Marco Gallo

Nonlocal elliptic PDEs with general nonlinearities

Dottorato in Informatica e Matematica, Bari (2023).

<http://www.bdim.eu/item?id=tesi_2023_GalloMarco_1>

Marco Gallo

Nonlocal Elliptic PDEs with General Nonlinearities

Supervised by

Prof. Silvia Cingolani

Prof. Denis Bonheure

Ph.D. Thesis

Università
Degli Studi di Bari
Aldo Moro

Università degli Studi di Bari Aldo Moro
Department of Mathematics

Ph.D. in Computer Science and Mathematics
XXXV Cycle

Mathematical Analysis MAT/05

Nonlocal Elliptic PDEs with General Nonlinearities

Ph. Doctor:
Marco Gallo

Supervisor:
Prof. Silvia Cingolani

Co-Supervisor:
Prof. Denis Bonheure

Ph.D. Thesis, March 2023

Abstract

In this thesis we investigate how the nonlocalities affect the study of different PDEs coming from physics, and we analyze these equations under almost optimal assumptions of the nonlinearity. In particular, we focus on the fractional Laplacian operator and on sources involving convolution with the Riesz potential, as well as on the interaction of the two, and we aim to do it through variational and topological methods.

We examine both quantitative and qualitative aspects, proving multiplicity of solutions for nonlocal nonlinear problems with free or prescribed mass, showing regularity, positivity, symmetry and sharp asymptotic decay of ground states, and exploring the influence of the topology of a potential well in presence of concentration phenomena. On the nonlinearities we consider general assumptions which avoid monotonicity and homogeneity: this generality obstructs the use of classical variational tools and forces the implementation of new ideas.

Throughout the thesis we develop some new tools: among them, a Lagrangian formulation modeled on Pohozaev mountains is used for the existence of normalized solutions, annuli-shaped multidimensional paths are built for genus-based multiplicity results, a fractional chain rule is proved to treat concave powers, and a fractional center of mass is defined to detect semiclassical standing waves. We believe that these tools could be used to face problems in different frameworks as well.

Contents

Introduction **ii**

1 Some facts about nonlocalities **1**
1.1 Notations . . . 1
1.2 The fractional Laplacian . . . 3
1.2.1 Fractional Sobolev spaces . . . 6
1.2.2 Some computations: hypergeometric Gaussian functions . . . 12
1.2.3 Definitions of solutions: weak, viscosity, strong, classical . . . 15
1.2.4 A concave Chain rule . . . 17
1.2.5 Regularity: tail functions and De Giorgi classes . . . 19
1.2.6 Existence theorems and comparison principles . . . 24
1.3 The Riesz potential . . . 27
1.3.1 The Riesz potential as the inverse of the fractional Laplacian . . . 30
1.4 Some manipulations: absolute value and polarization . . . 33
1.5 Berestycki-Lions type assumptions: some convergences . . . 35
1.5.1 Local nonlinearities . . . 36
1.5.2 Nonlocal nonlinearities . . . 39

2 Fractional Schrödinger equations: prescribed and free mass problems **43**
2.1 The fractional Schrödinger equation: a long-range interaction . . . 43
2.2 The unconstrained problem . . . 47
2.3 Lagrangian formulation and Pohozaev geometry . . . 50
2.4 Compactness by scaling . . . 56
2.4.1 A limiting Pohozaev identity . . . 57
2.4.2 A functional in an augmented space . . . 59
2.5 A deformation flow by projections . . . 61
2.6 Minimax critical points in the product space . . . 61
2.7 Multiple normalized solutions . . . 64
2.7.1 Symmetric deformation theorems . . . 64
2.7.2 Minimax values . . . 71
2.7.3 Multiplicity theorem . . . 75
2.8 L^2-minimum . . . 76
2.9 Relation between constrained and unconstrained problems . . . 78

3 Choquard-Hartree-Pekar equations: multiplicity of solutions **79**
3.1 Convolution with Riesz potential: a self-interaction . . . 79
3.2 Multidimensional annuli-shaped paths: even and odd nonlinearities . . . 85
3.3 Asymptotic analysis of mountain pass values . . . 95
3.4 The Pohozaev mountain . . . 97
3.5 The Palais-Smale-Pohozaev condition . . . 100
3.6 Genus-shaped critical points . . . 102
3.6.1 Augmented functional . . . 102

3.6.2 Deformation theory . . . 103
3.6.3 Multiple critical points . . . 104
3.7 The unconstrained problem . . . 105

4 Doubly nonlocal equations: qualitative and quantitative results 107
4.1 An example of double nonlocality: collapse of boson stars . . . 107
4.2 Different approaches for the existence problem . . . 111
4.2.1 Dealing with the boundary . . . 113
4.2.2 Existence of L^2-ground states . . . 117
4.3 Preliminary properties of Pohozaev energy levels . . . 119
4.4 Regularity . . . 123
4.4.1 Boundedness by splitting . . . 123
4.4.2 Hölder regularity: strong solutions . . . 130
4.4.3 L^1-summability: fixed point maps . . . 131
4.4.4 C^γ-regularity: classical solutions . . . 133
4.4.5 C^1 and C^2 regularity . . . 135
4.5 Shape of ground states . . . 138
4.5.1 Positivity through fibers . . . 138
4.5.2 Radial symmetry . . . 140
4.6 Asymptotic decay . . . 142
4.6.1 The (super)linear case . . . 143
4.6.2 The sublinear case: fractional Laplacian versus Riesz potential . . . 145
4.6.3 Fractional auxiliary functions . . . 150
4.6.4 A preliminary range . . . 153
4.6.5 Estimate from above . . . 156
4.6.6 Estimate from below . . . 158
4.6.7 An s-sublinear threshold . . . 159
4.7 The Pohozaev identity . . . 162

5 Concentration phenomena: the effect of the fractional operator 168
5.1 From classical to quantum: semiclassical states . . . 168
5.1.1 A tail-controlling mixed norm . . . 172
5.2 Limiting equation . . . 172
5.2.1 A single equation . . . 173
5.2.2 A family of equations: minimal radius map . . . 174
5.2.3 Fractional center of mass . . . 179
5.3 Singularly perturbed equation . . . 181
5.3.1 A mass-concentrating penalization . . . 182
5.3.2 Critical points and truncated Palais-Smale condition . . . 184
5.3.3 Deformation lemma on a neighborhood of expected solutions . . . 194
5.3.4 Maps homotopic to the embedding . . . 196
5.4 Existence of multiple solutions . . . 200
5.4.1 Concentration in the potential well . . . 201
5.5 The critical case . . . 207
5.5.1 Uniform L^∞-bound . . . 209
5.5.2 The truncated problem . . . 213
5.5.3 The local case . . . 217

A Appendix 219
A Some algebraic topology: the relative cup-length . . . 219
A.1 The singular cohomology . . . 219
A.2 Other cohomologies . . . 223
A.3 Properties of the cup-length . . . 225

A.4 Relation with the Ljusternik-Schnirelmann category 226
A.5 Application to multiplicity of solutions 228
A.6 The Krasnoselskii genus: a particular category 230

Bibliography **231**

Introduction

Nonlinear phenomena pervade natural and social sciences, and lots of them are modeled by nonlinear equations: there has been an enormous progress in the study of the structure and in the qualitative understanding of these equations in recent years, and many astonishing interrelations have been found. In this thesis we aim to contribute to these studies.

In particular, the goal is to detect local and nonlocal effects in some nonlinear partial differential equations, having as a common feature a variational structure. Mathematically, *nonlocality* is an intrinsic feature of integral operators and of associated energy functionals, which have the peculiarity – contrary to the classical local ones – of capturing long-range interactions or self-interactions. In the context of functional variational principles and associated inequalities, nonlocal energy functionals are currently receiving great attention since they are closely related to problems in geometry, physics, engineering, biology, finance and many others, manifesting both in the operator and in the source. In this setting, classical PDE theory fails because of the presence of the nonlocality.

A first goal of our research is the study of some generalized nonlinear Schrödinger equations (here the Planck's constant and the mass are normalized $\hbar = m = 1$)

$$i\partial_t u = P(D)u - h(|u|)u, \quad x \in \mathbb{R}^N,\, t > 0$$

where $P(D)$ denotes a pseudo-differential operator with constant coefficients, defined by multiplication in Fourier spaces as $\widehat{P(D)u}(\xi) = p(\xi)\widehat{u}(\xi)$, and $h \in C(\mathbb{R}_+)$. In particular, we are interested to the case $p(\xi) \equiv |\xi|^{2s}$, $s \in (0,1)$, and to the study of standing waves solutions

$$u(t,x) = e^{i\mu t} Q(x)$$

with some nontrivial profile Q, depending on the frequency $\mu > 0$: this leads to investigate the so called *fractional nonlinear Schrödinger equation* (fNLS),

$$(-\Delta)^s Q + \mu Q = h(|Q|)Q, \quad x \in \mathbb{R}^N$$

where $P(D) \equiv (-\Delta)^s$ is known as *fractional Laplacian.* In 1948 Feynman [182] proposed indeed a new suggestive description of the evolution of the state of a non-relativistic quantum particle: according to Feynman, the wave function solution of the Schrödinger equation should be given by a sum over all possible histories of the system, that is by a heuristic integral over an infinity of quantum-mechanically possible trajectories. Following this approach, Laskin [249–252] derived the fractional Schrödinger equation (fNLS): numerous applications of these equations in the physical sciences could be mentioned, ranging from image reconstruction to water wave dynamics, passing through jump processes in probability theory with applications to financial mathematics.

In this thesis we are interested in detecting existence of one or more solutions of (fNLS) equations, or more generally problems related to equations of the type

$$(-\Delta)^s u + \mu u = g(u), \quad x \in \mathbb{R}^N, \tag{I.1}$$

where $g \in C(\mathbb{R})$, and in studying their qualitative properties. We aim to do it by looking at solutions as critical points of suitable real-valued functionals, as well as by exploiting methods coming from both algebraic topology and geometry. Here, the influence of an external potential $V = V(x)$ may be considered as well.

Another target of this thesis is the analysis of the so-called *Pekar nonlinear problem*, which describes a polaron – namely a quantum electron in a polar crystal – at rest. This problem was raised by Pekar [313] in 1954: the atoms of the crystal are displaced due to the electrostatic force induced by the charge of the electron and the resulting deformation is then felt by the electron itself. Afterwards, Choquard [106] (see also Lieb [264, 265] and Lions [271]) developed a similar theory to study steady states of the one-component plasma approximation in the Hartree-Fock theory; the same model was then also derived by Penrose in his discussion about the self-gravitational collapse of a quantum-mechanical wave function [314–316], coupling together the Schrödinger equation with the Newton law. Mathematically, these models belong to the class of equations

$$-\Delta u + \mu u = (W * F(u))F'(u), \quad x \in \mathbb{R}^N$$

where W is a radially symmetric potential, $\mu > 0$ and $F \in C^1(\mathbb{R})$. In particular, the above-mentioned physics problems are set in the case $N = 3$, F power and $W(x) \equiv \frac{1}{4\pi|x|}$ *Newton potential*.

We address to study existence, multiplicity and qualitative results for these integro-differential equations, in the wider (model) class of *Riesz potentials* $W(x) \equiv I_\alpha(x) := \frac{C_{N,\alpha}}{|x|^{N-\alpha}}$, with $\alpha \in (0, N)$ and $C_{N,\alpha} > 0$ constant, that is

$$-\Delta u + \mu u = (I_\alpha * F(u))F'(u), \quad x \in \mathbb{R}^N \tag{I.2}$$

also known as *Choquard-Hartree-Pekar equation*.

When dealing with the mathematical description of the gravitational collapse of exotic stars, *double nonlocalities* arise naturally, both in the operator and in the source: this was observed already by Chandrasekhar [93] in 1931, and then developed by Lieb, Thirring and Yau [268–270, 360]. Other applications can be found for example in quantum chemistry and in the study of graphene. This is why part of the thesis will be devoted to the study of equations of the type

$$(-\Delta)^s u + \mu u = (I_\alpha * F(u))F'(u), \quad x \in \mathbb{R}^N, \tag{I.3}$$

highlighting especially how the two nonlocalities interact.

The approach of this thesis will be mainly of variational type: in the last thirty years, the study of abstract variational methods and their applications to nonlinear differential equations have greatly developed. In the past, variational methods have been applied to solve nonlinear differential equations, both ordinary and partial, taking advantage of a related functional with some specific features: among them we can find compactness properties (typically the Palais-Smale condition), natural constraints of Nehari type, use of integral identities (such as the Pohozaev identity), presence of a local operator, restriction to bounded domains, and others. In the subsequent years, the study of nonlinear differential equations arising in geometry, physics and applied mathematics has suggested developments in which at least one of the previous assumptions is not satisfied.

The substantial progress made in the last years allows now to tackle equations with particular features, as nonlocal PDEs. The greatly increased interest in nonlocal operators has motivated a systematic study of the properties of the fractional Laplacian and pseudo-differential operators in general [84, 86–88, 158, 177, 190, 201, 328, 329, 346, 347]; variational techniques have been employed also to obtain quantitative and qualitative results for elliptic PDEs with nonlocal nonlinearities [109, 128, 205, 206, 257, 279, 299, 300, 332, 372, 378, 384].

A key aspect in the study of partial differential equations consists also on the hypothesis assumed on the nonlinearity: considering very general ones allows to include different models coming from different frameworks. In 1983 Berestycki and Lions [50, 51] proposed a set of assumptions which relies, essentially, only on the growth of the nonlinearity in zero and at infinity: these assumptions may be considered, from a variational point of view, *almost optimal*, and include for instance the most common power type functions $g(t) \sim t^p$, but also combined powers representing cooperation $g(t) \sim t^p + t^q$ and competition $g(t) \sim t^p - t^q$, as well as asymptotically linear saturable sources arising in nonlinear optics $g(t) \sim \frac{t^3}{1+t^2}$ and in the study of semiconductors $g(t) \sim t - \frac{t}{\sqrt{1+t^2}}$, and many others. The generality of these assumptions, which do not include regularity, homogeneity, Ambrosetti-Rabinowitz-type or monotonicity conditions, precludes the possibility of using classical tools of the variational analysis, such as minimization on Nehari manifolds and fibering methods [74, 244, 306, 319, 379], use of Pohozaev identities [318], as well as boundedness of standard Palais-Smale sequences and classical Mountain Pass geometries [18].

Goal of this thesis is to investigate the abovementioned PDEs avoiding the use of these additional assumptions, examining especially how the geometry and the compactness of the problems can be tackled in this generality. In particular, we solve here also some problems which were left open in literature, and their resolution requires the implementation of new ideas.

Studying equations (I.1) and (I.2), the research has been pursued essentially in two main directions: the first is to assign the frequency $\mu \in (0, +\infty)$, and let the mass (given by the L^2-norm of u) to be free. This *unconstrained approach* has been extensively studied in the literature [49–51, 79, 95, 229, 230, 237, 290, 302, 304]. A second approach is to prescribe the mass $\int_{\mathbb{R}^N} u^2 = m > 0$ and let instead the frequency to be an unknown [37, 224, 271, 278, 343]: this *constrained approach* is also significantly meaningful in physics, for instance in quantum mechanics due to the normalization of probability.

In this thesis we aim to find existence and multiplicity results for L^2-constrained problems, that is

$$\begin{cases} (-\Delta)^s u + \mu u = g(u), \quad x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 = m, \end{cases} \qquad \begin{cases} -\Delta u + \mu u = (I_\alpha * F(u))F'(u), \quad x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 = m. \end{cases}$$

When dealing with nonlocalities, the classical minimization approach on the L^2-sphere is rather involved, since the techniques require a delicate control on the tails of the functions; moreover, this approach is less suitable for the research of multiple solutions. Here we propose instead a minimax approach, related to a *Lagrangian formulation* of the problem, and modeled on suitable *mountains on the product space*: we believe that this method may be applied to a wider class of equations. A posteriori, we show also that the found solution with minimal energy is indeed an L^2-minimum. Even though the approach to the two problems is similar, the study of the two abovementioned equations gives rise to different problems.

