1,1,1} = \nu(Y_{1,1,1})$ (cf. Remark 1.1.23). Since $q \in \tau(X_{q;1,1,1})$, there exists some $p = [u \otimes v \otimes w] \in X_{1,1,1}$ such that $q \in T_p X_{1,1,1} = \mathbb{P}(V \otimes v \otimes w + u \otimes V \otimes w + u \otimes v \otimes V)$, where we denoted by V the affine vector space of dimension two such that $\mathbb{P}V \times \mathbb{P}V \times \mathbb{P}V = Y_{1,1,1}$. Therefore there exists some $a, b, c \in V$ such that q can be written as

$$q = [a \otimes v \otimes w + u \otimes b \otimes w + u \otimes v \otimes c].$$

Now it is straightforward to see that q is actually non-identifiable.

In the following remark we explain the behaviour on $\sigma_3((\mathbb{P}^1)^4)$.

Remark 2.3.3. It has been shown in Theorem 1.1.24 that the third secant variety of a Segre variety $X_{n_1,\ldots,n_k}$ is never defective unless either $X_{1,1,1,1} = \nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ or $X_{1,1,a} = \nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^a)$, with $a \geq 3$. The case in which q is a rank-3 tensor in $\langle \nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^a) \rangle$ with $a \geq 3$ corresponds to a not-concise tensor (cf. Remark 2.2.3) therefore it won't play a role in our further discussion.

The case of $X_{1,1,1,1}$ and $q \in \langle X_{1,1,1,1} \rangle$ can also be easily handled. Indeed the fact that $\dim(\sigma_3(X_{1,1,1,1}))$ is strictly smaller than the expected dimension proves that the generic element of $\sigma_3(X_{1,1,1,1})$ has an infinite number of rank-3 decompositions. By definition of dimension, there is no element of $\sigma_3(X_{1,1,1,1})$ such that its tangent space has dimension equal to the expected one: $\dim(T_q(\sigma_3(X_{1,1,1,1}))) < \dim \sigma_3(X_{1,1,1,1})$ for all $q \in \sigma_3(X_{1,1,1,1})$. This does not exclude the existence of certain special rank-3 tensors q such that $\dim(T_q(\sigma_3(X_{1,1,1,1}))) = \dim(T_{q'}(AbSec_3(X_{1,1,1,1}))) < 14$ where $AbSec_3(X_{1,1,1,1})$is the 3-th abstract secant of $X_{1,1,1,1}$ as in Definition 1.1.25 and q' is the preimage of q via the projection on the first factor. The impossibility of the existence of such a point is guaranteed by Proposition 2.1.4. This proves that all the tensors of $\sigma_3^0(X_{1,1,1,1})$ have an infinite number of rank-3 decompositions.

Remarks 2.3.1, 2.3.2 and 2.3.3 correspond respectively to items (a),(b) and (c) of Theorem 2.6.1 and they are all well known cases of non-identifiable rank-3 tensors.

Before describing new instances of non-identifiability, we point out the behaviour of the projection $\pi_i(A) \subset \mathbb{P}^{n_i}$ of a solution $A \in \mathcal{S}(Y_{n_1,\ldots,n_k}, q)$ of a rank-3 tensor q when $n_i = 2$, for some $i = 1, \ldots, k$.

Remark 2.3.4. Let $Y_{n_1,\ldots,n_k}$ be a multiprojective space with at least two factors where at least one of them is of projective dimension 2. By relabeling, if necessary, we can assume that the first factor is a $\mathbb{P}^2$. Let $q \in \sigma_3^0(\nu(Y_{2,n_2,\ldots,n_k}))$, with $\nu(Y_{2,n_2,\ldots,n_k})$ being the concise Segre of q and let $A, B \in \mathcal{S}(Y_{2,n_2,\ldots,n_k}, q)$ be two disjoint subsets evincing the rank of q. By Autarky $\langle \pi_1(A) \rangle = \langle \pi_1(B) \rangle = \mathbb{P}^2$. Moreover when considering the restrictions of the projections $\pi_{1|A}$ and $\pi_{1|B}$ to the subsets A and B respectively, they are both injective and both $\pi_1(A)$ and $\pi_1(B)$ contain linearly independent points.

2.3.1 Two new examples of non-identifiability

For the first two new examples of non-identifiable tensors, we work on $Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Before going into details, we need some preliminary results.

Remark 2.3.5. Let $Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ and consider an irreducible divisor $G \in |\mathcal{O}_{Y_{2,1,1}}(0,1,1)|$. Then

$$\sigma_2(\nu(G)) \subsetneq \sigma_3(\nu(G)) = \langle \nu(G) \rangle = \mathbb{P}^8.$$

Indeed G is nothing else than the Segre-Veronese variety of $\mathbb{P}^2 \times \mathbb{P}^1$ embedded in bi–degree (1,2), i.e. $G \cong \mathbb{P}^2 \times \mathbb{P}^1$, $\mathcal{O}_{Y_{2,1,1}}(1,1,1)_{|G} \cong \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(1,2)$ and $\mathcal{O}_{Y_{2,1,1}}(1,0,0) \cong \mathcal{O}_{Y_{2,1,1}}(1,1,1)(-G)$. When considering the inclusion exact sequence, we get

$$0 \longrightarrow \mathcal{O}_{Y_{2,1,1}}(1,0,0) \longrightarrow \mathcal{O}_{Y_{2,1,1}}(1,1,1) \longrightarrow \mathcal{O}_{Y_{2,1,1}}(1,1,1)_{|G} \longrightarrow 0.$$

By direct computation one has that $\langle \nu(G) \rangle = \mathbb{P}^8$ and consequentially that $\sigma_2(\nu(G)) \subsetneq \langle \nu(G) \rangle$. The only non trivial fact to be proved is $\sigma_3(\nu(G)) = \langle \nu(G) \rangle \cong \mathbb{P}^8$.
Fix three general points $p_1, p_2, p_3 \in \mathbb{P}^2 \times \mathbb{P}^1$. By Terracini's Lemma, it is sufficient to prove that $h^0\big(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{I}_{2p_1\cup 2p_2\cup 2p_3}(1,2)\big) = 0$. This is done with the multiprojective-affine-projective technique explained at the end of Section 1.2.1 (cf. Theorem 1.2.6). Let $Z \subset G$ be the 0-dimensional scheme union of the three fat points $2p_1$, $2p_2$, $2p_3$ and consider line $W_1 \simeq \mathbb{P}^1$ and subscheme $W_2 \subset \mathbb{P}^1$ of the second factor of G defined by $W_2 = 2p$, where $p \in \mathbb{P}^1$ is a generic point. Let $Z' = 2p_1 + 2p_2 + 2p_3 \subset \mathbb{P}^3$ be the corresponding zero dimensional scheme of Z and consider $W + Z' \subset \mathbb{P}^3$ where we denoted by $W = W_1 + W_2$. Then we have that $\dim(\mathcal{I}_{2p_1\cup 2p_2\cup 2p_3})_{(1,2)} = \dim(\mathcal{I}_{W+Z})_3$, which is zero.

Proposition 2.3.6. *For the Segre embedding of $Y_{2,1,1}$, fix $G_1 \in |\mathcal{O}_{Y_{2,1,1}}(0,1,0)|$ and $G_2 \in |\mathcal{O}_{Y_{2,1,1}}(0,0,1)|$ and define $G := G_1 \cup G_2$ to be their union. We have that for $\{i,j\} = \{1,2\}$, $\dim\langle \nu(G_i)\rangle = 5$, $\dim\langle \nu(G)\rangle = 8$, $\sigma_2(\nu(G_i)) = \langle \nu(G_i)\rangle$ and $\langle \nu(G)\rangle$ is the join of $\sigma_2(\nu(G_i))$ and $\nu(G_j)$.*

Proof. First of all remark that, for $i = 1, 2$, $G_i \cong \mathbb{P}^2 \times \mathbb{P}^1$, $\mathcal{O}_{Y_{2,1,1}}(1,1,1)_{|G_i} \cong \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(1,1)$ and G is a reducible element of $|\mathcal{O}_{Y_{2,1,1}}(0,1,1)|$. With an analogous computation of the one in Remark 2.3.5, one sees that $\dim\langle \nu(G)\rangle = 8$ and $\sigma_2(\nu(G_i)) = \langle \nu(G_i)\rangle$. It remains to show that $\langle \nu(G)\rangle = \mathcal{J}$, where $\mathcal{J}$ denotes the join of $\sigma_2(\nu(G_i))$ and $\nu(G_j)$ with $\{i,j\} = \{1,2\}$ as in Definition 1.1.13. We remark that since $\sigma_2(\nu(G)) = \mathbb{P}^5$, then $\mathcal{J} = \mathrm{Join}(\mathbb{P}^5, \nu(G_i))$. In order to show that $\mathcal{J} = \mathbb{P}^8$ it is sufficient to see that $\dim(\sigma_2(\nu(G_i) \cap \nu(G_j))) = 1$ and this is a straightforward computation since the elements of $\nu(G_1)$ are tensors with a second factor fixed, while the elements of $\nu(G_2)$ have the third factor fixed, and in order to have the equality between an element of $\sigma_2(\nu(G_1))$ and an element of $\nu(G_2)$ it is sufficient to impose two linear independent conditions. Therefore since $\dim(\nu(G_2)) = 3$ we have that the intersection has dimension 1. □

We are now ready to present the first new example of a non-identifiable rank-3 tensor.

Example 2.3.7. Take $Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, consider the Segre embedding on the last two factors and take a hyperplane section which intersects $\nu(\mathbb{P}^1 \times \mathbb{P}^1)$ in a conic $\mathcal{C}$, then take a point $q \in \langle \nu(\mathbb{P}^2 \times \mathcal{C})\rangle$. Such a construction is equivalent to consider an irreducible divisor $G \in |\mathcal{O}_{Y_{2,1,1}}(0,1,1)|$, so $G \cong \mathbb{P}^2 \times \mathbb{P}^1$ embedded via $\mathcal{O}(1,2)$, then $\dim \sigma_2(\nu(G)) = 7$, thus $\sigma_2(\nu(G)) \subsetneq \langle \nu(G)\rangle \cong \mathbb{P}^8$. As a direct consequence we get that a general point $q \in \langle \nu(G)\rangle$ has $\nu(G)$-rank 3 and it is non-identifiable because of the non-identifiability of the points on $\langle \mathcal{C}\rangle$ and by Proposition 2.1.4. Thus $\dim(\mathcal{S}(G,q)) = 3$. See Figure 2.1.

Let us rewrite in coordinates the above example.

Example 2.3.8 (Coordinate description of Example 2.3.7)**.** Let $\mathcal{C}$ be an irreducible conic arising from the above description. This is equivalent to consider the following framework. Let $L \subset \nu(\mathbb{P}^1 \times \mathbb{P}^1)$ be a rational curve. So $L \cong \mathbb{P}^1$ and we consider $G \cong \mathbb{P}^2 \times L$ embedded via $\mathcal{O}(1,2)$. The rank-3 tensor we are looking for is $[T] \in \langle \nu(G) \rangle$. Therefore, there exist bases $\{u_1, u_2, u_3\} \subset \mathbb{C}^3$, $\{v_1, v_2\} \subset \mathbb{C}^2$ such that

$$T = u_1 \otimes v_1^2 + u_2 \otimes v_2^2 + u_3 \otimes (\alpha v_1 + \beta v_2)^2.$$

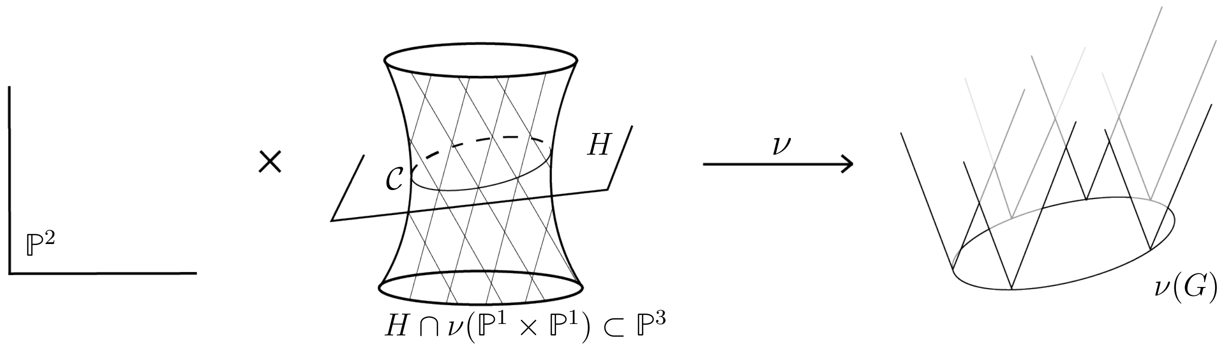


Figure 2.1: Pseudo-picture of Example 2.3.7.

The following example is in the same setting of the previous one, but in this case we deal with a reducible conic and in such a case we get a 4-dimensional family of solutions.

Example 2.3.9. Fix $Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Consider $G_1 \in |\mathcal{O}_{Y_{2,1,1}}(0,0,1)|$, $G_2 \in |\mathcal{O}_{Y_{2,1,1}}(0,1,0)|$ and call $G = G_1 \cup G_2$ which is a reducible element of $|\mathcal{O}_{Y_{2,1,1}}(0,1,1)|$. By Proposition 2.3.6, $\dim\langle \nu(G) \rangle = 8$ and $\langle \nu(G_i) \rangle = \sigma_2(G_i)$, for $i = 1, 2$, both having dimension 5. By Proposition 2.3.6 we also have that

$$\langle \nu(G) \rangle = \mathcal{J}_1 = \mathcal{J}_2,$$

where $\mathcal{J}_1 = \mathrm{Join}(\sigma_2(\nu(G_1)), \nu(G_2))$ and $\mathcal{J}_2 = \mathrm{Join}(\sigma_2(\nu(G_2)), \nu(G_1))$. A general $q \in \langle \nu(G) \rangle$ has rank 3 and for the subsets evincing its rank we have a 4-dimensional family of sets A such that

$$\#(A) = 3,\ \#(A \cap G_1) = 2,\ \#(A \cap G_2) = 1,\ A \cap G_1 \cap G_2 = \emptyset \text{ and } q \in \langle \nu(A) \rangle.$$

Such a family has dimension 4 since G_1 is a non defective threefold in $\mathbb{P}^5$, therefore there exists a 2-dimensional family of sets of cardinality 2 in G_1 spanning a general point of $\mathbb{P}^5$; moreover q sits in a 2-dimensional family of lines joining points of G_1 and G_2. Analogously, by looking at q as an element of $\mathcal{J}_2$, we get the existence of a 4-dimensional family of sets B such that

$$\#(B) = 3,\ \#(B \cap G_2) = 2,\ \#(B \cap G_1) = 1,\ A \cap G_1 \cap G_2 = \emptyset \text{ and } q \in \langle \nu(B) \rangle.$$

So we proved that $\mathcal{S}(G, q)$ contains at least two dimensional families of solution. Thus $\dim(\mathcal{S}(G, q)) \geq 4$.

Let us rewrite in coordinates the above example.

Example 2.3.10 (Coordinate description of Example 2.3.9)**.** Let $\mathcal{C}$ be a reducible conic. In this case, we take $G_1 \in |\mathcal{O}_{Y_{2,1,1}}(0,0,1)|$, $G_2 \in |\mathcal{O}_{Y_{2,1,1}}(0,1,0)|$ and set $G := G_1 \cup G_2$. So $G_1 \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \{[\tilde{p}]\}$ and $G_2 \cong \mathbb{P}^2 \times \{[\tilde{q}]\} \times \mathbb{P}^1$, for some $[\tilde{p}], [\tilde{q}] \in \mathbb{P}^1$. We consider

$$[T] \in \langle \nu(G) \rangle = Join(\sigma_2(\nu(G_1)), \nu(G_2)) = Join(\sigma_2(\nu(G_2)), \nu(G_1)).$$

Since $[T] \in \langle \nu(G) \rangle = Join(\sigma_2(\nu(G_1)), \nu(G_2))$, there exist bases $\{u_1, u_2, u_3\}$ of $\mathbb{C}^3$, $\{v_1, v_2\}$ of the second factor and $\{\tilde{p}, w\}$ of the third factor such that T can be written as

$$T = u_1 \otimes v_1 \otimes \tilde{p} + u_2 \otimes v_2 \otimes \tilde{p} + u_3 \otimes \tilde{q} \otimes w.$$

In Examples 2.3.7 and 2.3.9 we computed the dimension of $\mathcal{S}(G, q)$ in the case of G being either an irreducible or a reducible element of $|\mathcal{O}_{Y_{2,1,1}}(0,1,1)|$. The only thing left to do in order to conclude both examples, is to show that the space of solutions of q with respect to G coincide with the space of solutions of q with respect to $Y_{2,1,1}$.

Proposition 2.3.11. *Let $q \in \sigma_3^0(\nu(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1))$ and suppose that there exist $A, B \in \mathcal{S}(Y_{2,1,1}, q)$ s.t. $\#(A \cup B) = 6$. Then there exist a unique $G \in |\mathcal{O}_{Y_{2,1,1}}(0,1,1)|$ containing $S = A \cup B$. For such a G we have that $\mathcal{S}(Y_{2,1,1}, q) = \mathcal{S}(G, q)$.*

Proof. Call $S := A \cup B$, by Remark 2.3.4, both $\pi_{1|A}$ and $\pi_{1|B}$ are injective and both $\pi_1(A)$ and $\pi_1(B)$ are sets containing linearly independent points. So $h^1(\mathcal{I}_A(1,0,0)) = h^1(\mathcal{I}_B(1,0,0)) = 0$. Now $h^0(\mathcal{O}_{Y_{2,1,1}}(0,1,1)) = 4$, so there exists $G \in |\mathcal{O}_{Y_{2,1,1}}(0,1,1)|$ containing B. Moreover $S \setminus S \cap G \subseteq A$ but since $h^1(\mathcal{I}_A(1,0,0)) = 0$ we have that $S \subset G$. This holds for any $G \in |\mathcal{I}_B(0,1,1)|$, so $\langle \nu_1(\eta_1(A)) \rangle \subset \langle \nu_1(\eta_1(B)) \rangle$. The same holds exchanging the roles of A and B, thus $\langle \nu_1(\eta_1(A)) \rangle = \langle \nu_1(\eta_1(B)) \rangle$.

Assume G is irreducible, then B contains three linearly independent points on G, so the points of B are uniquely determined by G.

Assume G is reducible, i.e. $G = G_1 \cup G_2$, with $G_1 \in |\mathcal{O}_{Y_{2,1,1}}(0,1,0)|$ and $G_2 \in |\mathcal{O}_{Y_{2,1,1}}(0,0,1)|$. Remark that, by Autarky, it does not exist any $E \in \mathcal{S}(Y_{2,1,1}, q)$ which is all contained in G_i, for $i = 1, 2$, because G is a multiprojective subspace of $Y_{2,1,1}$. Without loss of generality, we may assume that two points of E lies in G_1; then the three points of E are uniquely determined by a reducible conic, i.e. by the reducible element $G = G_1 \cup G_2$ that contains them. □

Corollary 2.3.12. *If $q \in \sigma_3^0(\nu(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1))$ is such that there exist two disjoint sets $A, B \in \mathcal{S}(Y_{2,1,1}, q)$, then q can be either as in Example 2.3.7 and $\dim(\mathcal{S}(Y_{2,1,1}, q)) = 3$ or as in Example 2.3.9 and $\dim(\mathcal{S}(Y_{2,1,1}, q)) = 4$.*

Proof. This is a direct consequence of the uniqueness of the $G \in |\mathcal{O}_{Y_{2,1,1}}(0,1,1)|$ such that $\mathcal{S}(Y_{2,1,1}, q) = \mathcal{S}(G, q)$ as shown in Proposition 2.3.11. □

2.3.2 A new family of non-identifiable rank-3 tensors

We build now a new family of non-identifiable rank-3 tensors.
Let $Y' := \mathbb{P}^1 \times \mathbb{P}^1 \times \{u_3\} \times \cdots \times \{u_k\}$ be a proper subset of $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where we assumed $k \geq 2$. Take

$$q' \in \langle \nu(Y') \rangle \setminus \nu(Y'),\ A \in \mathcal{S}(Y', q') \text{ and } p \in Y_{n_1,\dots,n_k} \setminus Y'.$$

Assume also that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing $A \cup \{p\}$ and take $q \in \langle \{q', \nu(p)\} \rangle \setminus \{q', \nu(p)\}$ as shown in Figure 2.2.

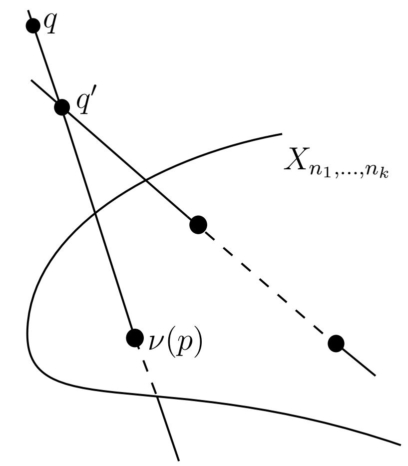


Figure 2.2: A new family of non-identifiable rank-3 tensors

Before proving under which conditions this construction leads to a non-identifiable rank-3 tensor, we remark the following.

Remark 2.3.13. With notation as above, observe that

- $\sum_{i=1}^{k} n_i \geq 3$;
- $n_1, n_2 \leq 2$, $n_3, \dots, n_k \leq 1$;
- if $k \geq 3$ then $r_{\nu(Y_{n_1,\dots,n_k})}(q) > 1$.

Indeed $r_{\nu(Y_{n_1,\dots,n_k})}(q) > 1$, otherwise there exists $o \in Y_{n_1,\dots,n_k}$ such that $q = \nu(o)$. Therefore one can look at q' as $q' \in \langle \nu(\{o,p\}) \rangle$ and since $r_{\nu(Y_{n_1,\dots,n_k})}(q') = 2$, we would have $\{o,p\} \in \mathcal{S}(Y_{n_1,\dots,n_k}, q')$ and by Autarky we would get $\{o,p\} \subset Y'$, contradicting the assumption that $p \notin Y'$.
The fact that $n_1 + \cdots + n_k \geq 3$ is obvious from the fact that $p \notin Y'$ so $Y_{n_1,\dots,n_k} \neq Y'$.
Since q' is a 2×2 matrix of rank 2, we get that $\dim(\mathcal{S}(Y', q')) = 2$ and that Y' is the minimal multiprojective subspace of $Y_{n_1,\dots,n_k}$ containing A. The minimal multiprojective subspace containing $Y' \cup \{p\}$ is $Y_{n_1,\dots,n_k}$, so since $\mathbb{P}^{n_i} = \langle \pi_i(Y' \cup \{p\}) \rangle$, we get $1 \leq n_i \leq 2$ for $i = 1, 2$ and $n_i = 1$ for all $i > 2$.

We are now ready to prove that if $k \geq 3$ and $\sum_{i \leq k} n_i \geq 4$, any tensor q as above is actually a non-identifiable rank-3 tensor. Moreover we see that all possible decompositions of q are given by the union of all possible decompositions $A \in \mathcal{S}(Y', q')$ of q' and p.

Proposition 2.3.14. *Let $Y' := \mathbb{P}^1 \times \mathbb{P}^1 \times \{u_3\} \times \cdots \times \{u_k\}$ be a proper subset of $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where $k \geq 3$, $n_1, n_2 \leq 2$, all $n_i = 1$ for $i = 3, \dots, k$ and $\sum_{i=1}^{k} n_i \geq 4$. Take $q' \in \langle \nu(Y') \rangle \setminus \nu(Y')$, $A \in \mathcal{S}(Y', q')$ and $p \in Y_{n_1,\dots,n_k} \setminus Y'$. Assume that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing $A \cup \{p\}$ and take $q \in \langle \{q', \nu(p)\} \rangle \setminus \{q', \nu(p)\}$. Then*

1. *$r_{\nu(Y_{n_1,\dots,n_k})}(q) = 3$ and $\mathcal{S}(Y_{n_1,\dots,n_k}, q) = \{\{p\} \cup A\}_{A \in \mathcal{S}(Y',q')}$.*
2. *$\nu(Y_{n_1,\dots,n_k})$ is the concise Segre of q.*

Proof. Item 2 will be a consequence of item 1, in fact if the structure of the elements on $\mathcal{S}(Y_{n_1,\dots,n_k}, q)$ is of type $A\cup\{p\}$ with $A \in \mathcal{S}(Y', q')$, then Autarky and the fact that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing $A \cup \{p\}$ will imply that $\nu(Y_{n_1,\dots,n_k})$ is the concise Segre of q. So let us prove item 1.

The proof is by induction on the number of factors. Step (A) is the basis of induction for the case in which $Y_{n_1,\dots,n_k}$ has at least one factor of projective dimension 2 ($k = 3$), Step (B) is the basis of induction for the case in which all the factors of $Y_{n_1,\dots,n_k}$ have projective dimension 1 ($k = 4$), Steps (C) and (D) are the induction processes of Step (B) and Step (A) respectively.
Let $E \in \mathcal{S}(Y_{n_1,\dots,n_k}, q)$, if we will show that $E \supset \{p\}$ and that there exists $B \in \mathcal{S}(Y', q')$, such that $E = B \cup \{p\}$, we will be done. Assume that there is no $B \in \mathcal{S}(Y', q')$ such that $E = B \cup \{p\}$. Fix any $A \in \mathcal{S}(Y', q')$ and set $S := A \cup \{p\} \cup E$.

(A) [Case $k = 3$, $n_1 = 2$, $n_2 = n_3 = 1$]
First assume $p \in E$ and set $E' := E \setminus \{p\}$ and $F = A \cup E'$. Since by Remark 2.2.6

$$\bigcap_{B\in\mathcal{S}(Y_{211},q')} \eta_3(B) = \emptyset,$$

taking another $A \in \mathcal{S}(Y_{2,1,1}, q')$ if necessary we may assume $\eta_3(A) \cap \eta_3(E') = \emptyset$. Set $\{D\} := |\mathcal{I}_p(0,0,1)|$, i.e. $D \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \{\pi_3(p)\}$. By Lemma 1.2.9, we have $h^1\big(\mathcal{I}_{S\setminus S\cap D}(1,1,0)\big) > 0$, where we note that actually $S \setminus S \cap D = F$. Therefore by the inclusion exact sequence, since $\#F \le 4$, we get $h^0\big(\mathcal{I}_{S\setminus S\cap D}(1,1,0)\big) \ge 3$. This must be true for all $A \in \mathcal{S}(Y', q')$ and hence we have $h^0\big(Y_{2,1;3}, \mathcal{I}_{\eta_3(Y')\cup\eta_3(E')}(1,1)\big) \ge 3$. Since $\eta_3(Y') \in |\mathcal{O}_{Y_{2,1;3}}(1,1)|$ we have $h^0\big(Y_{2,1;3}, \mathcal{I}_{\eta_3(Y')}(1,1)\big) = 1$, contradicting the previous inequality.

From now on suppose $p \notin E$. As above we may assume $\eta_3(A) \cap \eta_3(E) = \emptyset$.

Fix $o \in E$. Since $h^0(\mathcal{O}_{Y_{2,1,1}}(1,1,0)) = 6$ and $\#(A \cup \{p\} \cup \{o\}) = 4$ there is $G \in |\mathcal{O}_{Y_{2,1,1}}(1,1,0)|$ containing $A \cup \{p\} \cup \{o\}$. Assume for the moment $S \nsubseteq G$, i.e. $E \nsubseteq G$. By Lemma 1.2.9 we have $h^1\big(\mathcal{I}_{S\setminus S\cap G}(0,0,1)\big) > 0$, which means that $\#E \ge 3$. Since by construction $\#E \le 3$, we get $\#E = 3$ and hence q has rank 3 and $\nu(Y_{2,1,1})$ is the concise Segre of q. Moreover we get that $S \setminus S \cap G = E \setminus \{o\}$ and $\#\pi_3(E \setminus \{o\}) = 1$. By applying the same argument taking a different element of E we get that actually $\#\pi_3(E) = 1$, i.e. $\nu(Y_{2,1,1})$ is not the concise Segre of q, a contradiction.

Now assume $S \subset G$. Since this must be true for all $G \in |\mathcal{I}_{A\cup\{p,o\}}(1,1,0)|$, we get $|\mathcal{I}_{A\cup\{p,o\}}(1,1,0)| \supseteq |\mathcal{I}_{\{p\}\cup E}(1,1,0)| \ne \emptyset$. Note that $\eta_3(Y') \in |\mathcal{O}_{Y_{2,1;3}}(1,1)|$ and hence $h^0\big(Y_{2,1;3}, \mathcal{I}_{\eta_3(Y')}(1,1)\big) = 1$. Since the first factor of $Y_{2,1,1}$ has projective dimension 2 and $Y_{2,1,1}$ is the minimal multiprojective space containing q, we have $\eta_3(p) \notin \eta_3(Y')$. Thus $h^0\big(Y_{2,1;3}, \mathcal{I}_{\eta_3(Y')\cup\{\eta_3(p)\}}(1,1)\big) = 0$, a contradiction since $|\mathcal{I}_{A\cup\{p,o\}}(1,1,0)| \ne \emptyset$.

(B) [Case $k = 4$, $n_1 = n_2 = n_3 = n_4 = 1$]
Fix $G \in |\mathcal{O}_{Y_{1,1,1,1}}(0,0,1,1)|$ containing E. Assume $S \nsubseteq G$. Since $S \setminus E = A \cup \{p\}$, by Lemma 1.2.9, we have $h^1\big(\mathcal{I}_{A\cup\{p\}}(1,1,0,0)\big) > 0$. Call p' the projection of p via $Y_{1,1,1,1} \longrightarrow Y'$. Since $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ is very ample we get that either $p' \in A$ or that $\#(\pi_i(A \cup \{p'\})) = 1$ for some $i \in \{1,2\}$. The second possibility is excluded, because $\#(\pi_1(A)) = \#(\pi_2(A)) = 2$ for any $A \in \mathcal{S}(Y', q')$. The first possibility is excluded taking instead of A another general $A_1 \in \mathcal{S}(Y', q')$. Now assume $S \subset G$, hence in particular $A \subset G$. This is ruled out taking another $A \in \mathcal{S}(Y', q')$ since a general

$a \in Y'$ is contained in some $B \in \mathcal{S}(Y', q')$. Thus we would have that $Y' \subset G$ which is a contradiction.

(C) [Case $k \geq 5$, $n_i = 1$ for all i's]
We exclude this case by induction on k, the base case $k = 4$ being excluded in (B). Fix $o \in \mathbb{P}^1 \setminus \{p_k, u_k\}$, set $M := \pi_k^{-1}(o)$, i.e. $M = (\mathbb{P}^1)^{k-1} \times \{o\}$ and call $\Lambda := \langle \nu(M) \rangle$. Note that $(Y' \cup \{p\}) \cap M = \emptyset$. Denote by $r = 2^k - 1$ and define $r' := \dim \Lambda = 2^{k-1} - 1$.

Consider the following linear projection form Λ:

$$\ell : \mathbb{P}^r \setminus \Lambda \longrightarrow \mathbb{P}^{r'}. \tag{2.3.1}$$

Note that $\nu(Y_{1^k}) \cap \Lambda = \nu(Y_{1^{k-1};k}) \times \{o\}$ and that $\ell_{|\nu(Y_{1^k}) \setminus M} = \nu_k(\eta_k(Y_{1^k} \setminus M))$. We identify $\mathbb{P}^{r'}$ with the target projective space of $Y_{1^{k-1};k}$. Since $(Y' \cup \{p\}) \cap M = \emptyset$, we get that ℓ is well-defined on $Y' \cup \{p\}$ and it acts as the composition of η_k and the Segre embedding.

By the inductive assumption $\mathcal{S}(Y_{1^{k-1};k}, \ell(q)) = \{B \cup \eta_k(p)\}_{B \in \mathcal{S}(\eta_k(Y'), \eta_k(q'))}$. Thus for any $E \in \mathcal{S}(Y_{1^k}, q)$ there is $B \in \mathcal{S}(Y', q')$ such that $\eta_k(E) = \eta_k(B \cup \{p\})$. Since $\eta_{k|E}$ is injective, by Remark 2.2.4 and $\mathcal{S}(Y_{1^k}, q) \supseteq \{B \cup \{p\}\}_{B \in \mathcal{S}(Y', q')}$, we get $\mathcal{S}(Y_{1^k}, q) = \{B \cup \{p\}\}_{B \in \mathcal{S}(Y', q')}$.

(D) [Case $k \geq 3$, $n_1 = 2$, $n_1 + \cdots + n_k \geq 5$]
If only one of the factors is a $\mathbb{P}^2$ we use Step (A) as base of the induction and then we construct a projection similar to (2.3.1). Indeed $Y_{2,1^{k-1}} = \mathbb{P}^2 \times (\mathbb{P}^1)^{k-1}$, where $k \geq 4$. Fix $o \in \mathbb{P}^1 \setminus \{p_k, u_k\}$, set $M := \pi_k^{-1}(o)$ and define $\Lambda := \langle \nu(M) \rangle$. Denote $r = 3 \cdot 2^{k-1} - 1$ and $r' = \dim \Lambda := 3 \cdot 2^{k-2} - 1$. We consider the linear projection $\ell : \mathbb{P}^r \setminus \Lambda \longrightarrow \mathbb{P}^{r'}$ which acts as the composition of η_k and the Segre embedding. By the inductive assumption $\mathcal{S}(Y_{2,1^{k-2};k}\ell(q)) = \{B \cup \eta_k(p)\}_{B \in \mathcal{S}(\eta_k(Y'), \eta_k(q'))}$. Thus for any $E \in \mathcal{S}(Y_{2,1^{k-1}}, q)$ there is $B \in \mathcal{S}(Y', q')$ such that $\eta_k(E) = \eta_k(B \cup \{p\})$. Since $\eta_{k|E}$ is injective by Remark 2.2.4 and $\mathcal{S}(Y_{2,1^{k-1}}, q) \supseteq \{B \cup \{p\}\}_{B \in \mathcal{S}(Y', q')}$, we get $\mathcal{S}(Y_{2,1^{k-1}}, q) = \{B \cup \{p\}\}_{B \in \mathcal{S}(Y', q')}$.
Now assume also $n_2 = 2$, so that we must have $k \geq 3$. Let $Y_{2,2,1^{k-2}} = \mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^1)^{k-2}$ and fix $o \in \mathbb{P}^2 \setminus \pi_2(Y')$. Set $M := \pi_2^{-1}(o)$, and $\Lambda := \langle \nu(M) \rangle$. Then $r = 9 \cdot 2^{k-2} - 1$, $\dim \Lambda = 9 \cdot 2^{k-3} - 1$. Let $r' := 9 \cdot 2^{k-3} - 1$ and consider the linear projection $\ell : \mathbb{P}^r \setminus \Lambda \longrightarrow \mathbb{P}^{r'}$ from Λ which acts on $\nu(Y)$ as the composition of the Segre embedding and the map $\mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^1)^{k-2} \setminus \mathbb{P}^2 \times \{o\} \times (\mathbb{P}^1)^{k-2} \longrightarrow \mathbb{P}^2 \times (\mathbb{P}^1)^{k-1}$, which is the linear projection $\mathbb{P}^2 \setminus \{o\} \longrightarrow \mathbb{P}^1$ on the second factor and the identity on any other factor. Since $(Y' \cup \{p\}) \cap M = \emptyset$, $\ell(q)$ is well-defined. We conclude since we already proved the statement in the case where only one of the factors is a $\mathbb{P}^2$. □

We conclude this part with a coordinate description of the family of non-identifiable rank-3 tensors presented in the above proposition adapted for the three factors case.

Example 2.3.15 (Coordinate description 3-factors case)**.** Let $Y' = \mathbb{P}^1 \times \mathbb{P}^1 \times \{w\} \subset Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Take $q' \in \langle \nu(Y') \rangle \setminus \nu(Y_{2,1,1})$ and $p \in Y_{2,1,1} \setminus Y'$. Then $[T] \in \langle q', \nu(p) \rangle$ is a rank-3 tensor and it is not identifiable. Therefore there exist bases $\{u_1, u_2, u_3\} \subset \mathbb{C}^3$ of the first factor, $\{v_1, v_2\} \subset \mathbb{C}^2$ of the second factor and $\{w, \tilde{w}\} \subset \mathbb{C}^2$ of the third factor such that T can be seen as

$$\begin{aligned} T =& u_1 \otimes v_1 \otimes w + u_2 \otimes v_2 \otimes w + (\alpha_1 u_1 + \alpha_2 u_2 + u_3) \otimes (\beta_1 v_1 + \beta_2 v_2) \otimes \tilde{w} \\ =& (u_1 \otimes v_1 + u_2 \otimes v_2) \otimes w + (\alpha_1 u_1 + \alpha_2 u_2 + u_3) \otimes (\beta_1 v_1 + \beta_2 v_2) \otimes \tilde{w}. \end{aligned}$$

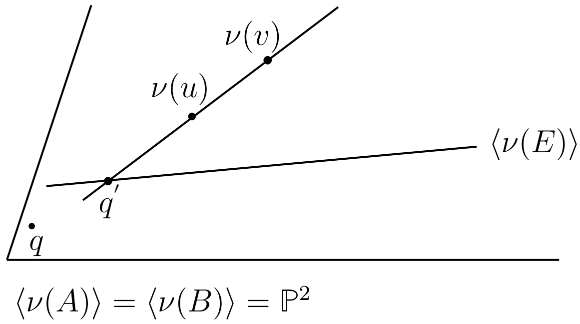


Figure 2.3: Two distinct decompositions with 2 common points

2.3.3 Two different decompositions of a non-identifiable rank-3 tensor can share at most one common point

Now we prove that all the previous examples are the only exceptions. More precisely, for any non-identifiable rank-3 tensor q we either show that q belongs to one of the above examples, or that the non-identifiability assumption leads to a contradiction.
Before proceeding, we can make some reduction by looking at the geometry of the union of two distinct solutions of a rank-3 tensor. In particular, the following two lemmas describe two very basic properties that two different sets A and B evincing the rank of the same rank-3 point q have to satisfy.