A particular feature of the fractional Laplacian, indeed, is the lack of a regularizing effect: this fact does not allow to prove the well known Pohozaev identity, a quite useful tool in the framework of PDEs. This lack of regularity is here tackled by implementing a suitable modification of the Palais-Smale condition, that we call *Palais-Smale-Pohozaev condition*, and a deformation argument around the set of critical points satisfying the Pohozaev identity. Here we face for the first time the problem of the existence of a normalized solution for a fractional framework, where the Pohozaev identity is no more ensured; moreover, we highlight that the multiplicity result presented is new even in the power setting $g(u) = |u|^{p-2}u$. This is done in Chapter 2.

In the case of Choquard nonlinearities, instead, a delicate issue is the research of multiple solutions: indeed, this is typically based on the construction of suitable multidimensional Mountain Pass paths. On the other hand, when the nonlinearity is not local, this is not obvious, and that is why we need to implement a delicate construction based on *multidimensional annuli* which takes into account the interaction of far components. In particular, as a peculiar feature of the nonlocal setting we are allowed to consider *odd* (and not only even) functions F, which make the energy functional symmetric as well: this possibility has not been developed in the common literature. Nevertheless, the case F odd makes much more involved the control of far nonlocal contributions; here we include this case in our study. Moreover, as a byproduct of this construction, we find existence of infinitely many solutions for the unconstrained Choquard problem (I.3), solving a problem which was left open in the literature [302] and extending to nonlocal nonlinearities the seminal paper by Berestycki and Lions [51]. We do this in Chapter 3.

When studying *fractional Choquard equations* [138] of the type (I.3), the combination of the two nonlocalities and of the nonhomogeneous nonlinearity heavily influences the investigation of qualitative properties of the solutions. The lack of explicit computations, the absence of a proper chain rule and the singularities of the Fourier symbol and of the convolution kernel obstruct classical approaches in the study of boundedness, L^1-summability and regularity of solutions, as well as positivity and asymptotic decay of ground states. Again, also here we consider the possibility of F to be odd in the study of some symmetry properties: all the abovementioned difficulties require new ideas and the implementation of more delicate arguments. Some of the cited results are, in addition, new even for the case $s = 1$, improving some results in [302].

The nonlocal interaction of the fractional Laplacian and of the Choquard term gives rise moreover to new phenomena: for instance, when F has a subquadratic growth in the origin, the asymptotic behaviour at infinity of the solutions seems to be connected to a new *growth threshold*, differently from the local case $s = 1$. All these properties are examined in Chapter 4.

Finally, another problem we aim to investigate is the *concentration* of solutions in fractional nonlinear Schrödinger equations. Indeed, given an external potential $V = V(x)$, physicists are interested in studying the effect of this potential on the solutions of the equation

$$\hbar^{2s}(-\Delta)^s u + V(x)u = g(u), \quad x \in \mathbb{R}^N$$

as long as the term $\hbar$ goes to zero, which somehow describes the passage from quantum to classical mechanics [73, 337]; this is why solutions of this equation for $\hbar > 0$ small are also called *semiclassical*. In particular it has been proved that, if a family of solutions has maxima which concentrate in a point, then that point is critical for V [174, 370]. This is the reason why a huge literature is focused on studying concentration on different types of critical points, both in a local framework [81, 119] and nonlocal [10, 96, 123, 183, 338]. Our aim is to investigate concentration phenomena on local minima of V, in the framework of fractional equations: in this case, the spreading of the mass carried by the fractional Laplacian strongly opposes the research of solutions *localized* in a prescribed domain of $\mathbb{R}^N$. Despite this obstruction, we find the existence of multiple solutions with this behaviour, whose number is related to some algebraic-topological information on the set of local minima of V.

In order to achieve this, some careful analysis is needed: indeed, the possible degeneracy of the local minimum of V does not allow to implement finite-dimensional reduction arguments, while the generality of the function g hinders the possibility of working on natural constraints, such as Nehari manifolds. In order to study sets of local minima we combine perturbation and penalization arguments and implement delicate deformation theorems on some set of expected solutions. In this discussion, we include a posteriori the case of a lost of compactness given by a Sobolev-critical growth of g, through the use of a truncation argument and suitable a priori estimates.

As already highlighted, the presence of a nonlocality makes the whole study much more involved: the lack of a proper Leibniz rule and of the preservation of the supports prevents the use of classical cut-off functions and standard penalization arguments. Moreover, a strong control on the tails of the functions is needed, especially when trying to localize their *fractional center of mass*, and we do this by means of a suitable mixed fractional seminorm. This study is made in Chapter 5.

The general spirit of this thesis is thus to investigate how the nonlocalities – both in the operator and in the source – comes into account in the study of different PDEs, and analyze these equations under almost optimal assumptions on the nonlinearity.

The thesis is organized as follows. In Chapter 1 we recall and revisit some known results in literature, furnishing the proofs whenever it was not possible to find a precise reference, and we present some new results as well. Chapter 2 is dedicated to the study of autonomous fractional equations: after having recalled what is known for the unconstrained problem, we focus on the study of the mass-constrained problem, obtaining both existence and multiplicity of solutions for general nonlinearities. Then, in Chapter 3 we research for multiple solutions to the Choquard problem: in this case, one of the main issues is the construction of suitable multidimensional paths, since the general and nonlocal nonlinearity heavily affects the geometry of the problem. In Chapter 4 we move to study the case of doubly nonlocal equations, where we mainly focus on the qualitative properties of the solutions, investigating how the interaction of the two nonlocalities influences both the techniques and the results. Finally, we face the fractional semiclassical problem in Chapter 5, by studying how the nonlocality of the fractional operator comes into play while searching for multiple solutions concentrating to a local minimum of the potential. Appendix A is dedicated to a little survey on the algebraic and topological tools used throughout the thesis.

This thesis is mainly based on the papers [111–117, 197, 198].

Chapter 1

Some facts about nonlocalities

In this Chapter we introduce some preliminary results about the fractional Laplacian (Section 1.2) and the Riesz potential (Section 1.3), as well as some considerations about nonlinear functionals (Section 1.5). Here we collect and revisit some known results in literature, furnishing some proofs whenever it was not possible to find a precise reference.

Moreover, we present here some new results: in particular, in Section 1.2.4 we deal with a fractional chain rule in presence of concave compositions, by working with a viscosity formulation; this can be found in paper [198]. In Section 1.2.5 instead, we present an L^∞-bound for non-positive solutions of fractional nonautonomous elliptic – possibly critical – equations, which adapts also to the Choquard framework; this has been developed in papers [115, 197].

1.1 Notations

We start by writing down some notations used throughout the thesis. We write $\mathbb{R}_+ := (0, +\infty)$ and

$$B_r(x_0) := B(x_0, r) := \{x \in \mathbb{R}^N \mid |x - x_0| < r\} \quad \text{for } x_0 \in \mathbb{R}^N \text{ and } r > 0,$$
$$D_N := \{\xi \in \mathbb{R}^N \mid |\xi| \leq 1\} \quad \text{for } N \in \mathbb{N}^*,$$
$$A(R, h) := \{x \in \mathbb{R}^N \mid |x| \in [R - h, R + h]\}, \quad \text{for } R > 0,\ h > 0$$

for balls, disks, annuli; in particular, $B_r := B_r(0)$, and $\chi(R, h; \cdot) := \chi_{A(R,h)}$. In addition,

$$A_\delta := \{x \in X \mid d(x, A) \leq \delta\}$$

denotes a neighborhood for any $A \subset (X, d)$ metric space. Sometimes we will write

$$\complement(A) := A^c := X \setminus A$$

for $A \subset X$ to avoid cumbersome notation, if the ambient space is clear from the context. The function P_i will denote, generally, the projection on the i-th component (in some product space).

We write

$$\|u\|_r := \left(\int_{\mathbb{R}^N} |u|^r \, dx \right)^{1/r} \quad \text{for } r \in [1, \infty) \text{ and } u \in L^r(\mathbb{R}^N),$$
$$\|u\|_\infty := \operatorname{ess\,sup}_{\mathbb{R}^N} |u| \quad \text{for } u \in L^\infty(\mathbb{R}^N),$$

the classical L^p-norm in the entire space, $p \in [1, +\infty]$; we will use also the following notation

$$\|f\|_{\infty,\theta} := \|f(\cdot)(1 + |\cdot|^\theta)\|_\infty$$

for any $\theta > 0$. By $\mathcal{F}(u)$ or $\widehat{u}$ we will denote, moreover, the Fourier transform of a function u, and by $u_\pm$ its positive and negative parts, $u = u_+ - u_-$.

The function $\Gamma(\cdot)$ will denote the standard Gamma function, while ${}_2F_1(\cdot,\cdot,\cdot\,;\,\cdot)$ will denote the Gauss hypergeometric function.

We write $\mathcal{S}$ for the Schwartz function space. For any $k \in \mathbb{N}$ and $\sigma \in (0,1)$, we denote by $C_0(\mathbb{R}^N)$ the space of continuous functions decaying to zero at infinity, by $C_b^k(\mathbb{R}^N)$ (resp. $C_c^k(\mathbb{R}^N)$) the space of k times differentiable functions with bounded (resp. compactly supported) and continuous j-derivative, $j = 0, \dots, k$, by $C^{k,\sigma}(\mathbb{R}^N)$ the space of k times differentiable functions with σ-Hölder continuous k-derivatives (on $\mathbb{R}^N$), where

$$[u]_{C^{0,\sigma}(A)} := \sup_{\substack{x,y\in A\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\sigma}$$

denotes the usual seminorm in Hölder spaces for $\sigma \in (0,1]$ and $A \subseteq \mathbb{R}^N$. By $C^{k,\sigma}_{loc}(\mathbb{R}^N)$ we consider functions whose k-derivatives are locally σ-Hölder continuous; if $\sigma = 1$ we also write $Lip(\mathbb{R}^N) := C^{0,1}(\mathbb{R}^N)$ and similarly $Lip_{loc}(\mathbb{R}^N)$ and $Lip_c(\mathbb{R}^N)$. More briefly we will sometimes write

$$C^\beta(\mathbb{R}^N) := C^{[\beta],\beta-[\beta]}(\mathbb{R}^N)$$

for any $\beta > 0$, observing that this notation throws out spaces $C^{k,1}(\mathbb{R}^N)$, usually subsituted by Zygmund spaces (see Remark 1.1.2 below); similarly $C^\beta_{loc}(\mathbb{R}^N)$.

Remark 1.1.1. *In [164] it is defined, for $\sigma \in (0,1]$, $u \in Lip(\sigma)$ if there exist $C > 0$ and $\delta > 0$ such that, for each $x, y \in \mathbb{R}^N$,*

$$0 < |x-y| \le \delta \implies \frac{|u(x)-u(y)|}{|x-y|^\sigma} \le C.$$

We notice that

$$C^{0,\sigma}(\mathbb{R}^N) \subset Lip(\sigma) \subset C^{0,\sigma}_{loc}(\mathbb{R}^N)$$

and moreover

$$Lip(\sigma) \cap L^\infty(\mathbb{R}^N) \subset C^{0,\sigma}(\mathbb{R}^N);$$

indeed, for each $x, y \in \mathbb{R}^N$,

$$|x-y| > \delta \implies \frac{|u(x)-u(y)|}{|x-y|^\sigma} \le \frac{2\|u\|_\infty}{\delta^\sigma}.$$

Remark 1.1.2. *To state some results it is useful to introduce also the* Zygmund space $\Lambda_1(\mathbb{R}^N)$ *[352, Section 6] as the space of the continuous functions u such that*

$$\sup_{x,h\in\mathbb{R}^N} \frac{|u(x+h)-2u(x)+u(x-h)|}{|h|} < \infty.$$

We notice that $u \in C^{0,\sigma}(\mathbb{R}^N)$ for $\sigma \in (0,1)$ if equivalently

$$\sup_{x,h\in\mathbb{R}^N} \frac{|u(x+h)-2u(x)+u(x-h)|}{|h|^\sigma} < \infty,$$

but the same does not hold true for $\sigma = 1$; indeed

$$C^{0,1}(\mathbb{R}^N) \subsetneq \Lambda_1(\mathbb{R}^N).$$

We can further define $\Lambda_2(\mathbb{R}^N)$ as the space of functions in $C^1(\mathbb{R}^N)$ with partial derivatives in $\Lambda_1(\mathbb{R}^N)$; also in this case $C^{1,1}(\mathbb{R}^N) \subsetneq \Lambda_2(\mathbb{R}^N)$. The following relations hold true [351, Propositions 5.5.8, 5.5.9 and 5.5.10]:

$$C^{k,1} \cap L^\infty \subset \Lambda_{k+1} \cap L^\infty \subset C^{k,\sigma_2} \cap L^\infty \subset C^{k,\sigma_1} \cap L^\infty$$

for $k = 0, 1$ and each $0 < \sigma_1 < \sigma_2 < 1$.

Here we write $f \sim g$ as $x \to x_0 \in \overline{\mathbb{R}}$ if there exist constants C_1, $C_2 > 0$ independent of x such that

$$C_1 g(x) \leq f(x) \leq C_2 g(x) \quad \text{for } x \text{ near } x_0,$$

while by $f \dot{\sim} g$ as $x \to x_0$ we mean that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

Moreover, by $\approx$ we will mean *approximately equal to* (in a sense clear from the context) or *isomorphic to*. Symbols $\lesssim$, $\simeq$ and $\gtrsim$ will mean less, equal or greater up to (positive) constants.

Finally, for every $A \subset B \subset \mathbb{R}^N$, we will write

$$A \prec \phi \prec B$$

to indicate a Urysohn-type regular function $\phi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\phi_{|A} = 1 \quad \text{and} \quad \phi_{|\mathbb{R}^N \setminus B} = 0.$$

We introduce the following terminology: if $\mathbb{G}$ is a group acting on a set X, we say that $A \subset X$ is *invariant* under $\mathbb{G}$ if $gA = A$ for each $g \in \mathbb{G}$, while we say that a function $f : X \to Y$ (Y another set) is *invariant* under $\mathbb{G}$ if $f(g\cdot) = f$ for each $g \in \mathbb{G}$; finally we say that $f : X \to X$ is *equivariant* if $f(g\cdot) = g \cdot f$ for each $g \in \mathbb{G}$. When $\mathbb{G} = \mathbb{Z}_2 \equiv \{\pm 1\}$ acting on some vectorial space X, we have that $A \subset X$ invariant means symmetric with respect to the origin ($A = -A$), f invariant means even ($f(-\cdot) = f$), f equivariant means odd ($f(-\cdot) = -f$).

We highlight that, all throughout the thesis, we will actually assume $N \geq 2$ when dealing with the fractional framework $s \in (0,1)$ (despite the beginning of the preliminaries, where generally $N > 2s$), and $N \geq 3$ in the local framework $s = 1$. Moreover the constants C, C' appearing in inequalities may change from a passage to another; to avoid cumbersome notations, we will not stress the dependence of such constants, which will be based only on the fixed quantities in play.

1.2 The fractional Laplacian

Let $s \in (0,1)$ and $N > 2s$. For this Section we mainly refer to [153, 201], together with [6, 79, 177, 346]; other interesting references are [22, 56, 76, 84, 150, 339] (see also [99]). For motivations and a physical introduction we refer to Sections 2.1 and 4.1.

Let the fractional Laplacian be defined by [153]

$$(-\Delta)^s u(x) := C_{N,s} \mathrm{PV} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy$$

where

$$C_{N,s} := \frac{4^s \Gamma(\frac{N+2s}{2})}{\pi^{N/2} |\Gamma(-s)|} > 0$$

is a normalization constant with Γ the Gamma function, and the integral is in the Principal Value sense, that is

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy;$$

notice that, when $s \in (0, \frac{1}{2})$, we actually do not need to employ the Principal Value formulation (when u belongs, for instance, to $C^{0,\sigma}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $\sigma \in (2s, 1]$ [153, Remark 3.1], see also the proof of Proposition 1.2.2 below).

A sufficient condition in order to have $(-\Delta)^s u$ well defined pointwise is given by [346, Proposition 2.4] (see also [201, Proposition 2.15] and [79, Lemma 2.4]).

Proposition 1.2.1 (Fractional well posedness)**.** *Let $x_0 \in \mathbb{R}^N$. Then, if*

- $u \in L^p(\mathbb{R}^N) \cap C^\gamma(U)$ *for some* $p \in [1, +\infty]$, $\gamma > 2s$ *and* U *open neighborhood of* x_0,

then $(-\Delta)^s u(x_0)$ *is well defined; in this case, actually,* $(-\Delta)^s u \in C(U)$. *In particular,* $(-\Delta)^s u$ *is everywhere well defined pointwise if*

- $u \in L^p(\mathbb{R}^N) \cap C^\gamma_{loc}(\mathbb{R}^N)$ *for some* $p \in [1, +\infty]$ *and* $\gamma > 2s$,

and we have $(-\Delta)^s u \in C(\mathbb{R}^N)$.

Actually the condition $u \in L^p(\mathbb{R}^N)$ can be substituted by the more general condition

$$\int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}} < \infty. \tag{1.2.1}$$

A different pointwise representation is given in the following Proposition [153, Lemma 3.2] (see also [201, Proposition 2.8]).

Proposition 1.2.2. *Assume* $u \in L^p(\mathbb{R}^N) \cap C^\gamma_{loc}(\mathbb{R}^N)$ *for some* $p \in [1, +\infty]$ *and* $\gamma > 2s$, *Then*

$$(-\Delta)^s u(x) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}} dy,$$

and the integral is absolutely convergent.

Proof. We check only the absolute convergence. Indeed, let $x \in \mathbb{R}^N$ and $R > 2|x| + 1$. Notice that, for $|y| \geq R$, we have, for $|y| \geq R$,

$$|x+y| \geq |y| - |x| \geq R - |x| > |x| + 1$$

and

$$|x+y| - |x| \geq \frac{|x+y|+1}{|x|+2}$$

thus

$$\begin{aligned}
&\int_{B_R^c} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{N+2s}} dy \\
&\leq \int_{B_R^c} \frac{2|u(x)|}{|y|^{N+2s}} dy + \int_{B_R^c} \frac{|u(x+y)|}{|y|^{N+2s}} dy + \int_{B_R^c} \frac{|u(x-y)|}{|y|^{N+2s}} dy \\
&\leq 2|u(x)| \int_{B_R^c} \frac{1}{|y|^{N+2s}} dy + 2 \int_{B_{|x|+1}^c} \frac{|u(z)|}{(|z| - |x|)^{N+2s}} dz \\
&\leq C_R |u(x)| + 2(2+|x|)^{N+2s} \int_{B_{|x|+1}^c} \frac{|u(z)|}{(|z|+1)^{N+2s}} dz < \infty.
\end{aligned}$$

Let now $s \in (0, \frac{1}{2})$. Then, being $u \in C^{0,\gamma}_{loc}(\mathbb{R}^N)$ for some $\gamma > 2s$,

$$\int_{B_R} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{N+2s}} dy \leq 2C \int_{B_R} \frac{1}{|y|^{N+2s-\gamma}} dy < \infty;$$

notice that a similarly argument shows also that the integral in the definition of the fractional Laplacian does not need the Principal Value, being absolute convergent.

If instead $s \in [\frac{1}{2}, 1)$, then, being $u \in C^{1,\gamma}_{loc}(\mathbb{R}^N)$ for some $\gamma > 2s - 1$, for each $x, y \in \mathbb{R}^N$ there exists $\sigma = \sigma(x,y) \in (0,1)$ such that

$$\int_{B_R} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{N+2s}} dy = \int_{B_R} \frac{|\nabla u(x+\sigma y) \cdot y - \nabla u(x - \sigma y) \cdot y|}{|y|^{N+2s}} dy$$

$$\leq \int_{B_R} \frac{2\sigma}{|y|^{N+2s-\gamma-1}} dy < \infty.$$

Joining the pieces, we have the claim. ∎

It is well known that the fractional Laplacian is a *nonlocal* operator. This means, for example, that

$$\operatorname{supp}\left((-\Delta)^s u\right) \not\subset \operatorname{supp}(u);$$

in particular, if ψ is a cut-off function with support in some $A \subset \mathbb{R}^N$, we cannot localize $(-\Delta)^s(\psi u)$ inside A as well. Notice that the fact that $(-\Delta)^s u$ is expressed through an integral does not directly implies that the operator is nonlocal (see, for instance, [1, Section 2.1]); anyway we can see this considering, for example, a nonnegative $u \in C^2_c(\mathbb{R}^N)$ with $u \geq 1$ on $B_1(0)$, and a point $x \in \mathbb{R}^N$ far from the support: we thus have

$$\begin{aligned}(-\Delta)^s u(x) &\leq -\int_{B_1(0)} \frac{u(y)}{|x-y|^{N+2s}} dy \leq -\int_{B_1(0)} \frac{1}{(1+|x|)^{N+2s}} dy \\ &= -\frac{C}{(1+|x|)^{N+2s}} < 0.\end{aligned}$$

Moreover, a proper Leibniz rule lacks in this framework, thus in general

$$(-\Delta)^{s/2}(\psi u) \neq (-\Delta)^{s/2} u \psi + (-\Delta)^{s/2} \psi u,$$

formula which instead holds when $(-\Delta)^{s/2}$ is substituted with the gradient ∇ in the local framework $s = 1$ (see Remark 1.2.11). In the fractional framework a correction term is needed [54, Proposition 1.5]

$$(-\Delta)^{s/2}(\psi u) = (-\Delta)^{s/2} u \psi + (-\Delta)^{s/2} \psi u + C_{N,s} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{N+s}} dy$$

or different approaches, like error estimates [208] or approximation arguments [336, Lemma 2.6] must be employed. All these issues create problems, for example, in concentration arguments (see Chapter 5). A proper chain rule lacks as well, and we will make some comments in Section 1.2.4.

The operator anyway enjoys some trivial but useful scaling properties

$$(-\Delta)^s(\lambda u) = \lambda(-\Delta)^s u, \quad (-\Delta)^s\big(u(\beta \cdot)\big) = |\beta|^{2s}\big((-\Delta)^s u\big)(\beta \cdot).$$

for any $\lambda, \beta \in \mathbb{R}$, as well as linearity.

We further have the following relation with the Fourier transform [153, Proposition 3.3] (see also [201, Proposition 2.8]) whenever $u \in \mathcal{S}$

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)); \tag{1.2.2}$$

this relation can be extended to the setting of Proposition 1.2.1, that is for functions $u \in C^{\gamma}_{loc}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ for $\gamma > 2s$, see [346, proof of Proposition 2.4] (see also [79, Lemma 2.4]).

When u is not regular enough, relation (1.2.2) might be taken as a definition, whenever for example $|\xi|^{2s}\mathcal{F}(u) \in L^2(\mathbb{R}^N)$; in this case the fractional Laplacian is defined up to a set of zero Lebesgue measure. Notice moreover that (1.2.2) could be interpreted more generally also in the sense of tempered distributions $\mathcal{S}'$.

Remark 1.2.3. *We notice that relation* (1.2.2)*, i.e.*

$$(-\Delta)^s u(x) = \int_{\mathbb{R}^N} (|\xi|^2)^s (u, e^{i\xi \cdot})_2 \, e^{i\xi \cdot x} d\xi$$

for almost every $x \in \mathbb{R}^N$, can be interpreted in terms of the spectral theorem by considering the continuum of eigenvalues $\xi \in \mathbb{R}^N \mapsto \lambda_\xi := |\xi|^2$ with eigenfunctions $e_\xi(x) := e^{i\xi \cdot x} \in L^\infty(\mathbb{R}^N)$, and applying the power function $h(t) := t^s$. This is indeed how the spectral fractional Laplacian *is defined on bounded sets (see [6, Section 2.3]).*

Remark 1.2.4. *Actually there are several equivalent ways to define the fractional Laplacian [245]. One of the most used is the Caffarelli-Silvestre* s-harmonic extension*, where the fractional Laplacian in* $\mathbb{R}^N$ *is seen as the trace of a divergence-form operator (possibly singular) in* $\mathbb{R}^{N+1}$ *[86]: this formulation is widely used in order to bring the computations from a nonlocal framework to a local framework. Anyway we stress that we will not make use of the s-harmonic extension in this thesis, by mean of working directly with integral quantities. This has the further advantage of possibly extending our results to other nonlocal frameworks where the harmonic extension is no more available, see e.g. [176].*

Relation (1.2.2) shows, informally, that

$$(-\Delta)^s u \overset{s\to 0^+}{\to} u, \quad (-\Delta)^s u \overset{s\to 1^-}{\to} -\Delta u$$

which motivates the symbol with a fractional power of the Laplacian; see [352, Theorems 3 and 4] for a precise statement (see also [153, Proposition 4.4]).

Moreover, (1.2.2) is suitable to extend the notion of fractional Laplacian to every $s > 0$ [32,98,330,346] (see also [3, Proposition 3.1]); see [1] for an overview on the topic (see in particular [2–5] and [334, Section 3.1] for a hypersingular integral definition, [89] for a recursive pointwise definition, [200] for a harmonic-extension definition).

Another feature of the fractional Laplacian is its *polynomial decay*, that is, whenever u is good enough (for example, Schwartz), then [201, Proposition 2.9] (see also Remark 5.2.2)

$$|(-\Delta)^s u(x)| \leq \frac{C}{1+|x|^{N+2s}} \quad x \in \mathbb{R}^N; \tag{1.2.3}$$

generally, one can not expect a faster decay: this is the case, for example, of $u(x) = \frac{1}{(1+|x|^2)^{\frac{N-2s}{2}}}$ (see Section 1.2.2, and also [201, Lemma 8.6] and [346, Proposition 2.12]). Even when u is a Schwartz function, by (1.2.2) we notice that $(-\Delta)^s u$ has generally not a fast decay, since $|\xi|^{2s}$ is not regular enough near zero when $s < 1$; thus

$$(-\Delta)^s \mathcal{S} \not\subset \mathcal{S} \quad \text{ for } s \in (0,1).$$

On the other hand, one can show [201, Lemma 8.1] that, for every $u \in \mathcal{S}$, one has $(-\Delta)^s u \in C^\infty(\mathbb{R}^N)$ with

$$|D^\beta((-\Delta)^s u)(x)| \leq \frac{C}{1+|x|^{N+2s}} \quad x \in \mathbb{R}^N$$

for each multi-index β. We find the asymptotic decay (1.2.3) also in fundamental solutions of the operator $(-\Delta)^s + id$ (see Lemma 1.2.29) and actually it will be a key feature of the solutions of fractional PDEs (see Section 5.2), at least when there is not a too strong nonlocal effect coming from the nonlinearity (see Section 4.6.2).

1.2.1 Fractional Sobolev spaces

We introduce, for any $\Omega \subseteq \mathbb{R}^N$ and $s \in (0,1)$, the fractional Sobolev space

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \mid [u]^2_{H^s(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dy < +\infty \right\},$$

endowed with

$$\|u\|^2_{H^s(\Omega)} := \|u\|^2_{L^2(\Omega)} + [u]^2_{H^s(\Omega)}.$$

The finite quantity $[u]_{H^s(\Omega)}$ is said Gagliardo seminorm. We will denote the dual space by $(H^s(\Omega))^*$.

We recall, whenever $\Omega = \mathbb{R}^N$ or Ω has a Lipschitz and bounded boundary, the continuous embedding [153, Theorem 6.7]

$$H^s(\Omega) \hookrightarrow L^p(\Omega) \tag{1.2.4}$$

for every $p \in [2, 2_s^*]$ with

$$2_s^* = \frac{2N}{N-2s}$$

the fractional Sobolev critical exponent, and, if $p \in [2, 2_s^*)$, the compact embedding [153, Corollary 7.2]

$$H^s(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^p_{loc}(\mathbb{R}^N)$$

in the sense that for every $(u_n)_n$ bounded in $H^s(\mathbb{R}^N)$, and for every $A \subset \mathbb{R}^N$ bounded and regular enough (e.g. ∂A Lipschitz), we have that $(u_n)_n$ restricted to A admits a convergent subsequence in $L^p(A)$.

Moreover we set

$$H^s_{loc}(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \in H^s(\Omega) \text{ for each } \Omega \subset\subset \mathbb{R}^N \right\}$$

and, for any $\Omega \subset \mathbb{R}^N$, [363, Section 4.3.2]

$$\begin{aligned} X_0^s(\Omega) &:= \left\{ w \in H^s(\mathbb{R}^N) \mid w = 0 \text{ on } \Omega^c \right\} \\ &= \left\{ w \in H^s(\mathbb{R}^N) \mid \operatorname{supp}(w) \subset \overline{\Omega} \right\}. \end{aligned}$$

Remark 1.2.5. *The following density result holds in $\mathbb{R}^N$ [150, Proposition 4.27] (see also [150, Proposition 4.11]):*

$$H^s(\mathbb{R}^N) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{H^s(\mathbb{R}^N)}}.$$

Assume now Ω, with $\partial\Omega$ compact, to be a Lipschitz domain [289, Definition 3.28]. Then [289, Theorem 3.29]

$$X_0^s(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^N)}}.$$

If moreover $s \neq \frac{1}{2}$, then [289, Theorem 3.33]

$$X_0^s(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}.$$

See also [363, Theorem 1 in Section 4.3.2] for more results on these spaces.