Lemma 2.3.16. *Let q be a non-identifiable tensor and let A, B be two distinct sets evincing the rank of q. Define $S := A \cup B$. If $\#(S) \geq 5$ and $\dim\langle \nu(S) \rangle = 2$, then the rank of q cannot be 3.*

Proof. Assume the existence of such a rank-3 tensor q with 2 distinct decompositions A and B such that $\#(A \cup B) \geq 5$. The plane $\langle \nu(S) \rangle$ contains at least five not-collinear points. Note that $\langle \nu(S) \rangle \nsubseteq X$, otherwise also $q \in X$ which contradicts $r_X(q) = 3$. So $\langle \nu(S) \rangle \cap X$ contains a conic $\mathcal{C}$. Either if it is reduced or not, the two secant variety of $\mathcal{C}$ fills $\langle \nu(S) \rangle = \mathbb{P}^2$. So $r_X(q) \leq 2$, which is a contradiction. $\square$

Lemma 2.3.17. *Let q be a not identifiable rank-3 tensor and let $A, B \in \mathcal{S}(Y_{n_1,\dots,n_k}, q)$ be distinct. Then $\#(A \cap B) \leq 1$.*

Proof. Suppose by contradiction that A and B have 2 distinct points in common and call the set of these two points E. Let $A = E \cup \{u\}$ and $B = E \cup \{v\}$. Since the rank of q is 3 we know that $q \notin \langle \nu(E) \rangle$, but since by definition $q \in \langle \nu(A) \rangle \cap \langle \nu(B) \rangle$ we have that $\langle \nu(E) \rangle \subsetneq \langle \nu(A) \rangle \cap \langle \nu(B) \rangle$. Clearly $\langle \nu(E) \rangle$ is a line, therefore $\dim\langle \nu(A) \rangle \cap \langle \nu(B) \rangle > 1$, but since $\langle \nu(A) \rangle$ and $\langle \nu(B) \rangle$ are both planes we get that $\langle \nu(A) \rangle = \langle \nu(B) \rangle$. In the plane $\langle \nu(A) \rangle$ we have two different lines: $\nu(E)$ and $\langle \nu(u), \nu(v) \rangle$, which mutually intersect in at most a point q', as shown in Figure 2.3. Remark that $q' \notin X$ because otherwise the line $\langle \nu(E) \rangle$ would have at least 3 points of rank 1 and so we would have $\langle \nu(E) \rangle \subset X$, contradicting Remark 2.2.4. This means that $r_X(q') = 2$ and $\#\mathcal{S}(Y_{n_1,\dots,n_k}, q') \geq 2$, therefore by Proposition 2.2.5 we get that actually $q' \in \langle \nu(Y'_{1,1}) \rangle$, where $Y'_{1,1} = \mathbb{P}^1 \times \mathbb{P}^1$. But also $E, \{u, v\} \subset Y'_{1,1}$, so $q \in \langle \nu(Y'_{1,1}) \rangle$, which contradicts the fact that q has rank 3. $\square$

An immediate corollary of Lemma 2.3.17 is the following.

Corollary 2.3.18. *If q is a rank-3 tensor and A and B are two distinct sets evincing its rank, then the cardinality of $A \cup B$ can only be either 5 or 6.*

This corollary turns out to be extremely useful for the proof of our main result, Theorem 2.6.1. We will be allowed to focus only on the structure of not-identifiable points of rank-3 with at least two decompositions A and B as in Corollary 2.3.18. This is the reason why we will study separately the case $\# A \cup B = 5$ in Section 2.4 form the case $\# A \cup B = 6$ in Section 2.5.

2.4 Two different solutions with one common point

We have seen in Corollary 2.3.18 that if a rank-3 tensor q is not identifiable and A, B are two sets of points on the Segre variety computing its rank, then $\# A \cup B$ can only be either 5 or 6. This section is fully devoted to the case in which $\# A \cup B = 5$, i.e. A and B share only one point that we denote p:

$$S := A \cup B, \quad \# S = 5, \quad A \cap B = \{p\} \text{ and } A' = A \setminus \{p\}, \quad B' = B \setminus \{p\}. \tag{2.4.2}$$

The matrix case is well known, therefore we will always assume that q is an order-$k \geq 3$ tensor, i.e. $q \in \langle \nu(Y_{n_1,\ldots,n_k}) \rangle$ with $Y_{n_1,\ldots,n_k} = \prod_{i=1}^{k} \mathbb{P}^{n_i}$ and $k \geq 3$.

We will study separately the cases in which:

- the multiprojective space contains at least one factor of projective dimension 2 and all the others of dimension either 1 or 2 (Proposition 2.4.1);
- the multiprojective space is a product of $\mathbb{P}^1$'s only (Proposition 2.4.2).

More precisely, in the first case we show that a non-identifiable rank-3 tensor with two distinct solutions sharing a common point is actually as in Proposition 2.3.14. Working with $Y_{1^k} = (\mathbb{P}^1)^k$, we will prove that two distinct solutions with one common point exist only if $k = 3, 4$.

This will completely cover the case of non-identifiable rank-3 tensors satisfying condition (2.4.2) since, by Remark 2.2.3, the concise Segre of a rank-3 point q is $X_{q,n_1,\ldots,n_k} = \nu(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$, where $n_1, \ldots, n_k \in \{1, 2\}$.

Proposition 2.4.1. *Let $Y_{2,n_1,\ldots,n_k}$ be a multiprojective space with at least 3 factors and at least one of them of projective dimension 2, i.e. $Y_{2,n_2,\ldots,n_k} = \mathbb{P}^2 \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$ with $n_i \in \{1, 2\}$ for $i = 1, \ldots, k$ and $k \geq 3$. Let $q \in \sigma_3^0(\nu(Y_{2,n_1,\ldots,n_k}))$, with $\nu(Y_{2,n_2,\ldots,n_k})$ the concise Segre of q. If there exist two sets $A, B \in \mathcal{S}(Y_{2,n_1,\ldots,n_k}, q)$ evincing the rank of q such that $\# A \cap B = 1$ then q is as in Proposition 2.3.14.*

Proof. Let $M \in |\mathcal{O}_{Y_{2,n_2,\ldots,n_k}}(\varepsilon_1)|$ be a divisor containing $A' = A \setminus \{p\}$, therefore $M \cong \mathbb{P}^1 \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$. By Concision/Autarky $S \not\subset M$, so, by Lemma 1.2.9, either $h^1\big(\mathcal{I}_{S \setminus S \cap M}(\hat{\varepsilon}_1)\big) > 0$ or $p \notin M$ and $A' \cup B' \subset M$. We study separately the two cases.

1. First assume $h^1\big(\mathcal{I}_{S \setminus S \cap M}(\hat{\varepsilon}_1)\big) > 0$.

   The divisor M contains A' by definition so $\#(S \setminus S \cap M) \leq 3$. Moreover, if we define $Y_1 := \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$ with $n_i = 1, 2$ for $i = 2, \ldots, k$, we have that $\mathcal{O}_{Y_1}(1, \ldots, 1)$ is very ample, therefore we can apply Lemma 1.2.11 and say that one of the following occurs:

(i) $\#(S \setminus S \cap M) = 3$ and there exists an index $j \in \{2, \ldots, k\}$ such that $\#(\pi_i(S \setminus S \cap M)) = 1$ for all $i \neq j$;

(ii) There exist $u, v \in (S \setminus S \cap M)$ such that $u \neq v$ and $\eta_1(u) = \eta_1(v)$.

We remark that case (ii) implies that $\pi_i(u) = \pi_i(v)$ for all $i > 1$, i.e. that u and v are contained in a line $L \subset X$. Since M contains A', we have that $S \setminus S \cap M = \{u, v\} \subseteq B$, we can exclude case (ii) thanks to Remark 2.2.4.

Therefore only case (i) is possible. Since $\#(S \setminus S \cap M) = 3$ we have that $S \setminus S \cap M = B$. Case (i) implies that for at least $k-2$ indices the projections of the points of B coincide, where we assumed $k \geq 3$. A direct consequence is that B only depends by 2 factors of $Y_{2,n_2,\ldots,n_k}$ at most, contradicting Autarky.

2. Now assume $A' \cup B' \subset M$.

Let Y'' be the minimal multiprojective space containing $A' \cup B'$ and contained in M. Since $q \in \langle \langle \nu(Y'') \rangle \cup \{\nu(p)\} \rangle$ and $p \notin Y''$, there is a unique $o \in \langle \nu(Y'') \rangle$ such that $q \in \langle \{\nu(p), o\} \rangle$. Since $\langle \nu(A) \rangle$ (resp. $\langle \nu(B) \rangle$) is a plane containing $\nu(p)$ and q, there is a unique $o_1 \in \langle \nu(A') \rangle$ (resp. $o_2 \in \langle \nu(B') \rangle$) such that $q \in \langle \{\nu(p), o_1\} \rangle$ (resp. $q \in \langle \{\nu(p), o_2\} \rangle$). The uniqueness of o gives $o = o_1 = o_2$. Therefore o is a rank-2 tensor with A' and B' as solutions. By Proposition 2.2.7 this means that $o \in \langle \nu(\mathbb{P}^1 \times \mathbb{P}^1) \rangle$. Moreover, since $p \notin M \supset Y''$, the minimal multiprojective space containing q is $Y_{n_1,n_2,1^{k-2}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, where $n_1, n_2 \in \{1, 2\}$. Therefore q is as described in Proposition 2.3.14. $\square$

We just treated the case in which the multiprojective space contains at least one factor of projective dimension 2, so from now on we can work with a multiprojective space made by $\mathbb{P}^1$'s only.

Proposition 2.4.2. *Let $Y_{1^k} = (\mathbb{P}^1)^k$ with $k \geq 3$ and let $q \in \sigma_3^0(\nu(Y_{1^k}))$ be such that there exist two different sets $A, B \in \mathcal{S}(Y_{1^k}, q)$ with $\#(A \cup B) = 5$, where $\nu(Y_{1^k})$ is the concise Segre of q. Then k can only be either 3 or 4. If $k = 3$ then q belongs to a tangent space of $\nu((\mathbb{P}^1)^3)$ and $\dim(\mathcal{S}(Y_{1,1,1}, q)) \geq 2$. If $k = 4$ then $\dim(\mathcal{S}(Y_{1,1,1,1}, q)) \geq 1$.*

Proof. Cases $k = 3, 4$ are covered by Remarks 2.3.2 and 2.3.3 respectively.

Assume $k > 4$ and write $Y_{1^k} = \prod_{i=1}^k \mathbb{P}^1_i$. Let $S = A \cup B$ be as in (2.4.2), we build now a recursive set of divisors as follows:

1st divisor: $M_4 \in |\mathcal{O}_{Y_{1^k}}(\varepsilon_4)|$ containing $A \cap B = \{p\}$;

2nd divisor: let $o_3 \in \mathbb{P}^1$ be such that $\pi_3^{-1}(o_3) \cap (S \setminus (S \cap M_4)) \neq \emptyset$ ad call $M_3 := \pi_3^{-1}(o_3)$;

3rd divisor:
- If $M_3 \cup M_4$ already covers the whole S (i.e. $S \subset M_3 \cup M_4$), set M_2 to be any divisor $M_2 \in |\mathcal{O}_{Y_{1^k}}(\varepsilon_2)|$.
- Otherwise $S \not\subset M_3 \cup M_4$. In this case choose $o_2 \in \mathbb{P}^1_2$ such that $\pi_2^{-1}(o_2) \cap (S \setminus S \cap (M_3 \cup M_4)) \neq \emptyset$ and set $M_2 := \pi_2^{-1}(o_2)$.

Now it may happen that either $S \subset M_2 \cup M_3 \cup M_4$ or not. We study those two cases in (a) and (b) respectively.

(a) Here we assume that $S \subset M_2 \cup M_3 \cup M_4$. Since $\#(S) = 5$ there is at least one of the M_i's containing at least two points of S and there are two of the M_i's whose union contains at least 4 points of S. More precisely, it is easy to see that $\#(S \cap (M_i \cup M_4)) \geq 4$ for some $i = 2, 3$, otherwise we would have $\#(S \cap (M_2 \cup M_3)) = 4$ and by Lemma 1.2.9 we would get $h^1\big(\mathcal{I}_{S \setminus S \cap M_2 \cup M_3}(1, 0, 0, 1, \dots)\big) > 0$, which is impossible. Therefore, by relabeling if necessary, we may assume $\#(S \cap (M_3 \cup M_4)) \geq 4$.

- If $\#(S \cap (M_3 \cup M_4)) = 4$, since $\mathcal{O}_Y(1, 1, 0, 0, \dots)$ is globally generated, we have that $h^1\big(\mathcal{I}_{S \setminus S \cap (M_3 \cup M_4)}(1, 1, 0, 0, 1, 1, \dots)\big) = 0$, which is in contradiction with Lemma 1.2.9.
- Assume $S \subset M_3 \cup M_4$. In this case, one of the M_i's contains at least 3 points of S for $i = 3, 4$ and we may assume $i = 4$. Indeed if $\#S \cap M_3 \geq 3$, by Lemma 1.2.9 we have $h^1\big(\mathcal{I}_{S \setminus (S \cap M_3)}(\hat{\varepsilon}_3)\big) > 0$ which is equivalent to say that there exists $u \in S$ such that $\pi_i(p) = \pi_i(u)$ for all $i \neq 3$, contradicting Remark 2.2.4. Therefore we may assume $\#(M_4 \cap S) \geq 3$. Since $S \not\subset M_4$, we get $h^1\big(\mathcal{I}_{S \setminus S \cap M_4}(\hat{\varepsilon}_4)\big) > 0$ (by Lemma 1.2.9), hence $\#(S \setminus S \cap M_4) = 2$ and

$$S \setminus S \cap M_4 = \{u, v\} \text{ with } \pi_i(u) = \pi_i(v),\ \forall i \neq 4. \tag{2.4.3}$$

Since $h^1\big(\mathcal{I}_{S \setminus S \cap M_3}(\hat{\varepsilon}_3)\big) > 0$ (again by Lemma 1.2.9), we get that either there are $w, z \in S \setminus S \cap M_3$ such that $w \neq z$, $\pi_i(w) = \pi_i(z)$ for all $i \neq 3$ or $\nu_4(\eta_4(S \cap M_4))$ (remind Notation 1.1.6) is made by 3 collinear points, say with a line corresponding to the i-th factor. The latter case cannot arise because S does not depend only on the third, fourth and i-th factor of Y. Thus there exist

$$w, z \in S \setminus S \cap M_3 \text{ such that } w \neq z,\ \pi_i(w) = \pi_i(z)\ \forall i \neq 3. \tag{2.4.4}$$

In (2.4.3) and (2.4.4) we have 4 distinct points u, v, w, z such that $\#(\pi_5(\{u, v, w, z\})) = 1$. Take $M_5 \in |\mathcal{O}_Y(\varepsilon_5)|$ containing $\{u, v, w, z\}$. Since $h^1\big(\mathcal{I}_{S \setminus S \cap M_5}(\hat{\varepsilon}_5)\big) = 0$, Autarky and Lemma 1.2.9 give a contradiction.

(b) Assume $S \not\subset M_2 \cup M_3 \cup M_4$. By Lemma 1.2.9 we get

$$h^1\big(\mathcal{I}_{S \setminus S \cap (M_2 \cup M_3 \cup M_4)}(1, 0, 0, 0, 1, 1, \dots)\big) > 0.$$

Thus $\#(S \setminus (M_2 \cup M_3 \cup M_4)) = 2$, say $S \setminus (M_2 \cup M_3 \cup M_4) = \{u, v\}$ and $\pi_i(u) = \pi_i(v)$ for all $i \neq 2, 3, 4$. Let $M_1 \in |\mathcal{O}_{Y_{1^k}}(\varepsilon_1)|$ contain u, v and note that by definition of the M_i's each one of them contains at least one point of S. Therefore $\#(S \cap (M_1 \cup M_3 \cup M_4)) \geq 4$. Since the case $S \subset M_1 \cup M_3 \cup M_4$ has already been excluded in step (a), we may assume $\#(S \cap (M_1 \cup M_3 \cup M_4)) = 4$. But then by Lemma 1.2.9 we would have $h^1\big(\mathcal{I}_{S \setminus S \cap (M_1 \cup M_3 \cup M_4)}(0, 1, 0, 0, 1, \dots)\big) > 0$ which is a contradiction. □

From the above result we deduce that all the solutions of a non-identifiable rank-3 tensor $q \in \langle \nu(Y_{1^k}) \rangle$ must be disjoint if $k > 4$. This concludes the case in which two distinct solutions of a rank-3 tensor share a common point.

2.5 Two disjoint solutions

We have seen in Corollary 2.3.18 that if a rank-3 tensor q is not identifiable and A, B are two sets of points on the Segre variety computing its rank, then $\#A \cup B$ can only be either 5 or 6. This section is fully devoted to the case in which $\#A \cup B = 6$, i.e. A and B are disjoint:

$$S := A \cup B, \quad \#S = 6, \quad A := \{a_1, a_2, a_3\}, \quad B := \{b_1, b_2, b_3\} \text{ and } A \cap B = \emptyset. \tag{2.5.5}$$

More precisely, let $k \geq 3$ and let $Y_{n_1,\ldots,n_k}$ be such that all $n_i \in \{1,2\}$. Let $q \in \langle \nu(Y_{n_1,\ldots,n_k}) \rangle$ be a non-identifiable rank-3 tensor such that $\nu(Y_{n_1,\ldots,n_k})$ is the concise Segre of q.
In Proposition 2.5.2 we manage the case in which the multiprojective space has at least two factors of projective dimension 2, proving that two distinct solutions of a non-identifiable rank-3 tensor cannot be disjoint. Then we focus on the number of factors $k \geq 3$. Since cases $Y_{1,1,1}$ and $Y_{2,1,1}$ are treated in Remark 2.3.2 and Corollary 2.3.12 respectively, we can focus on the 4-factors case. The case in which $Y_{1,1,1,1} = (\mathbb{P}^1)^4$ is considered in Remark 2.3.3. Therefore we may focus on $Y_{2,1,1,1} = \mathbb{P}^2 \times (\mathbb{P}^1)^3$ (Proposition 2.5.3 below), proving that there are no two disjoint solutions in this case. We conclude the discussion with Proposition 2.5.5 in which we prove that there do not exist two disjoint solutions if $Y_{n,1^{k-1}} = \mathbb{P}^n \times (\mathbb{P}^1)^{k-1}$ with $n \in \{1,2\}$, $k \geq 5$.

Before starting the discussion, let us see how to apply Lemma 1.2.9 in its contrapositive formulation when dealing with a set of 6 distinct points, 4 of which are contained in a divisor of the corresponding Segre.

Remark 2.5.1. Let $Y_{2^{k_1},1^{k_2}} = (\mathbb{P}^2)^{k_1} \times (\mathbb{P}^1)^{k_2}$ and $S \subset Y_{2^{k_1},1^{k_2}}$ a set of 6 distinct points. Consider $I \subseteq \{k_1+1, \ldots, k_1+k_2\}$ and $\varepsilon := \sum_{i \in I} \varepsilon_i$. Suppose there exists a divisor $M \in |\mathcal{O}_Y(\varepsilon)|$ intersecting S in 4 points. Call $\{u, v\} := S \setminus (S \cap M)$. In this setting one can apply Lemma 1.2.9 and get that $h^1\big(\mathcal{I}_{\{u,v\}}(\widehat{\varepsilon})\big) > 0$ (where $\widehat{\varepsilon}$ is a (k_1+k_2)-uple with 0's in position of the indices appearing in ε of I and 1's everywhere else, as described in Notation 1.2.1) and $\pi_h(u) = \pi_h(v)$ for any $h \in \{1, \ldots, k_1+k_2\} \setminus I$.

We prove now that if q is a non-identifiable rank-3 tensor whose concise Segre has at least two factors of projective dimension 2, then two different solutions of q cannot be disjoint.

Proposition 2.5.2. *Let $Y_{2,2,n_3,\ldots,n_k}$ be a multiprojective space with at least three factors and at least two of them of projective dimension 2, i.e. $Y_{2,2,n_3,\ldots,n_k} = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{n_3} \times \cdots \times \mathbb{P}^{n_k}$ with $n_i \in \{1,2\}$ for $i = 1, \ldots, k$ and $k \geq 3$. Let $q \in \sigma_3^0(\nu(Y_{2,2,n_3,\ldots,n_k}))$, with $\nu(Y_{2,2,n_3,\ldots,n_k})$ the concise Segre of q. If $A, B \in \mathcal{S}(Y_{2,2,n_3,\ldots,n_k}, q)$ evince the rank of q, then A and B cannot be disjoint.*

Proof. By contradiction, assume that there exist $A, B \in \mathcal{S}(Y_{2,2,n_3,\ldots,n_k}, q)$ with $A \cap B = \emptyset$. By Remark 2.3.4 we have that $\langle \pi_i(A) \rangle = \langle \pi_i(B) \rangle = \mathbb{P}^2$ for $i = 1, 2$. Fix $W \in |\mathcal{I}_B(\varepsilon_2 + \varepsilon_3)|$, it exists because $h^0\big(\mathcal{O}_{Y_{2,2,n_3,\ldots,n_k}}(\varepsilon_2 + \varepsilon_3)\big) = h^0\left(\mathbb{P}^2 \times \mathbb{P}^{n_3}, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{n_3}}(1,1)\right) = 3(n_3+1) > 4$. Since $\pi_{1|A}$ is injective, $h^1\big(\mathcal{I}_A(1,1,0,0,1\ldots,1)\big) = 0$. Thus $S \subset W$ by Lemma 1.2.9. In this way we have shown that

$$\text{any divisor } D \in |\mathcal{O}_{Y_{2,2,n_3,\ldots,n_k}}(\varepsilon_2 + \varepsilon_3)| \text{ containing } B \text{ contains also } A. \tag{*}$$

Claim 1. $\pi_3(a_i) = \pi_3(b_i)$ where a_i, b_i are as in (2.5.5), for $i = 1, 2, 3$.

The proof of this claim can be repeated verbatim for all the other projections with only one caution that we will highlight in the sequel. Therefore, by repeating the argument for all the projections, we will get that $\pi_j(a_i) = \pi_j(b_i)$ for $i = 1, 2, 3$ and for $j = 1, \ldots, k$ which is a contradiction with A and B being distinct. This will conclude the proof.

Proof of the Claim 1. Take a general hyperplane $J_{3,i} \subset \mathbb{P}^{n_3}$ containing $\pi_3(b_i)$, (where the b_i's are as in (2.5.5), $i = 1, 2, 3$) by genericity we may assume that if $n_3 = 2$ then $J_{3,i}$ is a line which does not contain any other point of that projection. Set $M_{3,i} := \pi_3^{-1}(J_{3,i})$. Take a line

$$L_{2,j,k} \subset \mathbb{P}^2 \text{ containing } \{\pi_2(b_j), \pi_2(b_k)\} \text{ with } j, k \neq i \qquad (**)$$
$$\text{and set } M_{2,j,k} := \pi_2^{-1}(L_{2,j,k}).$$

We have $B \subset M_{2,j,k} \cup M_{3,i} \in |\mathcal{O}_Y(\varepsilon_2 + \varepsilon_3)|$. Thus from (*) we get that $M_{2,j,k} \cup M_{3,i}$ contains also A. Since $A \nsubseteq M_{2,j,k}$ by Autarky, there is $a \in A \cap M_{3,i}$, i.e. there is $a \in A$ such that

$$\pi_3(a) = \pi_3(b_i), \tag{2.5.6}$$

in fact if $n_3 = 1$ it is trivial, if $n_3 = 2$ then we have already remarked that $\pi_3(b_i)$ is the only point of $J_{3,i}$ belonging to $\pi_3(S)$.
Remark that at most one point of A projects onto $\pi_3(b_i)$. Indeed assume by contradiction that $\pi_3(a) = \pi_3(b_i) = \pi_3(\tilde{a})$ for some $\tilde{a}, a \in A$. By Autarky $A \not\subset M_{3,i}$ and moreover $B \cap M_{3,i} = \{b_i\}$, hence $S \setminus (S \cap M_{3,i}) = \{\hat{a}, b_j, b_k\}$ with $\hat{a} \in A \setminus \{a, \tilde{a}\}$. Therefore, by Lemma 1.2.9, we get $h^1(\mathcal{I}_{S \setminus (S \cap M_{3,i})}(\hat{\varepsilon}_3)) > 0$, which is equivalent to say that there exists $j \in \{1, \ldots, k\} \setminus \{3\}$ such that $\#\pi_h(\{\hat{a}, b_i, b_j\}) = 1$ for all $h \notin \{3, h\}$ (cf. Lemma 1.2.11). This leads to a contradiction since $\pi_{i|B}$ is injective for $i = 1, 2$ (cf. Remark 2.3.4). By repeating all the above argument permuting i with j and then i with k respectively, we get that the points of A projecting on $\pi_3(b_\ell)$ are different for different $\ell's$ except if there are $b_i \neq b_j$ such that $\pi_3(b_i) = \pi_3(b_j)$.
Suppose that this is the case. By Lemma 1.2.11 we get that $\#(S \setminus S \cap M_{3,i}) = 2$. Thus if $\pi_3(b_i) = \pi_3(b_j)$ for $i \neq j$, there are 2 points of A and 2 points of B in $M_{3,i}$, i.e. $\#(S \cap M_{3,i}) = 4$. To fix ideas take $i = 3$ and $j = 2$, i.e. suppose that $S \cap M_{3,i} = \{a_3, b_3, a_2, b_2\}$. By Lemma 1.2.9 $h^1(\mathcal{I}_{S \setminus S \cap M_3}(\hat{\varepsilon}_3)) > 0$, i.e. $\pi_i(a_1) = \pi_i(b_1)$ for all $i \neq 3$. This is a contradiction since we already know that $\pi_3(a_2) = \pi_3(b_2)$ and we would have $a_2 = b_2$, which contradicts the assumption that $A \cap B = \emptyset$. Again, taking j instead of i (k instead of i) one can repeat an analogous argument.

Therefore the points $a \in A$ of (2.5.6) are all different for different choices of i's. So we may assume that $\pi_3(a_i) = \pi_3(b_i)$ for $i = 1, 2, 3$ and that $\pi_3(b_i) \neq \pi_3(b_j)$ for $i \neq j$. □

The argument of the proof of Claim 1 can be repeated verbatim for all the others π_ℓ's with the only caution that when we do the case $\ell = 2$ we have to use a line $L_{1,j,k} \subset \mathbb{P}^2$ containing $\{\pi_1(b_j), \pi_1(b_k)\}$ with $j, k \neq i$ and set $M_{1,j,k} := \pi_1^{-1}(L_{1,j,k})$ instead of $M_{2,j,k}$ and $L_{2,j,k}$ in (**). Moreover (*) clearly holds if we replace the ε_2 with ε_1 and ε_3 with ε_j for any $j = 3, \ldots, k$. As already highlighted this concludes the proves since $\pi_j(a_i) = \pi_j(b_i)$ for $i = 1, 2, 3$ and for $j = 1, \ldots, k$ which is a contradiction with A and B being distinct. □

This shows that under the assumption (2.5.5), we can exclude the case where the Segre variety has at least two factors of projective dimension 2.

Let us focus on the 4-factors case.

Proposition 2.5.3. *Let $Y_{2,1,1,1} = \mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$. Let $q \in \sigma_3^0(\nu(Y_{2,1,1,1}))$, with $\nu(Y_{2,1,1,1})$ the concise Segre of q. There do not exist two disjoint sets $A, B \in \mathcal{S}(Y_{2,1,1,1}, q)$ evincing the rank of q.*

Proof. Assume by contradiction that there exist two disjoint sets $A, B \in \mathcal{S}(Y_{2,1,1,1}, q)$ evincing the rank of q.

We want to fix the maximal number of points of S that share the same component, i.e. define

$$\alpha_4 := \max\{\#(\pi_i^{-1}(o)\cap S)\}_{o\in\mathbb{P}^1; i=2,\dots,4}. \tag{2.5.7}$$

By rearranging if necessary, we can assume that the index $i = 2,\dots,4$ realizing α_4 is $i = 4$. Denote by $o_4 \in \mathbb{P}^1$ the point realizing such value and call $K_4 := \pi_4^{-1}(o_4)$. Note that $1 \le \alpha_4 \le 5$, where the first inequality holds by definition and the second one by Autarky. Moreover, it is easy to see that α_4 cannot be 5. In fact if $\alpha_4 = 5$, then $\#(S \setminus S\cap K_4) = 1$ which implies that $h^1\big(\mathcal{I}_{S\setminus S\cap K_4}(1,1,1,0)\big) = 0$, contradicting Lemma 1.2.9.

Since $\alpha_4 \le 4$ we can look for another integer such that the preimages of points $o \in \mathbb{P}^1$ intersect maximally the points in $S \setminus (S \cap K_4)$. Define

$$\alpha_3 := \max\{\#(\pi_i^{-1}(o)\cap (S\setminus(S\cap K_4)))\}_{o\in\mathbb{P}^1; i=2,3}. \tag{2.5.8}$$

By rearranging if necessary, we can assume that the index $i = 2,3$ realizing α_3, is $i = 3$. Call $K_3 := \pi_3^{-1}(o_3)$, where $o_3 \in \mathbb{P}^1$ is the point for which we reach α_3. Therefore we have

$$1 \le \alpha_3 \le \alpha_4 \le 4.$$

In order to show that $S \subset K_3 \cup K_4$ or equivalently that $\alpha_3 + \alpha_4 = 6$, define

$$\alpha_2 := \max\{\#(\pi_2^{-1}(o)\cap (S\setminus(S\cap K_4\cup K_3)))\}_{o\in\mathbb{P}^1}. \tag{2.5.9}$$

Therefore, if we denote by $o_j \in \mathbb{P}^1$, $j = 2,3,4$ the points realizing $\alpha_2, \alpha_3, \alpha_4$ respectively then we call

$$K_j := \pi_j^{-1}(o_j) \text{ for } j = 2,3,4. \tag{2.5.10}$$

Let us prove that acually $\alpha_2 = 0$. Assume by contradiction that $\alpha_2 \ne 0$. Since $1 \le \alpha_2 \le \alpha_3 \le \alpha_4 \le 4$, the only possibility is

$$(\alpha_2,\alpha_3,\alpha_4) \in \{(1,1,4),(1,1,3),(1,1,2),(1,1,1),(1,2,3),(1,2,2),(2,2,2)\}.$$

If $(\alpha_2,\alpha_3,\alpha_4) = (1,1,4)$, by Lemma 1.2.9 we get $h^1\big(\mathcal{I}_{S\setminus(S\cap(K_4\cup K_3))}(1,1,0,0)\big) > 0$ which is impossible since $\#(S\setminus(S\cap(K_4\cup K_3))) = 1$. An analogous argument holds for the case $(\alpha_2,\alpha_3,\alpha_4) = (1,2,3)$. For the same reason $(\alpha_2,\alpha_3,\alpha_4) = (1,2,2)$ is also impossible because $\#(S\setminus(S\cap(K_4\cup K_3\cup K_2))) = 1$ and by Lemma 1.2.9 we would have $h^1\big(\mathcal{I}_{S\setminus(S\cap(K_4\cup K_3\cup K_2))}(1,0,0,0)\big) > 0$.
Now assume that $(\alpha_2,\alpha_3) = (1,1)$. Then $\pi_{3|S}$ is injective. The idea is to build a divisor $F \in |\mathcal{O}_{Y_{2,1,1,1}}(\varepsilon)|$ with $\varepsilon = \sum_{i\in I}\varepsilon_i$, for some finite $I \in \{1,\dots,k\}$, such that $\#(S\setminus F\cap S) = 2$ and apply Remark 2.5.1 to F: the existence of such a F will contradict the injectivity of $\pi_{3|S}$. Let $H_i \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $H_i \cap (S\setminus S\cap K_4) \ne \emptyset$ for $i = 2,3$. The divisor F is either $F = K_4 \cup H_3$ or $K_4\cup H_2\cup H_3$ if $\alpha_4 = 3,2$ respectively.
Assume that $(\alpha_2,\alpha_3,\alpha_4) = (1,1,1)$. In such a case the divisor $K_2\cup K_3\cup K_4 \in |\mathcal{O}_Y(\hat{\varepsilon}_1)|$ would contain exactly 3 points of S. Therefore, by Lemma 1.2.9, we get that

$$h^1\big(\mathcal{I}_{S\setminus(S\cap K_2\cup K_3\cup K_4)}(\varepsilon_1)\big) > 0,$$

which is equivalent to say that there exists $j \in \{2,3,4\}$ such that $\#\pi_h(S \setminus (S \cap (K_2 \cup K_3 \cup K_4))) = 1$ for all $h \neq j$ (cf. Lemma 1.2.11), contradicting $(\alpha_3, \alpha_4) = (1,1)$.
Let $\alpha_2 = \alpha_3 = \alpha_4 = 2$. By the definition of the K_i's in (2.5.10) for $i = 2,3,4$, we note that

$$S = \coprod_{i=2}^{4} S \cap K_i$$

So, since $S = \coprod_{i=2}^{4} S \cap K_i$ and $\#(S \cap K_i) = 2$ for $i = 2,3,4$, we can apply Remark 2.5.1 separately to the divisors $K_i \cup K_j$ with $i \neq j$ and get that $h^1\big(\mathcal{I}_{S \cap K_i}(\varepsilon_1 + \varepsilon_i)\big) > 0$ for $i = 2,3,4$ and so $\pi_1(S \cap K_i) = 1$ for $i = 2,3,4$. In order to get a contradiction it is sufficient to apply again Remark 2.5.1 to $\pi_1^{-1}(\langle \pi_1(S \cap K_3), \pi_1(S \cap K_2)\rangle)$. This shows that $\#(\pi_i(S \cap K_4)) = 1$ for $i = 2,3,4$. Now since also $\#(\pi_1(S \cap K_4)) = 1$, then $\#(S \cap K_4) = 1$, which is a contradiction with the assumption $\alpha_3 = 2$.

Thus we proved that $\alpha_2 = 0$, i.e. $\alpha_3 + \alpha_4 = 6$.

The case $(\alpha_3, \alpha_4) = (2,4)$ can be excluded using the same argument of the case $(\alpha_2, \alpha_3, \alpha_4) = (2,2,2)$ above applying Remark 2.5.1 and in this case K_4 plays the role of M in the remark.

We are therefore left with the unique possibility of $(\alpha_3, \alpha_4) = (3,3)$.

Claim 2. $\#(\pi_2(S \cap K_4)) = 1$.

Proof of Claim 2: Assume by contradiction that $\#(\pi_2(S \cap K_4)) \neq 1$, since we are in the hypothesis $\alpha_4 = 3$, the projection of $S \cap K_4$ onto the second factor is made by either 2 or 3 points.

If $\#(\pi_2(S \cap K_4)) = 2$, there exist at least two points, $u, v \in S \cap K_4$ such that they share the same image under the projection. Let $H \in |\mathcal{O}_{Y_{2,1,1,1}}(\varepsilon_2)|$ contain u, v, then by Lemma 1.2.9 $h^1\big(\mathcal{I}_{S \setminus (S \cap (H \cup K_3))}(1,0,0,1)\big) > 0$, but this is impossible since $\#(S \setminus (S \cap (H \cup K_3))) = 1$.

If $\#(\pi_2(S \cap K_4)) = 3$, fix $x \in S \cap K_4$ and take $H \in |\mathcal{O}_{Y_{2,1,1,1}}(\varepsilon_1)|$ containing x. By applying Remark 2.5.1 with $M = H \cup K_3$ we would get that $\#(\pi_2(S \cap K_4 \setminus \{x\})) = 1$ which is absurd. $\square$

Using the third factor instead of the second one, one gets $\#(\pi_3(K_4 \cap S)) = 1$ and since we assumed that α_4 is reached on the fourth factor we also have $\#(\pi_4(K_4 \cap S)) = 1$. The same argument can be applied to $S \cap K_3$ which leads to $\#(\pi_2(K_3 \cap S)) = \#(\pi_4(K_3 \cap S)) = 1$. Thus $\#(\pi_i(K_4 \cap S)) = \#(\pi_i(K_3 \cap S)) = 1$ for all $i > 1$ which contradicts Autarky. $\square$

Since the identifiability of rank-3 tensors in $\langle \nu((\mathbb{P}^1)^4)\rangle$ is already fully described by Remark 2.3.3, we are therefore done with the order-4 tensors and we can focus on tensors of order bigger or equal than 5. So we will deal with $Y = \mathbb{P}^n \times (\mathbb{P}^1)^{k-1}$, with $n = 1,2$ and $k \geq 5$.

Lemma 2.5.4. *Let q be a rank-3 tensor of order at least 5 and let $\nu(Y_{n_1,\dots,n_k})$ be its concise Segre where all $n_i \in \{1,2\}$. If there exist two disjoint sets $A, B \in \mathcal{S}(Y_{n_1,\dots,n_k}, q)$ as in (2.5.5), then there exists at least an index $i \in \{1,\dots,k\}$ such that $\eta_{i|S}$ and $\pi_{i|S}$ are injective.*

Proof. [Injectivity of $\eta_{i|S}$.] We remark that

1. If $\eta_i(u) = \eta_i(v)$ for some distinct $u, v \in S$ then, by Remark 2.2.4, u and v cannot be points of the same decomposition, i.e. $u \in A$ and $v \in B$ (or equivalently $u \in B$ and $v \in A$).

2. For all $i = 1, \dots, k$ if $\eta_i(u) = \eta_i(v)$ for some $u \in A, v \in B$, then $\eta_j(u) \neq \eta_j(v)$ if $i \neq j$, otherwise we would have $u = v$, contradicting $A \cap B = \emptyset$.

3. Let $u \in A$ and $v, v' \in B$ with $v \neq v'$. If $\eta_i(u) = \eta_i(v')$ and $\eta_j(u) = \eta_j(v'')$ for some $i \neq j$ then $\eta_k(u) \neq \eta_k(v)$, where v is the other element of B, i.e $B \setminus \{v', v''\} = \{v\}$. Otherwise the minimal multiprojective space containing $B = \{v, v', v''\}$ (and therefore containing q) would have at most 3 factors, but this is in contradiction with the assumption $k \geq 5$.

Before proceeding, we need to prove one last result on the behaviour of the maps $\eta_{i|S}$'s.

Claim 3. Let $\alpha, \beta, \gamma \in \{1, \dots, 5\}$ be distinct indices. Take $u_\alpha, u_\gamma \in A$ and $v_\alpha, v_\beta, v_\gamma \in B$. If $\eta_\alpha(u_\alpha) = \eta_\alpha(v_\alpha)$ and $\eta_\beta(u_\alpha) = \eta_\beta(v_\beta)$, then we cannot have neither that $\eta_\gamma(v_\alpha) = \eta_\gamma(u_\gamma)$ nor that $\eta_\gamma(v_\beta) = \eta_\gamma(u_\gamma)$.