In the case $\Omega = \mathbb{R}^N$ we also have the following relation [153, Proposition 3.4]

$$[u]^2_{H^s(\mathbb{R}^N)} = \tfrac{2}{C_{N,s}} \| |\xi|^s \widehat{u} \|_2^2;$$

by interpreting the fractional Laplacian through the Fourier transform definition (1.2.2) we may also write

$$[u]^2_{H^s(\mathbb{R}^N)} = \tfrac{2}{C_{N,s}} \| (-\Delta)^{s/2} u \|_2^2. \tag{1.2.5}$$

Moreover, by polarization

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v dx &= \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u}\widehat{v} d\xi \\ &= \tfrac{1}{2} C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(u))}{|x-y|^{N+2s}} dx dy \end{aligned} \tag{1.2.6}$$

for every u, $v \in H^s(\mathbb{R}^N)$. Relation (1.2.5) leads also to an equivalent definition for the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \mid |\xi|^s \widehat{u} \in L^2(\mathbb{R}^N) \right\}$$

$$= \left\{u \in L^2(\mathbb{R}^N) \mid (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\right\}$$

endowed with

$$\begin{aligned}\|u\|_{H^s}^2 &\equiv \|u\|_2^2 + \||\xi|^s\widehat{u}\|_2^2\\ &= \|u\|_2^2 + \|(-\Delta)^{s/2}u\|_2^2.\end{aligned}$$

Together with $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \cap L^{2_s^*}(\mathbb{R}^N)$ we have the following embedding of the homogeneous fractional space $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ [153, Theorem 6.5] (see also [98]), where

$$\dot{H}^s(\mathbb{R}^N) := \{u \text{ measurable} \mid (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\};$$

here the fractional Laplacian is intended in the sense of tempered distributions. That is, for some optimal constant $\mathcal{S} > 0$,

$$\|u\|_{2_s^*} \leq \mathcal{S}^{-1/2}\|(-\Delta)^{s/2}u\|_2. \tag{1.2.7}$$

Moreover, we recall the fractional version of the Gagliardo-Nirenberg inequality [312] (see also [42])

$$\|u\|_r \leq C\|(-\Delta)^{s/2}u\|_2^{\beta}\ \|u\|_2^{1-\beta} \tag{1.2.8}$$

for $u \in H^s(\mathbb{R}^N)$, $r \in [2, 2_s^*]$ and β satisfying

$$\frac{1}{r} = \frac{\beta}{2_s^*} + \frac{1-\beta}{2}.$$

Extension to $p \in [1, \infty]$ and $s > 0$

Consider now again the relation

$$\begin{aligned}H^s(\mathbb{R}^N) &= \left\{u \in L^2(\mathbb{R}^N) \mid \mathcal{F}^{-1}(|\xi|^s\widehat{u}) \in L^2(\mathbb{R}^N)\right\}\\ &= \left\{u \in L^2(\mathbb{R}^N) \mid \mathcal{F}^{-1}\big((1+|\xi|^s)\widehat{u}\big) \in L^2(\mathbb{R}^N)\right\}.\end{aligned} \tag{1.2.9}$$

This last expression is suitable for defining the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ also for $s \geq 1$ and $p \geq 1$, by [177]

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) \mid \mathcal{F}^{-1}\big((1+|\xi|^s)\widehat{u}\big) \in L^p(\mathbb{R}^N)\}. \tag{1.2.10}$$

It has been proved in [177, Theorem 3.1] that this definition coincide with the following

$$\overline{W}^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) \mid \mathcal{F}^{-1}\left((1+|\xi|^2)^{s/2}\widehat{u}\right) \in L^p(\mathbb{R}^N)\right\}$$

that is

$$W^{s,p}(\mathbb{R}^N) \equiv \overline{W}^{s,p}(\mathbb{R}^N).$$

Remark 1.2.6. *We want to highlight that the last equality is actually not trivial a priori. Indeed, we can rewrite the spaces as*

$$W^{s,p}(\mathbb{R}^N) \equiv \{u \in L^p(\mathbb{R}^N) \mid u = \mathcal{K}_{2s} * g \text{ for some } g \in L^p(\mathbb{R}^N)\},$$

$$\overline{W}^{s,p}(\mathbb{R}^N) \equiv \{u \in L^p(\mathbb{R}^N) \mid u = \mathcal{G}_{2s} * g \text{ for some } g \in L^p(\mathbb{R}^N)\}$$

where

$$\mathcal{K}_{2s} := \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^{2s}}\right)$$

is the Bessel kernel, *and*

$$\mathcal{G}_{2s} := \mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^s}\right)$$

is the pseudorelativistic kernel. *The two functions are the fundamental solutions, respectively, of*

$$(-\Delta)^s \mathcal{K}_{2s} + \mathcal{K}_{2s} = \delta_0, \quad (-\Delta + id)^s \mathcal{G}_{2s} = \delta_0$$

in $\mathbb{R}^N$, *where* δ_0 *is the Dirac delta; the operator* $(-\Delta + id)^s$ *is also called* pseudorelativistic *operator (see Section 4.1). Even if*

$$\xi \mapsto \frac{1}{1+|\xi|^{2s}} \quad \text{and} \quad \xi \mapsto \frac{1}{(1+|\xi|^2)^s}$$

have same behaviour in zero and at infinity and same summability, the fact that the two functions have different regularity (the first is nonregular in the origin, the second is analytic) brings $\mathcal{K}_{2s}$ *and* $\mathcal{G}_{2s}$ *to be quite different kernels: for instance,* $\mathcal{K}_{2s}$ *has a polynomial decay at infinity (of order* $\frac{1}{|x|^{N+2s}}$, *see Lemma 1.2.29), while* $\mathcal{G}_{2s}$ *decays exponentially [9, equation (1.2.15)]. These properties influence the qualitative behaviours of the solutions of the linear equations*

$$(-\Delta)^s u + u = g, \quad (-\Delta + id)^s u = g$$

in $\mathbb{R}^N$, *given by* $u = \mathcal{K}_{2s} * g$ *and* $u = \mathcal{G}_{2s} * g$ *respectively (see e.g. Lemma 1.2.29). Because of these representation formulas, we also write*

$$\mathcal{K}_{2s}* \equiv ((-\Delta)^s + id)^{-1}, \quad \mathcal{G}_{2s}* \equiv (-\Delta + id)^{-s}.$$

These considerations also show that the pseudorelativistic operator is quite different from the fractional Laplacian by giving more regularity and decay to solutions, but without enjoying the same scaling properties; its study is an interesting line of research for the future.

Remark 1.2.7. *Notice that in (1.2.9) and (1.2.10) the request* $u \in L^p(\mathbb{R}^N)$ *is actually superfluous. This is the same for* $\overline{W}^{s,p}(\mathbb{R}^N)$ *as well, since by [9, Theorem 1.2.4] we have (if* $N > ps$*)*

$$\mathcal{F}^{-1}\left((1+|\xi|^2)^{s/2}\widehat{u}\right) \in L^p(\mathbb{R}^N) \implies u \in L^q(\mathbb{R}^N) \text{ for each } q \in [p, \tfrac{pN}{N-ps}];$$

this result is in accordance to the continuous embeddings (1.2.4) stated before for $p = 2$. *In particular the previous embedding is* continuous, *which means that (for* $q = p$*)*

$$\|u\|_p \leq C\|\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\widehat{u})\|_p;$$

this relation can be rephrased by saying that

$$\|(-\Delta + id)^{-s}u\|_p = \|\mathcal{G}_{2s} * u\|_p \leq C\|u\|_p$$

and this can be obtained directly by Young's inequality with $C = \|\mathcal{G}_{2s}\|_1$ *(indeed* $\mathcal{G}_{2s} \in L^1(\mathbb{R}^N)$, *see [9, equation (1.2.12)]). A similar argument holds for* $((-\Delta)^s + id)^{-1}$, *since*

$$\|((-\Delta)^s + id)^{-1}u\|_p = \|\mathcal{K}_{2s} * u\|_p \leq \|\mathcal{K}_{2s}\|_1\|u\|_p$$

being $\mathcal{K}_{2s} \in L^1(\mathbb{R}^N)$ *(see Lemma 1.2.29), thus*

$$\left((-\Delta)^s + id\right)^{-1} : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N) \tag{1.2.11}$$

is a continuous operator for every $p \in (1, +\infty)$ *and*

$$\|u\|_p \leq \|\mathcal{K}_{2s}\|_1\|\mathcal{F}^{-1}(1+|\xi|^{2s})\widehat{u}\|_p. \tag{1.2.12}$$

We observe, by (1.2.2), that if $u \in W^{2s,p}(\mathbb{R}^N)$ for some p, then $(-\Delta)^s u$ is well defined pointwise up to a set of zero Lebesgue measure.
Moreover, by [177, Theorem 3.2] we obtain the following embedding, for every $s \in (0,1)$,

$$H^{2s}(\mathbb{R}^N) \cap W^{2s,\infty}(\mathbb{R}^N) \hookrightarrow \begin{cases} C^{0,\gamma}(\mathbb{R}^N) \text{ for } \gamma \in (0,2s) & \text{if } 2s \leq 1, \\ C^{1,\gamma-1}(\mathbb{R}^N) \text{ for } \gamma \in (0,2s) & \text{if } 2s > 1. \end{cases} \tag{1.2.13}$$

Remark 1.2.8. *We observe that, if $s \geq s'$ and $p \in (1,\infty)$, then [363, equation (9) in Section 2.3.3]*

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow W^{s',p}(\mathbb{R}^N);$$

this is easily seen for $p = 2$: indeed, by the fact that $|\xi|^{2s'} \leq 1 + |\xi|^{2s}$ we have

$$\int_{\mathbb{R}^N} |(1+|\xi|^{2s'})\widehat{u}|^2 \leq \int_{\mathbb{R}^N} |(2+|\xi|^{2s})\widehat{u}|^2 \leq 4\int_{\mathbb{R}^N} |(1+|\xi|^{2s})\widehat{u}|^2.$$

In particular,

$$H^{2s}(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N) \quad \textit{for } 2s \geq 1.$$

Moreover, for every $s > 0$, since $H^{2s}(\mathbb{R}^N) \hookrightarrow H^{2[s]}(\mathbb{R}^N) \hookrightarrow H^{2([s]-s)}(\mathbb{R}^N)$, we notice that, for $u \in H^{2s}(\mathbb{R}^N)$,

$$\begin{aligned}(-\Delta)^s u &= \mathcal{F}^{-1}\left(|\xi|^{2([s]-s)}|\xi|^{2[s]}\widehat{u}\right) \\ &= \mathcal{F}^{-1}\left(|\xi|^{2([s]-s)}\mathcal{F}\Big(\mathcal{F}^{-1}(|\xi|^{2[s]}\widehat{u})\Big)\right) \\ &= (-\Delta)^{[s]-s}\left((-\Delta)^{[s]}u\right)\end{aligned}$$

and similarly

$$(-\Delta)^s u = (-\Delta)^{[s]}\left((-\Delta)^{[s]-s}u\right).$$

See also [32, Proposition 2.1], [3, Remark 3.2], [89] and [2, Theorems 1.2 and 1.8 and Corollary 1].

Remark 1.2.9. *By exploiting the Gagliardo seminorm one can define a fractional Sobolev space, for $p \in [1,\infty)$ and $s \in (0,1)$, by*

$$\widetilde{W}^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) \mid \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dy < +\infty \right\};$$

this is a possible good choice [153], but generally it does not coincide with $W^{s,p}(\mathbb{R}^N)$ for $p \neq 2$ [153, Remark 3.5]. See also [363, Remark 4 in Section 2.3.3] and [333, Remark 6 in Section 2.1.1]. The space $W^{s,p}(\mathbb{R}^N)$ is also known as Triebel-Lizorkin *space, or* Bessel-potential space*, o* Liouville space*, while $\widetilde{W}^{s,p}(\mathbb{R}^N)$ is also known as* Besov space *or* Slobodeckiĭ space*.*

Radially symmetric functions

In order to gain some compactness on the entire space, we consider also the subspace of radially symmetric functions

$$H^s_r(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N) \mid \exists\, v : \mathbb{R}_+ \to \mathbb{R} \ \text{ s.t. } \ u(x) = v(|x|)\};$$

to avoid cumbersom notation, we will alway write $u(x) \equiv u(|x|)$. We notice that the fractional Laplacian inherits the radial symmetry of the function (this is immediate by use of the Fourier transform (1.2.2), see also [201, Lemma 2.7]); anyway, it has not an easy representation in radial coordinates (see [181] and [201, Lemma 7.1]) based on Gaussian hypergeometric functions (see Section 1.2.2), and this obstructs, for example, ODE's methods for resolution of PDEs.

We recall that, whenever $N \geq 2$, Lions proved the compact embedding [272] (see also [91, Proposition 1.7.1] and [167])

$$H^s_r(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N) \tag{1.2.14}$$

for every $p \in (2, 2^*_s)$; however, as shown in [105] for general $s \in (0, \frac{1}{2})$, a result in the spirit of Radial Lemma by Strauss [353]

$$|u(x)|^2 \lesssim \frac{1}{|x|^{N-2s}} \|(-\Delta)^{s/2} u\|_2^2, \quad x \in \mathbb{R}^N \setminus \{0\}$$

is not available in the fractional framework $H^s_r(\mathbb{R}^N)$. We highlight that the embedding is not compact for $q = 2^*_s$ even on bounded subsets of $\mathbb{R}^N$. Sometimes we will write $\| \cdot \|_{H^s_r} := \| \cdot \|_{H^s}$.

Remark 1.2.10. *We observe that*

$$H^s_r(\mathbb{R}^N) = \mathrm{Fix}(\mathrm{O}(N)) = \{u \in H^s(\mathbb{R}^N) \mid \tau(Q, u) = u \ \textit{ for each } Q \in \mathrm{O}(N)\},$$

where $\mathrm{O}(N)$ *is the orthogonal group of rotation matrices and the isometric action is given by*

$$\tau : (Q, u) \in \mathrm{O}(N) \times H^s(\mathbb{R}^N) \mapsto u(Q\cdot) \in H^s(\mathbb{R}^N);$$

working with a variational formulation, we will often work with $\mathrm{O}(N)$*-invariant functionals: by the Principle of Symmetric Criticality of Palais [310] we will obtain that every critical point on* $H^s_r(\mathbb{R}^N)$ *is actually a critical point on the whole* $H^s(\mathbb{R}^N)$*, which justifies our restriction onto the radial setting.*

Remark 1.2.11. *Notice that, when* $s = 1$*, we have*

$$\begin{aligned}
\|(-\Delta)^{1/2} u\|_2^2 &= \int_{\mathbb{R}^N} \left|\mathcal{F}^{-1}(|\xi|\widehat{u})\right|^2 = \int_{\mathbb{R}^N} ||\xi|\widehat{u}|^2 = \sum_i \int_{\mathbb{R}^N} ||\xi_i|\widehat{u}|^2 \\
&= \sum_i \int_{\mathbb{R}^N} |\mathcal{F}(\partial_i u)|^2 = \sum_i \int_{\mathbb{R}^N} |\partial_i u|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 \\
&= \|\nabla u\|_2^2
\end{aligned}$$

and this justifies, for example, the use of $(-\Delta)^{s/2}$ *in the weak formulation of PDEs (see Definition 1.2.16). We highlight, anyway, the nontriviality of the relation, since* $(-\Delta)^{1/2}$ *is a nonlocal operator, while* ∇ *is a local operator (see also [201, Section 6]).*

When $s = 1$ *thus we will actually consider the classical Sobolev space* $H^1(\mathbb{R}^N)$ *endowed with*

$$\|u\|_{H^1} := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx \right)^{1/2} \qquad \textit{for } u \in H^1(\mathbb{R}^N)$$

and its subspace

$$H^1_r(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \mid u \textit{ radially symmetric}\}.$$

Tail-controlling mixed norms

In order to handle the long range interaction of the fractional norms, we will make use of the following *mixed* Gagliardo seminorm

$$[u]^2_{A_1, A_2} := \int_{A_1} \int_{A_2} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx\, dy, \quad [u]_A := [u]_{A,A}$$

for any $A_1, A_2, A \subset \mathbb{R}^N$ and $u \in H^s(\mathbb{R}^N)$; by using that $\varphi_u(x,y) := \frac{|u(x)-u(y)|}{|x-y|^{N/2+s}}$ satisfies $\varphi_{u+v} \leq \varphi_u + \varphi_v$ and $[u]_{A_1,A_2} = \|\varphi_u\|_{L^2(A_1 \times A_2)}$, we have that $[u]_{A_1,A_2}$ is actually a seminorm. This seminorm has been introduced in [111], although after the publication the authors discovered that similar tools were implemented in different frameworks [130, 178, 234].

For any $u \in H^s(\mathbb{R}^N)$ and $A \subset \mathbb{R}^N$ it will be useful to work also with the following norms:

$$\|u\|_A^2 := \|u\|_{L^2(A)}^2 + [u]_{A,\mathbb{R}^N}^2$$

and

$$|\!|\!|u|\!|\!|_A := \|u\|_{L^{p+1}(A)} + \|u\|_A,$$

for some suitable $p \in (2, 2_s^*)$. We highlight that $\|u\|_{\mathbb{R}^N} = \|u\|_{H^s(\mathbb{R}^N)}$, but generally $\|u\|_A \geq \|u\|_{H^s(A)}$ for $A \neq \mathbb{R}^N$. By $H^s(A) \hookrightarrow L^{p+1}(A)$ the norms $\|\cdot\|_A$ and $|\!|\!|\cdot|\!|\!|_A$ are equivalent: on the other hand, the constant such that $|\!|\!|u|\!|\!|_A \leq C_A \|u\|_A$ depends on A, thus not useful for ε-dependent sets $A = A(\varepsilon)$ (see Chapter 5). This is why we will make direct use also of $|\!|\!|\cdot|\!|\!|_A$.

Regarding ε-dependent norms, we will use also

$$\|u\|_{H^s_{V,\varepsilon}(\mathbb{R}^N)}^2 := \|(-\Delta)^{s/2}u\|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx$$

which is an equivalent norm on $H^s(\mathbb{R}^N)$ whenever $V \in L^\infty(\mathbb{R}^N)$ with $V \geq V_0 > 0$; the space $H^s_\varepsilon(\mathbb{R}^N)$ is defined straightforwardly.