Proof. Assume by contradiction that $\eta_\gamma(v_\alpha) = \eta_\gamma(u_\gamma)$ and call

$$S' = \{u_\alpha, u_\gamma, v_\alpha, v_\beta\}.$$

Note that $\#S' \geq 2$ since $A \cap B = \emptyset$. We want to prove that actually $\#S' = 4$. Indeed if $v_\alpha = v_\beta$ then we would have $u_\alpha = v_\alpha$ which is again in contradiction with the fact that $A \cap B = \emptyset$. Therefore $\#S' \geq 3$. Moreover, by item 2 of the above remark, we know that $u_\alpha \neq u_\beta$. Therefore $\#S' = 4$. Thus S' is given by 4 distinct points that share all the same image under the projection onto two factors, i.e. there exist two indices $\rho, \zeta \in \{1, \dots, 5\} \setminus \{\alpha, \beta, \gamma\}$ such that $\#(\pi_\rho(S')) = \#(\pi_\zeta(S')) = 1$.
Let $H \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_\rho)|$ contain S' and note that $\#(S \setminus (S' \cap H)) \leq 2$. By Lemma 1.2.9, $h^1\big(\mathcal{I}_{S \setminus (S' \cap H)}(\hat{\varepsilon}_\rho)\big) > 0$ and hence $\#(S \setminus (S' \cap H)) = 2$, which is equivalent to say that

$$\text{there exist } x, y \in S \setminus S' \text{ such that } \pi_i(x) = \pi_i(y) \text{ for all } i \neq \rho.$$

Since also $\#(\pi_\zeta(S')) = 1$ we can take $D \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_\zeta)|$ containing S'. Now take $M \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_\rho)|$ containing x and note that $S \setminus (S \cap (M \cup D)) = \{y\}$. By Lemma 1.2.9, we would get $h^1\big(\mathcal{I}_y(\widehat{\varepsilon_\rho + \varepsilon_\zeta})\big) > 0$, but this is impossible.

With a similar argument one can prove the second statement of the claim, i.e. that $\eta_\gamma(v_\beta) \neq \eta_\gamma(u_\gamma)$. □

Now we are ready to prove that $\eta_{i|S}$ is injective for at least one $i \in \{1, \dots, 5\}$. Assume by contradiction that none of the $\eta_{i|S}$ is injective, therefore

$$\text{for all } i = 1, \dots, 5, \text{ there exist } u_i \in A, v_i \in B \text{ such that } \eta_i(u_i) = \eta_i(v_i). \tag{2.5.11}$$

We show that this condition, applied to two disjoint sets of 3 points each, and at least five η_i's, imposes a contradiction. Since $\#(A) = \#(B) = 3$, at least two of the u_j must be equal (the same for the v_j's) where $j \in \{1, \dots, 5\}$. By relabeling if necessary, we may assume $u_1 = u_2$. For $i = 1$ we have

$$\eta_1(u_1) = \eta_1(v_1).$$

Let now $i = 2$. Since $u_1 = u_2$, we get

$$\eta_2(u_1) = \eta_2(v_2).$$

By item 2 we have $v_1 \neq v_2$. Now let $i = 3$. By item 3 u_1 cannot be used anymore and moreover, by the above Claim 3, neither v_1 nor v_2 can be used, therefore we have

$$\eta_3(u_2) = \eta_3(v_3),$$

where $v_3 \notin \{v_1, v_2\}$. Now let $i = 4$. As a point of B we are forced to consider again v_3. Moreover, items 2 and 3 tells us that we cannot use again neither u_2 nor u_1. Therefore we have

$$\eta_4(u_3) = \eta_4(v_3),$$

where $u_3 \notin \{u_1, u_2\}$. Now we finally reached a contradiction since we cannot use anymore neither points of A nor points of B for the non-injectivity of $\eta_{5|S}$.

[Injectivity of $\pi_{i|S}$.]

Assume that $\eta_{i|S}$ is injective and that $\pi_{i|S}$ is not injective. If the i-th factor of $Y_{n_1,\dots,n_k}$ is $\mathbb{P}^2$ take $H \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_i)|$ as the preimage of a general line that contains one point of S; otherwise take $H \in |\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_i)|$ as the preimage of a point of S. We remark that in both cases we get that $\#(\pi_i(S \cap H)) = 1$. Since by Autarky $S \not\subset H$, by Lemma 1.2.9 we have that

$$h^1\big(\mathcal{I}_{S \setminus S \cap H}(\hat{\varepsilon}_i)\big) > 0. \tag{2.5.12}$$

Note that $\#(S \setminus S \cap H) \leq 4$ otherwise we would have a contradiction with (2.5.12). To prove the result, we distinguish different cases depending on $\#(S \setminus S \cap H)$.

1. Assume $\#(S \setminus S \cap H) = 4$ and call $S' := \eta_i(S \setminus S \cap H)$; let $A' \subset S'$ be such that $\#A' = 2$ and call $B' := S' \setminus A'$, so $\#B' = 2$. Since $\eta_{i|S}$ is injective and $h^1\big(Y_{n_1,\dots,n_{i-1},n_{i+1},\dots,n_k;i}, \mathcal{I}_{S'}(\hat{\varepsilon}_i)\big) = h^1\big(\mathcal{I}_{S \setminus S \cap H}(\hat{\varepsilon}_i)\big) > 0$, then $\langle \nu_i(A') \rangle \cap \langle \nu_i(B') \rangle \neq \emptyset$, which means that we have at least a point $q' \in \langle \nu_i(Y_{n_1,\dots,n_{i-1},n_{i+1},\dots,n_k;i}) \rangle$ of rank 2 for which A' and B' are different subsets evincing its rank. Thus by Proposition 2.2.7, since $\#\mathcal{S}(Y_i, q') > 1$, the points in A' and B' only depend on two factors, i.e. $\#(\pi_j(S')) = 1$ for at least two indices $j \in \{1, \dots, k\}$. Without loss of generality assume it happens for $j = 1, 2$. If the first factor of Y is $\mathbb{P}^2$, let $M_1 \in |\mathcal{I}_{\pi_1(S')}(\varepsilon_1)|$ be the preimage of a general line containing $\pi_1(S')$ and let $\{M_2\} := |\mathcal{I}_{\pi_2(S')}(\varepsilon_2)|$. Otherwise let $\{M_j\} := |\mathcal{I}_{\pi_j(S')}(\varepsilon_j)|$, for $j = 1, 2$; in both cases then $h^1\big(\mathcal{I}_{S \setminus S \cap M_j}(\hat{\varepsilon}_j)\big) > 0$. So $S \setminus S \cap M_j = S \cap H$ and $\#(\eta_j(S \cap H)) = 1$, for $j = 1, 2$. If we call $S \cap H = \{u, v\}$, it follows that $\eta_1(u) = \eta_1(v)$ and $\eta_2(u) = \eta_2(v)$, so in particular we get that $\pi_j(u) = \pi_j(v)$ for any j, which is a contradiction.

2. Assume $\#(S \setminus S \cap H) = 3$. By Proposition 1.2.11 there exists $j \neq i$ such that $\#(\pi_h(S \setminus S \cap H)) = 1$ for all $h \neq i, j$. For all $h \neq i, j$, if $n_h = 2$ take $M_h \in |\mathcal{I}_{S \setminus S \cap H}(\varepsilon_h)|$ as the preimage of a general line. Otherwise, since $h^0\left(\mathcal{O}_{Y_{n_1,\dots,n_k}}(\varepsilon_h)\right) = 2$ we have $h^0\big(\mathcal{I}_{S \setminus S \cap H}(\varepsilon_h)\big) = 1$ and we set $\{M_h\} := |\mathcal{I}_{S \setminus S \cap H}(\varepsilon_h)|$. Since we took

H such that $\#(\pi_i(S \cap H)) = 1$, there exists at least an index $t \neq i$ such that $\#\pi_t(S \cap H) \geq 2$. Thus we can find $D \in |\mathcal{O}_Y(\varepsilon_t)|$ containing exactly one point of $S \cap H$.
For all $s \neq t$ set $W_s := M_s \cup D$, so $\#(S \setminus S \cap W_s) = 2$; we remark that $W_j \cap S = W_s \cap S$ for any j, s thus we may call $E := S \setminus S \cap W_s$.
By Lemma 1.2.9 we have that $h^1\big(\mathcal{I}_E(\widehat{\varepsilon_s + \varepsilon_t})\big) > 0$, so $\#(\pi_j(E)) = 1$ for all $j \neq s, t$. Since $E \subset H$ we have that $\pi_i(E) = 1$, moreover taking $s = 1, 2, 3$, if $t \neq j$, we get that $\#E = 1$, thus a contradiction. It remains to study what happens when $t = j$, i.e. if $\#(\pi_j(S \cap H)) \geq 2$. In such a case, when we let s vary in $\{1, \ldots, k\} \setminus \{i, j\}$, we get $\#(\pi_s(S \cap H)) = 1$. Thus $\eta_j(S \cap H) = 1$, i.e. the three points of $S \cap H$ actually lies on a line, which is a contradiction with Remark 2.2.4, because two of them are points of A or B.

3. Assume $\#(S \setminus S \cap H) \leq 2$. Since $h^1\big(\mathcal{I}_{S \setminus S \cap H}(\hat{\varepsilon}_i)\big) > 0$, we get that $\#(S \setminus S \cap H) = 2$ and that $\#\eta_i(S \setminus S \cap H) = 1$, which is a contradiction. □

With the above lemma we can conclude the case of two disjoint sets $A, B \in \mathcal{S}(Y_{n_1,\ldots,n_k}, q)$ with q of rank-3.

Proposition 2.5.5. *Let $q \in \sigma_3^0(\nu(Y_{n,1^{k-1}}))$ be a tensor of order $k \geq 5$ and let $\nu(Y_{n,1^{k-1}})$ be its concise Segre, where $n \in \{1, 2\}$. Then $\mathcal{S}(Y_{n,1^{k-1}}, q)$ does not contain two disjoint sets.*

Proof. By Lemma 2.5.4 there exists at least an index $i \in \{1, \ldots, k\}$ such that $\eta_{i|S}$ is injective, from which follows that the corresponding $\pi_{i|S}$ is also injective. Now if $\eta_{j|S}$ is not injective for some $j \neq i$ then $\pi_{i|S}$ is not injective, which is a contradiction with the assumption that $\eta_{i|S}$ is injective. Therefore thus $\eta_{j|S}$ and $\pi_{j|S}$ have to be injective for all $j = 1, \ldots, k$.

Write $A := \{a_1, a_2, a_3\}$ and $B := \{a_4, a_5, a_6\}$. If the first factor is a $\mathbb{P}^2$ take $L_1 \in \mathbb{P}^2$ as a general line containing $\pi_1(a_1)$ and define $H_1 \in |\mathcal{I}_{a_1}(\varepsilon_1)|$ as $H_1 := \pi_1^{-1}(L_1)$. For $i = 2, \ldots, 4$ take $\{H_i\} := |\mathcal{I}_{a_i}(\varepsilon_i)|$ (this is possible since by hypothesis $k \geq 5$). Otherwise for all $i = 1, \ldots, 4$ take $\{H_i\} := |\mathcal{I}_{a_i}(\varepsilon_i)|$. In both cases, since every $\pi_{i|S}$ is injective we get that $H_1 \cup \cdots \cup H_4$ contains exactly 4 points of S. Thus from Lemma 1.2.9 we get that $h^1\big(\mathcal{I}_{S \setminus (S \cap H_1 \cup \cdots \cup H_4)}(0, 0, 0, 0, 1, \ldots, 1)\big) > 0$ which is a contradiction since all $\pi_{i|S}$ are injective (cf. Remark 2.5.1). □

2.6 Identifiability of rank-3 tensors

We are now ready to state and prove the main result of the present chapter that completely characterizes the identifiability of any rank-3 tensor. Indeed the following theorem collects all the results proved in the previous sections by stating a comprehensive list of non-identifiable families of rank 3 tensors.

Theorem 2.6.1. *Let $Y_{n_1,\ldots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be the multiprojective space of the concise Segre of the projective class $q = [T]$ of a rank-3 tensor T. Denote with $\mathcal{S}(Y_{n_1,\ldots,n_k}, q)$ the set of all subsets of Y computing the rank of q. The rank-3 tensor T is identifiable except in the following cases:*

(a) T is a 3×3 matrix and in this case $\dim(\mathcal{S}(Y_{2,2}, q)) = 6$;

(b) *there exist* $v_1, v_2, v_3 \in \mathbb{C}^2$ *such that* $T \in \mathbb{C}^2 \otimes v_2 \otimes v_3 + v_1 \otimes \mathbb{C}^2 \otimes v_3 + v_1 \otimes v_2 \otimes \mathbb{C}^2$, *in this case* $\dim\left(\mathcal{S}\left(Y_{1,1,1}, q\right)\right) \geq 2$;

(c) $T \in (\mathbb{C}^2)^{\otimes 4}$, *in this case* $\dim\left(\mathcal{S}\left(Y_{1,1,1,1}, q\right)\right) \geq 1$;

(d) $T \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ *and there exists a basis* $\{u_1, u_2, u_3\} \subset \mathbb{C}^3$ *and a basis* $\{v_1, v_2\} \subset \mathbb{C}^2$ *such that* T *can be written as*

$$T = u_1 \otimes v_1^2 + u_2 \otimes v_2^2 + u_3 \otimes (\alpha v_1 + \beta v_2)^2,$$

for some $\alpha, \beta \neq 0$ *(cf. Example 2.3.7). In this case* $\dim\left(\mathcal{S}\left(Y_{2,1,1}, q\right)\right) = 3$;

(e) $T \in \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ *and there exists a basis* $\{u_1, u_2, u_3\} \subset \mathbb{C}^3$ *and a basis* $\{v_1, v_2\} \subset \mathbb{C}^2$ *such that* T *can be written as*

$$T = u_1 \otimes v_1 \otimes \tilde{p} + u_2 \otimes v_2 \otimes \tilde{p} + u_3 \otimes \tilde{q} \otimes w,$$

for some $\tilde{q} \in \langle v_1, v_2 \rangle$, *where* $\tilde{p}, w \in \mathbb{C}^2$ *must be linearly independent (cf. Example 2.3.9). In this case* $\mathcal{S}\left(Y_{2,1,1}, q\right)$ *contains two different* 4*-dimensional families;*

(f) $T \in \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes (\mathbb{C}^2)^{k-2}$, *where* $k \geq 3$, $m_1, m_2 \in \{2, 3\}$ *such that* $m_1 + m_2 + (k-2) \geq 4$. *Moreover there exist distinct* $a_1, a_2 \in \mathbb{C}^{m_1}$, *distinct* $b_1, b_2 \in \mathbb{C}^{m_2}$ *and for all* $i \geq 3$ *there exists a basis* $\{u_i, \tilde{u}_i\}$ *of the* i*-th factor such that* T *can be written as*

$$T = (a_1 \otimes b_1 + a_2 \otimes b_2) \otimes u_3 \otimes \cdots \otimes u_k + a_3 \otimes b_3 \otimes \tilde{u}_3 \otimes \cdots \otimes \tilde{u}_k,$$

where if $m_1 = 2$ *then* $a_3 \in \langle a_1, a_2 \rangle$ *otherwise* a_1, a_2, a_3 *are linearly independent. Similarly, if* $m_2 = 2$ *then* $b_3 \in \langle b_1, b_2 \rangle$, *otherwise* b_1, b_2, b_3 *form a basis of the second factor. In this case* $\dim\left(\mathcal{S}\left(Y_{m_1-1, m_2-1, 1^{k-2}}, q\right)\right) \geq 2$ *and if* $m_1 + m_2 + k - 2 \geq 6$ *then* $\dim\left(\mathcal{S}\left(Y_{m_1-1, m_2-1, 2^{k-2}}, q\right)\right) = 2$.

Proof. In Case (a) the point q is a rank-3 matrix therefore it is highly not identifiable. See Remark 2.3.1 for the computation of the dimension of $\mathcal{S}(Y_{2,2}, q)$.

Case (b) is also well known: see Remark 2.3.2.

Case (c) corresponds to the defective 3-rd secant variety of the Segre embedding of $Y_{1,1,1,1} = (\mathbb{P}^1)^4$ and it is treated in Remark 2.3.3.

Cases (d), (e) and (f) are treated in Examples 2.3.7 and 2.3.9 and in Proposition 2.3.14 respectively.

All the above considerations prove that the list of cases enumerated in the statement corresponds to non-indentifiable rank-3 tensors. We need to show that such a list is exhaustive. Since the matrix case is already fully covered by case (a), we only need to care about tensors of order at least 3.

First of all recall that by Remark 2.2.3, the concise Segre of a rank-3 tensor q is $\nu(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$, with $n_1, \ldots, n_k \in \{1, 2\}$. Then consider two distinct sets $A, B \in \mathcal{S}(Y_{n_1,\ldots,n_k}, q)$. By Corollary 2.3.18 it can only happen that $\#(A \cup B) = 5, 6$.

If $\#(A \cup B) = 5$, the fact that our list of non-identifiable rank-3 tensors is exhaustive is proved in Propositions 2.4.1 and 2.4.2.

If $\#(A \cup B) = 6$ we can firstly use Proposition 2.5.2 to exclude the all the cases in which $Y_{n_1,\ldots,n_k}$ has at least two factors of dimension 2. Then we start arguing by the number of factors of $Y_{n_1,\ldots,n_k}$.
If $Y_{1,1,1}$ has 3 factors and it is the product of $\mathbb{P}^1$'s only, then the unique tensors of rank-3

are those of the tangential variety to the Segre variety and this is case (b) of our theorem. The case of $Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ is completely covered by Proposition 2.3.11 together with Examples 2.3.7 and 2.3.9 (cf. Corollary 2.3.12).
If $Y_{2,1,1,1}$ has 4 factors and one of them is a $\mathbb{P}^2$, there is Proposition 2.5.3 assuring that $\mathcal{S}(Y_{2,1,1,1}, q)$ does not contain two disjoint sets. If $Y_{1,1,1,1}$ is a product of four $\mathbb{P}^1$'s we are in case (c). of our theorem.
To conclude, we proved in Proposition 2.5.5 that if $Y_{n_1,\dots,n_k}$ has at least 5 factors then $\mathcal{S}(Y_{n_1,\dots,n_k}, q)$ does not contain two disjoint sets. □

2.7 An algorithm for non-identifiable rank-3 tensors

In this section we present an algorithm aimed to recognize if a tensor $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ is a non-identifiable rank-3 tensor. In the first part of the section we recall some basic facts on matrix pencils that are used for the algorithm, while Subsection 2.7.2 is devoted to present the algorithm.

2.7.1 Matrix pencils

Let us review some basic facts on matrix pencils. In particular, we describe how to achieve the Kronecker normal form of any matrix pencil and we refer to [Gan59, Vol. 1, Ch. XII] for a detailed exposition. For the rest of this subsection, unless specified, we will work over an arbitrary field $\mathbb{K}$ of characteristic 0.

Fix integers $m, n > 0$. A *polynomial matrix* $A(\lambda)$ is a matrix whose entries are polynomials in λ, namely

$$A(\lambda) = (a_{i,j}(\lambda))_{i=1,\dots,m, j=1,\dots,n}, \text{ where } a_{i,j}(\lambda) := a_{i,j}^{(0)} + a_{i,j}^{(1)}\lambda + \cdots + a_{i,j}^{(l)}\lambda^l,$$

for some $l > 0$. If we set $A_k := (a_{i,j}^{(k)})$, then we can write $A(\lambda)$ as

$$A(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^l A_l.$$

The *rank* $r(A(\lambda))$ of $A(\lambda)$ is the positive integer r such that all $r+1$ minors of $A(\lambda)$ are identically zero as polynomials in λ and there exists at least one minor of size r which is not identically zero.

Definition 2.7.1. A *matrix pencil* is a polynomial matrix of type $A(\lambda) = A_0 + \lambda A_1$.

Given two matrix pencils $A(\lambda) = A_0 + \lambda A_1$ and $B(\lambda) = B_0 + \lambda B_1$, we say that $A(\lambda)$ and $B(\lambda)$ are *strictly equivalent* if there exist two invertible matrices P, Q such that

$$P(A_0 + \lambda A_1)Q = B_0 + \lambda B_1.$$

We shall see that the Kronecker normal form of a matrix pencil is determined by a complete system of invariants with respect to the strict equivalence relation defined above.
Any matrix pencil $A + \lambda B$ of size $m \times n$ can be either regular or singular:

Definition 2.7.2. Let $A, B \in M_{m,n}(\mathbb{K})$. A pencil of matrices $A + \lambda B$ is called *regular* if

1. both A and B are square matrices of the same order m;

2. the determinant $\det(A + \lambda B)$ does not vanish identically in λ.

Otherwise the matrix pencil is called *singular*.

Now we describe how to find the normal form of a pencil $A+\lambda B$ depending on whether it is regular or not.

2.7.1.1 Normal form of regular pencils

In the case of regular pencils, normal forms can be found by looking at the elementary divisors of a given matrix pencil. In order to introduce them, it is convenient to consider the pencil $A + \lambda B$ with homogeneous parameters λ, μ, i.e. $\mu A + \lambda B$.

Let $\mu A + \lambda B$ be the rank r homogeneous matrix pencil associated to $A + \lambda B$. For all $j = 1, \ldots, r$, denote by $D(\lambda, \mu)_j$ the greatest common divisor of all the minors of order j in $\mu A + \lambda B$ and set $D_0(\lambda, \mu) = 1$. Define the following polynomials

$$i_j(\lambda, \mu) := \frac{D_{r-j+1}(\lambda, \mu)}{D_{r-j}(\lambda, \mu)}, \text{ for all } j = 1, \ldots, r.$$

Note that all $i_j(\lambda, \mu) \in \mathbb{K}[\lambda, \mu]$ can be splitted into products of powers of irreducible homogeneous polynomials that we call *elementary divisors*. Elementary divisors of the form μ^q for some $q > 0$ are called *infinite elementary divisors*.

Example 2.7.3. Consider the following pencil

$$A + \lambda B = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By looking at the corresponding homogeneous pencil $\mu A + \lambda B$ we can compute $i_j(\lambda, \mu)$ for all $j = 1, 2, 3$, namely

$$\begin{aligned} i_3(\lambda, \mu) &= \frac{D_3(\lambda, \mu)}{D_2(\lambda, \mu)} = \frac{\lambda^2 \mu}{\lambda} = \lambda\mu, \\ i_2(\lambda, \mu) &= \frac{D_2(\lambda, \mu)}{D_1(\lambda, \mu)} = \lambda, \\ i_1(\lambda, \mu) &= \frac{D_1(\lambda, \mu)}{D_0(\lambda, \mu)} = 1. \end{aligned}$$

Therefore the elementary divisors of $A + \lambda B$ are $\{\lambda, \mu\}$, where μ is an infinite elementary divisor.

Theorem 2.7.4 ([Gan59, Vol. 2, Ch. XII, Theorem 2])**.** *Two regular pencils $A + \lambda B$ and $A_1 + \lambda B_1$ are strictly equivalent if and only if they have the same elementary divisors and infinite elementary divisors.*

Remark 2.7.5. From the above theorem we deduce that elementary divisors and infinite elementary divisors are invariant with respect to the strict equivalence relation. Moreover they form a complete system of invariants for the strict equivalence relation since they are irreducible elements with respect the fixed field $\mathbb{K}$. This is the reason why the polynomials $i_j(\lambda, \mu)$ defined above are actually called *invariant polynomials* for all $j = 1, \ldots, r$.

Example 2.7.6. Let $A + \lambda B$ be the pencil of the above Example 2.7.3 and let

$$A_1 + \lambda B_1 = \begin{bmatrix} -4\lambda & \frac{7}{3}\lambda & -\frac{17}{9} \\ 3x & -2x & \frac{5}{3} \\ -x & x & -\frac{5}{6}y \end{bmatrix}.$$

It is easy to see that $A + \lambda B$ and $A_1 + \lambda B_1$ have the same elementary divisors $\{\lambda, \mu\}$. Therefore they are strictly equivalent, indeed one can easily see that $A + \lambda B = P(A_1 + \lambda B_1)Q$, where

$$P = \begin{bmatrix} -4 & 7/3 & -17/9 \\ 3 & -2 & 5/3 \\ -1 & 1 & -5/6 \end{bmatrix}, Q = I_3.$$

Before going further, we recall how to find the Frobenius normal form of a square matrix, which is a generalization of the Jordan form of a matrix that does not require the ground field to contain all eigenvalues of the considered matrix.

Frobenius normal form of a square matrix

We briefly describe a procedure to find the Frobenius normal form of a square matrix, also called rational normal form. Let $A \in M_m(\mathbb{K})$, one can consider $\lambda I - A$ and look for its elementary divisors. This allows us to talk about elementary divisors of a matrix.

Theorem 2.7.7 ([Gan59, Vol. 1, Ch. VI, Theorem 7])**.** *Two matrices $A, B \in M_m(\mathbb{K})$ are similar if and only if they have the same elementary divisors, i.e. the pencils $\lambda I - A$ and $\lambda I - B$ have the same elementary divisors.*

Example 2.7.8. Consider the following two matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

To compute the elementary divisors of A we have to look at invariant polynomials of $\lambda I - \mu A$, namely

$$\begin{aligned} i_3(\lambda, \mu) &= \frac{(x-y)^2(x+y)}{x-y} = (x-y)(x+y), \\ i_2(\lambda, \mu) &= \frac{x-y}{1} = x - y, \\ i_1(\lambda, \mu) &= 1. \end{aligned}$$

It is easy to see that they are exactly the invariant polynomials of $\lambda I - \mu B$. Therefore the elementary divisors of A and B coincide. Moreover note that A and B are similar, since $A = M^{-1}BM$, where

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The knowledge of invariant polynomials and elementary divisors of A enables us to investigate better its structure. Indeed one can compute the Frobenius normal form of A, also called the rational canonical form of A.

We recall that, given a polynomial $g(\lambda) = a_0 + a_1\lambda + \cdots + a_{n-1}x^{n-1} + x^n$, the *companion matrix* of $g(\lambda)$ is

$$L = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{bmatrix}.$$

By computing $\det(I\lambda - L)$, one can easily see that actually $g(\lambda)$ is the characteristic polynomial of L and that $g(\lambda)$ is the only invariant polynomial of L different from 1.

Fix $A \in M_m(\mathbb{K})$ with elementary divisors $e_1(\lambda), \dots, e_u(\lambda)$ and denote by $L^{(1)}, \dots, L^{(u)}$ the corresponding companion matrices. The elementary divisors of the block diagonal matrix

$$L_f = \begin{bmatrix} L^{(1)} & & \\ & \ddots & \\ & & L^{(u)} \end{bmatrix},$$

coincide with the elementary divisors of A, therefore, by Theorem 2.7.7, we get that A and L_f are similar. L_f is called the *Frobenius normal form* of A.

Remark 2.7.9. Assume that A has an elementary divisor of the form $(\lambda - \tilde{\lambda})^p$, for some $\tilde{\lambda} \in \mathbb{K}$, and let $\tilde{J} \in M_p(\mathbb{K})$ be

$$\tilde{J} = \begin{bmatrix} \tilde{\lambda} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \tilde{\lambda} \end{bmatrix}.$$

One can easily see that the only elementary divisor of $\tilde{J}$ is $(\lambda - \tilde{\lambda})^p$ and hence by Theorem 2.7.7, $\tilde{J}$ is similar to the companion matrix associated to the block of $(\lambda - \tilde{\lambda})^p$. Therefore, if the base field $\mathbb{K}$ contains all the roots of the characteristic polynomial associated to A, the elementary divisors of A are $(\lambda - \lambda_1)^{p_1}, \dots, (\lambda - \lambda_u)^{p_u}$, with $p_1 + \cdots + p_u = m$. Moreover in this case the Frobenius normal form of A actually coincides with the Jordan form of A.

Computing the normal form of regular pencils

We have now all the necessary tools to completely determine the normal form of a regular pencil with the following theorem. We also provide an idea of a constructive proof since it involves all the tools defined above.

Theorem 2.7.10 ([Gan59, Vol. 2, Ch. XII, Theorem 3])**.** *Every regular pencil $A + \lambda B$ can be reduced to a (strictly equivalent) canonical form of the following type*

$$[N^{(u_1)}; \dots; N^{(u_s)}; J_{v_1}; \dots; J_{v_t}; L_{w_1}; \dots; L_{w_p}],$$

where

- *The first s diagonal blocks are related to infinite elementary divisors $\mu^{u_1}, \dots, \mu^{u_s}$ of the pencil $A + \lambda B$ and for all $i = 1, \dots, s$*

$$N^{(u_i)} = \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix} \in M_{u_i}(\mathbb{C}).$$

- *The blocks J_{v_i} are the Jordan blocks related to elementary divisors of type $(\lambda - \lambda_i)^{v_i}$.*

- *The last p diagonal blocks $L_{w_1}, \dots, L_{w_p}$ are the companion matrices associated to the remaining elementary divisors of $A + \lambda B$.*

Idea of the proof. Let $A + \lambda B$ be a regular pencil of size m. Since $A + \lambda B$ is regular, by definition there exists $c \in \mathbb{K}$ such that $\det(A + cB) \neq 0$. Rewrite the pencil as

$$A + \lambda B = A_1 + (\lambda - c)B, \text{ where } A_1 = A + cB.$$

Multiplying on the left by A_1^{-1} we get

$$A_1^{-1}(A_1 + (\lambda - c)B) = I + (\lambda - c)A_1^{-1}B.$$

Consider the following block diagonal matrix $J = [J_0; J_1]$ obtained from $A_1^{-1}B$ by similarity, where J_0 is the maximal order nilpotent block of the form

$$J_0 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

and J_1 is a non singular block. One can look at the pencil as

$$I + (\lambda - c)J = \begin{bmatrix} I + (\lambda - c)J_0 & \\ & I + (\lambda - c)J_1 \end{bmatrix} = [I + (\lambda - c)J_0; I + (\lambda - c)J_1].$$

By looking at the form of J_0 we note that $I - cJ_0$ is invertible. Multiplying on the right by the block diagonal matrix $[(I - cJ_0)^{-1}; I]$ we get $[I + \lambda(I - cJ_0)^{-1}J_0; I + (\lambda - c)J_1]$. Since also $(I - cJ_0)^{-1}J_0$ is nilpotent, we can consider its Jordan form $\hat{J}$, hence we get

$$\begin{bmatrix} I + \lambda \hat{J} & \\ & I + (\lambda - c)J_1 \end{bmatrix} = \begin{bmatrix} N^{(u_1)} & & & \\ & \ddots & & \\ & & N^{(u_s)} & \\ & & & I + (\lambda - c)J_1 \end{bmatrix},$$

where for all $i = 1, \dots, s$

$$N^{(u_i)} = \begin{bmatrix} 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \\ & & & 1 \end{bmatrix} \in M_{u_i}(\mathbb{C}).$$

Multiplying on the right by $[I; J_1^{-1}]$ and setting $\hat{J}_1$ the Frobenius normal form of $J_1^{-1} - cI$ we get

$$\begin{bmatrix} N^{(u_1)} & & & \\ & \ddots & & \\ & & N^{(u_s)} & \\ & & & \lambda I + \hat{J}_1 \end{bmatrix}.$$

□

We conclude the subsection with an example that illustrates how to build the Kronecker normal form of a regular pencil.

Example 2.7.11. Consider the pencil

$$A + \lambda B = \begin{bmatrix} \lambda+3 & 0 & \lambda+2 & 1 & 1 \\ 0 & \lambda & 1 & 0 & 0 \\ \lambda+3 & \lambda & 2\lambda & 1 & 2 \\ \lambda+3 & 0 & 1 & \lambda+4 & 0 \\ 0 & 0 & \lambda & \lambda+3 & 1 \end{bmatrix}.$$

The determinant $\det(\mu A + \lambda B) = -7(\mu^2)(\lambda+3)^2\lambda$ and this is the only invariant polynomial of $A + \lambda B$ different from one. Therefore the elementary divisors of the pencil are $\{\mu^2, (\lambda+3)^2, \lambda\}$ and the Kronecker normal form of $A + \lambda B$ is

$$\left[\begin{array}{cc|cc|c} 1 & \lambda & & & \\ & 1 & & & \\ \hline & & \lambda+3 & 1 & \\ & & & \lambda+3 & \\ \hline & & & & \lambda \end{array}\right].$$

2.7.1.2 Normal form of singular pencils

In the previous case, a complete system of invariants was made by both elementary divisors and infinite ones (cf. Remark 2.7.5). We shall see that, in case of singular pencils, this is not sufficient to determine a complete system of invariants with respect to the strict equivalence relation. Fix $m \leq n$ and let $A + \lambda B$ be a singular pencil of rank r, where $A, B \in M_{m,n}(\mathbb{K})$. Since the pencil is singular, the columns of $A + \lambda B$ are linearly dependent, therefore the system

$$(A + \lambda B)x = 0 \tag{2.7.13}$$

has a non-zero solution with respect to x. Note that any solution $\tilde{x}$ of the above system is a vector whose entries are polynomials in λ, i.e. $\tilde{x} = \tilde{x}(\lambda)$.

Theorem 2.7.12 ([Gan59, Vol. 2, Ch. XII, Theorem 4])**.** *If equation (2.7.13) has a solution of minimal degree $\varepsilon \neq 0$ with respect to λ, the singular pencil $A + \lambda B$ is strictly equivalent to*

$$\begin{bmatrix} L_\varepsilon & \\ & \hat{A} + \lambda \hat{B} \end{bmatrix},$$

where

$$L_\varepsilon = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix} \in M_{\varepsilon,\varepsilon+1}(\mathbb{K}),$$

and $\hat{A} + \lambda\hat{B}$ is a pencil of matrices for which the equation analogous to (2.7.13) *has no solution of degree less than ε.*

By applying the previous theorem iteratively, a singular pencil $A + \lambda B$ is strictly equivalent to $[L_{\varepsilon_1}; \dots; L_{\varepsilon_p}; A_p + \lambda B_p]$, where $0 \neq \varepsilon_1 \leq \cdots \leq \varepsilon_p$ and the last block is such that $(A_p + \lambda B_p)x = 0$ has no non zero solution, i.e. the columns of $A_p + \lambda B_p$ are linearly independent. The same idea can be applied to the rows of $A_p + \lambda B_p$ if they are linearly dependent, by considering the associated system of the transposed pencil. Therefore $A + \lambda B$ is strictly equivalent to

$$\begin{bmatrix} L_{\varepsilon_1} & & & & & & \\ & \ddots & & & & & \\ & & L_{\varepsilon_p} & & & & \\ & & & L^T_{\eta_1} & & & \\ & & & & \ddots & & \\ & & & & & L^T_{\eta_q} & \\ & & & & & & A_0 + \lambda B_0 \end{bmatrix},$$

were $0 \neq \varepsilon_1 \leq \cdots \leq \varepsilon_p$, $0 \neq \eta_1 \leq \cdots \leq \eta_q$ and both the columns and rows of $A_0 + \lambda B_0$ are linearly independent, i.e. $A_0 + \lambda B_0$ is a regular pencil.

Now let us treat the case in which there are some relations of degree zero (with respect to λ) between the rows and the columns of the given pencil $A + \lambda B$. Denote by g and h the maximal number of independent constant solutions of equations

$$(A + \lambda B)x = 0 \text{ and } (A^T + \lambda B^T)x = 0 \text{ respectively.}$$

Let $e_1, \dots, e_g \in \mathbb{K}^n$ be linearly independent solutions of the system $(A + \lambda B)x = 0$, completing them to a basis of $\mathbb{K}^n$ and rewriting the pencil with respect to this basis, we get $\tilde{A} + \lambda\tilde{B} = \begin{bmatrix} 0_{m\times g} & \tilde{A}_1 + \lambda\tilde{B}_1 \end{bmatrix}$. One can do the same by taking h linearly independent vectors that are solutions of the transpose pencil and hence the first h rows of $\tilde{A}_1 + \lambda\tilde{B}_1$ are zero with respect this new basis. Thus we obtain

$$\begin{bmatrix} 0_{h\times g} & \\ & A_0 + \lambda B_0 \end{bmatrix},$$

where $A_0 + \lambda B_0$ does not have any degree zero relation, and hence either $A_0 + \lambda B_0$ satisfies the assumptions of Theorem 2.7.12 or it is a regular pencil. There is a quicker way, due to Kronecker, to determine the canonical form of a given pencil, avoiding the iterative reduction just explained. It involves the notion of minimal indices. These last, together with elementary divisors (possibly infinite) will form a complete system of invariants for non singular pencils.

Let $A + \lambda B$ be a non singular pencil and let $x_1(\lambda)$ be a non zero solution of least degree ε_1 for $(A + \lambda B)x = 0$. Take $x_2(\lambda)$ as a solution of least degree ε_2 such that $x_2(\lambda)$ is linearly independent from $x_1(\lambda)$. Continuing this process, we get a so called *fundamental series of solutions* of the system

$$x_1(\lambda), \ldots, x_p(\lambda), \text{ of degrees } \varepsilon_1 \leq \cdots \leq \varepsilon_p, \text{ for some } p \leq n.$$

We remark that a fundamental series of solution is not uniquely determined, but one can show that the degrees $\varepsilon_1, \ldots, \varepsilon_p$ are the same for any fundamental series associated to a given system $(A + \lambda B)x = 0$. The *minimal indices for the columns* of $A + \lambda B$ are the integers $\varepsilon_1, \ldots, \varepsilon_p$. Similarly, the *minimal indices for the rows* are the degrees $\eta_1, \ldots, \eta_q$ of a fundamental series of solutions of $(A^T + \lambda B^T)x = 0$.

Proposition 2.7.13 ([Gan59, Vol. 2, Ch. XII, Sec. 5, Par. 2])**.** *Strictly equivalent pencils have the same minimal indices.*

Now let $A + \lambda B$ be a singular pencil and consider its normal form

$$\begin{bmatrix} 0_{h\times g} & & & & & & & \\ & L_{\varepsilon_{g+1}} & & & & & & \\ & & \ddots & & & & & \\ & & & L_{\varepsilon_p} & & & & \\ & & & & L^T_{\eta_{h+1}} & & & \\ & & & & & \ddots & & \\ & & & & & & L^T_{\eta_q} & \\ & & & & & & & A_0 + \lambda B_0 \end{bmatrix}. \tag{2.7.14}$$

Remark 2.7.14. The system of indices for the columns (rows) of the above block diagonal matrix is obtained by taking the union of the corresponding system of minimal indices of the individual blocks.

We want to determine minimal indices for the above normal form (2.7.14). By the previous remark, it is sufficient to determine the minimal indices for each block. Clearly the regular block $A_0 + \lambda B_0$ has no minimal indices, the zero block $0_{h\times g}$ has g minimal indices for columns and h minimal indices for rows all equal to zero respectively, namely $\varepsilon_1 = \cdots = \varepsilon_g = \eta_1 = \cdots = \eta_h = 0$. The block $L_{\varepsilon_i} \in M_{\varepsilon_i, \varepsilon_i+1}(\mathbb{K})$ has linearly independent rows, therefore it has just one minimal index for column ε_i for all $i = 1, \ldots, p$. Similarly, for all $j = 1, \ldots, q$ the block L_{η_j} has just one minimal index for rows η_j.

We conclude that the canonical form (2.7.14) is completely determined by both the minimal indices $\varepsilon_1, \ldots, \varepsilon_p, \eta_1, \ldots, \eta_q$ and the elementary divisors.