1.2.2 Some computations: hypergeometric Gaussian functions

In order to implement some comparison argument (see Section 4.6.3), we search for a function which behaves like $\sim \frac{1}{|x|^\beta}$, $\beta > 0$, and which lies in $H^s(\mathbb{R}^N)$; in order to handle the presence of a pole in the origin when $\beta \geq N$, we make the following choice, by considering, for any $\beta > 0$,

$$h_\beta(x) := \frac{1}{(1+|x|^2)^{\frac{\beta}{2}}};$$

notice that, when $\beta = N + 2s$, this function is related to the extremals of the fractional Sobolev inequality [98, 265] and to the solutions of the zero mass critical fractional Choquard equation [253] (see also Proposition 1.3.1 below). Chosen h_β in this way, we have [246, Table 1 page 168] (see also [181, Sections 4 and 6])

$$(-\Delta)^s h_\beta(x) = C_{\beta,N,s}\, {}_2F_1\left(\frac{N}{2} + s, \frac{\beta}{2} + s, \frac{N}{2}; -|x|^2\right) \tag{1.2.15}$$

where

$$C_{\beta,N,s} := 2^{2s} \frac{\Gamma(\frac{N}{2}+s)\Gamma(\frac{\beta}{2}+s)}{\Gamma(\frac{N}{2})\Gamma(\frac{\beta}{2})} > 0$$

and ${}_2F_1$ denotes the Gauss hypergeometric function (see also [166, Corollary 2], observed that $h_\beta(x) = {}_2F_1(\frac{N}{2}, \frac{\beta}{2}, \frac{N}{2}, -|x|^2)$). Notice that we will be interested in

$$\beta \in (0, N + 2s].$$

The asymptotic behaviour at infinity of the hypergeometric function appearing in (1.2.15) can be found in [7, pages 559-560] (see also [23, pages 78-79, 88] and [374, page 161]). Recall that the Gamma function $\Gamma(z)$ is well defined whenever $z \in \mathbb{R} \setminus (-\mathbb{N})$ and $|\Gamma(z)| \to +\infty$ as z approaches $-\mathbb{N}$ (so that the *reciprocal Gamma function* is well defined on $-\mathbb{N}$ and equals zero). Moreover, recall the symmetry property ${}_2F_1(a, b, c; x) = {}_2F_1(b, a, c; x)$ and the fact that ${}_2F_1(0, b, c; x) = 1$ and ${}_2F_1(-1, b, c; x) = 1 - \frac{b}{c}z$.

Lemma 1.2.12 ([7]). *Consider ${}_2F_1(a,b,c;\cdot)$. For the sake of simplicity, assume a priori that $a,b,c>0$ and*

$$a-c\in\mathbb{R}_+\setminus\mathbb{N},$$

$$a-b\in\mathbb{Z}\iff a-b\in\mathbb{N},$$

$$b-c\in\mathbb{N}\iff b-c\in\{0,1\};$$

in particular $a-b$ and $b-c$ do not lie in $\mathbb{Z}$ at the same time. We have the following asymptotic estimates as $x\to-\infty$.

- *If $a-b\notin\mathbb{Z}$ and $b-c\notin\mathbb{N}$, then*

$$_2F_1(a,b,c;x)\dot{\sim}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}\frac{1}{(-x)^a}+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}\frac{1}{(-x)^b};$$

- *If $b=c$ (and $a-b\notin\mathbb{Z}$), then*

$$_2F_1(a,b,b;x)=\frac{1}{(1-x)^a};$$

- *If $b=c+1$ (and $a-b\notin\mathbb{Z}$), then*

$$\begin{aligned}{}_2F_1(a,b,b-1;x)&=-\frac{\Gamma(b-1)\Gamma(b-a)}{\Gamma(b-a-1)\Gamma(b)}\frac{x}{(1-x)^{a+1}}+\frac{1}{(1-x)^{a+1}}\\&\dot{\sim}\frac{\Gamma(b-1)\Gamma(b-a)}{\Gamma(b-a-1)\Gamma(b)}\frac{1}{(-x)^a};\end{aligned}$$

- *If $a=b$ (and $b-c\notin\mathbb{N}$), then*

$$_2F_1(a,a,c;x)\dot{\sim}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\frac{\log(-x)}{(-x)^a}+\frac{C_1}{(-x)^a}\dot{\sim}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\frac{\log(-x)}{(-x)^a};$$

- *If $a-b\in\mathbb{N}^*$ (and $b-c\notin\mathbb{N}$), then*

$$_2F_1(a,b,c;x)\dot{\sim}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}\frac{1}{(-x)^b}+C_2\frac{\log(-x)}{(-x)^a}+\frac{C_3}{(-x)^a}\dot{\sim}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}\frac{1}{(-x)^b}.$$

Here C_i, $i=1,2,3$, are some strictly positive constants.

Notice that $a=\frac{N}{2}+s$, $b=\frac{\beta}{2}+s$, $c=\frac{N}{2}$ satisfy the assumptions of the previous Lemma, whenever $s\in(0,1)$ and $\beta\in(0,N+2s]$. Thus, exploiting the representation of $(-\Delta)^s h_\beta$ given in (1.2.15) and the results on Gauss hypergeometric functions, we come up with the following lemma.

Lemma 1.2.13. *Let $\beta\in(0,N+2s]$. Then $(-\Delta)^s h_\beta(x)$ is well defined for every $x\neq0$. Moreover, we have the following asymptotic behaviours:*

- *if $\beta\in(N,N+2s]$, then*

$$(-\Delta)^s h_\beta(x)\dot{\sim}C'_{\beta,N,s}\frac{1}{|x|^{N+2s}}\qquad as\ |x|\to+\infty$$

where $C'_{\beta,N,s}:=2^{2s}\frac{\Gamma\left(\frac{N}{2}+s\right)\Gamma\left(\frac{\beta}{2}-\frac{N}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma(-s)}<0$. This in particular includes the case $\beta=N-2s+2$ (possible if $s>\frac12$), with $C'_{N-2s+2,N,s}=-2^{2s+1}\frac{s}{N-2s}<0$. Notice moreover that $C'_{N+2s,N,s}=2^{2s}\frac{\Gamma(s)}{\Gamma(-s)}\to0$ as $s\to1^-$.

- *if* $\beta = N$, *then*
$$(-\Delta)^s h_N(x) \dot{\sim} C'_{N,N,s} \frac{\log(|x|)}{|x|^{N+2s}} \quad \text{as } |x| \to +\infty$$
where $C'_{N,N,s} := 2^{2s+1} \frac{\Gamma\left(\frac{N}{2}+s\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(-s)} < 0.$

- *if* $\beta \in (N-2s, N)$, *then*
$$(-\Delta)^s h_\beta(x) \dot{\sim} C'_{\beta,N,s} \frac{1}{|x|^{\beta+2s}} \quad \text{as } |x| \to +\infty$$
where $C'_{\beta,N,s} := 2^{2s} \frac{\Gamma\left(\frac{\beta}{2}+s\right)\Gamma\left(\frac{N}{2}-\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{N}{2}-\frac{\beta}{2}-s\right)} < 0.$

- *if* $\beta = N-2s$, *then*
$$\begin{aligned}(-\Delta)^s h_{N-2s}(x) &= C'_{N-2s,N,s} h_{N+2s}(x) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\} \\ &\dot{\sim} C'_{N-2s,N,s} \frac{1}{|x|^{N+2s}} \quad \text{as } |x| \to +\infty\end{aligned}$$
where $C'_{N-2s,N,s} := 2^{2s} \frac{\Gamma\left(\frac{N}{2}+s\right)}{\Gamma\left(\frac{N}{2}-s\right)} > 0.$

- *if* $\beta \in (0, N-2s)$, *then*
$$(-\Delta)^s h_\beta(x) \dot{\sim} C'_{\beta,N,s} \frac{1}{|x|^{\beta+2s}} \quad \text{as } |x| \to +\infty$$
where $C'_{\beta,N,s} := 2^{2s} \frac{\Gamma\left(\frac{\beta}{2}+s\right)\Gamma\left(\frac{N}{2}-\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{N}{2}-\frac{\beta}{2}-s\right)} > 0$. *This in particular includes the case* $\beta = N-2k$ *with* $k = 1, \dots, [\frac{N}{2}]$.

Remark 1.2.14. *Notice that, for* $\beta \in \{N-2s\} \cup [N, N+2s]$, *the asymptotic behaviour of* $|(-\Delta)^s h_\beta(x)|$ *does not depend on* β; *on the other hand, the sign and the precise constant depend on* β.

In the case $\beta \in (0, N) \setminus \{N-2s\}$, *we may use* $x \mapsto \frac{1}{|x|^\beta}$, *whose fractional Laplacian has a close (simple) representation:*
$$\left((-\Delta)^s \frac{1}{|\cdot|^\beta}\right)(x) = C_{\beta,N,s} \frac{1}{|x|^{\beta+2s}},$$
see [246, Table 1 and Theorem 3.1] (see also [173, Lemma 4.1], [366, Appendix 1, page 798] and [68, Lemma A.2]). In particular
$$(-\Delta)^s h_\beta(x) \dot{\sim} \left((-\Delta)^s \frac{1}{|\cdot|^\beta}\right)(x) \quad \text{as } |x| \to +\infty.$$
On the other hand, if $\beta = N-2s$, *we obtain, far from the origin,* $(-\Delta)^s \frac{1}{|\cdot|^\beta} \equiv 0$ *(recall that the Riesz potential* $\frac{1}{|\cdot|^{N-2s}} \equiv I_{2s}$ *is a fundamental solution, see Proposition 1.3.4); thus, in particular, the two functions have different asymptotic behaviours. This is the same reason why, for* h_β, *we have a discontinuity on the behaviour at infinity around* $\beta = N-2s$.

Finally we highlight that, when $\beta = N+2s$, *we may use the function found in Lemma 1.2.30.*

1.2.3 Definitions of solutions: weak, viscosity, strong, classical

In the majority of the thesis we will work with the notion of weak solutions, by exploiting a variational formulation. Anyway, sometimes we will need to exploit different formulations, in particular strong, classical and viscosity formulations; that is why we recall them here for the sake of clarity.

Definition 1.2.15 (Strong and classical solution). *Let $\Omega \subseteq \mathbb{R}^N$ and $g : \Omega \to \mathbb{R}$. We say that u is a* strong *solution to*

$$(-\Delta)^s u = g(x) \quad \text{in } \Omega$$

if u and $(-\Delta)^s u$ are almost everywhere defined (e.g. $u \in H^{2s}(\Omega)$) and u satisfies the relation for almost every $x \in \Omega$.

We say instead that u is a classical *solution if u and $(-\Delta)^s u$ are continuous (e.g. $u \in L^p(\mathbb{R}^N) \cap C^\gamma_{loc}(\mathbb{R}^N)$ for some $p \in [1, +\infty]$ and $\gamma > 2s$) and the relation is satisfied pointwise everywhere on Ω.*

Definition 1.2.16 (Weak solution). *Let $\Omega \subseteq \mathbb{R}^N$ and $g : \Omega \to \mathbb{R}$ be measurable. We say that $u \in H^s(\Omega)$ is a* weak *subsolution [supersolution] of*

$$(-\Delta)^s u = g(x) \quad \text{in } \Omega$$

if

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi dx \leq \int_{\mathbb{R}^N} g(x) \varphi dx \tag{1.2.16}$$

$$\Big[\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi dx \geq \int_{\mathbb{R}^N} g(x) \varphi dx \Big]$$

is well defined (finite) and holds for each positive $\varphi \in X^s_0(\Omega)$. We say that u is a weak solution if it is both a subsolution and a supersolution, i.e. if it satisfies the equality in (1.2.16) for every $\varphi \in X^s_0(\Omega)$. Notice that, when $\Omega = \mathbb{R}^N$, we have $X^s_0(\mathbb{R}^N) \equiv H^s(\mathbb{R}^N)$.

Remark 1.2.17. *By (1.2.6) and (1.2.2) we may interpret the left-hand side of (1.2.16) as*

$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi dx &\equiv \tfrac{1}{2} C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} dxdy \\ &\equiv \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u} \widehat{\varphi} d\xi.\end{aligned}$$

Moreover we see that the definition of weak solution is justified by the following integration by parts *rule*

$$\int_{\mathbb{R}^N} (-\Delta)^s u v = \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u} \widehat{v} = \int_{\mathbb{R}^N} |\xi|^s \widehat{u} |\xi|^s \widehat{v} = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v$$

which holds whenever $u \in H^{2s}(\mathbb{R}^N)$ and $v \in H^s(\mathbb{R}^N)$. In particular, if both $u, v \in H^{2s}(\mathbb{R}^N)$ we have (see also [201, Lemma 5.4])

$$\int_{\mathbb{R}^N} (-\Delta)^s u v = \int_{\mathbb{R}^N} u (-\Delta)^s v.$$

Remark 1.2.18. *If $\varphi \in C^\infty_c(\mathbb{R}^N)$ and $u \in L^\infty(\mathbb{R}^N)$, we can show the relation*

$$\int_{\mathbb{R}^N} u (-\Delta)^s \varphi = \tfrac{1}{2} C_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} dxdy$$

also by exploiting the pointwise definition of the fractional Laplacian. Indeed (assume for simplicity $s \in (0, \frac{1}{2})$ to avoid the technicality of the Principal Value) we have

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^s \varphi(x) dx = C_{N,s} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{u(x)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} dy \right) dx. \tag{1.2.17}$$

First, we rewrite (1.2.17) by applying Fubini-Tonelli theorem, possible because

$$\int_{\mathbb{R}^{2N}} \left| \frac{u(x)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} \right| \leq \|u\|_\infty \int_{\mathbb{R}^{2N}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{N+2s}} < \infty;$$

notice that we are actually using that $\varphi \in \widetilde{W}^{2s,1}(\mathbb{R}^N)$ (see Remark 1.2.9). Thus

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^s \varphi(x) dx = C_{N,s} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{u(x)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} dx \right) dy.$$

Secondly, we rewrite (1.2.17) by simply renaming the variables, that is

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^s \varphi(x) dx = C_{N,s} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{\big(-u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{N+2s}} dx \right) dy.$$

By summing the two expressions obtained, we get the claim.

For the following definition, see e.g. [339, page 136] or [97, Definition 2.1].

Definition 1.2.19 (Viscosity solution)**.** *Let $\Omega \subseteq \mathbb{R}^N$ and $g : \Omega \to \mathbb{R}$. We say that $u \in C(\mathbb{R}^N)$ is a* viscosity *subsolution [supersolution] of*

$$(-\Delta)^s u = g(x) \quad \text{in } \Omega$$

if, for any $x_0 \in \Omega$, every $U \subset \Omega$ open neighborhood of x_0, and every $\phi \in C^2(U)$ such that

$$\phi(x_0) = u(x_0), \quad \phi \geq u \ [\phi \leq u] \ \text{ in } U$$

set

$$v := \phi \chi_U + u \chi_{U^c}$$

we have

$$(-\Delta)^s v(x_0) \leq g(x_0) \quad \big[(-\Delta)^s v(x_0) \geq g(x_0)\big].$$

We say that u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

We observe that, generally, the function v appearing in the definition of viscosity solution might be discontinuous. More generally, this definition involves lower and upper semicontinuity of u (see for instance [87, Definition 2.2]). Furthermore, one can easily check that every classical solution is a viscosity solution, that the sum of two viscosity solutions is still a viscosity solution (with source the sum of the sources), and that the notion of viscosity solution is conserved on subdomains $\Omega' \subset \Omega$.

We refer to [328, Remark 2.11] and [339, Theorem 1] for some discussions on the relation between classical, weak and viscosity solutions on bounded domains.