Theorem 2.7.15 (Kronecker)**.** *Two arbitrary pencils $A+\lambda B$ and $A_1+\lambda B_1$ of rectangular matrices are strictly equivalent if and only if they have the same minimal indices and the same elementary divisors (possibly infinite).*

Example 2.7.16. Consider the pencil

$$A + \lambda B = \begin{bmatrix} 1 & 0 & \lambda & 3\lambda + 1 & 1 & 2 \\ 2\lambda & \lambda & \lambda & 3 & \lambda & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 2\lambda + 1 & \lambda & 2\lambda + 1 & 3\lambda + 4 & \lambda + 1 & 2 \end{bmatrix}.$$

The kernel of the system $(A+\lambda B)x = 0$ is generated by

$$\mathrm{Ker}(A+\lambda B) = \langle \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -\lambda^2 \\ 2\lambda^2 - \lambda - 1 \\ \lambda \\ 0 \\ 0 \\ 0 \end{bmatrix} \rangle.$$

Since the minimum integer of the non constant solution is $\varepsilon = 2$, we know that the normal form of the pencil contains the following block

$$L_2 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \end{bmatrix}.$$

Moreover, we see that there are $g = 2$ linearly independent constant solutions. Considering the transpose pencil, then

$$\mathrm{Ker}((A+\lambda B)^T) = \langle \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \rangle,$$

so there is just one constant solution. Therefore, keeping the above notation, $\eta = 0$ and $h = 1$. Moreover the invariant polynomials of the pencil are $i_4(\lambda, \mu) = 0$, $i_3(\lambda, \mu) = \mu$ and all the others are equal to 1. Therefore the Kronecker normal form of $A + \lambda B$ is

$$\left[\begin{array}{cc|ccc|c} 0 & 0 & & & & \\ \hline & & \lambda & 1 & 0 & \\ & & 0 & \lambda & 1 & \\ \hline & & & & & 1 \end{array}\right].$$

2.7.1.3 3-factors tensor spaces and matrix pencils

From now on we work over $\mathbb{C}$. Any tensor $T \in \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ can be seen as a matrix pencil via the isomorphism

$$\mathbb{C}^2 \otimes (\mathbb{C}^m)^* \otimes (\mathbb{C}^n)^* \xrightarrow{\sim} \{\mathbb{C}^m \times \mathbb{C}^n \xrightarrow{\Phi} \mathbb{C}^2\}.$$

We can easily pass from a tensor $T \in \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ to its associated matrix pencil (and viceversa) by fixing a basis on each factor and looking at T in its coordinates with respect to the fixed bases. For example, let us fix the canonical basis on each factor and let $T = (t_{ijk}) \in \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$. We can associate to T the map

$$\begin{aligned} \Phi_T : \mathbb{C}^m \times \mathbb{C}^n &\longrightarrow \mathbb{C}^2 \\ (v, w) &\mapsto (v^T A w, v^T B w) \end{aligned}$$

where

$$A = (t_{1ij})_{i=1,\dots,m, j=1,\dots,n} \text{ and } B = (t_{2ij})_{i=1,\dots,m, j=1,\dots,n}.$$

Example 2.7.17. To the tensor

$$T = e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + (e_1 + e_2) \otimes e_1 \otimes e_2 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

we can associate the following matrix pencil

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Fixing the integer m equal to either 2 or 3 in $\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ leads us to consider very special tensor formats, i.e. $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ and $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$, because in these cases there is a finite number of orbits with respect to the action of products of general linear groups.

Theorem 2.7.18 ([Kac80])**.** *The only spaces of tensors with a finite number of $GL_{n_1+1} \times \cdots \times GL_{n_k+1}$-orbits are*

1. $\mathbb{C}^n \otimes \mathbb{C}^m$,
2. $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$,
3. $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$.

The last two items of the above theorem have been widely studied. In [Par01], P. Parfenov gave a complete orbit classification working in the affine setting. Moreover, for any tensor belonging to any of these tensor spaces, one can consider the associated matrix pencil and, by computing its Kronecker normal form, it is possible to understand its rank. This last result comes from the following more general statement that is historically attributed to Grigoriev, JáJá and Teichert. We refer to [BL13, Remark 5.4] for a historical note on the theorem.

Theorem 2.7.19 ([Gri78], [JáJ79], [Tei86])**.** *Let $T \in \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ and let A be the corresponding pencil with minimal indices $\varepsilon_1, \dots, \varepsilon_p, \eta_1, \dots, \eta_q$ and regular part $C = A_0 + \lambda B_0$ of size N. Let $\delta(C)$ be the number of non-squarefree invariant polynomials of C. Then T is a tensor of rank*

$$\sum_{i=1}^{p} (\varepsilon_i + 1) + \sum_{i=1}^{q} (\eta_j + 1) + N + \delta(C). \tag{2.7.15}$$

In [BL13], J. Buczyński and J. M. Landsberg reviewed the orbits classification made in [Par01] for the last two items of Theorem 2.7.18 and gave a geometric interpretation of the projectivization of all the orbits closures appearing in both cases.

Since it will be useful in the sequel, we report here two tables taken from [BL13]. Table 2.1 represents the orbit classification of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$, while Table 2.3 contains all orbits in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. In each table, we presented the projective closure of any orbits, a tensor representative together with its rank and border rank. In both cases we will use the following notation.

Notation 2.7.20. Let A, B, C be $\mathbb{C}$-vector spaces of dimensions $2, m$ and n respectively and let $X_{1,m-1,n-1} = \nu(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ be the corresponding Segre variety. Let $Sub_{ijk} \subset \mathbb{P}(A \otimes B \otimes C)$ be the space of tensors $q = [T] \in \mathbb{P}(A \otimes B \otimes C)$ such that there exist linear subspaces $A' \subset A$, $B' \subset B$, $C' \subset C$ of dimension i, j, k respectively such that $T \in A' \otimes B' \otimes C'$, i.e. $A' \otimes B' \otimes C'$ is the concise tensor space of T. Denote by $X_{1,m-1,n-1*} \subset \mathbb{P}(A^* \otimes B^* \otimes C^*)$ the Segre variety in the dual projective space and by $X^{\vee}_{1,m-1,n-1*} \subset \mathbb{P}(A \otimes B \otimes C)$ its dual variety.

Orbits in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$

	Orbit closure	Kroneker normal form	brank	rank
1	X	$a_1 \otimes b_1 \otimes c_1$	1	1
2	Sub_{221}	$a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_1$	2	2
3	Sub_{122}	$a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2$	2	2
4	Sub_{212}	$a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_1 \otimes c_2$	2	2
5	$\tau(X_{1,1,1})$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes b_1 \otimes c_2$	2	3
6	$\sigma_2(X_{1,1,1}) = Sub_{222}$	$a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2$	2	2
7	$X^\vee_{1,1,2*}$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes b_1 \otimes c_2$	3	3
8	$\sigma_3(X_{1,1,2}) = Sub_{223}$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_3)$	3	3
9	$\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4)$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4)$	4	4

Table 2.1: Normal forms of tensors in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ (cf. [BL13, Table 1])

We remark that in Table 2.1 all $a_i \in \mathbb{C}^2$, $b_j \in \mathbb{C}^2$ and $c_k \in \mathbb{C}^n$ are considered as linearly independent vectors of the corresponding spaces.

In terms of matrix pencils and taking λ, μ as parameters of the first factor, each orbit representative can be seen as follows.

1	2	3	4	5	6	7	8	9
$\begin{bmatrix}\lambda\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & \\ & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda \\ \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu \\ & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda & \\ & \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu & \\ & & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu & \\ & \lambda & \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu & & \\ & & \lambda & \mu\end{bmatrix}$

Table 2.2: Matrix pencils associated to Kronecker normal forms of table 2.1

Orbits in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

We present in the following table all orbits of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ that are not contained in $Sub_{223} \subset \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. The unnamed orbits $13-16$ are components of the singular locus of $X^\vee_{1,2,2*} \subset Sub_{233}$.

	Orbit closure	Kroneker normal form	brank	rank
10	Sub_{133}	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3)$	3	3
11	$X^\vee_{1,2,1*} \subset Sub_{232}$	$a_1 \otimes (b_1 \otimes c_1 + b_3 \otimes c_2) + a_2 \otimes b_2 \otimes c_1$	3	3
12	Sub_{232}	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_2 \otimes c_1 + b_3 \otimes c_2)$	3	3
13		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes (b_1 \otimes c_2 + b_3 \otimes c_3)$	3	4
14		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes b_3 \otimes c_3$	3	3
15		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) a_2 \otimes b_1 \otimes c_2$	3	4
16		$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2 + b_3 \otimes c_3) + a_2 (\otimes b_1 \otimes c_2 + b_2 \otimes c_3)$	3	4
17	$X^\vee_{1,2,2*} \subset S_{233}$	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_1 \otimes c_2 + b_3 \otimes c_3)$	3	4
18	Sub_{233}	$a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_2) + a_2 \otimes (b_2 \otimes c_2 + b_3 \otimes c_3)$	3	3

Table 2.3: Normal forms of tensors in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ (cf. [BL13, Table 3])

In terms of matrix pencils, taking λ, μ as parameters of the first factor, each orbit representative can be seen as follows.

10	11	12	13	14	15	16	17	18
$\begin{bmatrix}\lambda & & \\ & \lambda & \\ & & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda & \\ \mu & \\ & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda & \\ \mu & \lambda \\ & \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu & \\ & & \lambda \\ & & \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & & \\ & \lambda & \\ & & \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu & \\ & \lambda & \\ & & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu & \\ & \lambda & \mu \\ & & \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & \mu & \\ & \lambda & \\ & & \mu\end{bmatrix}$	$\begin{bmatrix}\lambda & & \\ & \lambda+\mu & \\ & & \mu\end{bmatrix}$

Table 2.4: Matrix pencils associated to Kronecker normal forms of Table 2.3

For an analogous table representation of both orbits and tensors in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$ for all $n \geq 4$ we refer to [BL13, Tables 4,5].

2.7.2 The algorithm

All possible cases of non-identifiabile rank-3 tensors are collected in Theorem 2.6.1.

The **input** of the algorithm we propose is a tensor $T = (t_{i_1,i_2,\cdots,i_k}) \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$, where $k \geq 3$, all $n_\ell \geq 1$ and all $i_j = 1, \ldots, n_j$. The **output** of the algorithm is a statement telling if the given tensor is a rank-3 tensor that falls into one of the cases mentioned above or not.

The first step of the algorithm is to compute the *concise tensor space* $\mathcal{T}_{n'_1,\ldots,n'_{k'}} = \mathbb{C}^{n'_1} \otimes \cdots \otimes \mathbb{C}^{n'_{k'}}$ *of* T, that is the smallest tensor space containing the cone of the concise Segre of T (cf. Definition 2.2.1). We refer to Subsubsection 1.1.1.1 for a detailed description of the concision process in coordiantes. Based on the resulting concise tensor space $\mathcal{T}_{n'_1,\ldots,n'_{k'}}$, we split the algorithm into 2 different parts depending on whether $\mathcal{T}_{n'_1,\ldots,n'_{k'}}$ is made by three factors or not. Subsubsection 2.7.2.1 is devoted to the 3-factors case while we refer to Subsubsection 2.7.2.2 for the other case.

Remark 2.7.21. Fix a tensor $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ and compute the multilinear rank of T. By using the left inequality in (1.1.2) on each flattening φ_ℓ, we are able to exclude some of the cases in which $r(T)$ is higher than 3. In those cases the algorithm stops since we are interested in rank-3 tensors. Moreover, if the multilinear rank of T contains more than $k-3$ positions equal to 1 then T is either a rank-1 tensor or a matrix and we can also exclude these cases. Lastly, we remark that since the concise Segre of a rank-3 tensor is $\nu(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k})$ where all $m_i \in \{1,2\}$ for all $i = 1, \ldots, k$, if one of the values in $mr(T) = (\dim(\mathbb{C}^{m_i+1}))_{i=1,\ldots,k}$ is different from either 2 or 3 then we can immediately stop the algorithm. Therefore, at the end of the concision process, we deal with a

$$T' \in \mathbb{C}_1^{n'_1} \otimes \cdots \otimes \mathbb{C}_{k'}^{n'_{k'}}$$

such that

- $r(T') \geq 2$,
- $3 \leq k' \leq k$
- all $n'_i \in \{2,3\}$.

Now, depending on whether $k' = 3$ or $k' \geq 4$, we split the algorithm into 2 different parts. In Section 2.7.2.1 we treat the 3 factors case and Section 2.7.2.2 contains the remaining cases.

2.7.2.1 Three factors case

This subsubsection is devoted to treat the case in which the concise tensor space of the tensor T given in input has three factors. By Remark 2.7.21, the concise space $\mathcal{T}_{n_1,\dots,n_k} = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ of a tensor T is such that all $n_i \in \{2,3\}$. Moreover, if $k=3$ the only possibilities for $\mathcal{T}_{n_1,n_2,n_3}$ up to a reordering of the factors are

1. $\mathcal{T}_{2,2,2} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$;
2. $\mathcal{T}_{3,2,2} = \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$;
3. $\mathcal{T}_{3,3,2} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$;
4. $\mathcal{T}_{3,3,3} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

Remark 2.7.22. The presence of a $\mathbb{C}^2$ in $\mathcal{T}_{2,2,2}, \mathcal{T}_{3,2,2}, \mathcal{T}_{3,3,2}$ allows to see all their elements as a matrix pencil (cf. Section 2.7.1.3), in these cases we are also able to compute the rank of one of those tensors by classifying their at its associated matrix pencils (cf. Theorem 2.7.19).

All the considerations made in the following will be summed up in Algorithm 1 at the end of the subsubsection.

$$\boxed{\mathcal{T}_{2,2,2} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2}$$

The second secant variety of $X_{1,1,1} = \nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^7$ fills the ambient space, i.e. $\dim \sigma_2(X_{1,1,1}) = 7$. Consequently, any tensor $[T] \in \mathbb{P}^7 \setminus X_{1,1,1}$ is either an element of the open part $\sigma_2^0(X_{1,1,1})$ or an element of the tangential variety $\tau(X_{1,1,1})$ of $X_{1,1,1}$. Therefore if the concise tensor space of T is $\mathcal{T}_{2,2,2} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ rank-1 is excluded and T has rank either 2 or 3. We detect the rank of T with the Cayley's hyperdeterminant which is the defining equation of $\tau(X_{1,1,1})$ (cf. [GKZ08]).

Definition 2.7.23 ([GKZ08])**.** Let $A = (a_{ijk})_{i,j,k=0,1} \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The *Cayley's hyperdeterminant* $\operatorname{Hdet}(A)$ of A is

$$\operatorname{Hdet}(A) := \left(\begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{111} \end{vmatrix} + \begin{vmatrix} a_{010} & a_{001} \\ a_{110} & a_{101} \end{vmatrix} \right)^2 - 4 \begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix} \cdot \begin{vmatrix} a_{010} & a_{011} \\ a_{110} & a_{111} \end{vmatrix}.$$

Therefore if T is a concise tensor in $\mathcal{T}_{2,2,2}$ and $\operatorname{Hdet}(T) = 0$ then T has rank 3 and it is not identifiable, otherwise it has rank 2.

$$\boxed{\mathcal{T}_{3,2,2} = \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2}$$

The non-identifiable rank-3 tensors of $\mathcal{T}_{3,2,2} = \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ come from cases (d), (e) and (f) of Theorem 2.6.1.

If $\mathcal{T}_{3,2,2}$ is the concise tensor space of T, then obviously $r(T) \geq 3$. One can show that actually $r(T) = 3$ by using the following result.

Theorem 2.7.24 ([Lan12, Theorem 3.1.1.1])**.** *Let $T \in \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \mathbb{C}^{m_3}$. Then $r(T)$ equals the number of rank one matrices needed to span (a space containing) $T((\mathbb{C}^{m_1})^*) \subset \mathbb{C}^{m_2} \otimes \mathbb{C}^{m_3}$ (and similarly for the permuted statements).*

Therefore every concise $T \in \mathcal{T}_{3,2,2}$ is a rank-3 tensor. Moreover, since the dimension of the third secant variety of $X_{2,1,1} = \nu(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^{11}$ is $\min\{14, 11\}$, the generic fiber of the projection from the abstract secant variety to the secant variety has projective dimension 2, so the generic element of $\sigma_3(X_{2,1,1})$ has an infinite number of decompositions. Therefore, by Proposition 2.1.4, any rank-3 tensor in $\sigma_3(X_{2,1,1})$ is not identifiable, from which follows that any tensor whose concise tensor space is $\mathcal{T}_{3,2,2} = \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is a non-identifiable rank-3 tensor.

Remark 2.7.25. Note that rank-3 tensors can also live in $\sigma_2(X_{2,1,1})$ but a concise rank-3 tensor $T \in \mathcal{T}_{3,2,2}$ lies only on the third secant variety of $X_{2,1,1}$.

The distinction between cases (d), (e) and (f) of Theorem 2.6.1 arises by looking at the space of solutions of the given tensor in $\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ (cf. Definition 2.2.2). All the three examples can be treated by looking at the matrix pencil associated to the corresponding tensor. We refer to Subsection 2.7.1 for a brief review of matrix pencils and in particular to Subsubsection 2.7.1.3 for the connection with 3-way tensors.

Remark 2.7.26. In order to be consistent with the matrix pencil notation used in Subsection 2.7.1 in which the first factor is used as a parameter space for the pencil, we swap the first and third factor of $\mathcal{T}_{3,2,2}$, working now on $\mathcal{T}_{2,2,3} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$.

Table 2.1 offers a complete description of all orbits in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$, providing also the orbit closure in each case together with the Kronecker normal form of each orbit representative and its rank. Since we are working with concise rank-3 tensors of $\mathcal{T}_{2,2,3}$, we are interested in cases 7 and 8 of Table 2.1, i.e.

- $\begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix} \sim a_1 \otimes (b_1 \otimes c_1 + b_2 \otimes c_3) + a_2 \otimes b_1 \otimes c_2,$
- $\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \end{bmatrix} \sim a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_2 + a_2 \otimes b_2 \otimes c_3,$
  where all a_i, b_j, c_k are linearly independent elements of the corresponding factors.

Let us see which is the relation between the above Kronecker normal forms and our examples of non-identifiable rank-3 tensors in $\mathcal{T}_{2,2,3}$.

Lemma 2.7.27. *The matrix pencil associated to any tensor $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ belonging either to Example 2.3.10 or to Example 2.3.15 is of the following form:*

$$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix} \sim \begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Proof. Let $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ be as in Example 2.3.10, so

$$T = \tilde{p} \otimes v_1 \otimes u_1 + \tilde{p} \otimes v_2 \otimes u_2 + w \otimes (\alpha v_1 + \beta v_2) \otimes u_3.$$

The matrix pencil associated to T with homogeneous parameters λ, μ referred to the basis $\{\tilde{p}, w\} \subset \mathbb{C}^2$ is

$$A = \begin{bmatrix} \lambda & 0 & \alpha\mu \\ 0 & \lambda & \beta\mu \end{bmatrix}.$$

Since A is a singular pencil (cf. Definition 2.7.2), in order to achieve the normal form of A, we have to look at the minimum degree ε of the elements in

$$\operatorname{Ker}(A) = \langle \begin{bmatrix} -\alpha\mu \\ -\beta\mu \\ \lambda \end{bmatrix} \rangle$$

with respect to λ, μ (cf. Section 2.7.1.2). Since $\varepsilon = 1$, by Theorem 2.7.12, the normal form of A should contain a block of size $\varepsilon \times (\varepsilon + 1)$ of this type

$$\begin{bmatrix} \lambda & \mu & & \\ & \ddots & \ddots & \\ & & \lambda & \mu \end{bmatrix}.$$

Therefore we can conclude that

$$A = \begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

If $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ is as in Example 2.3.15, then

$$T = w \otimes (v_1 \otimes u_1 + v_2 \otimes u_2) + \tilde{w} \otimes (\beta_1 v_1 + \beta_2 v_2) \otimes (\alpha_1 u_1 + \alpha_2 u_2 + u_3).$$

The matrix pencil associated to T with homogeneous parameters λ, μ referred to the basis $\{w, \tilde{w}\} \subset \mathbb{C}^2$ is

$$A = \begin{bmatrix} \lambda + \alpha_1\beta_1\mu & \alpha_2\beta_1\mu & \beta_1\mu \\ \alpha_1\beta_2\mu & \lambda + \alpha_2\beta_2\mu & \beta_2\mu \end{bmatrix}.$$

The kernel of A is

$$\operatorname{Ker}(A) = \langle \begin{bmatrix} -\beta_1\mu \\ -\beta_2\mu \\ \lambda + (\alpha_1\beta_1 + \alpha_2\beta_2)\mu \end{bmatrix} \rangle.$$

As before, by Theorem 2.7.12, we know that T must contain a block of size 1×2 of the form $\begin{bmatrix} \lambda & \mu \end{bmatrix}$. Therefore

$$T \sim \begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

□

Corollary 2.7.28. *Let $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. The tensor T is a non-identifiable rank 3 tensor coming from either case (e) or case (f) of Theorem 2.6.1 if and only if the pencil associated to T is of the form*

$$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Proof. By Lemma 2.7.27, the matrix pencil associated to any tensor that belongs to either Example 2.3.10 or Example 2.3.15 is

$$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

The viceversa also holds since actually the above pencil corresponds to the tensor

$$a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_3 + a_2 \otimes b_1 \otimes c_2$$

which is as in Example 2.3.10. □

Lemma 2.7.29. *The matrix pencil associated to a tensor $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ that is as in Example 2.3.8 is*

$$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \end{bmatrix}.$$

Proof. Let $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ be as in Example 2.3.8, i.e. there is a basis $\{u_i\}_{i \leq 3} \subset \mathbb{C}^3$ and a basis $\{v_1, v_2\} \subset \mathbb{C}^2$ such that

$$T = v_1 \otimes v_1 \otimes u_1 + v_2 \otimes v_2 \otimes u_2 + (\alpha v_1 + \beta v_2) \otimes (\alpha v_1 + \beta v_2) \otimes u_3,$$

for some $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$. The matrix pencil associated to T with homogeneous parameters λ, μ referred to the basis $\{v_1, v_2\} \subset \mathbb{C}^2$ is

$$A = \begin{bmatrix} \lambda & 0 & \alpha^2\lambda + \alpha\beta\mu \\ 0 & \mu & \alpha\beta\lambda + \beta^2\mu \end{bmatrix}.$$

The kernel of A is

$$\mathrm{Ker}(A) = \langle \begin{bmatrix} \alpha^2\lambda\mu + \alpha\beta\mu^2 \\ \alpha\beta\lambda^2 + \beta^2\lambda\mu \\ -\lambda\mu \end{bmatrix} \rangle,$$

so the minimum degree ε of the elements in $\mathrm{Ker}(A)$ with respect to λ, μ is 2. Therefore, by Theorem 2.7.12, the normal form of A is

$$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \end{bmatrix}.$$

□

Corollary 2.7.30. *Let $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. The tensor T is a non-identifiable rank-3 tensor coming from case (d) of Theorem 2.6.1 if and only if the pencil associated to T is of the form*

$$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \end{bmatrix}.$$

Proof. By Lemma 2.7.29, the matrix pencil associated to any tensor that belongs to Example 2.3.8 is

$$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \end{bmatrix}.$$

The viceversa also holds since actually the above pencil corresponds to the tensor

$$e_1 \otimes e_1 \otimes e_1 + (e_1 \otimes e_2 + e_2 \otimes e_1) \otimes e_2 + e_2 \otimes e_2 \otimes e_3$$

which is as in Example 2.3.8. □

$$\boxed{\mathcal{T}_{3,3,2} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2}$$

Let $\mathcal{T}_{3,3,2}$ be the concise tensor space of the tensor T we have in input. We recall that the only non-identifiable rank-3 tensors in this case are the ones of Proposition 2.3.14, i.e. case (f) of Theorem 2.6.1. More precisely, let $Y' = \mathbb{P}^1 \times \mathbb{P}^1 \times \{w\} \subset Y_{2,2,1} = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. Take $q' \in \langle \nu(Y') \rangle \setminus \nu(Y_{2,2,1})$ and $p \in Y_{2,2,1} \setminus Y'$. Then $[T] \in \langle q', \nu(p) \rangle$ is a rank-3 tensor and it is not identifiable. If we take $\{u_i\}_{i \leq 3} \subset \mathbb{C}^3$ as a basis of the first factor, $\{v_i\}_{i \leq 3} \subset \mathbb{C}^3$

as a basis of the second factor and $\{w, \tilde{w}\} \subset \mathbb{C}^2$ as a basis of the third factor, then T is of the form

$$T = u_1 \otimes v_1 \otimes w + u_2 \otimes v_2 \otimes w + u_3 \otimes v_3 \otimes \tilde{w}. \tag{2.7.16}$$

Again we can look at this case by considering the associated matrix pencil of T. As before (cf. Remark 2.7.26), to be consistent with the matrix pencil notation we already introduced, we swap the first and third factor of $\mathcal{T}_{3,3,2}$, working now on $\mathcal{T}_{2,3,3} = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

Table 2.3 collects all Kronecker normal forms contained in $\mathcal{T}_{2,3,3}$. Since we are interested in rank-3 tensors having $\mathcal{T}_{2,3,3}$ as concise tensor space, the only possibilities in terms of matrix pencils are

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \text{ and } \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda+\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}. \tag{2.7.17}$$

Remark 2.7.31. The matrix pencil associated to (2.7.16) is the first one in (2.7.17) and it is easy to check that the tensor corresponding to the first matrix pencil in (2.7.17) is actually T.

Therefore, if the concise tensor space of T is $\mathcal{T}_{2,3,3,}$, it is sufficient to consider the normal form of the concise tensor T' related to T and check if it corresponds to

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Moreover, as in the previous case, we are able to detect the rank of any tensor having $\mathcal{T}_{2,3,3}$ as a concise tensor space (cf. Remark 2.7.22).

$\boxed{\mathcal{T}_{3,3,3} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3}$

By Theorem 2.6.1, all rank-3 tensors whose concise tensor space is $\mathcal{T}_{3,3,3}$ are identifiable. Therefore if the concise tensor space of T is as in 4 we can immediately say that T does not belong to one of the 6 families of non-identifiable rank-3 tensors.

We collect all the considerations made in this subsubsection in the following algorithm.

Algorithm 1 (Three factors case)
Input: Concise tensor $T = (t_{i_1,i_2,i_3}) \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, with $n_i = 2, 3$ for all $i = 1, 2, 3$.
Output: A statement on whether T belongs to one of the six cases of non-identifiable rank-3 tensors or not.

0. Permute the three factors in a reverse lexicographic order.
1. Case $(n_1, n_2, n_3) = (2, 2, 2)$.
   If T satisfies Cayley's hyperdeterminant equation

$$\mathrm{Hdet}(T) := \left(\begin{vmatrix} t_{0,0,0} & t_{0,0,1} \\ t_{1,0,0} & t_{1,1,1} \end{vmatrix} + \begin{vmatrix} t_{0,1,0} & t_{0,0,1} \\ t_{1,1,0} & t_{1,0,1} \end{vmatrix} \right)^2 - 4 \begin{vmatrix} t_{0,0,0} & t_{0,0,1} \\ t_{1,0,0} & t_{1,0,1} \end{vmatrix} \cdot \begin{vmatrix} t_{0,1,0} & t_{0,1,1} \\ t_{1,1,0} & t_{1,1,1} \end{vmatrix}$$

   the output is T *belongs to case (b) of Theorem 2.6.1 therefore it is not identifiable.* Otherwise the output is T *is an identifiable rank-2 tensor.*

2. Case $(n_1, n_2, n_3) = (2, 2, 3)$ (Remark that we already know that T is not identifiable, so we only need to classify it).
Compute the Kronecker normal form of T.

   - If the Kronecker normal form of T is
$$\begin{bmatrix} \lambda & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$
then T is either as in Example 2.3.10 or as in Example 2.3.15 and the output is T *belongs to either case (e) or (f) of Theorem 2.6.1, therefore it is not identifiable.*
   - Else, T is as in Example 2.3.8 and the output is T *belongs to case (d) of Theorem 2.6.1 and it is not identifiable.*

3. Case $(n_1, n_2, n_3) = (2, 3, 3)$.
Compute the normal form of T.

   - If the Kronecker normal form of T is
$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$
then the output is T *belongs to case (f) of Theorem 2.6.1, therefore it is not identifiable.*
   - Else the output will be the rank of T computed via (2.7.15) of Theorem 2.7.19 and T *is not on the list of non-identifiable rank-3 tensors.*

4. Otherwise $(n_1, n_2, n_3) = (3, 3, 3)$ and the output is T *is not on the list of non-identifiable rank-3 tensors.*

2.7.2.2 More than three factors

Before proceeding, we need to recall some results on secant varieties of Segre varieties that will be useful in the sequel.

Given a tensor $T \in \mathcal{T}_{n_1,\dots,n_k}$, the j-th flattening of T is a linear map $\varphi_j \colon (\mathbb{C}^{n_j})^* \to \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_j - 1} \otimes \mathbb{C}^{n_j + 1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ (cf. Definition 1.1.9). Note that the j-th flattening is referred to the partition $\{1, \dots, k\} = \{j\} \cup \{1, \dots, j-1, j+1, \dots, k\}$. We can generalize this notion as follows.

Definition 2.7.32. Let $T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$ and let $I, J \subset \{1, \dots, k\}$ be set of indices partitioning $\{1, \dots, k\}$. Fix a partition $I \cup J = \{1, \dots, k\}$, the *generalized flattening* of T is the linear map
$$\varphi_{I,J} \colon \bigotimes_{i \in I} (\mathbb{C}^{n_i})^* \to \bigotimes_{j \in J} \mathbb{C}^{n_j}.$$

Minors of generalized flattening are useful to compute equations of some secant varieties.

Remark 2.7.33. The equations of $\sigma_2(X_{n_1,\dots,n_k})$ for some Segre variety $X_{n_1,\dots,n_k}$ are given by the 3×3 minors of all generalized flattenings (cf. [LM04]).

Let us focus on the third secant variety of a given projective variety. In order to recall set theoretic defining equations of this variety, we need to recall the Strassen's equations.

Let A, B, C be $\mathbb{C}$-vector spaces of finite dimensions and let $T \in A \otimes B \otimes C$, i.e. $T\colon B^* \to A \otimes C$. Consider the linear map

$$Id_A \otimes T\colon A \otimes C \to A \otimes A \otimes C$$

and the projection

$$\pi\colon A \otimes A \to \bigwedge^2 A.$$

By composing the two maps we get

$$T^{\wedge}_{BA}\colon A \otimes B^* \to \bigwedge^2 A \otimes C.$$

Strassen's equations are given by minors of the map $T^{\wedge}_{BA}$.

Theorem 2.7.34 ([Qi13, Theorem 1.4])**.** *1 The third secant variety of the Segre product of k projective spaces $\sigma_3(X_{n_1,\dots,n_k})$ is set theoretically defined by Strassen's equations for all partitions $I \cup J \cup K = \{1,\dots,k\}$ and all 4×4 minors of generalized flattenings. More precisely, when $n_i \leq 2$ for each i, $\sigma_3(X_{n_1,\dots,n_k})$ is set theoretically defined by Strassen's equations of degree 4 for the partitions $\{i\} \cup \{j\} \cup \{1,\dots,\hat{i},\dots,\hat{j},\dots,k\}$ and all 4×4 minors of generalized flattenings.*

Moreover we recall the following result on the stratification by rank of the third secant variety of a Segre variety.

Theorem 2.7.35 ([BB19, Theorem 1.8])**.** *Let $X_{n_1,\dots,n_k}$ be a Segre variety with $k \geq 3$ factors. Denote by α the cardinality of $\{i \in \{1,\dots,k\} \mid n_i \geq 2\}$. Then for any $x \in \{3,\dots,\alpha+k-1\}$ there is an element $p \in \sigma_3(X_{n_1,\dots,n_k}) \setminus \sigma_2(X_{n_1,\dots,n_k})$ with $r(p) = x$.*

For completeness, since it will be mentioned in the sequel we also recall how to find equations for the tangential variety of a Segre variety. Given a tensor space $V_1 \otimes \cdots \otimes V_k$, these equations arise by looking at the decomposition of $Sym^n(V_1 \otimes \cdots \otimes V_k)$ in irreducibles $(GL(V_1) \times \cdots \times GL(V_k))$-modules. Given a partition $\lambda = (\lambda_1,\dots,\lambda_p)$ of an integer n and given a vector space V, we denote by $S_\lambda V$ the Schur module associated to λ. We refer to [Lan12, Ch. 6] for a detailed exposition of these notions.

Theorem 2.7.36 ([Oed, Theorem 1.3])**.** *The tangential variety $\tau(\nu(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_k^*))$ is cut out set-theoretically by the cubics in $Sym^3(V_1 \otimes \cdots \otimes V_k)$ with four $S_{(2,1)}$ factors and all other factors $S_{(3,0)}$, and the quartics in $Sym^4(V_1 \otimes \cdots \otimes V_k)$ with three $S_{(2,2)}$'s and all other factors $S_{(4,0)}$.*

We are now ready to develop the case in which a concise tensor space of a tensor has more than 3 factors, i.e.

$$\mathcal{T}_{n_1,\dots,n_k} = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$$

where $k > 3$ and all $n_i \in \{2,3\}$. We will first treat the case in which $k = 4$ and $n_1 = n_2 = n_3 = n_4 = 2$ and then we will treat all together the remaining cases.

Non-identifiable tensors with at least 4 factors

Consider for the moment the 4-factors case, i.e.

$$\mathcal{T}_{n_1,n_2,n_3,n_4} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \otimes \mathbb{C}^{n_4},$$

where all $n_i \in \{2,3\}$. Following our classification theorem (cf. Theorem 2.6.1), working with 4 factors there are only two families of non-identifiable tensors, namely items (c) and (f) of Theorem 2.6.1. Case (f) is referred to non-identifiable rank-3 tensors of Proposition 2.3.14 adapted to the 4-factors case, while case (c) contains any rank-3 tensor in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Let us first treat the case of $\mathcal{T}_{2^4} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

$\boxed{\mathcal{T}_{2^4} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2}$

We recall that the third secant variety of the Segre variety X_{1^4} is defective (cf. Remark 2.3.3). Moreover, by Theorem 2.7.35, any element of $\sigma_3(X_{1^4}) \setminus \sigma_2(X_{1^4})$ is a rank-3 tensor. Therefore any tensor in $\sigma_3(X_{1^4}) \setminus \sigma_2(X_{1^4})$ is a non-identifiable rank-3 tensor.

Thus, working over $\mathcal{T}_{2,2,2,2}$, to detect whether a given tensor $T \in \mathcal{T}_{2,2,2,2}$ is a non-identifiable rank-3 tensor it is sufficient to verify if $[T] \in \sigma_3(X_{1^4}) \setminus \sigma_2(X_{1^4})$, i.e. if T satisfies the equations of $\sigma_3(X_{1^4})$ given in Theorem 2.7.34 and T does not satisfies the equations of $\sigma_2(X_{1^4})$ given in Remark 2.7.33.

$\boxed{\mathcal{T}_{n_1,\dots,n_k} \neq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \text{ , with } k \geq 4,\ n_i = 2,3 \text{ for all } i = 1,\dots,k}$

Let now $k \geq 4$ with $\mathcal{T}_{n_1,\dots,n_k} \neq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. In this case, any non-identifiable rank-3 tensor comes from case (f) of Theorem 2.6.1. More precisely, let

$$Y' := \mathbb{P}^1 \times \mathbb{P}^1 \times \{u_3\} \times \cdots \times \{u_k\} \subset Y_{m_1,m_2,1^{k-2}} = \mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1,$$

with $m_1, m_2 \in \{1,2\}$. Let $q' \in \langle \nu(Y') \rangle \setminus \nu(Y_{m_1,m_2,1^{k-2}})$ and $p \in Y_{m_1,m_2,1^{k-2}} \setminus Y'$. We saw that any $[T] \in \langle q', \nu(p) \rangle$ is a non-identifiable rank-3 tensor. Let $\{u_i, \tilde{u}_i\}$ be a basis of the $\mathbb{C}^{n_i}$ arising from the i-th factor of $Y_{m_1,m_2,1^{k-2}}$ for all $i \geq 3$. Take distinct $a_1, a_2 \in \mathbb{C}^{m_1+1}$ and distinct $b_1, b_2 \in \mathbb{C}^{m_2+1}$ and if $m_1 = 1$ then let $a_3 \in \langle a_1, a_2 \rangle$ otherwise we let a_1, a_2, a_3 form a basis of the first factor. Let $b_3 \in \langle b_1, b_2 \rangle$ if $m_2 = 1$, otherwise b_1, b_2, b_3 form a basis of the second factor. With respect to these bases T can be written as

$$T = (a_1 \otimes b_1 + a_2 \otimes b_2) \otimes u_3 \otimes \cdots \otimes u_k + a_3 \otimes b_3 \otimes \tilde{u}_3 \otimes \cdots \otimes \tilde{u}_k. \tag{2.7.18}$$

Since the only type of tensors that we have to detect correspond to (2.7.18), we may restrict ourselves to consider the following tensor spaces:

- $\mathcal{T}_{3,2^{k-1}} = \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$;
- $\mathcal{T}_{3,3,2^{k-2}} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$;
- $\mathcal{T}_{2^k} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (with $k \geq 5$).

Definition 2.7.37. Let $\mathcal{T}_{n_1,\dots,n_k} = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$, fix integer $k' \leq k$ and let $I = \cup_{i=1}^{k'} I_i$ be a partition of $\{1,\dots,k\}$. A *reshape* of $\mathcal{T}$ of type $I_1,\dots,I_{k'}$ is a bijection

$$\vartheta_{I_1,\dots,I_{k'}} : \mathcal{T}_{n_1,\dots,n_k} \longrightarrow \mathbb{C}^{N_1} \otimes \cdots \otimes \mathbb{C}^{N_{k'}},$$

where $\mathbb{C}^{N_i} = \bigotimes_{j \in I_i} \mathbb{C}^{n_j}$ for all $i = 1,\dots,k'$, i.e. $N_i = \prod_{j \in I_i} n_i$ and $\mathbb{C}^{N_i}$ is the vectorization of $\bigotimes_{j \in I_i} \mathbb{C}^{n_j}$.

In other words a reshape of a tensor space $\mathcal{T}_{n_1,\dots,n_k}$ is a different way of grouping together some of the factors of $\mathcal{T}_{n_1,\dots,n_k}$ (eventually it is also necessary to reorder the factors of $\mathcal{T}_{n_1,\dots,n_k}$).