When dealing with equations with nonlinearities of the type $h = h(x, u)$, $h : \Omega \times \mathbb{R} \to \mathbb{R}$, we interpret the equation by saying that u is a classic/strong/weak/viscosity solution if u satisfies the equation with nonlinearity $g(x) := h(x, u(x))$. The same interpretation will be given in the case of nonlocal nonlinearities (see Section 1.3).

Remark 1.2.20. *In this preliminary Chapter, we will sometimes mention* distributional *solutions of equations of the type* $(-\Delta)^s u = T$*, with* T *distribution on some* Ω*. By this, we mean that* $u \in L^1_{loc}(\mathbb{R}^N)$ *satisfies* (1.2.1) *and*

$$\int_\Omega u(-\Delta)^{s/2}\varphi = T(\varphi)$$

for every $\varphi \in C_c^\infty(\Omega)$*. The extra condition required on* u *(differently form the usual definition of distributional solution) is due to the fact that* $(-\Delta)^{s/2}\varphi$ *has generally not compact support; here we use thus* (1.2.3) *to well define the integral.*

1.2.4 A concave Chain rule

We already pointed out how the fractional Laplacian does not satisfy a proper Lebiniz formula. The same conclusion is actually true looking at chain rule formulas. A first result is given by the following lemma.

Lemma 1.2.21. *Let* $\Omega \subseteq \mathbb{R}^N$*. If* $u \in H^s(\Omega)$ *and* $h : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz function with* $h(0) = 0$*, then* $h(u) \in H^s(\Omega)$*.*

Proof. The proof is straightforward. Indeed

$$\|h(u)\|^2_{L^2(\Omega)} = \int_\Omega |h(u) - h(0)|^2\,dx \le \int_\Omega \|h'\|^2_\infty |u - 0|^2\,dx = \|h'\|^2_\infty \|u\|^2_{L^2(\Omega)}$$

and

$$[h(u)]_{H^s(\Omega)} \le C_{N,s} \int_\Omega \int_\Omega \frac{\|h'\|^2_\infty |u(x) - u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = \|h'\|^2_\infty [u]_{H^s(\Omega)}.$$ ∎

We look now to proper pointwise chain rules. What one can prove is that, whenever φ is convex (and Lipschitz), then the following inequality holds (see [88, Theorem 1.1], [201, Theorem 19.1])

$$(-\Delta)^s \varphi(u) \le \varphi'(u)(-\Delta)^s u$$

in the weak sense. One may expect the inverse inequality when handling concave functions: and this is actually what we need in the study of the asymptotic behaviour of ground state in doubly nonlocal equations (see Section 4.6.6).

On the other hand, since we do not know if $u^\theta \notin H^s(\mathbb{R}^N)$ when $u \in H^s(\mathbb{R}^N)$ and $\theta \in (0,1)$, the weak formulation seems not to be the right choise; pointwise formulation seems not good as well, since $(-\Delta)^s u^\theta$ might be not well defined, even by assuming u regular. The idea is thus to take advantage of a viscosity formulation.

We prove hence the following inequality in the case of concave (not globally Lipschitz) function, in the framework of viscosity solutions. Notice that we do not require u to be in $L^2(\mathbb{R}^N)$.

Lemma 1.2.22 (Córdoba-Córdoba chain rule inequality)**.** *Let* $\varphi : I \to \mathbb{R}$ *be a concave function,* $I \subseteq \mathbb{R}$ *interval, such that* $\varphi \in C^1(I)$*. Let* $u : \mathbb{R}^N \to I$*.*

- *Let* $\Omega \subset \mathbb{R}^N$*, and assume* $\varphi \in Lip(u(\Omega))$*. Then*

$$[\varphi(u)]_{H^s(\Omega)} \le \|\varphi'\|_{L^\infty(u(\Omega))}[u]_{H^s(\Omega)}.$$

In particular, if $\varphi \in Lip(I)$ *and* $(-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)$*, then* $(-\Delta)^{s/2}\varphi(u) \in L^2(\mathbb{R}^N)$ *and*

$$\|(-\Delta)^{s/2}\varphi(u)\|_2 \le \|\varphi'\|_{L^\infty(I)} \|(-\Delta)^{s/2}u\|_2.$$

- *If* u *is defined pointwise, then*

$$(-\Delta)^s(\varphi(u))(x) \ge \varphi'(u(x))(-\Delta)^s u(x)$$

for every $x \in \mathbb{R}^N$ *such that* $(-\Delta)^s(\varphi(u))(x)$ *and* $(-\Delta)^s u(x)$ *are well defined.*

- *Assume in addition φ invertible, increasing, with $\varphi^{-1} \in C^2$ increasing. If u is a continuous viscosity supersolution of*
$$(-\Delta)^s u \geq g \quad \text{in } \Omega$$
for some function g and $\Omega \subseteq \mathbb{R}^N$, then $\varphi(u)$ is a viscosity supersolution of
$$(-\Delta)^s(\varphi(u)) \geq \varphi'(u)g \quad \text{in } \Omega.$$

Proof. The first claim is a direct consequence of the Lipschitz continuity
$$\int_\Omega\int_\Omega \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x-y|^{N+2s}}dxdy \leq \|\varphi'\|^2_{L^\infty(u(\Omega))}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy.$$
Secondly, by the concavity of φ, for each $t, r \in I$ we have
$$\varphi(t) - \varphi(r) \geq \varphi'(t)(t-r)$$
thus
$$\begin{aligned}(-\Delta)^s(\varphi(u))(x) &= C_{N,s}\int_{\mathbb{R}^N}\frac{\varphi(u(x)) - \varphi(u(y))}{|x-y|^{N+2s}}dy \\ &\geq C_{N,s}\int_{\mathbb{R}^N}\frac{\varphi'(u(x))(u(x)-u(y))}{|x-y|^{N+2s}}dy = \varphi'(u(x))(-\Delta)^s u(x).\end{aligned}$$

We move to the third part. Let $x_0 \in U \subset \Omega$ and $\phi \in C^2(U)$ be such that $\phi(x_0) = \varphi(u(x_0))$ and $\phi \leq \varphi(u)$ in U, and set $v := \phi\chi_U + \varphi(u)\chi_{U^c}$. Let now
$$\psi := \varphi^{-1}\circ\phi, \quad w := \varphi^{-1}\circ v = \psi\chi_U + u\chi_{U^c}.$$
By the assumptions on φ^{-1} we have $\psi \in C^2(U)$, $\psi(x_0) = u(x_0)$ and $\psi \leq u$ in U. Thus
$$(-\Delta)^s w(x_0) \geq g(x_0).$$
On the other hand, $w = \psi \in C^2$ on U and $\varphi(w) = \phi \in C^2$ on U, hence both the functions are regular enough in a neighborhood of x_0 to state that both the fractional Laplacians are well defined (see Proposition 1.2.1). Thus we may apply the previous point and obtain
$$(-\Delta)^s(\varphi(w))(x_0) \geq \varphi'(w(x_0))(-\Delta)^s w(x_0).$$
Since $w(x_0) = u(x_0)$, $\varphi(w) = v$ and φ' is positive, we obtain, by joining the two previous inequalities
$$(-\Delta)^s v(x_0) \geq \varphi'(u(x_0))g(x_0)$$
which is the claim. This concludes the proof. ∎

As a corollary, we obtain the following result.

Corollary 1.2.23. *Let $\theta \in (0,1)$, and let $u \in C(\mathbb{R}^N)$ be strictly positive. We have the following results.*

- *We have*
$$[u^\theta]_{H^s(\Omega)} \leq \frac{\theta}{\min_\Omega u^{1-\theta}}[u]_{H^s(\Omega)}$$
In particular, if $u \in H^s_{loc}(\mathbb{R}^N)$, then $u^\theta \in H^s_{loc}(\mathbb{R}^N)$.[1] *As a consequence, if $u \in H^s(\mathbb{R}^N)$, then*
$$[u^\theta]_{H^s(\Omega)} \leq \frac{\theta}{\min_\Omega u^{1-\theta}}\|(-\Delta)^{s/2}u\|_2.$$

[1]Indeed, if $u \in L^2_{loc}(\mathbb{R}^N)$, then $u^\theta \in L^2_{loc}(\mathbb{R}^N)$ can be deduced by the inverse Hölder inequality: $\int_\Omega u^2 = \int_\Omega u^2 \cdot 1 \geq \int_\Omega u^{\frac{2}{p}} \int_\Omega 1^{-\frac{1}{p-1}} = \int_\Omega u^{2\theta}\cdot m(\Omega)$, if $p := \frac{1}{\theta} > 1$ and Ω is bounded (with positive measure).

- *If $(-\Delta)^s u$ is well defined pointwise, then*

$$(-\Delta)^s u^\theta(x) \geq \frac{\theta}{(u(x))^{1-\theta}}(-\Delta)^s u(x)$$

for every $x \in \mathbb{R}^N$ such that $(-\Delta)^s u^\theta(x)$ is well defined.

- *If u is a viscosity supersolution of*

$$(-\Delta)^s u \geq g \quad \text{in } \Omega$$

for some function g and $\Omega \subseteq \mathbb{R}^N$, then u^θ is a viscosity supersolution of

$$(-\Delta)^s u^\theta \geq \frac{\theta}{u^{1-\theta}} g \quad \text{in } \Omega.$$

1.2.5 Regularity: tail functions and De Giorgi classes

We gain now some L^∞-bound for sign-changing solutions, in a fractional, possibly critical, framework. We adapt some arguments from the papers [115, 197]. This result will be then implemented in the study of sign-changing solutions for doubly nonlocal equations (Theorem 4.4.1), and in the study of uniform bounds for semiclassical critical problems (Proposition 5.5.5). Notice that we avoid the use of the Caffarelli-Silvestre s-harmonic extension, and this allows to extend our proof to different frameworks where this tool is not available.

Proposition 1.2.24. *Let $u \in H^s(\mathbb{R}^N)$ be a weak subsolution of*

$$(-\Delta)^s u \leq g(x,u) \quad \text{in } \mathbb{R}^N$$

with

$$|g(x,t)| \leq C\left(|t| + |t|^{2_s^*-1}\right) \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}$$

for some uniform $C > 0$. Then $u \in L^\infty(\mathbb{R}^N)$.

Proof. We already know that $u \in L^2(\mathbb{R}^N) \cap L^{2_s^*}(\mathbb{R}^N)$. Let us introduce $\gamma > 1$, to be fixed, and an arbitrary $T > 0$, and set a *γ-linear (positive) truncation at T*

$$h(t) \equiv h_{T,\gamma}(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t^\gamma & \text{if } t \in (0,T], \\ \gamma T^{\gamma-1} t - (\gamma-1)T^\gamma & \text{if } t > T. \end{cases}$$

We have that $h \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, it is positive (increasing and convex), zero on the negative halfline, and by direct computations it satisfies the following properties

$$0 \leq h(t) \leq |t|^\gamma, \quad t \in \mathbb{R}, \tag{1.2.18}$$

$$0 \leq t h'(t) \leq \gamma h(t), \quad t \in \mathbb{R}, \tag{1.2.19}$$

$$\lim_{T \to +\infty} h_{T,\gamma}(t) = t^\gamma, \quad t \geq 0. \tag{1.2.20}$$

The goal is to estimate $\|h(u)\|_{2_s^*}$ and give thus a bound of u in $L^{2_s^*\gamma}(\mathbb{R}^N)$, where $2_s^*\gamma > 2_s^*$. In order to handle the weak formulation of the notion of solution we introduce

$$\tilde{h}(t) := \int_0^t (h'(r))^2\, dr, \quad t \in \mathbb{R}$$

and observe that $\tilde{h} \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ is positive, increasing, convex and zero on the negative halfline. In particular

$$\tilde{h}'(t) = (h'(t))^2, \quad t \in \mathbb{R} \tag{1.2.21}$$

by definition and

$$\tilde{h}(t) - \tilde{h}(r) \le \tilde{h}'(t)(t-r), \quad t, r \in \mathbb{R} \tag{1.2.22}$$

by convexity, and we gain also the Lipschitz continuity

$$|\tilde{h}(t) - \tilde{h}(r)| \le \|\tilde{h}'\|_\infty |t-r|, \quad t, r \in \mathbb{R}.$$

Combining the definition of $\tilde{h}$, (1.2.19) and (1.2.18) we obtain

$$0 \le \tilde{h}(t) \le \|h'\|_\infty |t|^\gamma, \quad t \in \mathbb{R}. \tag{1.2.23}$$

Finally, by a direct application of Jensen inequality we gain

$$|h(t) - h(r)|^2 \le (\tilde{h}(t) - \tilde{h}(r))(t-r), \quad t, r \in \mathbb{R}. \tag{1.2.24}$$

We observe that $\tilde{h}(u) \in H^s(\mathbb{R}^N)$ since $\tilde{h}$ is Lipschitz continuous and $\tilde{h}(0) = 0$ (see Lemma 1.2.21); moreover, since 2_s^* is the best summability exponent, if we assume

$$1 < \gamma \le \frac{2_s^*}{2} \tag{1.2.25}$$

by (1.2.23) we obtain also

$$\tilde{h}(u) \le \|h'\|_\infty |u|^\gamma \in L^2(\mathbb{R}^N).$$

We use now the embedding (1.2.7) and combine (1.2.5), (1.2.24) and (1.2.6) to obtain

$$\begin{aligned}
\|h(u)\|_{2_s^*}^2 &\le \mathcal{S}^{-1} \|(-\Delta)^{s/2} h(u)\|_2^2 \\
&= (C'(N,s))^{-1} \mathcal{S}^{-1} \int_{\mathbb{R}^{2N}} \frac{|h(u(x)) - h(u(y))|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\le (C'(N,s))^{-1} \mathcal{S}^{-1} \int_{\mathbb{R}^{2N}} \frac{(\tilde{h}(u(x)) - \tilde{h}(u(y)))(u(x) - u(y))}{|x-y|^{N+2s}} \, dx \, dy \\
&= \mathcal{S}^{-1} \int_{\mathbb{R}^N} (-\Delta)^{s/2} u \, (-\Delta)^{s/2} \tilde{h}(u) \, dx.
\end{aligned}$$

Since $\tilde{h}(u) \in H^s(\mathbb{R}^N)$ we can choose it as a test function in the equation and gain

$$\|h(u)\|_{2_s^*}^2 \le \mathcal{S}^{-1} \int_{\mathbb{R}^N} g(x,u) \tilde{h}(u) \, dx.$$

By the assumptions on g and the positivity of $\tilde{h}(u)$ we get

$$\|h(u)\|_{2_s^*}^2 \le \mathcal{S}^{-1} \int_{\mathbb{R}^N} |g(x,u)| \tilde{h}(u) \, dx \le C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (|u| + |u|^{2_s^*-1}) \tilde{h}(u) \, dx.$$

Since $h(u)$ and $\tilde{h}(u)$ are zero when u is negative, we obtain

$$\|h(u_+)\|_{2_s^*}^2 \le C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (u_+ + u_+^{2_s^*-1}) \tilde{h}(u_+) \, dx.$$

Now we use (1.2.22) (with $r = 0$), (1.2.21), and (1.2.19)

$$\begin{aligned}
&\|h(u_+)\|_{2_s^*}^2 \le C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (u_+ + u_+^{2_s^*-1}) u_+ \tilde{h}'(u_+) \, dx \\
&\le \; C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (u_+ + u_+^{2_s^*-1}) u_+ (h'(u_+))^2 \, dx \le \gamma^2 C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (1 + u_+^{2_s^*-2}) (h(u_+))^2 \, dx \\
&\le \; \gamma^2 C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (h(u_+))^2 \, dx + \gamma^2 C\mathcal{S}^{-1} \int_{\mathbb{R}^N} u_+^{2_s^*-2} (h(u_+))^2 \, dx.
\end{aligned} \tag{1.2.26}$$