In the following we will be interested in the reshape grouping together two factors of a tensor space $\mathcal{T}_{n_1,\dots,n_k}$ and we will denote by $\vartheta_{i,j}$ the corresponding map for some $i,j = 1,\dots,k$, i.e.

$$\vartheta_{i,j}\colon \mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_k} \xrightarrow{\sim} (\mathbb{C}^{n_i}\otimes\mathbb{C}^{n_j})\otimes\mathbb{C}^{n_1}\otimes\cdots\otimes\widehat{\mathbb{C}^{n_i}}\otimes\cdots\otimes\widehat{\mathbb{C}^{n_j}}\otimes\cdots\otimes\mathbb{C}^{n_k}.$$

Example 2.7.38. Let $\mathcal{T}_{n_1,\dots,n_k} = \mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_k}$ and denote by $\vartheta_{1,2}$ the reshape grouping together the first two factors of $\mathcal{T}_{n_1,\dots,n_k}$

$$\vartheta_{1,2} : \mathcal{T}_{n_1,\dots,n_k} \longrightarrow (\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2})\otimes\mathbb{C}^{n_3}\otimes\cdots\otimes\mathbb{C}^{n_k}$$

$$T = \sum_{\substack{i_1,\dots,i_k\\ i_j=1,\dots,n_j, j=1,\dots,k}} t_{i_1,\dots,i_k} e_{i_1}\otimes\cdots\otimes e_{i_k} \mapsto \sum_{\substack{i_1,\dots,i_k\\ i_j=1,\dots,n_j, j=1,\dots,k}} t_{i_1,\dots,i_l}(e_{i_1}\otimes e_{i_2})\otimes e_{i_3}\otimes\cdots\otimes e_{i_k}.$$

Since $\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}\cong\mathbb{C}^{n_1n_2}$, by sending the basis $\{e_{i_1}\otimes e_{i_2}\}_{i_1=1,\dots,n_1,i_2=1,\dots,n_2}$ of $\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}$ to the basis $\{e_1,\dots,e_{n_1n_2}\}$ of $\mathbb{C}^{n_1n_2}$, then we write

$$\vartheta_{1,2}(T) = \sum_{\substack{j_1,i_3,\dots,i_k\\ j_1=1,\dots,n_1n_2,\ i_\ell=1,\dots,n_\ell,\ell=3,\dots,k}} t_{j_1,i_3,\dots,i_k} e_{j_1}\otimes e_{i_3}\otimes\cdots\otimes e_{i_k} \in \mathbb{C}^{n_1n_2}\otimes\mathbb{C}^{n_3}\otimes\cdots\otimes\mathbb{C}^{n_k}.$$

The following lemma tells us how to completely characterize non-identifiable rank-3 tensors lying on either $\mathcal{T}_{3,2^{k-1}}$ or $\mathcal{T}_{3,3,2^{k-2}}$ or $\mathcal{T}_{2^k}$.

Lemma 2.7.39. *Let $T\in\mathcal{T}_{n_1,n_2,2^{k-2}} = \mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}\otimes\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2$ be a concise tensor in $\mathcal{T}_{n_1,n_2,2^{k-2}}$, where $n_1,n_2\in\{2,3\}$, $k\geq 4$ and $\mathcal{T}_{n_1,n_2,2^{k-2}}\neq\mathcal{T}_{2^4}$. Then T is as in case (f) of Theorem 2.6.1 if and only if the following conditions hold:*

1. *the reshaped tensor $\vartheta_{1,2}(T)\in\mathbb{C}^{n_1n_2}\otimes(\mathbb{C}^2)^{\otimes(k-2)}$ is an identifiable rank-2 tensor with respect to $\mathbb{C}^{n_1n_2}\otimes(\mathbb{C}^2)^{\otimes(k-2)}$*

$$\vartheta_{1,2}(T) = T_1+T_2 = x\otimes u_3\otimes\cdots\otimes u_k + y\otimes v_3\otimes\cdots\otimes v_k \in \mathbb{C}^{n_1n_2}\otimes(\mathbb{C}^2)^{\otimes(k-2)}$$

for some independent $x,y\in\mathbb{C}^{n_1n_2}$ and some $u_i,v_i\in\mathbb{C}^2$ with u_i,v_i linearly independent for all $i=3,\dots,k$;

2. *looking at $x,y\in\mathbb{C}^{n_1n_2}$ as elements of $\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}$ then $\{r(x),r(y)\}=\{1,2\}$.*

Proof. Let $T\in\mathcal{T}_{n_1,n_2,2^{k-2}}$ be as in case (f) of Theorem 2.6.1, so T can be written as

$$T = a_1\otimes b_1\otimes u_3\otimes\cdots\otimes u_k + a_2\otimes b_2\otimes u_3\otimes\cdots\otimes u_k + a_3\otimes b_3\otimes v_3\otimes\cdots\otimes v_k,$$

where $u_i\neq v_i$ for all $i=3,\dots,k$, a_1,a_2,a_3 are linearly independent if $n_1=3$ and b_1,b_2,b_3 are linearly independent if $n_2=3$. Let $\vartheta_{1,2}$ be the reshape grouping together the first two factors of $\mathcal{T}_{n_1,\dots,n_k}$. Let $x := a_1\otimes b_1, y := a_2\otimes b_2$ and $z := a_3\otimes b_3$ and remark that $r(x+y)=2$ and $r(z)=1$. Therefore

$$\begin{aligned}\vartheta_{1,2}(T) =& x\otimes u_3\otimes\cdots\otimes u_k + y\otimes u_3\otimes\cdots\otimes u_k + z\otimes v_3\otimes\cdots\otimes v_k\\ =&(x+y)\otimes u_3\otimes\cdots\otimes u_k + z\otimes v_3\otimes\cdots\otimes v_k\\ =&T_1+T_2\in\mathbb{C}^{n_1n_2}\otimes\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2.\end{aligned}$$

Note that the rank of $(T_1+T_2) \in \mathcal{T}_{n_1n_2,2^{k-2}}$ is at most 2 and in fact $r(T_1+T_2)=2$ since u_i, v_i are linearly independent for all $i=3,\dots,k$. Moreover, we recall that the only non-identifiable rank-2 tensors are matrices (cf. Proposition 2.2.7). Therefore, since the concise tensor space of T_1+T_2 is made by at least 3 factors, then T_1+T_2 is an identifiable rank-2 tensor.

Viceversa let $T \in \mathcal{T}_{n_1,n_2,2^{k-2}}$ such that $\vartheta_{1,2}(T) \in \mathbb{C}^{n_1n_2} \otimes (\mathbb{C}^2)^{k-2}$ is an identifiable rank-2 tensor

$$\vartheta_{1,2}(T) = T_1 + T_2 = a \otimes u_3 \otimes \cdots \otimes u_k + b \otimes v_3 \otimes \cdots \otimes v_k,$$

for some unique $a, b \in \mathbb{C}^{n_1n_2}$ with $\langle a, b\rangle \cong \mathbb{C}^2$ and unique $u_i, v_i \in \mathbb{C}^2$ with $\langle u_i, v_i\rangle \cong \mathbb{C}^2$ for all $i=3,\dots,k$. By assumption $a, b \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ are such that $\{r(a), r(b)\} = \{1,2\}$ and by relabeling if necessary we may assume $r(a)=2$ and $r(b)=1$.

Let us see $\vartheta_{1,2}(T)$ as an element of $\mathcal{T}_{n_1,n_2,2^{k-2}} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. Since T_2 is a rank-1 tensor, there exist $v_1 \in \mathbb{C}^{n_1}$, $v_2 \in \mathbb{C}^{n_2}$ such that $b = v_1 \otimes v_2$, i.e.

$$T_2 = v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_k.$$

Moreover, since $r(a)=2$ then there exist independent $a_1, a_2 \in \mathbb{C}^{n_1}$ and independent $b_1, b_2 \in \mathbb{C}^{n_2}$ such that $a = a_1 \otimes b_1 + a_2 \otimes b_2$, i.e.

$$T_1 = a_1 \otimes b_1 \otimes u_3 \otimes \cdots \otimes u_k + a_2 \otimes b_2 \otimes u_3 \otimes \cdots \otimes u_k.$$

We remark that the concise space of T is $\mathcal{T}_{n_1,n_2,2^{k-2}}$, therefore if $n_1=3$ (or $n_2=3$) then a_1, a_2, v_1 are linearly independent (b_1, b_2, v_2 are linearly independent). Thus T is as in case (f). $\square$

Remark 2.7.40. In Lemma 2.7.39 we assumed that dealing with a tensor as in (2.7.18) the non-identifiable part of the tensor was in the first two factors because it is always possible to permute the factors of the tensor space in this way. This assumption cannot be made in the algorithm and we have to be careful if either $(n_1,n_2)=(3,2)$ or $(n_1,n_2)=(2,2)$. More precisely, for all $i,j=1,\dots,k$ denote by

$$\vartheta_{i,j}\colon \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k} \xrightarrow{\sim} (\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}) \otimes \mathbb{C}^{n_1} \otimes \cdots \otimes \widehat{\mathbb{C}^{n_i}} \otimes \cdots \otimes \widehat{\mathbb{C}^{n_j}} \otimes \cdots \otimes \mathbb{C}^{n_k}$$

the reshape grouping together the i-th and j-th factor of $\mathcal{T}_{n_1,\dots,n_k}$.
Dealing with $(n_1,n_2)=(3,2)$, we have to check if there exists $i=2,\dots,k$ such that $\vartheta_{1,i}(T)$ satisfies the conditions of Lemma 2.7.39.
Similarly, for the case of $(n_1,n_2)=(2,2)$ we have to check all reshaping of T if necessary, i.e. we have to check if there exist $i,j \in \{1,\dots,k\}$ with $i \neq j$ such that $\vartheta_{i,j}(T)$ satisfies the conditions of Lemma 2.7.39.

Recall that a concise tensor $T \in \mathbb{C}^{n_1n_2} \otimes (\mathbb{C}^2)^{\otimes(k-2)}$ is an element of $\sigma_2(X_{(n_1n_2-1),1^{k-2}}) \setminus \tau(X_{(n_2n_2-1),1^{k-2}})$ if and only if there is a specific change of basis on each factors $\tilde{g} = (g, g_3, \dots, g_k) \in GL_{n_1n_2} \times GL_2 \times \cdots \times GL_2$ such that

$$\tilde{g}T = x \otimes u_3 \otimes \cdots u_k + y \otimes v_3 \otimes \cdots \otimes v_k. \tag{2.7.19}$$

By Lemma 2.7.39, given an identifiable rank-2 tensor $T \in \mathcal{T}_{n_1n_2,2^{k-2}}$, in order to verify if T is as in case (f), we do not need to find an explicit decomposition of T as in (2.7.19) but it is enough made the following steps:

- distinguish $x, y \in \mathbb{C}^{n_1n_2}$ and look at them as elements of $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$;
- prove that either $r(x)=2$ and $r(y)=1$ or that $r(x)=1$ and $r(y)=2$.

Let us explain in detail how to do so.

Reshape procedure for an identifiable rank-2 tensor of $\mathcal{T}_{n_1n_2,2^{k-2}}$ (how to find $x, y \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$)

Let T be an identifiable rank-2 tensor in $\mathcal{T}_{n_1n_2,2^{k-2}} = \mathbb{C}^{n_1n_2} \otimes (\mathbb{C}^2)^{\otimes(k-2)}$. Remark that the first factor of $\mathcal{T}_{n_1n_2,2^{k-2}}$ is not concise. Indeed the rank of the first flattening $\varphi_1\colon (\mathbb{C}^2)^{\otimes(k-2)} \to (\mathbb{C}^{n_1n_2})^*$ of T is $r(\varphi_1) = 2$ and to complete the concision process, we can take as basis of the first new factor two independent elements $\widehat{x}, \widehat{y}$ of $\mathrm{Im}(\varphi_1)$. Therefore T can be written as

$$T = \widehat{x} \otimes u_3 \otimes \cdots \otimes u_k + \widehat{y} \otimes v_3 \otimes \cdots \otimes v_k \in \mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes k}.$$

If we reshape our tensor space by grouping together all factors from the 4-th one onwards, then we look at T as

$$\begin{aligned} \widehat{x} \otimes u_3 \otimes \overbrace{(u_4 \otimes \cdots \otimes u_k)}^{\widehat{u}} &+ \widehat{y} \otimes v_3 \otimes \overbrace{(v_4 \otimes \cdots \otimes v_k)}^{\widehat{v}} = \\ &\widehat{x} \otimes u_3 \otimes \widehat{u} + \widehat{y} \otimes v_3 \otimes \widehat{v} \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes ((\mathbb{C}^2)^{\otimes(k-3)}) \end{aligned}$$

We want to look at this three factors tensor as a pencil of matrices with respect to the second factor of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes(k-3)}$. Let $u_3 = (u_{3,1}, u_{3,2})$, $v_3 = (v_{3,1}, v_{3,2})$ and denote by

$$C_1 := \begin{bmatrix} u_{3,1}\widehat{x} \otimes \widehat{u} \\ v_{3,1}\widehat{y} \otimes \widehat{v} \end{bmatrix},\ C_2 := \begin{bmatrix} u_{3,2}\widehat{x} \otimes \widehat{u} \\ v_{3,2}\widehat{y} \otimes \widehat{v} \end{bmatrix} \in \mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes(k-3)}.$$

We can write T as

$$C_1\lambda + C_2\mu.$$

Call X_3 the matrix whose columns are given by $\widehat{x}$ and $\widehat{y}$ and denote by X_4 the matrix whose rows are given by $\widehat{u}$ and $\widehat{v}$. Therefore

$$\begin{aligned} C_1 &= \begin{bmatrix} \widehat{x} & \widehat{y} \end{bmatrix} \begin{bmatrix} u_{3,1} & 0 \\ 0 & v_{3,1} \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widehat{v} \end{bmatrix} = X_3 \begin{bmatrix} u_{3,1} & 0 \\ 0 & v_{3,1} \end{bmatrix} X_4, \\ C_2 &= \begin{bmatrix} \widehat{x} & \widehat{y} \end{bmatrix} \begin{bmatrix} u_{3,2} & 0 \\ 0 & v_{3,2} \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \widehat{v} \end{bmatrix} = X_3 \begin{bmatrix} u_{3,2} & 0 \\ 0 & v_{3,2} \end{bmatrix} X_4. \end{aligned}$$

Remark that C_2 is right invertible and denote by C_2^{-1} its right inverse. Moreover $r(X_3) = r(X_4) = 2$, therefore X_3 is invertible and there exists a right inverse of X_4 that we denote by X_4^{-1}. Thus

$$\begin{aligned} C_1C_2^{-1} &= \left(X_3 \begin{bmatrix} u_{3,1} & 0 \\ 0 & v_{3,1} \end{bmatrix} X_4 \right) \left(X_3 \begin{bmatrix} u_{3,2} & 0 \\ 0 & v_{3,2} \end{bmatrix} X_4 \right)^{-1} \\ &= X_3 \begin{bmatrix} \frac{u_{3,1}}{u_{3,2}} & 0 \\ 0 & \frac{v_{3,1}}{v_{3,2}} \end{bmatrix} X_3^{-1}. \end{aligned}$$

We have now an eigenvalue problem that we can easily solve to find $\widehat{x}, \widehat{y} \in \mathbb{C}^2$.

Remark 2.7.41. When computing the concision process of T with respect to the first factor of $\mathcal{T}_{n_1n_2,2^{k-2}}$, we concretely find a basis of $\mathrm{Im}\varphi_1$. Therefore, after we found $\widehat{x}, \widehat{y} \in \mathbb{C}^2$ with the above procedure, we can easily get back to $x, y \in \mathbb{C}^{n_1n_2} \cong \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ and compute the rank of both x, y seen as elements of $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$.

Let us sum up how to find a non-identifiable rank-3 tensor of at least 4 factors in the following algorithm.

Algorithm 2 (Non-identifiability with at least 4 factors)
Input: Concise tensor $T = (t_{i_1,\dots,i_k}) \in \mathcal{T}_{n_1,n_2,2^{k-2}}$, for some $k > 3$, $2 \le n_1, n_2 \le 3$.
Output: A statement on whether T belongs to one of the six cases of non-identifiable rank-3 tensors or not.

0. For all $i, j = 1, \dots, k$ with $i \neq j$ denote by $\vartheta_{i,j}$ the reshape grouping the i-th and j-th factor of $\mathcal{T}_{n_1,\dots,n_k}$.

1. Case $(n_1, n_2) = (2, 2)$.

   - Case $k = 4$. Test if $T \in \sigma_3(X_{1^4}) \setminus \sigma_2(X_{1^4})$, i.e. if T satisfies the equations of $\sigma_3(X_{1^4})$ given in Theorem 2.7.34 and T does not satisfy the equations of $\sigma_2(X_{1^4})$ given in Remark 2.7.33. In this case the output is *T is a non-identifiable rank-3 tensor*, otherwise the output is *T is not on the list of non-identifiable rank-3 tensors.*
   - Case $k \ge 5$. For all $i = 1, \dots, k-1$ and for all $j = i+1 \dots, k$ follow this procedure.
     - Test if $\vartheta_{i,j}(T)$ satisfies the equations of $\sigma_2(X_{3,1^{k-2}})$ and does not satisfy the equations of $\tau(X_{3,^{k-2}})$ (cf. Remark 2.7.33 and Theorem 2.7.36). If $\vartheta_{i,j}(T) \in \sigma_2(X_{3,1^{k-2}}) \setminus \tau(X_{3,1^{k-2}})$ then $\vartheta_{i,j}(T)$ is an identifiable rank-2 tensor. Make the concision process on the first factor of $\mathcal{T}_{3,1^{k-2}}$ and call T' the resulting tensor. Consider T' as a matrix pencil of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes ((\mathbb{C}^2)^{\otimes(k-2)})$ with respect to the second factor
     $$T' = \lambda C_1 + \mu C_2.$$
     Find the eigenvectors $x, y \in \mathbb{C}^2$ of $C_1 C_2^{-1}$ and then rewrite x, y as elements of $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$. If $\{r(x), r(y)\} = \{1, 2\}$ then the output is *T is a non-identifiable rank-3 tensor.*
     - Else, if one of the previous conditions is not satisfied, then stop and restart with another j (and another i when necessary).

   If the algorithm stops at some point when $i = k-1, j = k$ then stop and the output is *T is not on the list of non-identifiable rank-3 tensors.*

2. Case $(n_1, n_2) = (3, 2)$.
   For all $i = 2, \dots, k-1$ follow this procedure:

   - Test if $\vartheta_{1,i}(T)$ satisfies the equations of $\sigma_2(X_{5,1^{k-2}})$ and does not satisfy the equations of $\tau(X_{5,^{k-2}})$ (cf. Remark 2.7.33 and Theorem 2.7.36). If $\vartheta_{1,i}(T) \in \sigma_2(X_{5,1^{k-2}}) \setminus \tau(X_{5,1^{k-2}})$ then $\vartheta_{1,i}(T)$ is an identifiable rank-2 tensor. Reduce the first factor of $\mathcal{T}_{6,2^{k-2}}$ via concision, working now on $\mathcal{T}_{2^{k-1}}$ with T'. Consider T' as a matrix pencil with respect to the second factor of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes(k-3)}$, i.e.
   $$T' = \lambda C_1 + \mu C_2.$$
   Find the eigenvectors x, y of $C_1 C_2^{-1}$ and then rewrite x, y as elements of $\mathbb{C}^6 = \mathbb{C}^3 \otimes \mathbb{C}^2$. If $\{r(x), r(y)\} = \{2, 1\}$ the output is *T is a non-identifiable rank-3 tensor.*
   - Else, if one of the previous conditions is not satisfied then stop and restart with another i.

If the algorithm stops at some point when $i = k$ then stop and the output is T *is not on the list of non-identifiable rank-3 tensors.*

3. Case $(n_1, n_2) = (3, 3)$.

   - Test if $\vartheta_{1,2}(T)$ satisfies the equations of $\sigma_2(X_{8,1^{k-2}})$ and does not satisfy the equations of $\tau(X_{8,^{k-2}})$ (cf. Remark 2.7.33 and Theorem 2.7.36). If $\vartheta_{1,2}(T) \in \sigma_2(X_{8,1^{k-2}}) \setminus \tau(X_{8,1^{k-2}})$ then $\vartheta_{1,2}(T)$ is an identifiable rank-2 tensor. Reduce the first factor of $\mathcal{T}_{9,2^{k-2}}$ via the concision process, working now with T' on $(\mathbb{C}^2)^{\otimes(k-1)}$. Consider T' as a matrix pencil with respect to the second factor of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes(k-3)}$, i.e.
   $$T' = \lambda C_1 + \mu C_2.$$
   Find the eigenvectors x, y of $C_1 C_2^{-1}$ and then rewrite x, y as elements of $\mathbb{C}^9 \cong \mathbb{C}^3 \otimes \mathbb{C}^3$. If $\{r(x), r(y)\} = \{1, 2\}$ the output is T *is a non-idenfitiable rank-3 tensor as in case (f).*
   - If one of these conditions is not satisfied then stop and the output is T *is not on the list of non-identifiable rank-3 tensors.*

Example 2.7.42. Let $\mathcal{T}_{3,2,2,2} = \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and for all $j, k, \ell = 1, 2$ and for all $i = 1, 2, 3$ denote by $e_{i,j,k,\ell} = e_i \otimes e_j \otimes e_k \otimes e_\ell$. Similarly, we set $e_{i,j} = e_i \otimes e_j$. Consider the tensor

$$\begin{aligned} T =& 12e_{1,1,1,1} + 8e_{1,1,1,2} + 6e_{1,1,2,1} + 4e_{1,1,2,2} + 30e_{1,2,1,1} + 20e_{1,2,1,2} + 15e_{1,2,2,1} + \\ & 10e_{1,2,2,2} + 8e_{2,1,1,1} + 8e_{2,1,1,2} + 5e_{2,1,2,1} + 6e_{2,1,2,2} + 35e_{2,2,1,1} 38e_{2,2,1,2} + \\ & 23e_{2,2,2,1} + 30e_{2,2,2,2} + 16e_{3,1,1,1} + 16e_{3,1,1,2} + 10e_{3,1,2,1} + 12e_{3,1,2,2} + \\ & 52e_{3,2,1,1} + 64e_{3,2,1,2} + 37e_{3,2,2,1} + 54e_{3,2,2,2}. \end{aligned}$$

Let $\vartheta_{1,2} \colon \mathcal{T}_{3,2,2,2} \to \mathbb{C}^6 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ be the reshape grouping together the first two factors of $\mathcal{T}_{3,2,2,2}$. Let

$$f_1 = e_{1,1},\ f_2 = e_{1,2},\ f_3 = e_{2,1},\ f_4 = e_{2,2},\ f_5 = e_{3,1},\ f_6 = e_{3,2}$$

be a basis of $\mathbb{C}^6$ such that $\vartheta_{1,2}(T)$ can be written as

$$\begin{aligned} \vartheta_{1,2}(T) =& 12f_1 \otimes e_{1,1} + 8f_1 \otimes e_{1,2} + 6f_1 \otimes e_{2,1} + 4f_1 \otimes e_{2,2} + 30f_2 \otimes e_{1,1} + 20f_2 \otimes e_{1,2} + \\ & 15f_2 \otimes e_{2,1} + 10f_2 \otimes e_{2,2} + 8f_3 \otimes e_{1,1} + 8f_3 \otimes e_{1,2} + 5f_3 \otimes e_{2,1} + 6f_3 \otimes e_{2,2} + \\ & 35f_4 \otimes e_{1,1} + 38f_4 \otimes e_{1,2} + 23f_4 \otimes e_{2,1} + 30f_4 \otimes e_{2,2} + 16f_5 \otimes e_{1,1} + 16f_5 \otimes e_{1,2} + \\ & 10f_5 \otimes e_{2,1} + 12f_5 \otimes e_{2,2} + 52f_6 \otimes e_{1,1} + 64f_6 \otimes e_{1,2} + 37f_6 \otimes e_{2,1} + 54f_6 \otimes e_{2,2}. \end{aligned}$$

One can verify that $\vartheta_{1,2}(T) \in \sigma_2(X_{5,1^3}) \setminus \tau(X_{5,1^3})$, therefore we can continue our procedure by considering the matrix associated to the first flattening $\varphi_1 \colon (\mathbb{C}^2 \otimes \mathbb{C}^2)^* \to \mathbb{C}^6$ of T:

$$A = \begin{bmatrix} 12 & 8 & 6 & 4 \\ 30 & 20 & 15 & 10 \\ 8 & 8 & 5 & 6 \\ 35 & 38 & 23 & 30 \\ 16 & 16 & 10 & 12 \\ 52 & 64 & 37 & 54 \end{bmatrix}.$$

The rank of A is 2 and we take the first two columns $\widehat{x}, \widehat{y}$ of A as linearly independent vectors of $Im(\varphi_1)$ and rewrite all the others as a linear combinations of $\widehat{x}, \widehat{y}$. Denote by T' the resulting tensor

$$T' = \widehat{x} \otimes e_{1,1} + \widehat{y} \otimes e_{1,2} + \left(\frac{1}{4}\widehat{x} + \frac{3}{8}\widehat{y}\right) \otimes e_{2,1} + \left(-\frac{1}{2}\widehat{x} + \frac{5}{4}\widehat{y}\right) \otimes e_{2,2}.$$

Let us consider now $T' \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ as a matrix pencil with respect to the second factor

$$T' = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mu \begin{bmatrix} 1/4 & -1/2 \\ 3/8 & 5/4 \end{bmatrix} = \lambda C_1 + \mu C_2.$$

It is easy to see that the eigenvectors of

$$C_1 C_2^{-1} = \begin{bmatrix} 10/4 & 1 \\ -3/4 & 1/2 \end{bmatrix}$$

are $x = (-2, 1)$ and $y = (-2/3, 1)$, i.e.

$$\begin{aligned} x = &- 2\widehat{x} + \widehat{y} = -16f_1 - 40f_2 - 8f_3 - 32f_4 - 16f_5 - 40f_6 = \\ &-(16e_{1,1} + 40e_{1,2} + 8e_{2,1} + 32e_{2,2} + 16e_{3,1} + 40e_{3,2}) = - \begin{bmatrix} 16 & 40 \\ 8 & 32 \\ 16 & 40 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} y = &- 2/3\widehat{x} + \widehat{y} = 8/3f_3 + 44/3f_4 + 16/3f_5 + 88/3f_6 = \\ &8/3e_{2,1} + 44/3e_{2,2} + 16/3e_{3,1} + 88/3e_{3,2} = \begin{bmatrix} 0 & 0 \\ 8/3 & 44/3 \\ 16/3 & 88/3 \end{bmatrix}. \end{aligned}$$

It is easy to see that $r(x) = 2$ and $r(y) = 1$, therefore T is a non-identifiable rank-3 tensor as in case (f). Indeed by multiplying T with

$$g = \left(\begin{bmatrix} 1/2 & -1 & 1/2 \\ 0 & 2 & -1 \\ -1/2 & 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1/3 & 1/3 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 1/2 & -1/4 \\ -1/2 & 3/4 \end{bmatrix} \right)$$

we get

$$T = e_1 \otimes e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_1 \otimes e_1 + e_3 \otimes (2e_1 + 3e_2) \otimes e_2 \otimes e_2.$$

Remark 2.7.43. Since we already considered all concise spaces of tensors related to all non-identifiable rank-3 tensors of Theorem 2.6.1, any other concise tensor space will not be considered. Therefore, for any other concise space, the output of the algorithm will be T *is not on the list of non-identifiable rank-*3 *tensors.*

To conclude, it is sufficient to collect all together the steps made until now.

Algorithm (Non-identifiable rank-3 tensors)
Input: Tensor $T = (t_{i_1,\ldots,i_k}) \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$, for some $k \geq 3$.
Output: A statement on whether T belongs to one of the six cases of non-identifiable rank-3 tensors or not.

1. Compute the concise tensor space $\mathcal{T}_{n'_1,\ldots,n'_{k'}}$ of T.

$$2. \begin{cases} \text{If } k' = 3 \text{ run Algorithm 1.} \\ \\ \text{Else if } \mathcal{T}_{n'_1,\ldots,n'_{k'}} \in \{\mathcal{T}_{3,2^{k'-1}}, \mathcal{T}_{3,3,2^{k'-2}}, \mathcal{T}_{2^{k'}}\}, \text{ where } k' \geq 4, \text{ run Algorithm 2.} \\ \\ \text{Else the output will be } T \textit{ is not on the list of Theorem 2.6.1}. \end{cases}$$

Chapter 3

Terracini Locus

The celebrated Terracini's Lemma is a well known and extremely powerful result in Algebraic Geometry that allows to compute the dimensions of r-th secant varieties of a given variety X in terms of the dimensions of the sum of tangent spaces at r generic points of X. If X is the embedding of a variety Y into a projective space via a complete linear system $\mathcal{L}$, then the codimension of the r-th secant variety of X is equal to the value $h^0(Y, I_Z \otimes \mathcal{L})$ where Z is a 0-dimensional scheme of r double generic fat points (cf. Remark 1.2.3). A first complete classification of dimensions of all secant varieties has been made for the case of Veronese varieties. The well-known Alexander-Hirschowitz Theorem treats the case in which $Y = \mathbb{P}^n$ is embedded via $\mathcal{O}(d)$ (cf. [AH95]). Another complete classification is for secant varieties of Segre-Veronese embedding of products of $Y = (\mathbb{P}^1)$'s via $\mathcal{O}(d_1, \ldots, d_k)$ due to Laface-Postinghel (cf. [LP13]). Recently Galuppi-Oneto determined dimensions of secant varieties in the case of Segre-Veronese embedding of $Y = \mathbb{P}^m \times \mathbb{P}^n$ in bidegree (d_1, d_2) for all $d_1, d_2 \geq 3$ (cf. [GO21]).
All these classifications relates the study of generic 0-dimensional schemes of double fat points, but almost nothing has been said for the case in which the 0-dimensional scheme of double fat points is not necessarily supported on generic points. Clearly if the points are not generic, the equivalence between $h^0(Y, I_Z \otimes \mathcal{L})$ and the codimension of the secant variety of X is not valid anymore. Indeed the Terracini's Lemma states that the tangent space of an r-th secant variety of a variety X at a generic point $q \in \langle p_1, \ldots, p_r \rangle$, with $p_i \in X$ generic and hence $p_i \in X_{\text{reg}}$, is equal to the span of the tangent spaces of X at p_i's, but if the p_i's are not generic one can only say that $\langle T_{p_1}X, \ldots, T_{p_r}X \rangle \subseteq T_q(\sigma_r(X))$ where $T_q(\sigma_r(X))$ is the tangent space at $q \in \langle p_1, \ldots, p_r \rangle$ of the r-th secant variety of X. This phenomenon is related to the fact that $\operatorname{codim}\langle T_{p_1}X, \ldots, T_{p_r}X \rangle = h^0(Y, I_Z \otimes \mathcal{L})$ which may be higher than the one for generic points. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and tensorize it by $\mathcal{L}$:

$$0 \longrightarrow \mathcal{I}_Z \otimes \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_Z \longrightarrow 0.$$

If $h^1(Y, \mathcal{L}) = 0$, then we get the exact sequence

$$0 \longrightarrow H^0(Y, \mathcal{I}_Z \otimes \mathcal{L}) \longrightarrow H^0(Y, \mathcal{L}) \longrightarrow H^0(Z, \mathcal{L}|_Z) \longrightarrow H^1(Y, \mathcal{I}_Z \otimes \mathcal{L}) \longrightarrow 0.$$

We will always take $(Y, \mathcal{L})$ such that $h^1(Y, \mathcal{L}) = 0$. Therefore, one gets

$$h^0(Y, \mathcal{L}) - h^0(Y, \mathcal{I}_Z \otimes \mathcal{L}) = h^0(Z, \mathcal{L}|_Z) - h^1(Y, \mathcal{I}_Z \otimes \mathcal{L})$$

that is to say that the dimension of the span of embedding of Z via $\mathcal{L}$ can be computed as

$$h^0(Z, \mathcal{L}|_Z) - h^1(\mathcal{I}_Z \otimes \mathcal{L}) - 1.$$

From this one easily sees the role played by the $h^1(\mathcal{I}_Z \otimes \mathcal{L})$ in controlling the value of $h^0(Y, \mathcal{I}_Z \otimes \mathcal{L})$.

In this chapter we fix our attention to the case of Segre varieties, i.e. the embedding of $Y_{n_1,\ldots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ via $\mathcal{O}(1,\ldots,1)$ and Z a scheme of either 2 or 3 double fat points. The key object of the discussion is the r-th *Terracini locus* that will essentially contain all the sets of r points, seen in the smallest multiprojective space possible that contains them, for which

$$h^0(\mathcal{I}_Z(1,\ldots,1))h^1(\mathcal{I}_Z(1,\ldots,1)) > 0.$$

We like to point out the geometric importance of the r-th Terracini locus. Let $(X_{n_1,\ldots,n_k})^r_{\text{reg}}$ contain all the r-uple of non-singular points of X. Consider the open part of the r-th abstract secant variety $AbSec_r(X_{n_1,\ldots,n_k})$ of a Segre variety $X_{n_1,\ldots,n_k} \subset \mathbb{P}^N$, namely

$$AbSec^0_r(X_{n_1,\ldots,n_k}) := \{(q,(p_1,\ldots,p_r)) \in \mathbb{P}^N \times (X_{n_1,\ldots,n_k})^r_{\text{reg}} \,|\, q \in \langle p_1,\ldots,p_r\rangle \cong \mathbb{P}^{r-1}\}.$$

Remark that in the definition of $AbSec^0_r(X_{n_1,\ldots,n_k})$ we only take $p_1,\ldots,p_r$ linearly independent. Consider the projection T_r of $AbSec^0_r(X_{n_1,\ldots,n_k})$ onto $\mathbb{P}^N$, namely

$$T_r : AbSec^0_r(X_{n_1,\ldots,n_k}) \longrightarrow \sigma^0_r(X_{n_1,\ldots,n_k}),$$

where we recall that

$$\sigma^0_r(X_{n_1,\ldots,n_k}) := \{q \in \mathbb{P}^N \,|\, q \in \langle p_1,\ldots,p_r\rangle \cong \mathbb{P}^{r-1}, \text{ where all } p_i \in X_{n_1,\ldots,n_k}\}.$$

The projection T_r is the r-th *Terracini map*. The differential of the r-th Terracini map is defined on each point of $AbSec^0_r(X_{n_1,\ldots,n_k})$ and the r-th Terracini locus is nothing else than a measure of the degeneracy of such a linear map. Remark that any point of $AbSec^0_r(X_{n_1,\ldots,n_k})$ is smooth since $X_{n_1,\ldots,n_k}$ is smooth.

A numerical point of view on the Terracini locus

Working with tensors coming from applied problems measurement errors may occur. Moreover, working with a machine, one is forced to use non-exact arithmetic and, even if we start with an exact tensor, round-off errors may occur due to the possibly inexact representation of the given tensor into the machine.

Therefore, when running algorithms that involve tensors, our actual input is a perturbed tensor and the error representation we are starting with may be amplified when performing algorithms.

The condition number of a function measures the rate of error that happens to the output element conditioned to a small change on the element in the domain. Moreover, a problem is said to be well conditioned if it has a small condition number and it is ill-conditioned when the condition number is very high. In this case we say that the problem is sensitive to small perturbations.

Terracini loci are involved when measuring the sensitivity of a tensor rank decomposition, also called CPD canonical polyadic decomposition (cf. [Hit27], [DDL14]).

Denote by $T_{r,(p_1,\dots,p_r)}^{-1}$ the local inverse of T_r at $(p_1,\dots,p_r) \in (X_{n_1,\dots,n_k})^r$. If the differential $\mathrm{d}_{(p_1,\dots,p_r)}T_r$ of T_r at $(p_1,\dots,p_r)$ is invertible, then a local inverse exists at $(p_1,\dots,p_r)$. Moreover, we recall that $(\mathrm{d}_{(p_1,\dots,p_r)}T_r)^{-1} = \mathrm{d}_q T_{r,(p_1,\dots,p_r)}^{-1}$. To define the condition number of a r-uple $(p_1,\dots,p_r) \in (X_{n_1,\dots,n_k})^r$, we follow the spectral characterization of [BV18, Theorem 1.1].

Denote by U_i an orthonormal basis of the affine tangent space $T_{p_i}X_{n_1,\dots,n_k}$ for all $i = 1,\dots,r$ and let $U = [U_1 \cdots U_r]$. We recall that the spectral norm $\|U\|_2$ of U is the largest singular value $\zeta(U)$ of U, i.e. $\zeta(U)$ is the square root of the biggest eigenvalue of UU^*.
If T_r is locally invertible at $(p_1,\dots,p_r)$ then

$$\|(\mathrm{d}_{(p_1,\dots,p_r)}T_r)^{-1}\|_2 = \|U^{-1}\|_2 = \zeta(U^{-1}) = \frac{1}{\min\{\lambda \mid \lambda \text{ is a singular value of } U\}}.$$

In this case they define the condition number of $(p_1,\dots,p_r)$ as

$$\kappa(p_1,\dots,p_r) := \|(\mathrm{d}_{(p_1,\dots,p_r)}T_r)^{-1}\|_2.$$

Otherwise, if $\mathrm{d}T_r$ is not invertible at $(p_1\dots,p_r)$, then U has an eigenvalue equal to 0, which is also the smallest singular value of U, and in this case we set $\kappa(p_1,\dots,p_r) = \infty$. Therefore, the *condition number* of the r-uple $(p_1,\dots,p_r)$ is

$$\kappa(p_1,\dots,p_r) := \begin{cases} \|(\mathrm{d}_{(p_1,\dots,p_r)}T_r)^{-1}\|_2 & \text{if } \mathrm{d}T_r \text{ is invertible at } (p_1,\dots,p_r), \\ \infty & \text{otherwise.} \end{cases}$$

The condition number of a tensor rank decomposition is therefore a measure of the sensitivity of the decomposition itself under errors perturbations. One would like to avoid points of $(X_{n_1,\dots,n_k})^r$ whose condition number is infinite since in these cases to a unique element $q \in \mathbb{P}^N$ correspond different r-uples in $(X_{n_1,\dots,n_k})^r$ and this behaviour generates ambiguity in the interpretations of the results when performing algorithms of tensor rank decomposition.

In [BV18], the authors defined the *ill-posed set* of a decomposition $(p_1,\dots,p_r)$ as

$$\Sigma_{\mathbb{P}} = \{(p_1,\dots,p_r) \in (X_{n_1,\dots,n_k})^r \colon \kappa(p_1,\dots,p_r) = \infty\}.$$

This locus contains precisely all r-uples $(p_1,\dots,p_r) \in (X_{n_1,\dots,n_k})^r$ for which the differential of the map T_r has not maximal rank. Therefore, the distinction between the r-th Terracini locus of a multiprojective space and the ill-posed locus $\Sigma_{\mathbb{P}}$ relies on

- considering the r rank-1 tensors as a set of points instead of a tuple;
- working with the minimal multiprojective space containing the r points.