Let now $R > 0$ to be fixed; splitting the second piece of the right-hand side of (1.2.26) and by using the Hölder inequality we gain

$$\int_{\mathbb{R}^N} u_+^{2_s^*-2}(h(u_+))^2\,dx = \int_{u\le R} u_+^{2_s^*-2}(h(u_+))^2\,dx + \int_{u>R} u_+^{2_s^*-2}(h(u_+))^2\,dx$$
$$\le R^{2_s^*-2}\|h(u_+)\|_2^2 + \left(\int_{u>R} u^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}} \|h(u_+)\|_{2_s^*}^2.$$

Since $u \in L^{2_s^*}(\mathbb{R}^N)$, we can find a sufficiently large $R = R(\gamma, m_0, \mathcal{S}^{-1})$ such that

$$\left(\int_{u>R} u^{2_s^*}\,dx\right)^{\frac{2_s^*-2}{2_s^*}} < \frac{1}{2}\frac{1}{\gamma^2 C\mathcal{S}^{-1}}.$$

Thus, plugging this information into (1.2.26), and absorbing the second piece on the right-hand side into the left-hand side, we obtain by (1.2.18)

$$\|h(u_+)\|_{2_s^*}^2 \le 2\gamma^2 C\mathcal{S}^{-1}(1+R^{2_s^*-2})\|h(u_+)\|_2^2 \le 2\gamma^2 C\mathcal{S}^{-1}(1+R^{2_s^*-2})\|u_+\|_{2\gamma}^{2\gamma}.$$

Recalled that $h = h_{T,\gamma}$, by (1.2.20) and Fatou's Lemma we have

$$\|u_+\|_{2_s^*\gamma}^{2\gamma} = \left(\int_{\mathbb{R}^N} \liminf_{T\to+\infty} h_{T,\gamma}^{2_s^*}(u_+)\,dx\right)^{\frac{2}{2_s^*}} \le \left(\liminf_{T\to+\infty}\int_{\mathbb{R}^N} h_{T,\gamma}^{2_s^*}(u_+)\,dx\right)^{\frac{2}{2_s^*}}$$
$$\le 2\gamma^2 C\mathcal{S}^{-1}(1+R^{2_s^*-2})\|u_+\|_{2\gamma}^{2\gamma}.$$

By our choice (1.2.25) of γ we gain that $u_+ \in L^{2_s^*\gamma}(\mathbb{R}^N)$, which was the claim. By an iteration argument, with

$$\gamma_0 := \frac{1}{2}2_s^*, \quad \gamma_i := \frac{1}{2}2_s^*\gamma_{i-1}, \quad \gamma_i \to +\infty,$$

we obtain $u_+ \in L^r(\mathbb{R}^N)$ for each $r \in [2, +\infty)$. In order to achieve $u_+ \in L^\infty(\mathbb{R}^N)$ we need to be careful on the bound on the L^r-norms.

Knowing that u_+ lies in every Lebesgue space for $r < \infty$ we can implement a more precise iteration argument, where we drop the dependence of the constant on R. We exploit once more (1.2.26). Applying again Fatou's Lemma to (1.2.26) and using (1.2.18) we obtain

$$\|u_+\|_{2_s^*\gamma}^{2\gamma} \le \gamma^2 C\mathcal{S}^{-1}\int_{\mathbb{R}^N} (u_+^{2\gamma} + u_+^{2_s^*-2+2\gamma})\,dx. \tag{1.2.27}$$

Focusing on the second term on the right-hand side, exploiting first the generalized Hölder inequality with

$$\frac{1}{N/s} + \frac{1}{2} + \frac{1}{2_s^*} = 1,$$

possible since $u_+^{2_s^*-2} \in L^{\frac{N}{s}}(\mathbb{R}^N)$ because $(2_s^*-2)\frac{N}{s} = \frac{4N}{N-2s} \ge 2$, and the generalized Young's inequality then, we get

$$\int_{\mathbb{R}^N} u_+^{2_s^*-2+2\gamma}\,dx = \int_{\mathbb{R}^N} u_+^{2_s^*-2}u_+^\gamma u_+^\gamma\,dx \le \|u_+^{2_s^*-2}\|_{\frac{N}{s}}\,\|u_+^\gamma\|_2\,\|u_+^\gamma\|_{2_s^*}$$
$$\le \|u_+^{2_s^*-2}\|_{\frac{N}{s}}\left(\frac{1}{2\varepsilon}\|u_+^\gamma\|_2^2 + \frac{\varepsilon}{2}\|u_+^\gamma\|_{2_s^*}^2\right) = \|u_+\|_{\frac{4N}{N-2s}}^{2_s^*-2}\left(\frac{1}{2\varepsilon}\|u_+\|_{2\gamma}^{2\gamma} + \frac{\varepsilon}{2}\|u_+\|_{2_s^*\gamma}^{2\gamma}\right).$$

Plugging this into (1.2.27), set $a := \|u_+\|_{\frac{4N}{N-2s}}^{2_s^*-2}$, choosing $\varepsilon = \frac{1}{a\gamma^2 C\mathcal{S}^{-1}}$ and bringing the $L^{2_s^*\gamma}$-norm on the left hand side, we gain

$$\|u_+\|_{2_s^*\gamma}^{2\gamma} \le 2\gamma^2 C\mathcal{S}^{-1}(1 + \tfrac{1}{2}a^2\gamma^2 C\mathcal{S}^{-1})\|u_+\|_{2\gamma}^{2\gamma} \le C'\gamma^4\|u_+\|_{2\gamma}^{2\gamma}$$

for some γ-independent $C' > 0$. Choosing $2\gamma_i := 2_s^* \gamma_{i-1}$ we have

$$\|u_+\|_{2_s^*\gamma_i} \leq (C'\gamma_i^4)^{\frac{1}{2\gamma_i}} \|u_+\|_{2_s^*\gamma_{i-1}}$$

and thus

$$\begin{aligned}\|u_+\|_{2_s^*\gamma_i} &\leq \prod_{j=0}^{i} (C'\gamma_j^4)^{\frac{1}{2\gamma_j}} \|u_+\|_{2_s^*\gamma_0} = e^{\sum_{j=0}^{i} \frac{\log\left(C'\gamma_j^4\right)}{2\gamma_j}} \|u_+\|_{2_s^*\gamma_0} \\ &= e^{\sum_{j=0}^{i} \frac{\log\left(C'\left(\frac{2_s^*}{2}\right)^{4j}\gamma_0^4\right)}{2\left(\frac{2_s^*}{2}\right)^{j}\gamma_0}} \|u_+\|_{2_s^*\gamma_0}\end{aligned}$$

and finally, sending $i \to +\infty$ (recall that $\|\cdot\|_p \to \|\cdot\|_\infty$ as $p \to +\infty$),

$$\|u_+\|_\infty \leq e^{\sum_{j=0}^{\infty} \frac{\log\left(C'\left(\frac{2_s^*}{2}\right)^{4j}\gamma_0^4\right)}{2\left(\frac{2_s^*}{2}\right)^{j}\gamma_0}} \|u_+\|_{2_s^*\gamma_0}$$

where the constant is finite. Thus $u_+ \in L^\infty(\mathbb{R}^N)$.

To deal with u_- we consider

$$k(t) \equiv k_{T,\gamma}(t) := h_{T,\gamma}(-t), \quad \tilde{k}(t) := \int_t^0 (k'(r))^2\, dr = \tilde{h}(-t)$$

and choose $\tilde{k}(u)$ as test function. With the same passages as before we obtain

$$\|k(u)\|_{2_s^*}^2 \leq -\mathcal{S}^{-1} \int_{\mathbb{R}^N} g(x,u)\tilde{k}(u)\, dx$$

and thus

$$\|k(u)\|_{2_s^*}^2 \leq \mathcal{S}^{-1} \int_{\mathbb{R}^N} |g(x,u)|\tilde{k}(u)\, dx \leq C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (|u| + |u|^{2_s^*-1})\tilde{k}(u)\, dx$$

which implies

$$\|k(-u_-)\|_{2_s^*}^2 \leq C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (|-u_-| + |-u_-|^{2_s^*-1})\tilde{k}(-u_-)\, dx$$

and hence

$$\|h(u_-)\|_{2_s^*}^2 \leq C\mathcal{S}^{-1} \int_{\mathbb{R}^N} (|u_-| + |u_-|^{2_s^*-1})\tilde{h}(u_-)\, dx;$$

we then proceed as before to gain $u_- \in L^\infty(\mathbb{R}^N)$. This concludes the proof. ∎

Once obtained that $u \in L^\infty(\mathbb{R}^N)$, we can improve the regularity. The following result can be found in [352, Theorem 15] (see also [346, Propositions 2.8 and 2.9]); see Remark 1.1.2 for the definition of Λ_1 and Λ_2.

Proposition 1.2.25. *Let $s \in (0,1)$ and $u \in L^\infty(\mathbb{R}^N)$ be a strong solution of*

$$(-\Delta)^s u = g \quad \text{in } \mathbb{R}^N.$$

i) If $g \in L^\infty(\mathbb{R}^N)$, then

$$u \in \begin{cases} C^{0,\gamma}(\mathbb{R}^N) \ \text{ for } \gamma < 2s & \text{if } 2s \in (0,1), \\ \Lambda_1(\mathbb{R}^N), \text{ thus } C^{0,\gamma}(\mathbb{R}^N) \ \text{ for } \gamma < 1 & \text{if } 2s = 1, \\ C^{1,\gamma-1}(\mathbb{R}^N) \ \text{ for } \gamma < 2s & \text{if } 2s \in (1,2). \end{cases}$$

ii) If $g \in C^{0,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (0,1]$, then

$$u \in \begin{cases} C^{0,\sigma+2s}(\mathbb{R}^N) & \text{if } \sigma+2s \in (0,1), \\ \Lambda_1, \ \text{thus } C^{0,\gamma}(\mathbb{R}^N) \ \text{ for } \gamma<1 & \text{if } \sigma+2s = 1, \\ C^{1,\sigma+2s-1}(\mathbb{R}^N) & \text{if } \sigma+2s \in (1,2), \\ \Lambda_2, \ \text{thus } C^{1,\gamma}(\mathbb{R}^N) \ \text{ for } \gamma<1 & \text{if } \sigma+2s = 2, \\ C^{2,\sigma+2s-2}(\mathbb{R}^N) & \text{if } \sigma+2s \in (2,3); \end{cases}$$

the previous relations holds also if we substitute global spaces with local spaces.

Notice that the conclusion in $i)$ was partially contained in the embedding (1.2.13).

Remark 1.2.26. *We see that regularity theory of Proposition 1.2.25 extends to $s \geq 1$. Indeed, by Remark 1.2.8, if $u \in H^{2s}(\mathbb{R}^N) \cap W^{2[s],\infty}(\mathbb{R}^N)$ then*

$$(-\Delta)^s u = g \implies (-\Delta)^{[s]-s}((-\Delta)^{[s]} u) = g$$

and all the regularity results apply to $(-\Delta)^{[s]}u$. At this point it is sufficient to apply regularity theory for polyharmonic operators [381, Section 3.20]. See also [330, Theorem 1.2] and [5, Theorem 3.7].

We want now to investigate in a more detailed way the regularity of solutions. Set first

$$\mathrm{Tail}(u; x_0, R) := (1-s) R^{2s} \int_{\mathbb{R}^N \setminus B_R(x_0)} \frac{|u(x)|}{|x-x_0|^{N+2s}} dx \tag{1.2.28}$$

the *tail function* of $u \in H^s(\mathbb{R}^N)$, centered in $x_0 \in \mathbb{R}^N$ with radius $R>0$, introduced in [151,152]. We recall properties of the fractional De Giorgi class stated in [134], to which we refer for a complete introduction on the topic; we focus only on the linear case.

By [134, Paragraph 6.1] we have the following definition.

Definition 1.2.27. *Let $A \subset \mathbb{R}^N$ be open, $\zeta \geq 0$, $H \geq 1$, $k_0 \in \mathbb{R}$, $\mu \in (0, 2s/N]$, $\lambda \geq 0$ and $R_0 \in (0,+\infty]$. We say that u belongs to the* fractional De Giorgi class $DG^{s,2}_+(A,\zeta,H,k_0,\mu,\lambda,R_0)$ *if and only if*

$$\begin{aligned} &[(u-k)_+]^2_{B_r(x_0)} + \int_{B_r(x_0)} (u(x)-k)_+ \left(\int_{B_{2R_0}(x)} \frac{(u(y)-k)_-}{|x-y|^{N+2s}} dy \right) dx \\ &\leq \frac{H}{1-s} \Bigg(\left(R^\lambda \zeta^2 + \frac{|k|^2}{R^{N\mu}} \right) \left| \operatorname{supp}((u-k)_+) \cap B_R(x_0) \right|^{1-\frac{2s}{N}+\mu} + \\ &\quad + \frac{R^{2(1-s)}}{(R-r)^2} \|(u-k)_+\|^2_{L^2(B_R(x_0))} + \\ &\quad + \frac{R^N}{(R-r)^{N+2s}} \|(u-k)_+\|_{L^1(B_R(x_0))} \mathrm{Tail}((u-k)_+; x_0, r) \Bigg) \end{aligned}$$

for any $x_0 \in A$, $0<r<R<\min\{R_0, d(x_0,\partial A)\}$ and $k \geq k_0$.

We see now how this class of functions is related to the PDE setting. By a careful analysis of the proof of [134, Proposition 8.5] we obtain the following result.

Theorem 1.2.28. *Let $N \geq 2$ and let $u \in H^s(\mathbb{R}^N)$ be a weak subsolution of*

$$(-\Delta)^s u \leq g(x,u), \quad x \in \mathbb{R}^N$$

where $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *satisfies, for a. e.* $x \in \mathbb{R}^N$ *and every* $t \in \mathbb{R}$,

$$|g(x,t)| \le d_1 + d_2|t|^{q-1}$$

for some $q \in (2, 2_s^*)$. *Then there exist* $\alpha = \alpha(N,s,q) > 0$, $C = C(N,s,q,d_2) > 0$ *and* $H = H(N,s,q,d_2) \ge 1$ *such that, for each* $x_0 \in \mathbb{R}^N$ *and each* R_0 *verifying*

$$0 < R_0 \le C(N,s,q,d_2) \min\left\{1, \|u\|_{L^{2_s^*}(\mathbb{R}^N)}^{-\alpha(N,s,q)}\right\},$$

it results that

$$u \in \mathrm{DG}_+^{s,2}\Big(B_{R_0}(x_0), d_1, H, 0, 1 - \frac{q}{2_s^*}, 2s, R_0\Big).$$

As shown in [134, Proposition 6.1 and Theorem 8.2], the belonging to a De Giorgi class implies useful L^∞_{loc} and $C^{0,\sigma}_{loc}$ estimates.

For other regularity results we refer to [22, 79, 84, 190, 240, 328].

1.2.6 Existence theorems and comparison principles

We collect here some results regarding existence and comparison principles.

As a consequence of the Riesz representation theorem, we start by recalling the situation for linear equations in $\mathbb{R}^N$ [177, page 1241, Theorem 3.3 and Lemma 4.1] (see also [190, Lemma C.1]).