Remark 3.0.1. Even though we will work under minimality assumption for the multiprojective space containing a set of points, the result we will achieve in this chapter are interesting from a numerical point of view since in [DBV21] they proved that the condition number of a CPD does not change under Tucker compression of the CPD itself, which is the analogous of considering the minimal multiprojective space containing both the tensor and all its possible rank decompositions.

In this chapter we will work on both the second and third Terracini locus. We will prove that the second Terracini locus is always empty for any multiprojective space, while for $r = 3$ we will completely determine all cases for which the third Terracini locus is not empty in Theorem 3.3.14. More precisely, the chapter is organized as follows.
In the first section we introduce the notation and we show that the second Terracini locus is empty. Section 3.2 is a crucial section where we show all the examples that will turn out to be the only non trivial cases in which the 3-rd Terracini locus will be non-empty. Section 3.3 is devoted to the proof of the main theorem (Theorem 3.3.14) that essentially will be a discussion on why the already highlighted examples in Subsections 3.2.1 and 3.2.3 are the only non-trivial elements in the 3-rd Terracini locus. In the last section we prove that for any multiprojective space $Y_{n_1,\dots,n_k}$ of dimension $n \geq 3$, one can always find $r \geq 3$ points $S \subset Y_{n_1,\dots,n_k}$ belonging to the corresponding r-th Terracini locus. Moreover, we compute the maximal value $\max_{n>0,r\geq 2}\{h^1(\mathcal{I}_Z(1,\dots,1)) > 0\}$, where we denoted by $Z \subset Y_{n_1,\dots,n_k}$ a zero-dimensional scheme of r double fat points. We will show that

$$h^1(\mathcal{I}_Z(1,\dots,1)) \leq (r-1)(n+1)$$

and that equality holds if and only if $Y_n = \mathbb{P}^n$. Since if $Y_n = \mathbb{P}^n$ then $h^0(\mathbb{P}^n, \mathcal{I}_{\mathbb{P}^n}(1)) = 0$, we compute the maximal value of such dimension providing that also $h^0(\mathcal{I}_Z(1,\dots,1)) > 0$. A preprint [BBS20b] has already been extracted from this work.

3.1 Introduction to the problem and 2-nd Terracini locus

In this section we set up the framework of the present chapter. First, we introduce a useful notation for the integer $h^1(\mathcal{I}_Z(1,\dots,1))$, where $Z \subset Y_{n_1,\dots,n_k}$ is a zero-dimensional scheme of some multiprojective space $Y_{n_1,\dots,n_k}$. Then, we define the minimal multiprojective space containing a given set of points and lastly, we define the r-th Terracini locus of a given multiprojective space and we describe it in the case $r = 2$.

Notation 3.1.1. Let $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, for some $k \geq 1$ and all $n_i > 0$. For any zero-dimensional scheme $Z \subset Y_{n_1,\dots,n_k}$ set

$$\delta(Z, Y_{n_1,\dots,n_k}) := h^1(\mathcal{I}_Z(1,\dots,1)).$$

Let $W \subseteq Y_{n_1,\dots,n_k}$ be a multiprojective subspace, i.e. $W = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}$, where $0 \leq m_i \leq n_i$ for all $i = 1,\dots,k$. If $Z \subset W$, set $\delta(Z, W) := h^1(W, \mathcal{I}_{Z,W}(1,\dots,1))$. We remark that $\delta(Z, W) = \delta(Z, Y_{n_1,\dots,n_k})$ since $h^i(\mathcal{I}_W(1,\dots,1)) = 0$ for $i = 1, 2$.

For the specific case of double fat points, $\delta(2S, Y_{n_1,\dots,n_k})$ will be called the *Terracini defect* of S in $Y_{n_1,\dots,n_k}$ (see Definition 3.1.2 below).

We have now introduced all the necessary tools to define the **r-th Terracini locus** that will be the main actor of the present chapter.

Definition 3.1.2. For all positive integers r and for any multiprojective space $Y_{n_1,\dots,n_k}$, define

$$\mathbb{T}_1(Y_{n_1,\dots,n_k}, r) := \left\{ S \subset Y_{n_1,\dots,n_k} \mid \#(S) = r, h^0\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) > 0 \text{ and } \delta(2S, Y_{n_1,\dots,n_k}) > 0 \right\}.$$

We will call the *r-th Terracini locus* $\mathbb{T}(Y_{n_1,\dots,n_k}, r)$ of all r-uple of points of $Y_{n_1,\dots,n_k}$ the set

$$\mathbb{T}(Y_{n_1,\dots,n_k},r) := \left\{ S \in \mathbb{T}_1(Y_{n_1,\dots,n_k},r) \;\middle|\; \begin{array}{l} Y_{n_1,\dots,n_k} \text{ is the minimal multiprojective} \\ \text{space containing } S \end{array} \right\}.$$

If $Y' \subseteq Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing a finite set S, the integer

$$\delta(2S,Y') := h^1\big(Y', \mathcal{I}_{(2S,Y')}(1,\dots,1)\big)$$

is called the *r-th Terracini defect of S.*

3.1.1 The 2-nd Terracini locus is empty

In this subsection we prove that no sets of two distinct points $S \subset Y_{n_1,\dots,n_k}$ such that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing S, is contained in the 2-nd Terracini locus $\mathbb{T}(Y_{n_1,\dots,n_k},2)$.

Proposition 3.1.3. *The 2-nd Terracini locus $\mathbb{T}(Y_{n_1,\dots,n_k},2)$ is empty for any multiprojective space $Y_{n_1,\dots,n_k}$.*

Proof. Let $S \subset Y_{n_1,\dots,n_k}$ be a set of two distinct points such that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing S. Therefore $\#(\pi_i(S)) = 2$ for all $i = 1,\dots,k$ and hence all $n_i = 1$, i.e. we are working with $Y_{1^k} \cong (\mathbb{P}^1)^k$, for some $k \geq 1$. By definition of the second Terracini locus, we need to prove that either $h^0\big(\mathcal{I}_{(2S,Y_{1^k})}(1,\dots,1)\big) = 0$ or $h^1\big(\mathcal{I}_{(2S,Y_{1^k})}(1,\dots,1)\big) = 0$. Clearly, if $k=1$ then $h^0\left(\mathcal{I}_{(2S,\mathbb{P}^1)}(1)\right) = 0$.
If $k = 2$, then $h^0\left(\mathcal{I}_{(2S,Y_{1,1})}(1,1)\right) = 0$ since S can be seen as a general subset of 2 distinct points by the action of $(\mathrm{Aut}(\mathbb{P}^1))^2$ and a general 2×2 matrix has rank 2.
Let $k \geq 3$ and define

$$E := \{A \subset Y \mid \#A = \#(\pi_i(A)) = 2 \text{ for all } i = 1,\dots,k\}.$$

The group $(\mathrm{Aut}(\mathbb{P}^1))^k$ acts transitively on E. Thus S may be considered as a general subset of Y_{1^k} with cardinality 2. By Proposition 1.1.19 we know that $\dim \sigma_2(X_{1^k}) = 2k+1$ for all $k \geq 3$, hence by Terracini's Lemma we conclude that $h^1\big(\mathcal{I}_{(2S,Y_{1^k})}(1,\dots,1)\big) = 0$. $\square$

We remark that, since we are dealing with finite subsets S of two distinct points, the minimal multiprojective space containing S is $Y_{1^k} = (\mathbb{P}^1)^k$ for some $k \geq 1$, which is equivalent to say that $\#(\pi_i(S)) = 2$ for all $i = 1,\dots,k$. Thus we may look at $S := \{p_1,p_2\}$ as a general set of two distinct points using the action of $(\mathrm{Aut}(\mathbb{P}^1))^k$.
The emptiness of the 2-nd Terracini locus $\mathbb{T}(Y_{1^k},2)$ means that the differential of the map $T_2 : Abs_2^0(X_{1^k}) \longrightarrow \sigma_2^0(X_{1^k})$ has full rank for any $X_{1^k} = \nu((\mathbb{P}^1)^k)$, with arbitrary $k \geq 2$. Since in this case we can consider S as a general set of two distinct points, the condition

$$h^0\big(\mathcal{I}_{(2S,Y_{1^k})}(1,\dots,1)\big) > 0$$

corresponds to prescribe that the 2-nd secant variety $\sigma_2(X_{1^k})$ does not fill the ambient space. This condition together with

$$h^1\big(\mathcal{I}_{(2S,Y_{1^k})}(1,\dots,1)\big) > 0$$

are equivalent to say that the dimension of the tangent space $T_q\sigma_2(X_{1^k})$ at a general $q \in \mathbb{P}^{2^k-1}$ such that $q \in \langle \nu(p_1), \nu(p_2) \rangle$, is strictly less than $2(k+1)-1$.

3.2 Two families of points in the third Terracini locus

We start now working with three points. The aim of this section is to present two different families of sets $S \subset Y_{n_1,\ldots,n_k}$, with $\#(S) = 3$ for which $Y_{n_1,\ldots,n_k}$ is the minimal multiprojective space containing S, such that

$$h^0\left(\mathcal{I}_{(2S,Y_{n_1,\ldots,n_k})}(1,\ldots,1)\right) h^1\left(\mathcal{I}_{(2S,Y_{n_1,\ldots,n_k})}(1,\ldots,1)\right) > 0.$$

These two different families of points will turn out to be the only non-trivial examples of points in the 3-rd Terracini locus. Therefore, they will be crucial for the main theorem of the present chapter.
We point out that, by dimensional reasons, if we focus on Y_{1^4} then all $S \subset Y_{1^4}$ of cardinality three that have Y_{1^4} as minimal multiprojective space in which they are contained belong to $\mathbb{T}(Y_{1^4}, 3)$. We will always refer to this case as a trivial example of points lying in the third Terracini locus. Anyhow, we will discuss more accurately this case in Remark 3.3.4.

3.2.1 Example: Products of $\mathbb{P}^1$'s with at most one $\mathbb{P}^2$

In the first example we work over $Y_{m,1^{k-1}} = \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$, where $m \in \{1, 2\}$ and $k \geq 3$. We consider a set of three distinct points $S \subset Y_{m,1^{k-1}}$ with $S := \{a, b, c\}$ such that a and b share all the last $k-1$ coordinates and we request that $Y_{m,1^{k-1}}$ is the minimal multiprojective space containing S. In the following proposition we prove that a necessary and sufficient condition for S to lie in the third Terracini locus $\mathbb{T}(Y_{m,1^{k-1}}, 3)$ is that $k \geq 4$.

Example 3.2.1. Let $Y_{m,1^{k-1}} = \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$ for some $k \geq 3$, with $m \in \{1, 2\}$. Define $S := \{a, b, c\} \subset Y_{m,1^{k-1}}$ be such that

$$\begin{aligned} &a := (a_1, u_2, \ldots, u_k), b := (b_1, u_2, \ldots, u_k), c := (c_1, \ldots, c_k), \text{ with} \\ &a_1, b_1, c_1 \in \mathbb{P}^m \text{ such that } a_1 \neq b_1 \text{ and } u_i \neq c_i \text{ for all } i > 1. \end{aligned}$$

Moreover if $m = 2$ assume also $\dim\langle \pi_1(S) \rangle = 2$.

Proposition 3.2.2. *Let $Y_{m,1^{k-1}} = \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$ for some $k \geq 3$, where $m \in \{1, 2\}$. Let $S \subset Y_{m,1^{k-1}}$ be as in Example 3.2.1. Therefore*

$$S \in \mathbb{T}(Y_{m,1^k}, 3) \textit{ if and only if } k \geq 4.$$

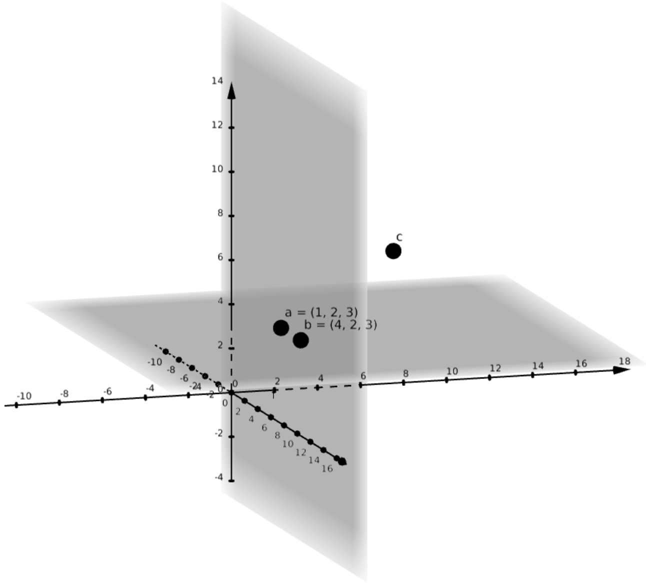


Figure 3.1: Picture of Example 3.2.1 with $m = 1$.

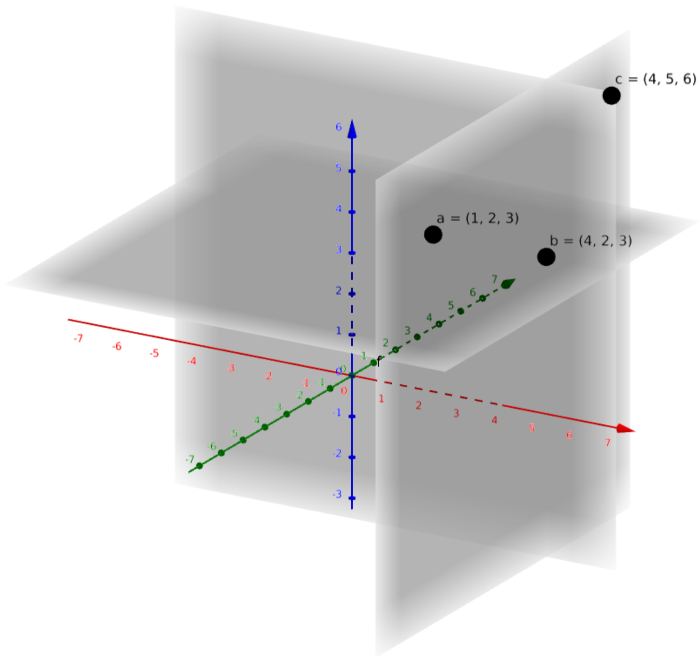


Figure 3.2: Pseudo-picture of Example 3.2.1 with $m = 2$ (the red axis is a $\mathbb{P}^2$)

Proof. We remark that since $\#(\pi_i(S)) \geq 2$ for all $i = 1, \ldots, k$ and $\#(\pi_1(S)) = 3$ if $m = 2$, then $Y_{m,1^{k-1}}$ is the minimal multiprojective space containing S. If we consider the subset

$$S' := \{a, b\}$$

of S we may apply Remark 1.2.13 and have that $\delta(2S, Y_{m,1^{k-1}}) \geq \delta(2S', Y_{m,1^{k-1}})$. Since $S' \subset \mathbb{P}^m \times \{u_2\} \times \cdots \times \{u_k\} \subset Y_{m,1^{k-1}}$, one can use case (b) of Lemma 1.2.14, with $W := \mathbb{P}^m$, to get

$$\delta(2S', Y_{m,1^{k-1}}) = m + 1.$$

Thus in order to see if $S \in \mathbb{T}(Y_{m,1^{k-1}}, 3)$, it suffices to understand whether

$$h^0\big(\mathcal{I}_{(2S, Y_{m,1^{k-1}})}(1, \ldots, 1)\big) > 0.$$

- If $k \geq 4$ then $h^0\big(\mathcal{O}_{Y_{m,1^{k-1}}}(1, \ldots, 1)\big) = (m+1)2^{k-1} > 3(m+k+1) = \deg(2S, Y_{m,1^{k-1}}) > 0$. Therefore, if $k \geq 4$ then $S \in \mathbb{T}(Y_{m,1^{k-1}}, 3)$.
- Let now $k = 3$. If we show that in this case none of the sets $S \subset Y_{m,1,1}$ as above belongs to $\mathbb{T}(Y_{m,1,1}, 3)$ we will be done.

  Remark that by assumption $u_i \neq c_i$ for $i = 2, 3$. To determine whether

  $$h^0\left(\mathcal{I}_{(2S, Y_{m,1,1})}(1, 1, 1)\right) > 0$$

  or not, we distinguish two cases depending on m being equal to either 1 or 2.

  (a) Assume $m = 2$, i.e. $Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.
  Since $h^0(\mathcal{O}_{Y_{2,1,1}}(\varepsilon_1)) = 3$, there exists $H \in |\mathcal{I}_{\{a,c\}}(\varepsilon_1)|$ and remark that

  $$H \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

  Since $\langle \pi_1(S) \rangle = \mathbb{P}^2$, then $H \cap S = \{a, c\}$. Consider the residual exact sequence of S with respect to H:

  $$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_H(2S, Y_{2,1,1})}(0, 1, 1) \longrightarrow \mathcal{I}_{(2S, Y_{2,1,1})}(1, 1, 1) \longrightarrow \mathcal{I}_{H \cap (2S, Y_{2,1,1})}(1, 1, 1) \longrightarrow 0.$$

  Since H is smooth then $H \cap (2S, Y_{2,1,1}) = (2(S \cap H), H) = (2\{a, c\}, H)$ and the residue of $(2S, Y_{2,1,1})$ with respect to H is

  $$\mathrm{Res}_H(2S, Y_{2,1,1}) = \{a, c\} \cup (2b, Y_{2,1,1}).$$

  Remark that $h^0\left(\mathcal{I}_{\mathrm{Res}_H(2S, Y_{2,1,1})}(0, 1, 1)\right) = h^0\left(Y_{1,1;1}, \mathcal{I}_{\eta_1(\mathrm{Res}_H(2S, Y_{2,1,1}))}(1, 1)\right)$ (cf. Notation 1.1.6).
  Since $\pi_i(a) = \pi_i(b) \neq \pi_i(c)$ for $i = 2, 3$, then $\eta_1(\mathrm{Res}_H(2S, Y_{2,1,1})) = \eta_1(\{a, c\} \cup (2b, Y_{2,1,1})) = \eta_1(c) \cup (2\eta_1(b), Y_{1,1;1})$.
  In order to compute $h^0\left(Y_{1,1;1}, \mathcal{I}_{\eta_1(c) \cup (2\eta_1(b), Y_{1,1,;1})}(1, 1)\right)$, we have to look at the hyperplanes of $\mathbb{P}^3$ containing both $\nu_1(\eta_1(c))$ and $T_{\nu_1(\eta_1(b))}\nu_1(Y_{1,1;1})$. Note that the tangent space $T_{\nu_1(b)}\nu_1(Y_{1,1;1})$ is the union of two lines through $\nu_1(b)$, i.e. the image by ν_1 of the set of all $x \in Y_1$ with $\pi_2(x) = \pi_2(v)$ and the set of all $y \in Y_{1,1;1}$ with $\pi_3(y) = \pi_3(v)$. Thus, since $u_i \neq c_i$ for $i = 2, 3$, there are no such hyperplanes, hence

  $$h^0\left(Y_{1,1;1}, \mathcal{I}_{\eta_1(\mathrm{Res}_H(2S, Y_{2,1,1})}(1, 1)\right) = 0.$$

  So by the residual sequence of S with respect to H recalled above, it is sufficient to prove that $h^0\left(H, \mathcal{I}_{(2\{a,c\}, H)}(1, 1, 1)\right) = 0$.

  Since $\langle \pi_1(S) \rangle = \mathbb{P}^2$ then $\pi_i(a) \neq \pi_i(c)$ for $i = 1, 2, 3$. Since $H \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ thus $\{a, c\}$ is in the open orbit of $\nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ for the action of $(\mathrm{Aut}(\mathbb{P}^1))^3$ on H. Since $\sigma_2(\nu(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)) = \mathbb{P}^7$, then

  $$h^0\left(H, \mathcal{I}_{(2\{a,c\}, H)}(1, 1, 1)\right) = 0.$$

(b) Assume $m = 1$, i.e. $Y_{1,1,1} = (\mathbb{P}^1)^3$.
Fix $H \in |\mathcal{I}_a(\varepsilon_3)|$. Since $u_i \neq c_i$ for $i = 2, 3$ and H is smooth we have that $H \cap S = \{a, b\}$ and $\mathrm{Res}_H(2S, H) = \{a, b\} \cup (2c, Y_{1,1,1})$. As in the last part of step (a), we remark that $h^0\big(\mathcal{I}_{\mathrm{Res}_H(2S,Y_{1,1,1})}(1,1,0)\big) = h^0\big(Y_{1,1;3}, \mathcal{I}_{\eta_3(\{a,b\} \cup (2c,Y_{1,1}))}\big)$ and in order to compute it we have to look at the hyperplanes of $\mathbb{P}^3$ containing both $T_{\nu_3(\eta_3(c))}\nu_3(Y_{1,1;3})$ and $\nu_3(\eta_3(\{a,b\}))$.
So $h^0\big(\mathcal{I}_{\mathrm{Res}_H(2S,Y_{1,1,1})}(1,1,0)\big) = 0$. Moreover, identifying $\nu(H)$ with a smooth quadric surface, by looking at case $k = 2$ of the proof of Proposition 3.1.3 we get

$$h^0\big(H, \mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\big) = 0.$$

Thus any set of points $S \subset Y$ constructed as above is in the 3-rd Terracini locus $\mathbb{T}(Y_{m,1^{k-1}}, 3)$ if and only if $k \geq 4$. □

3.2.2 Example: Products of $\mathbb{P}^1$'s with at most two $\mathbb{P}^2$'s

In this second example, we work over $Y_{n_1,n_2,1^{k-2}} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, where $n_1, n_2 \in \{1, 2\}$. We consider $S \subset Y_{n_1,n_2,1^{k-2}}$, with $S := \{u, v, o\}$ such that u and v share just the last $k-2$ components and we request that $Y_{n_1,n_2,1^{k-2}}$ is the minimal multiprojective space containing S.

Example 3.2.3. Let $Y_{n_1,n_2,1^{k-2}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, where $n_1, n_2 \in \{1, 2\}$ and $k \geq 3$. Let $S := \{o, u, v\}$ where

$$\begin{aligned} u = (u_1, u_2, u_3 \dots, u_n), v = (v_1, v_2, u_3, \dots, u_n), o = (o_1, \dots, o_n) \text{ with} \\ \langle u_i, v_i \rangle := L_i \cong \mathbb{P}^1 \text{ for } i = 1, 2 \text{ and } o_j \neq u_j \text{ for all } j = 3, \dots, k. \end{aligned}$$

Moreover if $n_i = 2$ assume also that $o_i \notin L_i$ for $i = 1, 2$.

Remark 3.2.4. Example 3.2.1 is not a particular case of Example 3.2.3. To fix the ideas let $Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ and take $S, S' \subset Y_{2,1,1}$ as in Examples 3.2.1 and 3.2.3 respectively. Then $S = \{a, b, c\}$ with

$$\begin{aligned} a = (a_1, a_2, a_3), b = (b_1, a_2, a_3), c = (c_1, c_2, c_3) \text{ such that} \\ a_i \neq c_i \text{ for all } i = 2, 3 \text{ and } \langle a_1, b_1, c_1 \rangle \cong \mathbb{P}^2, \end{aligned}$$

while $S' = \{o, u, v\}$ with

$$\begin{aligned} u = (u_1, u_2, u_3), v = (v_1, v_2, u_3), o = (o_1, o_2, o_3) \text{ such that} \\ u_3 \neq o_3 \text{ and } \langle u_1, v_1, o_1 \rangle \cong \mathbb{P}^2. \end{aligned}$$

Notice that S' cannot be as in Example 3.2.1 even if $o_2 \in \{u_2, v_2\}$.

Taking $S \subset Y_{n_1,n_2,1^{k-2}}$ as in Example 3.2.3, we will prove that

- If $k \geq 4$ then $S \in \mathbb{T}(Y_{n_1,n_2,1^{k-2}}, 3)$.
- If $k = 3$ and $n_1 = n_2 = 2$ then $S \in \mathbb{T}(Y_{2,2,1}, 3)$.
- If $k = 3$ and either $\{n_1, n_2\} = \{1, 2\}$ or $n_1 = n_2 = 1$ then we need to add more restrictive conditions to the points of S in order to get $S \in \mathbb{T}(Y_{n_1,n_2,1}, 3)$.

More precisely, we state and prove the following result.

Proposition 3.2.5. *Let $Y_{n_1,n_2,1^{k-2}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, where $n_1, n_2 \in \{1,2\}$ and $k \geq 3$. Let $S = \{u, v, o\} \subset Y_{n_1,n_2,1^{k-2}}$ be as in Example 3.2.3. Set*

$$Y' := L_1 \times L_2 \times \{\pi_3(u)\} \times \cdots \times \{\pi_k(u)\} \subset Y_{n_1,n_2,1^{k-2}}.$$

Therefore

(i) If $k \geq 4$ then $\delta(2S, Y_{n_1,n_2,1^{k-2}}) = 2$.

(ii) If $k = 3$ and $n_1 = n_2 = 2$ then $\delta(2S, Y_{2,2,1}) = 2$.

(iii) If $k = 3$ and $n_1 = n_2 = 1$ then $4 \leq \delta(2S, Y_{1,1,1}) \leq 5$ and $h^0(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)) > 0$ if and only if $\pi_i(u) = \pi_i(o)$ and $\pi_h(v) = \pi_h(o)$ for some $i, h \in \{1,2\}$.

(iv) If $k = 3$ and $\{n_1, n_2\} = \{1,2\}$ then $\delta(2S, Y_{2,1,1}) \geq 3$ and $h^0(\mathcal{I}_{(2S,Y_{2,1,1})}(1,1,1)) > 0$ if and only if $\pi_2(o) \in \pi_2(S')$.

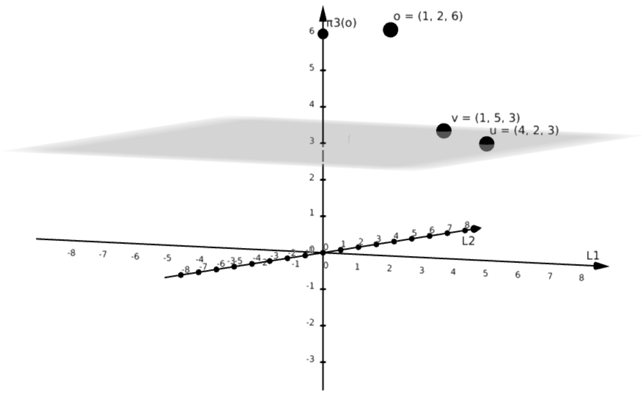


Figure 3.3: Picture of Proposition 3.2.3.

Proof. Remark that $S' := \{u, v\} \subset Y'$, and Y' is actually the minimal multiprojective space containing S' while $Y_{n_1,n_2,1^{k-2}}$ is the minimal multiprojective space containing S. Part (b) of Lemma 1.2.14 gives

$$h^1\big(Y_{n_1,n_2,1^{k-2}}, \mathcal{I}_{(2S',Y_{n_1,n_2,1^{k-2}})}(1,\ldots,1)\big) = h^1\big(Y', \mathcal{I}_{(2S',Y')}(1,1)\big)$$

and hence

$$h^1\big(\mathcal{I}_{(2S',Y_{n_1,n_2,1^{k-2}})}(1,\ldots,1)\big) = h^1\big(\mathcal{I}_{(2S',Y')}(1,1,0,\ldots,0)\big). \tag{3.2.1}$$

Proposition 3.1.3 (or rather its proof for $k = 2$) and Lemma 1.2.15 give

$$h^1\big(Y', \mathcal{I}_{(2S',Y')}(1,1)\big) = 2.$$

(i) Assume $k \geq 4$. Take $M \in |\mathcal{O}_{Y_{n_1,n_2,1^{k-2}}}(\varepsilon_k)|$ containing o. By looking at the residual exact sequence of M, we remark that $\mathrm{Res}_M(2S, Y_{n_1,n_2,1^{k-2}}) = \{o\} \cup (2S', Y_{n_1,n_2,1^{k-2}})$ and $M \cap (2S, Y_{n_1,n_2,1^{k-2}}) = (2o, M)$. Since $h^1\big(\mathcal{I}_{(2o,M)}(1,\ldots,1)\big) = 0$, by (3.2.1) we get

$$h^1\big(Y_{n_1,n_2,1^{k-2}}, \mathcal{I}_{(2S',Y_{n_1,n_2,1^{k-2}})\cup\{o\}}(\hat{\varepsilon}_k)\big) = h^1\big(Y_{n_1,n_2,1^{k-2}}, \mathcal{I}_{(2S',Y')}(\hat{\varepsilon}_k)\big)$$

and we conclude the proof of part (i).

(ii) Assume $k = 3$ and $n_1 = n_2 = 2$.

Consider the subgroup G of $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1)$ fixing pointwise Y'. The action of G on $Y_{2,2,1}$ has an open orbit U and $o \in U$. Since $h^1(\mathcal{I}_{S'}(1,1,0)) = 0$, case (a) of Lemma 1.2.14 gives $h^1\left(Y_{2,2,1}, \mathcal{I}_{(2S',Y_{2,2,1})}(1,1,1)\right) = h^1\big(Y', \mathcal{I}_{(2S',Y')}(1,1)\big)$. Obviously $h^1\big(Y', \mathcal{I}_{(2S',Y')}(1,1)\big) = h^1\left(Y_{2,2,1}, \mathcal{I}_{(2S',Y')}(\hat{\varepsilon}_3)\right)$. Since $h^0\left(Y_{2,2,1}, \mathcal{I}_{(2S',Y')}(\hat{\varepsilon}_3)\right) > 0$ and o is in the open orbit U, we have $h^0\big(Y_{2,2,1}, \mathcal{I}_{(2S',Y')\cup\{o\}}(\hat{\varepsilon}_3)\big) = h^0\big(Y_{2,2,1}, \mathcal{I}_{(2S',Y')}(\hat{\varepsilon}_3)\big) - 1$. Therefore

$$h^1\left(Y_{2,2,1}, \mathcal{I}_{(2S',Y')\cup\{o\}}(\hat{\varepsilon}_3)\right) = h^1\left(Y_{2,2,1}, \mathcal{I}_{(2S',Y')}(\hat{\varepsilon}_3)\right).$$

Thus to prove that $h^1\left(\mathcal{I}_{(2S',Y_{2,2,1})\cup\{o\}}(\hat{\varepsilon}_3)\right) = 2$ and hence to prove part (ii), it is sufficient to observe that $h^0\left(W, \mathcal{I}_{(2A,W)}(1,1)\right) > 0$, because not all 3×3 matrices have rank at most 2, where we took $W = \mathbb{P}^2 \times \mathbb{P}^2$ and $A \subset W$ as a general subset of two distinct points.

(iii) Assume $k = 3$ and $n_1 = n_2 = 1$.

Since $h^0\left(\mathcal{O}_{Y_{1,1,1}}(1,1,1)\right) = 8$ and $\deg(2S, Y_{1,1,1}) = 12$, we have $h^1\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right) = 4 + h^0\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right)$, so $h^1\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right) \geq 4$. To conclude this case it is sufficient to show the following claim.

Claim 4. With the notation as above $h^0\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right) > 0$ if and only if $\pi_i(u) = \pi_i(o)$ for some $i \in \{1,2\}$ and $\pi_j(v) = \pi_j(o)$ for some $j \in \{1,2\}$. In this case $h^0\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right) = 1$ and $h^1\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right) = 5$.

Proof. Take $H \in |\mathcal{I}_{\{u\}}(\varepsilon_3)|$. Since $\pi_3(u) = \pi_3(v)$, then $H \cap S = S'$. Since H is smooth, $(2S, Y_{1,1,1}) \cap H = (2S', H)$ and $\mathrm{Res}_H(2S, Y_{1,1,1}) = S' \cup \{2o\}$. We identify $\nu(H)$ with a smooth quadric surface $Q \subset \mathbb{P}^3$. Since a tangent plane to a smooth quadric surface Q is tangent to Q at a unique point, then we have the vanishing of $h^0\left(H, \mathcal{I}_{(2S,Y_{1,1,1})\cap H,H}(1,1,1)\right)$.
Consider the residual exact sequence of S with respect to H:

$$0 \longrightarrow \mathcal{I}_{S'\cup\{2o\}}(1,1,0) \longrightarrow \mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1) \longrightarrow \mathcal{I}_{(2S,Y_{1,1,1})\cap H,H}(1,1,1) \longrightarrow 0.$$

Since $h^0\left(H, \mathcal{I}_{(2S,Y_{1,1,1})\cap H,H}(1,1,1)\right) = 0$, then

$$h^0\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right) = h^0\left(\mathcal{I}_{S'\cup\{2o\}}(1,1,0)\right).$$

Moreover $h^0\left(\mathcal{I}_{S'\cup\{2o\}}(1,1,0)\right) = h^0\left(Y_{1,1;3}, \mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_{1,1;3})}(1,1)\right)$ and we can think of $\nu_3(Y_{1,1;3})$ as a smooth quadric surface. Since $T_{\nu_3(o)}\nu_3(Y_{1,1;3})$ is a plane $h^0\left(Y_{1,1;3}, \mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_{1,1;3})}(1,1)\right) \leq 1$.
Now $h^0\left(Y_{1,1;3}, \mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_{1,1;3})}(1,1)\right) = 1$ if and only if both $\nu_3(\eta_3(u))$ and

$\nu_3(\eta_3(u))$ are contained in $\nu_3(Y_{1,1;3}) \cap T_{\nu_3(\eta_3(o))}(\nu_3(Y_{1,1;3}))$. We remark that $T_{\nu_3(\eta_3(o))}(\nu_3(Y_{1,1;3}))$ is the union of two lines through $\nu_3(\eta_3(o))$, i.e. the image by ν_3 of the set of all $x \in Y_{1,1;3}$ with $\pi_1(x) = \pi_1(o)$ and the set of all $y \in Y_{1,1;3}$ with $\pi_2(y) = \pi_2(o)$. Hence Claim 4 is just a translation of this observation. □

(iv) Assume $\{n_1, n_2\} = \{2, 1\}$ and $k = 3$.

With no loss of generality we may assume $n_1 = 2$ and $n_2 = 1$. Since $h^0\left(\mathcal{O}_{Y_{2,1,1}}(1,1,1)\right) = 12$ and $\deg(2S, Y_{2,1,1}) = 15$, we have $h^1\left(\mathcal{I}_{(2S,Y_{2,1,1})}(1,1,1)\right) = 3 + h^0\left(\mathcal{I}_{(2S,Y_{2,1,1})}(1,1,1)\right)$. Hence $\delta(2S, Y_{2,1,1}) \geq 3$.
We remark that by assumption $\pi_1(o) \notin \langle \pi_1(u), \pi_1(v)\rangle$, $\pi_2(u) \neq \pi_2(v)$ and $\pi_3(o) \neq \pi_3(u) = \pi_3(v)$.
To conclude this case we have to show that $h^0\left(\mathcal{I}_{(2S,Y_{2,1,1})}(1,1,1)\right) > 0$ if and only if $\pi_2(o) \in \pi_2(S')$.

- Assume $\pi_2(o) \in \pi_2(S')$. Without loss of generality we may assume that $\pi_2(u) = \pi_2(o)$. Since $h^0\left(\mathcal{O}_{Y_{2,1,1}}(\varepsilon_2)\right) = 2$ then $|\mathcal{I}_o(\varepsilon_2)|$ is a singleton. Set $\{H\} := |\mathcal{I}_o(\varepsilon_2)|$. Since $H \cong \mathbb{P}^2 \times \mathbb{P}^1$ it is smooth, hence $(2S, Y_{2,1,1}) \cap H = (2\{o,u\}, H)$ scheme-theoretically and $\mathrm{Res}_H(2S, Y_{2,1,1}) = (2v, Y_{2,1,1}) \cup \{o, u\}$. Remark that
$$h^0\left(\mathcal{I}_{(2v,Y_{2,1,1})\cup\{o,u\}}(1,0,1)\right) = h^0\left(Y_{2,1;2}, \mathcal{I}_{(2\eta_2(v),Y_{2,1;2})\cup\{\eta_2(o),\eta_2(u)\}}(1,1)\right).$$
Moreover, $Y_{2,1;2} \cong \mathbb{P}^2 \times \mathbb{P}^1$ and
$$h^0\left(\mathcal{O}_{Y_{2,1;2}}(1,1)\right) = 6 = \deg((2\eta_2(v), Y_{2,1;2}) \cup \{\eta_2(o), \eta_2(u)\}).$$
This last equality implies that
$$h^0\left(Y_{2,1;2}, \mathcal{I}_{(2\eta_2(v),Y_{2,1;2})\cup\{\eta_2(o),\eta_2(u)\}}(1,1)\right) = h^1\left(Y_{2,1;2}, \mathcal{I}_{(2\eta_2(v),Y_{2,1;2})\cup\{\eta_2(o),\eta_2(u)\}}(1,1)\right).$$
To show that $h^0\left(Y_{2,1;2}, \mathcal{I}_{(2\eta_2(v),Y_{2,1;2})\cup\{\eta_2(o),\eta_2(u)\}}(1,1)\right) > 0$, we have to look at the hyperplanes of $\mathbb{P}^5 \supset \nu_2(Y_{2,1;2})$ that contain both the tangent space $T_{\nu_2(\eta_2(v))}\nu_2(Y_{2,1;2})$ and the points $\nu_2(\eta_2(\{o,u\}))$. Remark that $T_{\nu_2(\eta_2(v))}\nu_2(Y_{2,1;2}) \cap \nu_2(Y_{2,1;2})$ is the union of 2 linear spaces containing $\nu_2(\eta_2(v))$, one of dimension 2 and one of dimension 1, spanning the 3-dimensional projective space $T_{\nu_2(\eta_2(v))}\nu_2(Y_{2,1;2})$. Since $\pi_3(u) = \pi_3(v)$, then $\nu_2(\eta_2(u))$ is a point of the 2-dimensional irreducible component of the tangent space $T_{\nu_2(\eta_2(v))}\nu_2(Y_{2,1;2}) \cap \nu_2(Y_{2,1;2})$.
Thus $h^0\left(\mathcal{I}_{\mathrm{Res}_H(2S,Y_{2,1,1})}(1,0,1)\right) > 0$. Hence, by the long cohomology exact sequence induced by the exact sequence of the residue of S with respect to H, we get $h^0\left(\mathcal{I}_{(2S,Y_{2,1,1})}(1,1,1)\right) > 0$.
- Assume $\pi_2(o) \notin \pi_2(S')$. Since $h^0\left(\mathcal{O}_{Y_{2,1,1}}(\varepsilon_3)\right) = 2$ then $|\mathcal{I}_u(\varepsilon_3)|$ is a singleton. Set $\{M\} := |\mathcal{I}_u(\varepsilon_3)|$. Since M is smooth and $M \cap S = S'$, then $(2S, Y_{2,1,1}) \cap M = (2S', M)$ scheme-theoretically and $\mathrm{Res}_M(2S, Y_{2,1,1}) = S' \cup (2o, Y_{2,1,1})$. We have $h^0\left(\mathcal{I}_{S'\cup(2o,Y_{2,1,1})}(1,1,0)\right) = h^0\left(Y_{2,1;3}, \mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_{2,1;3})}(1,1)\right)$. Obviously
$$h^0\left(Y_{2,1;3}, \mathcal{I}_{(2\eta_3(o),Y_{2,1;3})}(1,1)\right) = 2.$$
Since $\eta_3(S)$ is in the open orbit of $S(Y_{2,1;3}, 3)$ for the action of $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1)$, $h^0\left(Y_{2,1;3}, \mathcal{I}_{\eta_3(S')\cup(2\eta_3(o),Y_{2,1;3})}(1,1)\right) = 0$. The set S' is in the open orbit of $\mathrm{Aut}(M)$ for its action in $S_M(3)$ (cf. Definition 3.4.1). Since any 3×2 matrix has rank at most 2, we know that $\sigma_2(\nu(M)) = \mathbb{P}^5$. Thus $h^0\left(H, \mathcal{I}_{(2S,Y_{2,1,1})\cap H,H}(1,1,1)\right) = 0$. The residual exact sequence of S with respect to M gives $h^0\left(\mathcal{I}_{(2S,Y_{2,1,1})}(1,1,1)\right) = 0$.

In summary if $k=3$ and $\{n_1,n_2\}=\{1,2\}$, then $S\in\mathbb{T}(Y_{2,1,1},3)$ if and only if $\pi_2(o)\in\pi_2(S')$. $\square$

3.3 Characterization of the third Terracini locus

In this section we prove the main theorem of the present chapter, i.e. we prove that the previous examples are the only non-trivial sets of points lying in the third Terracini locus. Let $Y_{n_1,\dots,n_k}=\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ and take $S\subset Y_{n_1,\dots,n_k}$ with $\#(S)=3$. Note that in this case, the minimal multiprojective space $Y'\subset Y_{n_1,\dots,n_k}$ containing S is $Y':=\mathbb{P}^{n'_1}\times\cdots\times\mathbb{P}^{n'_{k'}}$, where $n'_i\in\{1,2\}$ for all $i=1,\dots,k'$. We will treat all the possible cases for Y'. In particular, we will consider the case in which

- the multiprojective space is a product of $\mathbb{P}^1$'s only in Lemma 3.3.2, Remark 3.3.4 and Lemma 3.3.7 respectively.

Then we start arguing by number of factors, more precisely:

- the 3-factors case is treated in Lemmas 3.3.8,3.3.9 and 3.3.10 respectively;
- for $k=4$ we refer to Lemma 3.3.12;
- case $k\geq 5$ is done in Lemma 3.3.11.

In all these cases, if $Y_{n_1,\dots,n_k}$ is not equivalent neither to Y_{1^4} nor to Y_{2^k} we will prove that any $S\in\mathbb{T}(Y_{n_1,\dots,n_k},3)$ is either as in Example 3.2.3 or as in Example 3.2.1. If $Y\cong Y_{1^4}$ then we will see in Remark 3.3.4 that any set S of three points such that Y_{1^4} is the minimal multiprojective space containing S lies in $\mathbb{T}(Y_{1^4},3)$. While for the case of multiprojective spaces given by products of $\mathbb{P}^2$'s only, we will prove that $\mathbb{T}(Y_{2^k},3)=\emptyset$.

Therefore from now on we will work with multiprojective spaces $Y_{n_1,\dots,n_k}=\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ such that $n_i\in\{1,2\}$ for all $i=1,\dots,k$.

Let us begin our discussion with the case of products of $\mathbb{P}^1$'s and by recalling that the action of the group $(\mathrm{Aut}(\mathbb{P}^1))^k$ on sets of points S having all $\pi_{i|S}$ injective is transitive.

Remark 3.3.1. Let $Y_{1^k}=(\mathbb{P}^1)^k$, for some $k\geq 2$. Given any two subset $S,S'\subset Y_{1^k}$ of three distinct points such that $\#(\pi_i(S))=\#(\pi_i(S'))=3$ for all $i=1,\dots,k$, one can always find $f\in(\mathrm{Aut}(\mathbb{P}^1))^k$ such that $S=f(S')$. Since Y_{1^k} is the minimal multiprojective space containing both S and S', then $S\in\mathbb{T}(Y_{1^k},3)$ if and only if $S'\in\mathbb{T}(Y_{1^k},3)$.

Lemma 3.3.2. *Let $Y_{1,1,1}=\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ and let $S:=\{u,v,o\}\subset Y_{1,1,1}$, with $\#S=3$, be such that $Y_{1,1,1}$ is the minimal multiprojective space containing S. Then $S\in\mathbb{T}(Y_{1,1,1},3)$ if and only if there exist $h\in\{1,2,3\}$ and $i,j\in\{1,2,3\}\setminus\{h\}$, with $i<j$ such that*

$$\pi_h(u)=\pi_h(v)\neq\pi_h(o),\ \pi_i(o)=\pi_i(u)\ \textit{and}\ \pi_j(o)=\pi_j(v),$$
$$\textit{where both}\ \pi_i(u)\neq\pi_i(v)\ \textit{and}\ \pi_j(u)\neq\pi_j(v).$$

Proof. Up to a permutation of the index $h\in\{1,2,3\}$ we may assume $h=3$.
Take $S\subset Y_{1,1,1}$ with $\#S=3$ and let $X_{1,1,1}:=\nu(Y_{1,1,1})$. Since $\dim\sigma_3(X_{1,1,1})=7$ (cf. Theorem 1.1.24), if $\#(\pi_i(S))=3$ for all $i=1,2,3$, by Remark 3.3.1 we have that

$$h^0\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right)=0.$$

Hence $S \notin \mathbb{T}(Y_{1,1,1}, 3)$.
Now assume $\#(\pi_i(S)) \leq 2$ for some i. Remark that since $Y_{1,1,1}$ is the minimal multiprojective space containing S then $\#(\pi_i(S)) = 2$. We distinguish different cases depending on the number of indices $i \in \{1, 2, 3\}$ for which $\#(\pi_i(S)) = 2$.

- If there exists only an index i such that $\#(\pi_i(S)) = 2$ then S is as in Example 3.2.3 and by case (iii) of Proposition 3.2.5 we know that $h^0\left(\mathcal{I}_{(2S, Y_{1,1,1})}(1, 1, 1)\right) = 0$.
- If $\#(\pi_i(S)) = 2$ for two indices, then S is as in Example 3.2.3 or as in Example 3.2.1. For both cases we have $h^0\left(\mathcal{I}_{(2S, Y_{1,1,1})}(1, 1, 1)\right) = 0$.
- Finally, if $\#(\pi_i(S)) = 2$ for all $i \in \{1, 2, 3\}$, then S is as in Example 3.2.3 and by case (iii) of Proposition 3.2.5 we get that $h^0\left(\mathcal{I}_{(2S, Y_{1,1,1})}(1, 1, 1)\right) = 1$. □

Remark 3.3.3. Let $Y_{1,1,1} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let $S \in S(Y_{1,1,1}, 3)$ such that $Y_{1,1,1}$ is the minimal multiprojective space containing S. We remark that the characterization of the elements $\mathbb{T}(Y_{1,1,1}, 3)$ presented in Lemma 3.3.2 is the well known description of the general element of the tangential variety $\tau(X_{1,1,1})$, which is also called W-state in quantum information literature (cf. [Cab02]). Indeed let $S = \{u, v, o\} \in \mathbb{T}(Y_{1,1,1}, 3)$ and without loss of generality take $\{i, j, h\} = \{1, 2, 3\}$. Then

$$u = (\alpha, b, \gamma), v = (a, \beta, \gamma), o = (\alpha, \beta, c),$$

for some distinct $\alpha, \beta, \gamma, a, b, c \in \mathbb{P}^1$. Now it is straightforward to see that the general $q \in \langle \nu(S) \rangle$ is actually an element of $T_{\nu(p)} X_{1,1,1}$ where $p = (\alpha, \beta, \gamma)$.

Remark 3.3.4. Fix $Y_{1^4} = (\mathbb{P}^1)^4$ and let $A \subset Y_{1^4}$ be a general subset of three distinct points. Since the 3-rd secant variety of $X_{1^4} := \nu(Y_{1^4}) \subset \mathbb{P}^{15}$ is defective with defect 1 (cf. Theorem 1.1.24), then $\dim(\sigma_3(X_{1^4})) = 13$ and $h^0\left(\mathcal{I}_{(2A, Y_{1^4})}(1, 1, 1, 1)\right) = 2$, so by looking at the restriction exact sequence we get $h^1\left(\mathcal{I}_{(2A, Y_{1^4})}(1, 1, 1, 1)\right) = 1$. By the semicontinuity theorem for cohomology (cf. [Har77, Ch. III §12]) $h^1\left(\mathcal{I}_{(2S, Y_{1^4})}(1, 1, 1, 1)\right) \geq 1$ and $h^0\left(\mathcal{I}_{(2S, Y_{1^4})}(1, 1, 1, 1)\right) \geq 2$ for all $S \subset Y_{1^4}$ with $\#(S) = 3$. Moreover we remark that Y_{1^4} is the minimal multiprojective space containing a set S of three distinct points if and only if $\#(\pi_i(S)) \geq 2$ for all $i = 1, 2, 3, 4$.
Thus for all $S \subset Y_{1^4}$ such that $\#(S) = 3$ we have that

$$\delta(2S, Y_{1^4}) h^0\left(\mathcal{I}_{(2S, Y_{1^4})}(1, 1, 1, 1)\right) > 0$$

and the 3-rd Terracini locus $\mathbb{T}(Y_{1^4}, 3)$ contains all subsets S of Y_{1^4} of cardinality three such that $\#(\pi_i(S)) \geq 2$ for all $i = 1, 2, 3, 4$.

Remark 3.3.5. Let $Y_{1^k} = (\mathbb{P}^1)^k$ with $k \geq 5$. By Theorem 1.1.24 we know that $\dim(\sigma_3(X_{1^k})) = 3k + 2$. Take a general S of cardinality 3, by looking at the restriction exact sequence of $2S$ with respect to Y_{1^k}, we get that $h^1\left(\mathcal{I}_{(2S, Y_{1^k})}(1, \ldots, 1)\right) = 0$. So a general $S \subset Y_{1^k}$ with $\#(S) = 3$ is not in the 3-rd Terracini locus $\mathbb{T}(Y_{1^k}, 3)$. Thus by Remark 3.3.1, for all $S \subset Y_{1^k}$ with $\#(S) = 3$ such that $\#(\pi_i(S)) = 3$ for all $i = 1, \ldots, k$, then $S \notin \mathbb{T}(Y_{1^k}, 3)$.

Lemma 3.3.6. *Let $Y_{1^k} = (\mathbb{P}^1)^k$ with $k \geq 5$. Fix $S := \{a, b, c\} \subset Y_{1^k}$ such that Y_{1^k} is the minimal multiprojective space containing S. Assume that there are at least $k - 2$ indices i's for which $\pi_i(a) = \pi_i(b)$. Then S is either as in Example 3.2.3 or as in Example 3.2.1.*

Proof. Define $E := \{i \in \{1, \ldots, k\} \mid \pi_i(a) = \pi_i(b)\}$, by assumption $\#(E) \geq k-2$ and since $a \neq b$ then $\#(E) \leq k-1$. By permuting the factors of Y_{1^k} if necessary, one can always assume that E contains the last $k-2$ indices and that the index $1 \notin E$. If $2 \notin E$ then S is constructed as in Example 3.2.3 with $n_1 = n_2 = 1$, else S is as in Example 3.2.1 where we took $m = 1$. □

Lemma 3.3.7. *Let $Y_{1^k} = (\mathbb{P}^1)^k$ with $k \geq 5$. Fix $S \subset Y_{1^k}$ with $\#(S) = 3$ such that Y_{1^k} is the minimal multiprojective subspace containing S. If $S \in \mathbb{T}(Y_{1^k}, 3)$ then S is either as in Example 3.2.3 or as in Example 3.2.1.*

Proof. Write $S := \{u, v, z\}$. Since $S \in \mathbb{T}(Y_{1^k}, 3)$, by Remark 3.3.5 we may assume that $\pi_i(u) = \pi_i(v)$ for at least one $i \in \{1, \ldots, k\}$. With no loss of generality we may assume $i = 1$. Since $h^0\left(\mathcal{O}_{Y_{1^k}}(\varepsilon_i)\right) = 2$ for $i = 1, 2$, both $|\mathcal{I}_u(\varepsilon_1)|$ and $|\mathcal{I}_z(\varepsilon_2)|$ are singletons. Set $\{H\} := |\mathcal{I}_u(\varepsilon_1)|$ and $\{M\} := |\mathcal{I}_z(\varepsilon_2)|$. Since $v \in H$, then $S \subset H \cup M$. Moreover, since Y_{1^k} is the minimal multiprojective space containing S, then $z \notin H$ and $\#(S \cap M) \leq 2$.

Claim 5. $h^1(\mathcal{I}_S(0, 0, 1, \ldots, 1)) = 0$ unless S is either as in Example 3.2.3 or as in Example 3.2.1.

Proof. Call $\eta : Y_{1^k} \longrightarrow (\mathbb{P}^1)^{k-2}$ the projection onto the last $k-2$ factor of Y_{1^k} and set $Y' := (\mathbb{P}^1)^{k-2}$.
Assume $h^1(\mathcal{I}_S(0, 0, 1, \ldots, 1)) > 0$. Therefore either $\eta_{|S}$ is not injective or $\#(\eta(S)) = 3$ and $h^1\left(Y', \mathcal{I}_{\eta(S)}(1, \ldots, 1)\right) > 0$, which means that the points of $\eta(S)$ are collinear. In the first case S is either as in Example 3.2.3 or as in Example 3.2.1 by Lemma 3.3.6. In the second case we would have an $i \in \{3, \ldots, k\}$ such that $\#(\pi_h(S)) = 1$ for all $h \in \{3, \ldots, k\} \setminus \{i\}$, contradicting the minimality of Y' for $\eta(S)$, which is a consequence of the minimality of Y_{1^k} for S. □

Assume by contradiction that S is neither as in Example 3.2.3 nor as in Example 3.2.1.

(a) Assume $\#(S \cap M) = 1$, i.e. $S \cap (H \cap M) = \emptyset$. So S is contained in the smooth part of $H \cup M$ and $\mathrm{Res}_{H \cup M}(2S) = S$. Since S is not as in one of the examples, by Claim 5 we get $h^1\left(\mathcal{I}_{\mathrm{Res}_{H\cup M}(2S,Y)}(0, 0, 1, \ldots, 1)\right) = h^1(\mathcal{I}_S(0, 0, 1, \ldots, 1)) = 0$. Moreover, by the restriction exact sequence of S, we get $h^0(\mathcal{I}_S(0, 0, 1, \ldots, 1)) = 2^{k-2} - 3$. Since by assumption $S \in \mathbb{T}(Y_{1^k}, 3)$, then $h^0\left(\mathcal{I}_{(2S, Y_{1^k})}(1, \ldots, 1)\right) > 0$ and more precisely $h^0\left(\mathcal{I}_{(2S, Y_{1^k})}(1, \ldots, 1)\right) \geq 2^k - 3(k+1)$, where $k \geq 5$. Thus the residual exact sequence of $H \cup M$ gives $h^1\left(H \cup M, \mathcal{I}_{(2S, H \cup M), H \cup M}(1, \ldots, 1)\right) > 0$. Since by assumption $S \cap (H \cap M) = \emptyset$, $(2S, H \cup M)$ is equal to $(2u, H) \cup (2v, H) \cup (2z, M)$.

Denote by G the set of all $g \in (\mathrm{Aut}(\mathbb{P}^1))^k$ acting as the identity on the last $k-1$ factors of Y_{1^k}; we remark that the elements of G are 3-transitive on the first factor. Let G_u be the subgroup of G fixing also the first component $\pi_1(u)$ of $u \in S$. Hence, since we assumed $\pi_1(u) = \pi_1(v)$, any $g \in G_u$ fixes both u and v. Obviously $h^1\left(H \cup M, \mathcal{I}_{(2u,H)\cup(2v,H)\cup(2z,M), H\cup M}(1, \ldots, 1)\right) = h^1\left(H \cup M, \mathcal{I}_{(2u,H)\cup(2v,H)\cup(2g(z),M), H\cup M}(1, \ldots, 1)\right)$ for all $g \in G_u$. Thus it is sufficient to find a contradiction for a single $z' \in M \setminus H \cap M$ with $\pi_i(z') = \pi_i(z)$ for $i > 1$.

We may specialize z by considering a general $o \in H \cap M$. So it is sufficient to work on H rather than $H \cup M$. Denote by $Z := (2u, H) \cup (2v, H)$ and call A the union of Z and the double point $(2o, H \cap M)$. We want to use the Differential Horace Lemma with $H \cap M$ as a divisor of H (cf. Lemma 1.2.4). We remark that $Z \subset H$ satisfies the assumptions of the Differential Horace Lemma, i.e. both

$$h^1\big(H, \mathcal{I}_{\mathrm{Res}_{(H\cap M)}(Z)} \otimes \mathcal{L}(-H \cap M)\big) = 0$$

$$\text{and } h^1\left(H \cap M, \mathcal{I}_{Z\cap(H\cap M), H\cap M} \otimes \mathcal{L}_{|H\cap M}\right) = 0,$$

where $\mathcal{L} = \mathcal{O}(1, \ldots, 1)$. Indeed the latter is trivially zero since by assumption $\#(S \cap M) = 1$. The former is zero since H is the minimal multiprojective space containing Z and by Proposition 3.1.3 we know that $\mathbb{T}(H, 2) = \emptyset$ and in particular that $\delta(2\{u, v\}, H) = 0$. Thus by Lemma 1.2.4, in order to show that $h^1(H, \mathcal{I}_A(1, \ldots, 1)) = 0$, it suffices to show that both

$$h^1\left(H \cap M, \mathcal{I}_{(Z\cap(H\cap M))\cup\{o\}}(1, \ldots, 1)\right) = 0$$

$$\text{and } h^1\left(H, \mathcal{I}_{\mathrm{Res}_{H\cap M}(A)}(1, 0, 1, \ldots, 1)\right) = 0.$$

Clearly since $(Z \cap (H \cap M)) \cup \{o\} = \{o\}$ then $h^1\left(H \cap M, \mathcal{I}_{(Z\cap(H\cap M))\cup\{o\}}(1, \ldots, 1)\right)$ is trivially zero. The second equality follows from (1.2.3) of Remark 1.2.13 since we already pointed out that $\delta(2\{u, v\}, H) = 0$.

(b) Assume $\#(S \cap M) = 2$. Taking $\{M_i\} = |\mathcal{I}_z(\varepsilon_i)|$ for $i = 3, \ldots, k$ and applying step (a) to $H \cup M_i$, we see that it is sufficient to handle the case with $\#(\pi_i(S)) = 2$ for all i.

Write $S = \{a, b, c\}$. Since $\mathrm{Aut}(\mathbb{P}^1)$ is 2-transitive, by composing with an element of $(\mathrm{Aut}(\mathbb{P}^1))^k$, we may assume $\pi_i(S) = \{\alpha, \beta\}$ for all i.

Without loss of generality we may also assume $\pi_i(a) = \alpha$ for all i. Thus $\beta \in \{\pi_i(b), \pi_i(c)\}$ for all i. Moreover, since S is neither as in Example 3.2.3 nor as in Example 3.2.1, for all $A \subset S$ with $\#(A) = 2$ then $\#(\pi_i(A)) = 2$ for at least 3 indices i's. We define the maximum number of common components that any two points of S can have as

$$t := \max\{\#(I) \,|\, I \subset \{1, \ldots, k\} \text{ and } \exists A \subset S \text{ with } \#(A) = 2 \text{ such that } \forall i \in I\ \pi_i(A) = 1\}.$$

By relabeling if necessary, we may assume that $\{a, b\}$ is one of the subsets of S reaching such t. By assumption $t \leq k - 3$.

We distinguish different cases depending on the integer $k \geq 5$. In particular, for $k = 5, 6$, we will get to a contradiction with the assumption $\delta(2S, Y_{1^k}) > 0$ by direct computation with Macaulay2 (cf. [GS02]).

(i) Assume $k = 5$. So $t \leq 2$ and since $\#(\pi_i(S)) = 2$ for all i and $k > 3$ then $t = 2$. Permuting the factors of Y we may assume $\pi_i(b) = \alpha$ for $i = 1, 2$ and $\pi_i(b) = \beta$ for $i = 3, 4, 5$. Since $\#(\pi_i(S)) = 2$ for all i, then $\pi_1(c) = \pi_2(c) = \beta$. Since a and c have at most 2 common projections, then we may assume $\pi_i(c) = \alpha$ for $i = 3, 4$ and $\pi_5(c) = \beta$. Thus $S = \{a, b, c\}$ is such that

$$a = (\alpha, \alpha, \alpha, \alpha, \alpha),\ b = (\alpha, \alpha, \beta, \beta, \beta),\ c = (\beta, \beta, \alpha, \alpha, \beta)$$

and up to a permutation of the elements of S and of the factors of Y_{1^5}, there is a unique such S.

By direct computation one can see that $h^0\left(\mathcal{I}_{(2S, Y_{1^5})}(1, 1, 1, 1, 1)\right) = 14$ and consequentially $h^1\left(\mathcal{I}_{(2S, Y_{1^5})}(1, 1, 1, 1, 1)\right) = 0$ contradicting the assumption.

(ii) Assume $k = 6$. We have $t \leq 3$. Moreover, since $\#(\pi_i(S)) = 2$ for all i, then $t \geq 2$. We distinguish two different cases in dependence on the value $t \in \{2, 3\}$.

Assume $t = 3$. Permuting if necessary the factors of Y_{1^6}, we may assume $\pi_1(b) = \pi_2(b) = \pi_3(b) = \alpha$ and $\pi_4(b) = \pi_5(b) = \pi_6(b) = \beta$. Thus since $\#(\pi_i(S)) = 2$ for all i's, then $\pi_1(c) = \pi_2(c) = \pi_3(c) = \beta$. Moreover c and a can have 2 or 3 common components. In the first case $S = \{a, b, c\}$ is such that

$$a = (\alpha, \alpha, \alpha, \alpha, \alpha, \alpha),\ b = (\alpha, \alpha, \alpha, \beta, \beta, \beta),\ c = (\beta, \beta, \beta, \beta, \alpha, \alpha).$$

In the second case $S = \{a, b, c\}$ is such that

$$a = (\alpha, \alpha, \alpha, \alpha, \alpha, \alpha),\ b = (\alpha, \alpha, \alpha, \beta, \beta, \beta),\ c = (\beta, \beta, \beta, \alpha, \alpha, \alpha).$$

We remark that up to permuting the factors of Y_{1^6} and relabeling the elements of S, these are the only cases for $t = 3$. As before, by direct computation, one gets for both cases $\delta(2S, Y_{1^6}) = 0$ contradicting the assumption.

Assume $t = 2$. Permuting the factors of Y we may assume $\pi_1(b) = \pi_2(b) = \alpha$ (and hence $\pi_1(c) = \pi_2(c) = \beta$) and $\pi_3(b) = \pi_4(b) = \pi_5(b) = \pi_6(b) = \beta$. Since $\#\{\pi_i(a), \pi_i(c)\} = \#\{\pi_i(b), \pi_i(c)\} = 1$ for at most 2 indices, among the set $\{3, 4, 5, 6\}$ exactly 2 i's have $\pi_i(c) = \beta$, while the other ones have $\pi_i(c) = \alpha$. Thus $S = \{a, b, c\}$ is such that

$$a = (\alpha, \alpha, \alpha, \alpha, \alpha, \alpha),\ b = (\alpha, \alpha, \beta, \beta, \beta, \beta),\ c = (\beta, \beta, \beta, \beta, \alpha, \alpha).$$

Up to relabeling the points of S and a permutation of the factors of Y_{1^6}, there is a unique such S. By direct computation one gets $h^0\left(\mathcal{I}_{(2S, Y_{1^6})}(1,1,1,1,1,1)\right) = 20$, so $\delta(2S, Y_{1^6}) = 0$ contradicting the assumption.

(iii) Now assume $k \geq 7$. Exchanging if necessary the names of the points of S we may assume $\pi_1(a) = \pi_1(b) = \alpha$ and hence $\pi_1(c) \neq \alpha$. For any $t \in \mathbb{P}^1$ set $S_t := \{a, b, c_t\}$, where $\pi_1(c_t) := t$ and $\pi_i(c_t) := \pi_i(c)$ for all $i > 1$. Since any two of points of S differ in at least 3 coordinates, $\#(S_t) = 3$ for all t. Since $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive, for each $t \in \mathbb{P}^1 \setminus \{\pi_1(a)\}$ there is

$$g_t \in ((\mathrm{Aut}(\mathbb{P}^1))^k) \subset \mathrm{Aut}(Y)$$

such that $g_t(S_t) = S$. Thus $\delta(2S) = \delta(2S_t)$ for all $t \in \mathbb{P}^1 \setminus \{\pi_1(a)\}$. Denote by $a_1 := \pi_1(a)$, by the semicontinuity theorem for cohomology it is sufficient to prove $\delta(2S_{a_1}, Y_{1^k}) = 0$.

To show that $\delta(2S_{a_1}, Y_{1^k}) = 0$, we proceed by induction on the integer $n := k - 7$.

Assume $n = 0$, i.e. $k = 7$. Since $h^0(\mathcal{O}_{Y_{1^k}}(\varepsilon_1)) = 2$, $|\mathcal{I}_a(\varepsilon_1)|$ is a singleton. Set $\{H\} := |\mathcal{I}_a(\varepsilon_1)|$, so $H \supset S_{a_1}$ by definition. Since any two points of S differs in at least 3 coordinates, by Lemma 1.2.11 we know that $h^1\left(H, \mathcal{I}_{S_{a_1}}(\hat{\varepsilon}_1)\right) = 0$. By case (b) of Lemma 1.2.14 we know that $\delta(2S_{a_1}, Y_{1^k}) = \delta(2S_{a_1}, H)$. In item (ii) we proved that for any subset $S \subset (\mathbb{P}^1)^6$ of three points such that any two of them have at least 3 distinct components, then $\delta(2S, (\mathbb{P}^1)^6) = 0$. Thus $\delta(2S_{a_1}, H) = 0$, and hence $\delta(2S_{a_1}, Y_{1^k}) = 0$.

Assume now $n > 0$, i.e. $k > 7$. As before, we set $\{H\} := |\mathcal{I}_a(\varepsilon_1)|$, so $H \supset S_{a_1}$ by definition. By the same argument we get $\delta(2S_{a_1}, Y_{1^k}) = \delta(2S_{a_1}, H)$. If c_{a_1} differs from a and from b in at least 3 coordinates, then the inductive assumption gives

$\delta(2S_{a_1}, H) = 0$ and hence $\delta(2S, Y_{1^k}) = 0$. We conclude since $k > 7$ and $\#(\pi_i(S)) = 2$ for all i, so not all pairs of points of S may differ in only 3 coordinates.

Thus we proved that for all $k \geq 7$, then $\delta(2S_{a_1}, Y_{1^k}) = 0$, so by the semicontinuity theorem for cohomology, for all $k \geq 7$ we get $\delta(2S, Y_{1^k}) = 0$ contradicting the assumption. □

This concludes the case of $Y_{1^k} = (\mathbb{P}^1)^k$ for all $k \geq 3$. Therefore let $Y_{n_1,\dots,n_k} \ncong Y_{1^k}$ and let us distinguish different cases based on the number of factors of $Y_{n_1,\dots,n_k}$. The following three lemmas treat the 3-factors case depending on whether the integer $n_1 + n_2 + n_3$ is equal to $4, 5$ or 6 respectively.

Lemma 3.3.8. *Let $Y_{2,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. If $S \in \mathbb{T}(Y_{2,1,1}, 3)$ then S is either as in Example 3.2.1 or as in Example 3.2.3.*

Proof. Let $S \subset Y_{2,1,1}$, with $\#(S) = 3$, be such that $Y_{2,1,1}$ is the minimal multiprojective space containing S, i.e. $\pi_{1|S}$ is injective, $\dim\langle\pi_1(S)\rangle = \mathbb{P}^2$ and $\#(\pi_i(S)) \geq 2$ for all $i \in \{2, 3\}$. We remark that S is as in Example 3.2.3 or as in Example 3.2.1 if and only if there exists an index $i \in \{2, 3\}$ such that $\#(\pi_i(S)) = 2$.

Assume by contradiction that S is neither as in Example 3.2.3 nor Example 3.2.1, i.e. assume that $\#(\pi_i(S)) = 3$ for $i = 2, 3$. Since $\mathrm{Aut}(\mathbb{P}^2)$ is transitive on the set of triples of linearly independent points of $\mathbb{P}^2$ and $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive, S is in the open orbit for the action of $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$ on the set of three points of $Y_{2,1,1}$. So we can deal with a general set $S \subset Y_{2,1,1}$ of cardinality three. By Theorem 1.1.24 we know that $\sigma_3(X_{2,1,1})$ is not defective, so $\sigma_3(X_{2,1,1}) = \mathbb{P}^{11}$ and hence $h^0\left(\mathcal{I}_{(2S, Y_{2,1,1})}(1,1,1)\right) = 0$, contradicting the assumption. □

Lemma 3.3.9. *Let $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. Then each $S \in \mathbb{T}(Y_{2,2,1}, 3)$ is as in Example 3.2.3 for $k = 3$ and $n_1 = n_2 = 2$.*

Proof. Set

$$\mathcal{U} := \{S \subset Y_{2,2,1} \mid \#(S) = 3 \text{ and } Y_{2,2,1} \text{ is the minimal multiprojective space containing S}\}.$$

So any $S \in \mathcal{U}$ is such that $\#(\pi_3(S)) \geq 2$, $\pi_{1|S}$ and $\pi_{2|S}$ are injective and $\dim\langle\pi_1(S)\rangle = \dim\langle\pi_2(S)\rangle = 2$. The group $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathcal{U}$ with exactly 2 orbits:

1. $\#(\pi_3(S)) = 3$;
2. $\#(\pi_3(S)) = 2$.

Call O_1 the first orbit and O_2 the second one. Obviously for all $S, \tilde{S}$ in the same orbit we have $h^1(\mathcal{I}_{(2S, Y_{2,2,1})}(1,1,1)) = h^1(\mathcal{I}_{(2\tilde{S}, Y_{2,2,1})}(1,1,1))$. Among the elements of O_1 there is the general subset of $Y_{2,2,1}$ with cardinality 3. Since $\sigma_3(\nu(Y_{2,2,1})) = \mathbb{P}^{17}$ (cf. Theorem 1.1.24) $h^1\left(\mathcal{I}_{(2S, Y_{2,2,1})}(1,1,1)\right) = 0$ for all $S \in O_1$. We conclude since the elements of O_2 are exactly the sets S described in Example 3.2.3 for $n_1 = n_2 = 2$ and $k = 3$. □

Lemma 3.3.10. *Let $Y_{2,2,2} = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. The 3-rd Terracini locus $\mathbb{T}(Y_{2,2,2}, 3)$ is empty.*

Proof. Let $S \subset Y_{2,2,2}$, with $\#(S) = 3$, be such that $Y_{2,2,2}$ is the minimal multiprojective space that contains S, i.e. $\pi_{i|S}$ is injective and $\dim\langle\pi_i(S)\rangle = 2$ for all $i = 1, 2, 3$. By the action of $(\mathrm{Aut}(\mathbb{P}^2))^3$, we can reduce to work with a general set $S \subset Y_{2,2,2}$ of cardinality three. Since $\sigma_3(X_{2,2,2})$ is not defective (cf. Theorem 1.1.24) we know that $\dim(\sigma_3(X_{2,2,2})) = 20$, so $h^0(\mathcal{I}_{(2S, Y_{2,2,2})}(1,1,1)) = 6$. Hence, by the restriction exact sequence, $\delta(2S, Y_{2,2,2}) = 0$. □

With the previous proposition we are done with multiprojective spaces of $k = 3$ factors. Let us focus for the moment on the integer $k \geq 5$. In this case we deal with $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ where all $n_i \in \{1,2\}$. We will prove by induction on the integer $t := \dim Y_{n_1,\dots,n_k} - k$ that any $S \subset Y_{n_1,\dots,n_k}$, with $\#(S) = 3$, that belongs to $\mathbb{T}(Y_{n_1,\dots,n_k}, 3)$ is either as in Example 3.2.1 or as in Example 3.2.3, by using Lemma 3.3.7 as a base case $t = 0$.

Lemma 3.3.11. *Let $Y_{n_1,\dots,n_k} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where $k \geq 5$ and $n_i \in \{1,2\}$ for all i's. If $S \in \mathbb{T}(Y_{n_1,\dots,n_k}, 3)$ then S is either as in Example 3.2.1 or as in Example 3.2.3. In particular $\mathbb{T}(Y_{n_1,\dots,n_k}, 3) = \emptyset$, unless $n_i = 1$ for at least $k-2$ indices i.*

Proof. We proceed by induction on the integer $t := \dim Y_{n_1,\dots,n_k} - k$.
The base case $t = 0$ corresponds to Lemma 3.3.7. Assume $t > 0$ and that the lemma is true for any multiprojective space $Y_{n_1,\dots,n_k}$ of dimension at most $k + t - 1$. Since $t > 0$, there exists at least an index i such that $n_i = 2$, without loss of generality we may assume $i = 1$. Fix $S \in \mathbb{T}(Y_{2,n_2,\dots,n_k}, 3)$. So we know that $\delta(2S, Y_{2,n_2,\dots,n_k}) > 0$ and $Y_{2,n_2,\dots,n_k}$ is the minimal multiprojective space containing S. Thus $\pi_{1|S}$ is injective and $\langle \pi_1(S) \rangle = \mathbb{P}^2$. Fix $o \in \mathbb{P}^2 \setminus \pi_1(S)$. Choose a system of homogeneous coordinates $\{x_0, x_1, x_2\}$ of $\mathbb{P}^2$ such that $o = [1:0:0]$, the line $L := \{x_0 = 0\}$ contains no point of $\pi_1(S)$ and o is not contained in one of the 3 lines spanned by 2 of the points of $\pi_1(S)$. Let $\ell_o : \mathbb{P}^2 \setminus \{o\} \longrightarrow L$ denote the linear projection from o, i.e. the rational map defined by $[a_0 : a_1 : a_2] \mapsto [0 : a_1 : a_2]$.

Write $Y_{2,n_2,\dots,n_k} = \mathbb{P}^2 \times Y'$ with $Y' = \prod_{i>1} \mathbb{P}^{n_i}$ and set $H := L \times Y' \in |\mathcal{O}_{Y_{2,n_2,\dots,n_k}}(\varepsilon_1)|$. The morphism ℓ_o extends to a morphism

$$\begin{aligned} f_o : \left(\mathbb{P}^2 \setminus \{o\}\right) \times Y' &\longrightarrow H \\ (a,b) &\mapsto (\ell_o(a), b), \end{aligned}$$

We remark that $\#(f_o(S)) = 3$ and that H is the minimal multiprojective subspace of $Y_{2,n_2,\dots,n_k}$ containing $f_o(S)$.
For each $\lambda \in \mathbb{K} \setminus \{0\}$ let $u_\lambda : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ denote the automorphism of $\mathbb{P}^2$ defined by the formula $[a_0 : a_1 : a_2] \mapsto [\lambda a_0 : a_1 : a_2]$. Let $\mathbb{K}' \subseteq \mathbb{K} \setminus \{0\}$ be the set of all $\lambda \in \mathbb{K} \setminus \{0\}$ such that no line spanned by 2 of the points of $u_\lambda(\pi_1(S))$ contains o. For each $\lambda \in \mathbb{K}'$ we have $\#(u_\lambda(\pi_1(S))) = 3$ and $u_\lambda(\pi_1(S))$ spans $\mathbb{P}^2$.
For each $\lambda \in \mathbb{K}'$ define

$$\begin{aligned} g_\lambda : Y_{2,n_2,\dots,n_k} &\longrightarrow Y_{2,n_2,\dots,n_k} \\ (a,b) &\mapsto (u_\lambda(a), b). \end{aligned}$$

Composing f_o with the inclusion $j : H \subset Y_{2,n_2,\dots,n_k}$ we see that the rational map $j \circ f_o$ is a limit for λ going to 0 of the family $\{g_\lambda\}_{\lambda \in \mathbb{K}'}$ of automorphisms of $Y_{2,n_2,\dots,n_k}$. By the semicontinuity theorem for cohomology $\delta(2(j \circ f_o(S)), Y_{2,n_2,\dots,n_k}) \geq \delta(2S, Y_{2,n_2,\dots,n_k}) > 0$.

Claim 6. $\delta(2g_0(S), H) = \delta(2(j \circ f_o(S)), Y_{2,n_2,\dots,n_k})$.

Proof. Since $\dim Y_{2,n_2,\dots,n_k} = \dim H + 1$, part (a) of Lemma 1.2.14 gives

$$\delta(2g_0(S), H) \leq \delta(2(j \circ f_o(S)), Y_{2,n_2,\dots,n_k}) \leq \delta(2g_0(S), H) + h^1\left(\mathcal{I}_S(\hat{\varepsilon}_1)\right).$$

To conclude the proof of Claim 6 it is sufficient to prove that $h^1(\mathcal{I}_S(\hat{\varepsilon}_1)) = 0$. Assume $h^1(\mathcal{I}_S(\hat{\varepsilon}_1)) > 0$. By Lemma 1.2.11 either there are $u, v \in S$ such that $u \neq v$ and $\eta_1(u) = \eta_1(v)$ or there is $i \in \{2, \dots, k\}$ such that $\#(\pi_h(S)) = 1$ for all $h \in \{2, \dots, k\} \setminus \{i\}$. In the former case, i.e. if $\pi_i(u) = \pi_i(v)$ for all $i > 1$, S is as in Example 3.2.1. In the second case we are either in Example 3.2.3 or in Example 3.2.1 and for both cases we have $h^1(\mathcal{I}_S(\hat{\varepsilon}_1)) = 0$. □

By Claim 6 and the inequality $h^0(\mathcal{O}_H(1, \dots, 1)) > 3 \dim H$ (true because $k \geq 5$) $f_o(S) \in \mathbb{T}(H, 3)$. By the inductive assumption $f_o(S)$ is as in one of the Examples 3.2.3 or 3.2.1 and in particular $n_h = 1$ for at least $k-2$ of the last $(k-1)$ indices h, say for $h \in \{3, \dots, k\}$. Moreover there is $A \subset f_o(S)$ such that $\#(A) = 2$ and $\#(\pi_h(A)) = 1$ for all $h > 2$. Since f_o act as the identity on the last $(k-1)$ components of any $p \in Y_{2,n_2,\dots,n_k} \setminus H$, we get that S is described by the same Example which describes $f_o(S)$. □

The only case left in the discussion is the 4-factors case, where we work with mulit-projective spaces $Y_{n_1,\dots,n_4} \not\cong Y_{1^4}$.

Lemma 3.3.12. *Take $Y_{n_1,\dots,n_4} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3} \times \mathbb{P}^{n_4}$ with $n_i \in \{1, 2\}$ for all i and $n_1 + n_2 + n_3 + n_4 \geq 5$. If $S \in \mathbb{T}(Y_{n_1,\dots,n_k}, 3)$, then S is either as in Example 3.2.3 or as in Example 3.2.1.*

Proof. We will show the result by induction on the integer $t = n_1 + \cdots + n_4 - 5 \geq 0$. First assume $t = 0$, i.e. $n_1 + n_2 + n_3 + n_4 = 5$. With no loss of generality we may assume $Y_{2,1,1,1} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since $Y_{2,1,1,1}$ is the minimal multiprojective space containing S and $n_1 = 2$, then $\pi_{1|S}$ is injective. Assume for the moment $\pi_{i|S}$ injective for $i = 2, 3, 4$. Since $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive, S is in the same orbit for the action of $\mathrm{Aut}(\mathbb{P}^2) \times (\mathrm{Aut}(\mathbb{P}^1))^3$ of 3 general points of $Y_{2,1,1,1}$. We know that $\dim \sigma_3(\nu(Y_{2,1,1,1})) = 17$ (cf. Theorem 1.1.24), so $\delta(2S, Y_{2,1,1,1}) = 0$ contradicting the assumption. Thus we may assume $\#(\pi_i(S)) = 2$ for some $i \in \{2, 3, 4\}$. With no loss of generality we may assume that at least $\#(\pi_3(S)) = 2$. Since $\pi_{1|S}$ is injective, $\eta_{4|S}$ is injective. The set $\eta_4(S)$ is as in case (iv) of Proposition 3.2.5. Using η_2 and η_3 instead of η_4 we see the existence of at least two indices $h \in \{2, 3, 4\}$ such that $\#(\pi_h(S)) = 2$. With no loss of generality we may assume $\#(\pi_3(S)) = \#(\pi_4(S)) = 2$, i.e. neither $\pi_{3|S}$ nor $\pi_{4|S}$ are injective. If there is $S' \subset S$ such that $\#(S') = 2$ and $\#(\pi_3(S')) = \#(\pi_4(S')) = 1$, then we are in Example 3.2.3 or Example 3.2.1. The non-existence of such S' shows that we may name $S = \{a, b, c\}$ so that $\pi_4(a) = \pi_4(b)$, $\pi_3(a) = \pi_3(c)$. We distinguish two cases:

(i) $\#(\pi_2(S)) = 2$;

(ii) $\#(\pi_2(S)) = 3$.

Write $a = [a_1, a_2, a_3, a_4]$, $b = [b_1, b_2, b_3, b_4]$ and $c = [c_1, c_2, c_3, c_4]$. Since $\mathrm{Aut}(\mathbb{P}^2)$ is transitive on the set of all triples of linearly independent points, we may assume $a_1 = [1 : 0 : 0]$, $b_1 = [0 : 1 : 0]$ and $c_1 = [0 : 0 : 1]$. Since $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive we may assume $a_2 = a_3 = a_4 = \alpha$, $b_3 = \beta$, $b_4 = \alpha$, $c_3 = \alpha$ and $c_4 = \beta$, for some $\alpha \neq \beta \in \mathbb{P}^1$. Moreover, in case (i) we may assume $b_2 = c_2 = \beta$, while in case (ii) we may assume $b_2 = \beta$ and $c_2 = \gamma$, for some $\gamma \in \mathbb{P}^1$ with $\gamma \neq \alpha, \beta$. For both cases, by direct computation one gets $h^0\left(\mathcal{I}_{(2S, Y_{2,1,1,1})}(1, 1, 1, 1)\right) = 17$, so $\delta(2S, Y_{2,1,1,1}) = 0$ contradicting the assumption.

Now assume $t > 0$, i.e. $n_1 + n_2 + n_3 + n_4 \geq 6$. As in the proof of Lemma 3.3.11, we will use a linear projection from a general point of a 2-dimensional factor of $Y_{n_1,\dots,n_4}$ to conclude by induction on the integer $n_1 + n_2 + n_3 + n_4$. □

The previous proposition concludes the analysis on the study of the third Terracini locus of any multiprojective space. Now we collect all together the above results in the upcoming theorem. Before proceeding, note that the case of $Y_{2^k} = (\mathbb{P}^2)^k$ with $k \geq 4$ is already contained in both Lemma 3.3.12 and Lemma 3.3.11 but it can be easily treated as follows.

Remark 3.3.13. Let $S \subset Y_{2^k}$, with $\#(S) = 3$, be such that Y_{2^k} is the minimal multiprojective space containing S, i.e. $\pi_{i|S}$ is injective for all $i \leq k$. We can look at S as a general set of three distinct points by the action of $(\mathrm{Aut}(\mathbb{P}^2))^k$. By Theorem 1.1.24 $\sigma_3(X_{2^k})$ is never defective, therefore $h^1(\mathcal{I}_{(2S,Y_{2^k})}(1,\dots,1)) = 0$ and hence $\mathbb{T}(Y_{2^k}, 3) = \emptyset$.

We are ready to state the main theorem of the present chapter that completely describes the third Terracini locus of any set of three points.

Theorem 3.3.14. *Let $Y_{n_1,\dots,n_k}$ be the minimal multiprojective space of $k \geq 1$ factors containing a set S of 3 points, where all $n_i \in \{1,2\}$. Then the following characterization of the 3-rd Terracini locus holds.*

$$\mathbb{T}(Y_{n_1,\dots,n_k}, 3) \text{ is empty if and only if either } k = 1,2 \text{ or } Y_{2^k} = (\mathbb{P}^2)^k, \text{ for all } k \geq 3.$$

Moreover the non-empty $S \in \mathbb{T}(Y_{n_1,\dots,n_k}, 3)$ can only be either as in Example 3.2.1 or as in Example 3.2.3 or $Y_{1^4} = (\mathbb{P}^1)^4$, in this last case all $S \subset Y_{1^4}$ with $\#(S) = 3$ that have Y_{1^k} as minimal multiprojective space lie in $\mathbb{T}(Y_{1^4}, 3)$.

Proof. Let $S \in \mathbb{T}(Y_{n_1,\dots,n_k}, 3)$ be such that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing S, so $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is such that $n_i \in \{1,2\}$ for all $i = 1,\dots,k$.

If $k = 1$ we always have $h^0(\mathcal{I}_{2S}(1)) = 0$, thus the case of either $Y_2 = \mathbb{P}^2$ or $Y_1 = \mathbb{P}^1$ is clear.

Assume $k = 2$. In this case $Y_{n_1,n_2} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ with $1 \leq n_1 \leq 2$ and $1 \leq n_1 \leq 2$. If $n_1 = n_2 = 1$, then obviously $h^0(\mathcal{I}_{2S}(1,1)) = 0$. If $n_i = 2$, then $\pi_{i|S}$ is injective and $\pi_i(S)$ is linearly independent. Thus if $n_1 = n_2 = 2$, then S is in open orbit for the action of $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^2)$ on the set of three points of $Y_{2,2}$. Since a general 3×3 matrix has rank 3 we get $\sigma_3(\nu(Y_{2,2})) = \mathbb{P}^8$. Hence $h^0(\mathcal{I}_{(2S,Y_{2,2})}(1,1)) = 0$, contradicting the assumption $S \in \mathbb{T}(Y_{2,2}, 3)$. Now assume $n_i = 1$ for exactly one i, say for $i = 1$. Since $Y_{1,2}$ is the minimal multiprojective space containing containing S, $\#(\pi_1(S)) \geq 2$ and $\#(\pi_2(S)) = 3$. Thus there is $S' \subset S$ such that $\#(S') = \#(\pi_1(S')) = 2$. S' is in the open orbit for the action of $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^2)$ on $S_{Y_{1,2}}(2)$. Since a general 2×3 matrix has rank 2, $\sigma_2(\nu(Y_{1,2})) = \mathbb{P}^5$. Thus $h^0(\mathcal{I}_{(2S',Y_{1,2})}(1,1)) = 0$. Hence $h^0(\mathcal{I}_{(2S,Y_{1,2})}(1,1)) = 0$, contradicting the assumption $S \in \mathbb{T}(Y_{1,2}, 3)$. This concludes the case of two factors.

The case of $k = 3$ is completely covered by Lemmas 3.3.2, 3.3.10, 3.3.8 and 3.3.9.

In the case of $k = 4$ there is the defective 3-rd secant variety of the Segre embedding of $Y_{1^4} = (\mathbb{P}^1)^4$ (cf. Remark 3.3.4).

For any other couple $(S, Y_{n_1,\dots,n_4})$ where $Y_{n_1,\dots,n_k} \ncong (\mathbb{P}^1)^4$, Lemma 3.3.12 shows that S must be either as in Example 3.2.1 or as in Example 3.2.3.

If $k \geq 5$ it is sufficient to use Lemma 3.3.11. □

3.4 Computing the maximal Terracini defect

An interesting question related to the discussion we faced, is to determine the maximal value of defect $\delta(2S, Y_{n_1,\dots,n_k})$ that a set of points $S \subset Y_{n_1,\dots,n_k}$ could have and, as a consequence, to also understand for which multiprojective space it happens. The present section is dedicated to compute the maximal defect $\delta(2S, Y_{n_1,\dots,n_k})$, for all $S \subset Y_{n_1,\dots,n_k}$ and all $Y_{n_1,\dots,n_k}$. More precisely, note that, if we fix any multiprojective space $Y_{n_1,\dots,n_k}$ of dimension $n > 0$, then for any $p \in Y_{n_1,\dots,n_k}$ the very ampleness of $\mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)$ implies $h^1\left(\mathcal{I}_{(2p,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) = 0$. For any integer $r \geq 2$ there are many $S \subset Y_{n_1,\dots,n_k}$ with $\#S = r$ that have $\delta(2S, Y_{n_1,\dots,n_k}) > 0$. In the following we compute the maximal value of all $\delta(2S, Y_{n_1,\dots,n_k})$ for some multiprojective space $Y_{n_1,\dots,n_k}$ of dimension n. This maximal value is obtained when $n_1 = n$ and all $n_2 = \cdots = n_k = 0$ i.e. for $Y_n = \mathbb{P}^n$ (cf. Proposition 3.4.4). But of course in this case $h^0\left(\mathbb{P}^n, \mathcal{I}_{(2S,\mathbb{P}^n)}(1)\right) = 0$ for any finite set $S \subset \mathbb{P}^n$ with $S \neq \emptyset$. Therefore we will compute the maximal value of $\delta(2S, Y_{n_1,\dots,n_k})$ requesting that also $h^0\big(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\big) > 0$.

Before proceeding, we need to introduce the following objects that will be used only in the present section. They encode coulpes $(Y_{n_1,\dots,n_k}, S)$ of multiprojective spaces $Y_{n_1,\dots,n_k}$ and set of points S such that $S \subset Y_{n_1,\dots,n_k}$ once fixed the integers $n := \sum_{i\leq k} n_i$ and $r := \#(S)$.

Notation 3.4.1. Let $Y_{n_1,\dots,n_k}$ be any multiprojective space of k factors. For all positive integers r define

$$S_{Y_{n_1,\dots,n_k}}(r) := \{S \subset Y_{n_1,\dots,n_k} \mid \#S = r\}.$$

Definition 3.4.2. For any integer $n > 0$, denote by $\mathcal{U}(n)$ the set of all isomorphism classes of multiprojective spaces $Y_{n_1,\dots,n_k}$ such that $\dim Y_{n_1,\dots,n_k} = n$.

For any integer $r \geq 2$, $n \geq 2$ define

$$\begin{aligned}
\mathcal{E}(n,r) :=& \{(Y_{n_1,\dots,n_k}, S) \in \mathcal{U}(n) \times S_{Y_{n_1,\dots,n_k}}(r) \mid \delta(2S, Y_{n_1,\dots,n_k}) h^0\big(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\big) > 0\},\\
\mathbb{E}(n,r) :=& \{(Y_{n_1,\dots,n_k}, S) \in \mathcal{U}(n) \times S_{Y_{n_1,\dots,n_k}}(r) \mid S \in \mathbb{T}(Y_{n_1,\dots,n_k}, r)\}.
\end{aligned}$$

We remark that in the definition of $\mathcal{E}(n,r)$ we do not require the points of S to lie in the minimal multiprojective space containing them.

The set of all (n,r) such that $\mathcal{E}(n,r) \neq \emptyset$ will be easily computed in Lemma 3.4.7 and we will show that $\mathcal{E}(n,r) \neq \emptyset$ if and only if $n \geq 3$ and $r \geq 2$. We introduce now a notation for the maximal value of $\delta(2S, Y_{n_1,\dots,n_k})$ once fixed integers n, r.

Notation 3.4.3. Fix integers $n, r > 0$. Denote by

$$\delta_1(n,r) := \max\{\delta(2S, Y_{n_1,\dots,n_k}) \mid (Y_{n_1,\dots,n_k}, S) \in \mathcal{E}(n,r)\}.$$

If we prescribe that $(Y_{n_1,\dots,n_k}, S) \in \mathbb{E}(n,x)$, i.e. if we assume that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing S, then we get the definition of the integer $\delta(n,x)$.

In Proposition 3.4.8 we will show that

$$\delta_1(n,r) = (r-1)(n+1) - 1.$$

In order to do so, let us start by finding an upper bound of $\delta(2S, Y_{n_1,\dots,n_k})$, for any couple $(Y_{n_1,\dots,n_k}, S)$ such that $S \subset Y_{n_1,\dots,n_k}$.

Proposition 3.4.4. *Fix integers $n > 0$ and $r \geq 2$. Fix $Y_{n_1,\dots,n_k} \in \mathcal{U}(n)$ and $S \in S_{Y_{n_1,\dots,n_k}}(r)$. Then*

$$h^1\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) \leq (r-1)(n+1)$$

and equality holds if and only if the multiprojective space is $Y_n = \mathbb{P}^n$.

Proof. Fix $Y_{n_1,\dots,n_k} \in \mathcal{U}(n)$, with $n_i > 0$ for all i's and $n_1+\cdots+n_k = n$ and assume $k \geq 2$, i.e. assume $Y_{n_1,\dots,n_k} \ncong \mathbb{P}^n$. Fix $S \in S_{Y_{n_1,\dots,n_k}}(r)$ and take $o \in S$. Since $\mathcal{O}_{Y_{n_1,\dots,n_k}}(1,\dots,1)$ is very ample, we have $h^1\left(\mathcal{I}_{(2o,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) = 0$. Therefore we can easily bound $\delta(2S, Y_{n_1,\dots,n_k})$ as follows

$$h^1\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) \leq \deg\left(2(S\setminus\{o\}), Y_{n_1,\dots,n_k}\right) = (r-1)(n+1)$$

(cf. (1.2.4) of Remark 1.2.13). This concludes the proof of the inequality.

The "if" part of the equality is clear, so we just need to prove the "only if" part. We will use induction on the integer n starting with the case $n = 2$.

Let $n = 2$ and assume by contradiction that $Y_{n_1,\dots,n_k} \ncong \mathbb{P}^2$, so we are working with $Y_{1,1} = \mathbb{P}^1 \times \mathbb{P}^1$. Thus $h^0\left(\mathcal{O}_{Y_{1,1}}(1,1)\right) = 4$. Since each tangent plane to $\nu(\mathbb{P}^1\times\mathbb{P}^1)$ is tangent at a unique point of the smooth quadric $\nu(\mathbb{P}^1\times\mathbb{P}^1)$ and $r \geq 2$, we have $h^0\left(\mathcal{I}_{(2S,Y_{1,1})}(1,1)\right) = 0$ and hence $h^1\left(\mathcal{I}_{(2S,Y_{1,1})}(1,1)\right) = 3(r-1)-1 \neq 3(r-1)$. Now assume $n > 2$. We distinguish two different cases depending on whether $r = 2$ or not.

(a) Assume $r = 2$ and write $S = \{u, v\}$. Assume by contradiction that $Y_{n_1,\dots,n_k} \ncong \mathbb{P}^n$. We remark that by assumption $h^1\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) = n+1$. This implies that the Zariski tangent spaces $T_{\nu(u)}\nu(Y_{n_1,\dots,n_k})$ and $T_{\nu(v)}(Y_{n_1,\dots,n_k})$ of $\nu(Y_{n_1,\dots,n_k})$ at $\nu(u)$ and $\nu(v)$ are the same. Since $\nu(v) \in T_{\nu(u)}\nu(Y_{n_1,\dots,n_k})$, the line $L := \langle\{\nu(v),\nu(u)\}\rangle$ contains two points of $T_{\nu(u)}\nu(Y_{n_1,\dots,n_k})$ and hence it is contained in $T_{\nu(u)}\nu(Y_{n_1,\dots,n_k})$. Since $\nu(u) \in L$, L is tangent to $\nu(Y_{n_1,\dots,n_k})$ at $\nu(u)$. Hence $L \cap \nu(Y_{n_1,\dots,n_k})$ contains a zero-dimensional scheme of degree strictly greater than 2. Since $\nu(Y_{n_1,\dots,n_k})$ is scheme-theoretically cut out by quadrics, we get $L \subset \nu(Y_{n_1,\dots,n_k})$, i.e. there is $D \subset Y_{n_1,\dots,n_k}$, such that $\nu(D) = L$, $D \cong \mathbb{P}^1$ and $\#(\pi_i(D)) = 1$ for $k-1$ indices i. Let $i \in \{1,\dots,k\}$ be the index such that $\#(\pi_i(S)) \neq 1$. Since $\#(\pi_j(S)) = 1$ for all $j \neq i$ and $S \subset D$, by case (b) of Lemma 1.2.14, we know that $\delta(2S, Y_{n_1,\dots,n_k}) = \delta(2S, \mathbb{P}^{n_i})$, where $n_i < n$. By the inductive assumption we get $\delta(2S,\mathbb{P}^{n_i}) = n_i + 1 < n+1$ which is absurd since by assumption $\delta(2S, Y_{n_1,\dots,n_k}) = n+1$.

(b) Assume $r > 2$. Write $S = A \cup B$ with $\#(A) = 2$ and $\#(B) = r-2$. By part (a) we have $h^1\left(\mathcal{I}_{(2A,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) \leq n$. Thus by 1.2.4 of Remark 1.2.13 we get $h^1(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)) \leq h^1(\mathcal{I}_{(2A,Y_{n_1,\dots,n_k})}(1,\dots,1)) + \deg(2B, Y_{n_1,\dots,n_k}) \leq n + (r-2)(n+1)$, which is absurd since by assumption $h^1\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) = (r-1)(n+1)$. □

Remark 3.4.5. Let $n > 0$ and $r \geq 2$. By Proposition 3.4.4, for all $Y_{n_1,\dots,n_k} \in \mathcal{U}(n)$ and $S \in S_{Y_{n_1,\dots,n_k}}(r)$ the maximum value of $h^1\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right)$ is achieved when the multiprojective space is $\mathbb{P}^n$. Clearly, in this case, $h^0\left(\mathcal{I}_{(2S,Y_n)}(1,\dots,1)\right) = 0$. Thus the couple $(Y_{n_1,\dots,n_k}, S) \in \mathcal{U}(n) \times S_{Y_{n_1,\dots,n_k}}(r)$ evincing $\delta_1(n,r)$ is such that $Y_{n_1,\dots,n_k}$ is a multiprojective space with $k \geq 2$ factors.

The following example shows that for a precise family of couples $(Y_{n_1,\dots,n_k}, S)$, where we take $Y_{n_1,\dots,n_k}$ as a multiprojective space with at least two factors, the value $(r-1)(n+1)-\mu$ is attained for $\delta(2S, Y_{n_1,\dots,n_k})$ for any positive integer $\mu \leq n-1$. Moreover we will show in Theorem 3.4.8 below that this example is the only one with maximal $\delta_1(n,r)$.

Example 3.4.6. Let $n \geq 3$, fix an integer $1 \leq \mu \leq n-1$ and let $r \geq \mu+1$. Let $L \subset \mathbb{P}^{n-1}$ be a μ-dimensional linear subspace and let $Y_{n-1,1} := \mathbb{P}^{n-1} \times \mathbb{P}^1$. Fix $o \in \mathbb{P}^1$ and a finite set $S \subset L \times \{o\}$ with $\#S = r$ and such that $\langle \pi_1(S) \rangle = L$. The aim of this example is to show that

$$\delta(2S, Y_{n-1,1}) = (r-1)(n+1) - \mu.$$

Take $H := \pi_2^{-1}(o) \in |\mathcal{O}_{Y_{n-1,1}}(\varepsilon_2)|$. Note that $S \subset H$. Thus the residual exact sequence of $(2S, Y_{n-1,1})$ with respect to H is

$$0 \longrightarrow \mathcal{I}_S(1,0) \longrightarrow \mathcal{I}_{(2S,Y_{n-1,1})}(1,1) \longrightarrow \mathcal{I}_{(2S,H),H}(1,1) \longrightarrow 0. \tag{3.4.2}$$

We remark that $S \neq \emptyset$ and in particular $\#(S) \geq 2$. Moreover, since$H \cong \mathbb{P}^{n-1}$ we get $h^0\left(H, \mathcal{I}_{(2S,H)}(1,1)\right) = 0$. Since by assumption $\langle \pi_1(S) \rangle = L$, where $\dim L = \mu$, we get $h^0\left(\mathcal{I}_S(1,0)\right) = n-1-\mu$. So by (3.4.2) we get $h^0\left(\mathcal{I}_{(2S,Y_{n-1,1})}(1,1)\right) = n-1-\mu$. Thus

$$\delta(2S, Y) = r(n+1) - 2n + n - \mu - 1 = (r-1)(n+1) - \mu.$$

In particular for $\mu = 1$, i.e. if L is a line, we obtain $\delta(2S, Y_{n-1,1}) = (r-1)(n+1) - 1$. Since $h^0(\mathcal{O}_{Y_{n-1,1}}(1,1)) = 2n$ and $\deg(2S, Y_{n-1,1}) = r(n+1)$, when $\mu = 1$ we get $h^0(\mathcal{I}_{(2S,Y_{n-1,1})}(1,1)) = 2n - r(n+1) + (r-1)(n+1) - 1 = n-2 > 0$.
Thus if $\mu = 1$ then $\delta(2S, Y_{n-1,1}) h^0(\mathcal{I}_{(2S,Y_{n-1,1})}(1,1)) > 0$ and in particular $\delta_1(S, Y_{n-1,1}) = (r-1)(n+1) - 1$.
Obviously also $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ gives an example, taking an L in the second factor of $Y_{1,n-1}$.

In Remark 3.4.5 we noted that we should work with a multiprojective space $Y_{n_1,\dots,n_k}$ of at least $k \geq 2$ factors in order to have both $h^0(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1))$ and $\delta(2S, Y_{n_1,\dots,n_k})$ non-zero. In addition to this, we have to consider multiprojective spaces of dimension $n \geq 3$.

Lemma 3.4.7. *Fix integers $n \geq 2$ and $r \geq 2$. $\mathcal{E}(n,r) \neq \emptyset$ if and only if $n \geq 3$.*

Proof. For $n = 2$ we remark that $\mathcal{U}(2) = \{[\mathbb{P}^2], [\mathbb{P}^1 \times \mathbb{P}^1]\}$. For both cases we get $h^0\left(\mathcal{I}_{(2S,\mathbb{P}^2)}(1)\right) = h^0(\mathcal{I}_{(2S,\mathbb{P}^1\times\mathbb{P}^1)}(1,1)) = 0$ (cf. e.g. proof of Proposition 3.4.4). Viceversa, if $n \geq 3$ we may take $Y_{n-1,1} = \mathbb{P}^{n-1} \times \mathbb{P}^1$ and S as in Example 3.4.6. $\square$

Let us finally prove what is the maximum value that $\delta(2S, Y_{n_1,\dots,n_k})$ can achieve, providing that also $h^0(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)) > 0$.

Theorem 3.4.8. *Fix integers $n \geq 3$ and $r \geq 2$. Then $\delta_1(n,r) = (r-1)(n+1) - 1$ and any $(Y_{n_1,\dots,n_k}, S)$ evincing $\delta_1(n,r)$ is as in Example 3.4.6 with $\mu = 1$.*

Proof. By Remark 3.4.5 we may work with multiprojective spaces $Y_{n_1,\dots,n_k}$'s of $k \geq 2$ factors. So, by Proposition 3.4.4, for all $(Y_{n_1,\dots,n_k}, S)$

$$\delta(2S, Y_{n_1,\dots,n_k}) \leq (r-1)(n+1) - 1.$$

The case $\mu = 1$ of Example 3.4.6 gives the inequality $\delta_1(n,r) \geq (r-1)(n+1) - 1$. Thus it remains to prove that this is the only case.

Fix $(Y_{n_1,\dots,n_k}, S)$ evincing $\delta_1(n,r)$. Thus $Y_{n_1,\dots,n_k} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ where all $n_i > 0$ and are such that $n_1 + \cdots + n_k = n$. The finite set $S \in S_{Y_{n_1,\dots,n_k}}(r)$, is such that $h^0\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) > 0$ and $h^1\left(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}(1,\dots,1)\right) \geq (r-1)(n+1) - 1$.

We will show the result by induction on $n \geq 3$.

If $n=3$ then $\mathcal{U}(3)=\{[\mathbb{P}^3],[\mathbb{P}^2\times\mathbb{P}^1],[(\mathbb{P}^1)^3]\}$. Clearly the case $Y_3=\mathbb{P}^3$ is excluded by Remark 3.4.5. If $Y_{2,1}=\mathbb{P}^2\times\mathbb{P}^1$, it suffices to show that for any other r-uple of points $\hat{S}\in S(Y_{2,1},r)$ that is not as in Example 3.4.6, we get $\delta(2\hat{S},Y_{2,1})<4(r-1)-1$. If $r=2$ this is true since $\delta(2S,Y_{2,1})=2$ unless $S\in S_{Y_{2.1}}(2)$ is as in Example 3.4.6. If $r\geq 3$ then $h^0\left(\mathcal{I}_{(2S,Y_{2,1})}(1,1)\right)=0$ for all $S\in S_{Y_{2,1}}(r)$.
Let $Y_{1,1,1}=(\mathbb{P}^1)^3$. By Proposition 3.1.3 we exclude the case $r=2$ since either $\delta(2S,Y_{1,1,1})$ or $h^0\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right)$ is zero. If $r=3$, Lemma 3.3.2 gives the only cases for which $S\in\mathbb{T}(Y_{1,1,1},3)$ and for such cases we already proved that $\delta(2S,Y_{1,1,1})=5<\delta_1(3,3)$ and $h^0\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right)=1$. Thus for $r\geq 4$ we get $h^0\left(\mathcal{I}_{(2S,Y_{1,1,1})}(1,1,1)\right)=0$ for all $S\in S_{Y_{1,1,1}}(r)$ that are not as in Example 3.4.6.

Assume that the proposition is true for all $n'<n$. We will prove the inductive step by induction on $r\geq 2$. Case (a) will be the base case and in case (b) we will show the inductive step.

(a) Assume $r=2$ and let $L:=\langle\nu(S)\rangle$.

First assume that we are dealing with a multiprojective space of $k=2$ factors, i.e. $Y_{n_1,n_2}=\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}$. With no loss of generality we may assume $n_1\geq n_2$. To conclude this case it is sufficient to prove that $n_2=1$ and $\#(\pi_2(S))=1$ and we will do it by contradiction.

First assume $n_2\geq 2$. Since $h^0\big(\mathcal{O}_{Y_{n_1,n_2}}(0,1)\big)=n_2+1>2$, there is $M\in|\mathcal{I}_S(0,1)|$. Thus $S\subset M$. If (S,M) is as in Example 3.4.6 there is nothing to prove, otherwise by the inductive step we get $h^1\big(M,\mathcal{I}_{(2S,M)}(1,1)\big)\leq n-2$. Since $\dim Y_{n_1,n_2}=\dim M+1$, part (a) of Lemma 1.2.14 gives $h^1\big(\mathcal{I}_{(2S,Y_{n_1,n_2})}(1,1)\big)\leq n-2+1<n$ which is absurd since we took (Y_{n_1,n_2},S) evincing $\delta_1(2,n)=n$.

Assume now that $\#(\pi_2(S))=2$. Again if $\#(\pi_1(S))=1$ then S is as in Example 3.4.6, so assume also $\#(\pi_1(S))=2$. Thus the minimal multiprojective space containing S is $Y_{1,1}=\mathbb{P}^1\times\mathbb{P}^1$. So S is in the open orbit for the action of $(\mathrm{Aut}(\mathbb{P}^1))^2$ on $S(Y_{1,1},3)$. Hence $h^0\left(\mathcal{I}_{(2S,Y_{1,1})}(1,1)\right)=0$ and consequently, since $\deg(2S,Y_{1,1})=15$ and $h^0\left(\mathcal{O}_{Y_{1,1}}(1,1)\right)=9$, we get $\delta(2S,Y_{1,1})=6<\delta_1(4,3)$.

Assume now that we are dealing with a multiprojective space $Y_{n_1,\dots,n_k}$ of $k>2$ factors. By Lemma 1.2.14 and the equality $\delta_1(n',2)=(r-1)(n'+1)-1$ for all $n'<n$, we know that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing S. Thus we are working with $Y_{1^k}=(\mathbb{P}^1)^k$. Fix $H\in|\mathcal{O}_{Y_{1^k}}(\varepsilon_k)|$ containing at least on point of S. Since $S\nsubseteq H$, $\#(S\cap H)=\#(S\setminus S\cap H)=1$. Denote by $S:=\{a,b\}$ and by relabeling if necessary, assume $S\cap H=\{a\}$ and $S\setminus S\cap H=\{b\}$.

Consider the residual exact sequence of H:

$$0\longrightarrow\mathcal{I}_{(2b,Y_{1^k})\cup(a,Y_{1^k})}(\hat{\varepsilon}_k)\longrightarrow\mathcal{I}_{(2S,Y_{1^k})}(1,\dots,1)\longrightarrow\mathcal{I}_{(2a,H),H}(1,\dots,1)\longrightarrow 0.\qquad(3.4.3)$$

Since $\#(S\cap H)=1$ and $\mathcal{O}_H(1,\dots,1)$ is very ample, $\delta(2a,H)=0$. Since $\#(S\setminus S\cap H)=1$, $\mathcal{O}_{Y_{1^{k-1};k}}(1,\dots,1)$ is very ample and $\dim Y_{1^k}-\dim Y_{1^{k-1};k}=1$, we have $h^1\big(\mathcal{I}_{(2b,Y_{1^k})}(\hat{\varepsilon}_k)\big)=0$. Since $\#(S\cap H)=1$, $h^1\big(H,\mathcal{I}_{(2b,Y_{1^k})\cup(a,Y_{1^k})}(\hat{\varepsilon}_k)\big)\leq 1$. Thus (3.4.3) gives $h^1\big(\mathcal{I}_{(2S,Y_{1^k})}(1,\dots,1)\big)\leq 1<n$, a contradiction.

(b) Assume now $r\geq 3$. Fix any $A\subset S$ such that $\#(A)=r-1$. Since $\delta(2S,Y)\leq\delta(2A,Y)+n+1$ (cf. Remark 1.2.13), the inductive assumption tells us that the pair $(Y_{n_1,\dots,n_k},A)$ is as in Example 3.4.6. Thus either $Y_{n_1,\dots,n_k}\cong\mathbb{P}^{n-1}\times\mathbb{P}^1$ or $Y_{n_1,\dots,n_k}\cong$

$\mathbb{P}^1 \times \mathbb{P}^{n-1}$. With no loss of generality we may assume $\mathbb{P}^{n-1} \times \mathbb{P}^1$. The inductive assumption gives the existence of a line $L_A \subset \mathbb{P}^{n-1}$ and a point $o_A \in \mathbb{P}^1$ such that $A \subset L \times \{o_A\}$. Since $r \geq 3$ there is $B \subset S$ with $\#(B) = r-1$, $B \cap A \neq \emptyset$ and $B \neq A$. We get $\{o_A\} = \pi_2(A) = \pi_2(B) = \{o_B\}$. Thus $\#(\pi_2(S)) = 1$.

To conclude the proof it is sufficient to show that $\pi_1(S)$ spans a line and we will do it by induction on $r \geq 3$. Take for the moment $r = 3$, assume that $\langle \pi_1(S) \rangle$ is a plane and set $M := \langle \pi_1(S) \rangle \times \mathbb{P}^1$. By part (a) of Lemma 1.2.14 and the assumption $\delta(2S, Y_{n-1,1}) = 2(\dim Y_{n-1,1} + 1) - 1$, we have $\delta(2S, M) \geq 2(\dim M + 1) - 1$. Moreover $\delta(2S, M) = 2(\dim M + 1) - 1 = 7$, because M is not a projective space. However, by direct computation, one gets $\delta(2S, M) = 3(\dim M + 1) - 6 = 6$.

Let $r \geq 4$. Take any 2 distinct subsets A, B of r with $\#(A) = \#(B) = r-1$. Since $\#(A \cap B) = r - 2 \geq 2$, the lines L_A and L_B have at least 2 common points. Thus $L_A = L_B$. Hence $\pi_1(S)$ spans a line. $\square$

Example 3.4.6 gives the following result, the last equality being true by Theorem 3.4.8.

Theorem 3.4.9. *Fix integers $n > \mu \geq 2$ and $r \geq \mu + 1$. Then there is $(Y_{n_1,\dots,n_k}, S) \in \mathcal{E}(n,r)$ such that $\delta(2S, Y_{n_1,\dots,n_k}) = (r-1)(n+1) - \mu = \delta_1(n,r) - \mu + 1$.*

Untill now we worked without minimality assumption on the multiprojective space we were considering. We recall that if $S \in S_{Y_{n_1,\dots,n_k}}(2)$ is such that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing S, then $Y_{n_1,\dots,n_k} \cong (\mathbb{P}^1)^k$, for some $k \geq 1$. In this case, Proposition 3.1.3 gives $\mathbb{E}(n,2) = \emptyset$. If we set $r = 3$, Theorem 3.3.14 gives that $\mathbb{E}(n,3)$ is the set of all $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{n-n_1-n_2}$ with $1 \leq n_2 \leq n_1 \leq 2$ and $n > n_1 + n_2$. Therefore, we wonder if it always exists a couple $(S, Y_{n,\dots,n_k})$ with $S \subset Y_{n_1,\dots,n_k}$ such that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing S, that have both $\delta(2S, Y_{n_1,\dots,n_k})$ and $h^0\big(\mathcal{I}_{(2S,Y_{n_1,\dots,n_k})}\big)$ non-zero. In other words, we ask whether $\mathbb{E}(n,r)$ is not empty for $r \geq 3$, $n \geq 3$.

In this example we present a family of sets $S \subset Y_{n_1,\dots,n_k}$ such that $Y_{n_1,\dots,n_k}$ is the minimal multiprojective space containing S, that always belong to the r-th Terracini locus of $Y_{n_1,\dots,n_k}$, for some $r, n \geq 3$.

Example 3.4.10. Let $r, n \geq 3$ and let $Y_{1^n} := (\mathbb{P}^1)^n$. Take $A \subset \mathbb{P}^1$ such that $\#(A) = r-1$ and define $S := \{p_1, \dots, p_r\} \subset Y_{1^n}$ where

$$p_i = (a_i, u_2, \dots, u_n) \text{ for } i = 1, \dots, r-1 \text{ with all } a_i \in A \text{ and all } u_j \in \mathbb{P}^1$$
$$p_r = (o_1, \dots, o_r), \text{ with } o_1 \in \mathbb{P}^1 \setminus A,\ o_k \in \mathbb{P}^1 \ :\ o_k \neq u_k \text{ for all } k = 2, \dots, n$$

Set $S' = S \setminus \{p_r\}$ and let $Y' := \mathbb{P}^1 \times \{u_2\} \times \dots \times \{u_n\}$. Note that $Y' \cong \mathbb{P}^1$ is the minimal multiprojective subspace containing S' and that Y_{1^n} is the minimal multiprojective subspace containing S. From (1.2.4) of Remark 1.2.13 we know that $\delta(2S, Y_{1^n}) \geq \delta(2S', Y_{1^n}) \geq \delta(2S', Y') = \delta(2A, \mathbb{P}^1) = 2(r-2) > 0$. Take $H := \pi_n^{-1}(u_n) \in |\mathcal{O}_{Y_{1^n}}(\varepsilon_n)|$. Since $S' \subset H$, the residual exact sequence of $2S'$ with respect to H gives

$$0 \longrightarrow \mathcal{I}_{S'}(1, \dots, 1, 0) \longrightarrow \mathcal{I}_{(2S', Y_{1^n})}(1, \dots, 1) \longrightarrow \mathcal{I}_{(2S', H), H}(1, \dots, 1) \longrightarrow 0 \qquad (3.4.4)$$

Thus (3.4.4) gives $h^0\left(\mathcal{I}_{(2S',Y_{1^n})}(1,\dots,1)\right) \geq h^0\left(\mathcal{I}_{S'}(1,\dots,1)\right)$. Since $\nu(S')$ spans a line, $h^0\left(\mathcal{I}_{S'}(1,\dots,1)\right) = 2^n - 2$. Since $n \geq 3$, $h^0\left(\mathcal{I}_{(2S',Y_{1^n})}(1,\dots,1)\right) \geq n+2$. Thus $\delta(2S, Y_{1^n}) > 0$ and $h^0\left(\mathcal{I}_{(2S,Y_{1^n})}(1,\dots,1)\right) > 0$.

We conclude the chapter by proving that if $n, r \geq 3$ then $\mathbb{E}(n,r) \neq \emptyset$. This conclusion might open to further investigation of the introduced locus.

Proposition 3.4.11. $\mathbb{E}(n,r) \neq \emptyset$ *if and only if* $n \geq 3$ *and* $r \geq 3$.

Proof. If $n \geq 3$ and $r \geq 3$, Example 3.4.10 shows that $\mathbb{E}(n,r) \neq \emptyset$. The other implication follows from Lemma 3.4.7 since $\mathbb{E}(n,r) \subseteq \mathcal{E}(n,r)$ and $\mathbb{T}(Y_{n_1,\dots,n_k}, 2) = \emptyset$ (cf. Proposition 3.1.3). $\square$

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