Lemma 1.2.29 (Representation in $\mathbb{R}^N$). *Consider the equation in the weak sense*

$$(-\Delta)^s u + \lambda u = g \quad \text{in } \mathbb{R}^N$$

where $\lambda > 0$ *and* $g \in L^2(\mathbb{R}^N)$. *Then* u *is given by*

$$u = \mathcal{K}_{2s,\lambda} * g$$

where $\mathcal{K}_{2s,\lambda}$ *is the* Bessel Kernel *(see Remark 1.2.6)*

$$\mathcal{K}_{2s,\lambda} := \mathcal{F}^{-1}\left(\frac{1}{\lambda + |\xi|^{2s}}\right).$$

Moreover

- $\mathcal{K}_{2s,\lambda}$ *is non-negative, radially symmetric and decreasing,*
- $\frac{C_1}{|x|^{N+2s}} \le \mathcal{K}_{2s,\lambda}(x) \le \frac{C_2}{|x|^{N+2s}}$ *for* $|x| \ge 1$ *and some* $C_1, C_2 > 0$, *while* $|\mathcal{K}(x)| \le \frac{C_3}{|x|^{N-2s}}$ *for* $|x| \le 1$ *and some* $C_3 > 0$,
- $\mathcal{K}_{2s,\lambda} \in L^q(\mathbb{R}^N)$ *for every* $q \in [1, 1 + \frac{2s}{N-2s})$,
- $\mathcal{K}_{2s,\lambda}$ *solves* $(-\Delta)^s \mathcal{K}_{2s,\lambda} + \lambda \mathcal{K}_{2s,\lambda} = \delta_0$ *(in a distributional sense), where* δ_0 *is the standard Dirac delta.*

We notice that the fundamental solution of $(-\Delta)^s u = \delta_0$, instead, is given (up to constants) by $I_{2s} := \mathcal{F}^{-1}\left(\frac{1}{|\xi|^{2s}}\right) = \frac{1}{|x|^{N-2s}}$, which lies in $L^q_{loc}(\mathbb{R}^N)$ for every $q < \frac{N}{N-2s}$ but in no $L^p(\mathbb{R}^N)$. This *Riesz potential* will be better studied in Section 1.3.

The Bessel kernel allows also to find suitable comparison function with no restriction on the boundary; the result can be found in [111, Lemma A.2] (see also [177, Lemmas 4.2 and 4.3]).

Lemma 1.2.30 (Comparison function)**.** *Let $b > 0$. Then there exists a strictly positive continuous function $W_b \in H^s(\mathbb{R}^N)$ such that, for some positive constants C'_b, C''_b, it verifies*

$$(-\Delta)^s W_b + \frac{b}{2} W_b = 0, \quad x \in \mathbb{R}^N \setminus B_{r_b}$$

pointwise, with $r_b := \left(\frac{2}{b}\right)^{1/2s}$, and

$$\frac{C'_b}{|x|^{N+2s}} < W_b(x) < \frac{C''_b}{|x|^{N+2s}}, \qquad \text{for } |x| > 2r_b. \tag{1.2.29}$$

The constants r_b, C'_b, C''_b remain bounded by letting b vary in a compact set far from zero.

Proof. Let $B_{1/2} \prec \varphi \prec B_1$, and define $\tilde{W} := \mathcal{K}_{2s} * \varphi$, where $\mathcal{K}_{2s}$ is the Bessel potential. Arguing as in [177] (see also [79, Theorem 1.3]) we obtain

$$(-\Delta)^s \tilde{W} + \tilde{W} = \varphi, \quad x \in \mathbb{R}^N$$

and

$$\frac{C'}{|x|^{N+2s}} < \tilde{W}(x) \leq \frac{C''}{|x|^{N+2s}} \quad \text{for } |x| \geq 2.$$

By scaling $W := \tilde{W}(r_b \cdot)$ we reach the claim. ∎

We give now an existence result (see also [344, Corollary 1.15]).

Lemma 1.2.31 (Existence for weak solutions)**.** *Let $\Omega \subset \mathbb{R}^N$ be of class $C^{0,1}$ with bounded boundary, $\lambda > 0$, $\psi \in H^s(\Omega^c)$, and $g \in L^q(\Omega)$, for some $q \in [\frac{2N}{N+2s}, 2]$. Then there exists a (unique) function $v \in H^s(\mathbb{R}^N)$ such that*

$$\begin{cases} (-\Delta)^s v + \lambda v = g & \text{in } \Omega, \\ v = \psi & \text{on } \Omega^c, \end{cases}$$

in the weak sense, which in particular means $v \in X_0^s(\Omega) + \psi$. If moreover $g \in L^q_{loc}(\mathbb{R}^N)$ for some $q > \frac{N}{2s}$, then $v \in L^\infty_{loc}(\mathbb{R}^N)$. If instead $g \in C^{0,\sigma}_{loc}(\mathbb{R}^N)$ for some $\sigma \in (0,1]$, then $v \in C^{2s+\sigma}_{loc}(\mathbb{R}^N)$.

Remark 1.2.32. *The result is still valid in a whatever Ω^c* extension domain *(see [153]).*

Proof. By [153, Theorem 5.4] we know that there exists $\tilde{\psi} \in H^s(\mathbb{R}^N)$ such that $\tilde{\psi}_{|\Omega^c} \equiv \psi$. The problem is thus equivalent to

$$\begin{cases} (-\Delta)^s v + \lambda v = g & \text{in } \Omega, \\ v = \tilde{\psi} & \text{on } \Omega^c. \end{cases}$$

Consider $u = v - \tilde{\psi}$ and rewrite the weak formulation as

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi + \lambda \int_{\mathbb{R}^N} u\varphi = \int_{\mathbb{R}^N} (g - \lambda\psi)\varphi - \int_{\mathbb{R}^N} (-\Delta)^{s/2} \tilde{\psi} (-\Delta)^{s/2} \varphi.$$

It is easy to see that the left-hand side is a bilinear, continuous coercive map on $X_0^s(\Omega)$, while

$$\varphi \in X_0^s(\Omega) \mapsto \int_{\mathbb{R}^N} (g - \lambda\psi)\varphi - \int_{\mathbb{R}^N} (-\Delta)^{s/2} \tilde{\psi} (-\Delta)^{s/2} \varphi$$

belongs to the dual space $(X_0^s)^*(\Omega)$. By Lax-Milgram theorem, we obtain a solution $u \in X_0^s(\Omega)$, which implies that $v := u + \tilde{\psi}$ is the desired function.

Finally, the regularity results are a consequence of De Giorgi-Nash-Moser estimates [240, Proposition 2.6] and Schauder estimates [240, Theorem 2.11]. ∎

The following existence result can be found in [97, Lemma 2.2 and Remark 4.1] for bounded domains, and in [349, Theorem A.1] for the homogeneous case $\psi \equiv 0$.

Lemma 1.2.33 (Existence for viscosity solutions)**.** *Let $\Omega \subset \mathbb{R}^N$ be a C^2-domain, $\lambda > 0$, $\psi \in L^\infty(\Omega^c) \cap C(\Omega^c)$, and $g \in L^\infty(\Omega) \cap C(\overline{\Omega})$. Then there exists a function $v \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that*

$$\begin{cases} (-\Delta)^s v + \lambda v = g & \text{in } \Omega, \\ v = \psi & \text{on } \Omega^c, \end{cases}$$

in the viscosity sense. If $g \in C^\sigma_{loc}(\Omega)$ for some $\sigma \in (0,1)$, then $v \in C^\gamma_{loc}(\Omega)$, for some $\gamma > 2s$ is a pointwise solution. If $\psi \equiv 0$, we further have $v \in C^s(\mathbb{R}^N) \cap C^\gamma_{loc}(\Omega)$, for some $\gamma > \max\{1, 2s\}$ and $\frac{w}{(\mathrm{dist}(\cdot,\partial\Omega))^s} \in C^{0,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$.

Proof. First notice that, by extension, we may assume $g \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Since Ω is a C^2-domain, $g \in C(\mathbb{R}^N)$, $\psi \in C(\Omega^c) \cap L^\infty(\Omega^c)$, by [31, Theorem 4] with $b \equiv c \equiv 0$, we obtain the existence of a (unique) viscosity solution $v \in C(\mathbb{R}^N)$, satisfying the boundary condition pointwise (see also [87, page 615]). Since the cited theorem is a corollary of [31, Theorem 1], with $F(x,u,p,X,l) \equiv F(x,u,l) = l + \lambda u - g(x)$, $l = \mathcal{I}[u] \equiv (-\Delta)^s u$, $d\mu_x(z) = \frac{dz}{|z|^{N+\alpha}}$, one can notice, looking carefully at the proof, that the found solution is actually bounded (see also [339, Corollary 4]). Thus v is a bounded viscosity solution.

By [322, Theorem 2.6], since $(-\Delta)^s v = -\lambda v + g \in L^\infty(\Omega)$ with $v \in C(\overline{\Omega})$, we have $v \in C^{\gamma_1}_{loc}(\mathbb{R}^N)$ for some $\gamma_1 > 0$. Since $\psi \in L^\infty(\Omega^c)$ and $g - \lambda v \in C^{\min\{\sigma,\gamma_1\}}_{loc}(\Omega)$, by [322, Theorem 2.5] we have that $v \in C^\gamma_{loc}(\Omega)$ for some $\gamma > 2s$; thus $(-\Delta)^s v$ is pointwise defined (actually Hölder continuous). As observed in [322, Remark 2.3], we conclude that v is a pointwise solution. ∎

We write down now the following two maximum principles (for unbounded domains). See [111, Lemma A.1] for the first (see also [339, Lemma 6] and [233]).

Lemma 1.2.34 (Maximum Principle (weak))**.** *Let $\Omega \subset \mathbb{R}^N$, $\lambda > 0$, and let $u \in H^s(\mathbb{R}^N)$ be a weak subsolution of*

$$(-\Delta)^s u + \lambda u \le 0 \quad \text{in } \Omega.$$

Assume moreover that

$$u(x) \le 0 \quad \text{for a. e. } x \in \Omega^c.$$

Then

$$u(x) \le 0 \quad \text{for a. e. } x \in \mathbb{R}^N. \tag{1.2.30}$$

Proof. By the assumption we have $u^+ = 0$ on Ω^c, thus $u^+ \in X^s_0(\Omega)$ is a suitable test function (see Lemma 1.4.1) and we obtain, using $u = u^+ - u^-$ and $u^+ u^- \equiv 0$,

$$\begin{aligned} 0 &\ge \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u^+|^2 \, dx + \lambda \int_{\mathbb{R}^N} |u^+|^2 \, dx - \int_{\mathbb{R}^N} (-\Delta)^{s/2} u^- (-\Delta)^{s/2} u^+ \, dx \\ &= \|(-\Delta)^{s/2} u^+\|_2^2 + \lambda \|u^+\|_2^2 + C \int_{\mathbb{R}^{2N}} \frac{u^-(x)u^+(y) + u^-(y)u^+(x)}{|x-y|^{N+2s}} \, dx \, dy \\ &\ge \|(-\Delta)^{s/2} u^+\|_2^2 + \lambda \|u^+\|_2^2 \end{aligned}$$

which implies $u^+ = 0$ on $\mathbb{R}^N$. ∎

Remark 1.2.35. *We point out that if u is assumed continuous, then (1.2.30) is actually pointwise. Moreover, the constant $\lambda > 0$ may be substituted by a more general $V(x) > 0$ which gives sense to the integrals.*

Lemma 1.2.36 (Maximum Principle (viscosity))**.** *Let $\Omega \subset \mathbb{R}^N$ be open, $\lambda > 0$, and let u be a viscosity, continuous subsolution of*

$$(-\Delta)^s u + \lambda u \le 0 \quad \text{in } \Omega$$

such that

$$\lim_{|x|\to+\infty} u(x) \leq 0.$$

Assume moreover that

$$u(x) \leq 0 \quad on\ \Omega^c.$$

Then

$$u(x) \leq 0 \quad on\ \mathbb{R}^N. \tag{1.2.31}$$

The result applies, in particular, to pointwise solutions.

Proof. We first observe that $u \in L^\infty(\mathbb{R}^N)$ and set $M := \sup_{x\in\mathbb{R}^N} u(x)$. By contradiction, assume $M > 0$. Let $(x_n)_n$ be a maximizing sequence, i.e. $u(x_n) \to M$ as $n \to +\infty$; we can assume that $x_n \in \Omega$. We observe that $(x_n)_n$ is bounded (up to a subsequence) since, if not, we would have $|x_n| \to +\infty$ and thus $\lim_n u(x_n) \leq 0$, which is an absurd. Thus $x_n \to x_0 \in \overline{\Omega}$, and by continuity $u(x_0) = M > 0$; since $u(x) \leq 0$ on $\overline{\Omega^c} \supset \partial\Omega$, we have $x_0 \in \Omega$. In particular, x_0 is a point of maximum for u.

We can thus choose a whatever $U \subset \Omega$ neighborhood of x_0 and set $\phi \equiv u(x_0)$ as contact function in the definition of viscosity solution: indeed $\phi \in C^2(U)$, $\phi(x_0) = u(x_0)$ and $\phi \geq u$ in U. Hence, set $v := \phi\chi_U + u\chi_{U^c}$ we have

$$\begin{aligned} 0 &\geq (-\Delta)^s v(x_0) + \lambda v(x_0) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x_0) - v(y)}{|x_0 - y|^{N+2s}} dy + \lambda u(x_0) \\ &= C_{N,s} \int_{U^c} \frac{M - u(y)}{|x_0 - y|^{N+2s}} dy + \lambda M > 0, \end{aligned}$$

which is a contradiction. This concludes the proof. ∎

1.3 The Riesz potential

Let $\alpha \in (0, N)$. We recall here some results on the Riesz kernel [304, Appendix]

$$I_\alpha(x) := \frac{C_{N,\alpha}}{|x|^{N-\alpha}} \tag{1.3.32}$$

where

$$C_{N,\alpha} := \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{N/2} \Gamma(\frac{\alpha}{2})} > 0$$

is a normalization constant. For motivations and a physical introduction we refer to Sections 3.1 and 4.1.

We are interested in studying the behaviour of the convolution

$$I_\alpha * g$$

for some g. We will use the following notation, whenever well defined for some g and h:

$$\mathcal{D}_\alpha(g,h) := \int_{\mathbb{R}^N} (I_\alpha * g)h = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^{N-\alpha}} dxdy.$$

We start observing that the operator enjoys a trivial but useful scaling property

$$\mathcal{D}_\alpha(g(\theta\cdot), h(\theta\cdot)) = |\theta|^{-(N+\alpha)} \mathcal{D}_\alpha(g,h).$$

for any $\theta \in \mathbb{R}$.

Well posedness

The following theorem ensures the well posedness of the Riesz potential: see [266, Theorem 4.3], [248, pages 61-62] and [297, Section 4.2] for a proof.

Proposition 1.3.1 (Hardy-Littlewood-Sobolev inequality)**.** *Let $\alpha \in (0, N)$.*

- *Let g be a measurable function. Then $I_\alpha * g$ is finite almost everywhere if and only if*

$$\int_{\mathbb{R}^N} \frac{|g(x)|}{(1+|x|)^{N-\alpha}} < \infty. \tag{1.3.33}$$

  In particular, $I_\alpha * g$ is well defined if $g \in L^1_{loc}(\mathbb{R}^N) \cap L^r(B_R^c)$ for some $R \geq 0$ and some $r \in [1, \frac{N}{\alpha})$. Moreover, if (1.3.33) does not hold, then $I_\alpha * |g| \equiv \infty$.

- *Let $r \in (1, \frac{N}{\alpha})$. Then, for some $C = C(N, \alpha, r) > 0$ we have*

$$\|I_\alpha * g\|_{\frac{Nr}{N-\alpha r}} \leq C\|g\|_r$$

  for all $g \in L^r(\mathbb{R}^N)$, thus the map

$$g \in L^r(\mathbb{R}^N) \mapsto I_\alpha * g \in L^{\frac{Nr}{N-\alpha r}}(\mathbb{R}^N)$$

  is continuous. In particular, since the operator is linear,

$$g_n \rightharpoonup g \text{ in } L^r(\mathbb{R}^N) \implies I_\alpha * g_n \rightharpoonup I_\alpha * g \text{ in } L^{\frac{Nr}{N-\alpha r}}(\mathbb{R}^N).$$

- *Let $r, t \in (1, +\infty)$ be such that $\frac{1}{r} + \frac{1}{t} = \frac{N+\alpha}{N}$. Then there exists a constant $C = C(N, \alpha, r, t) > 0$ such that*

$$|\mathcal{D}_\alpha(g, h)| \leq C\|g\|_r\|h\|_t$$

  for all $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Thus the bilinear map

$$(g, h) \in L^r(\mathbb{R}^N) \times L^t(\mathbb{R}^N) \mapsto \mathcal{D}_\alpha(g, h) \in \mathbb{R}$$

  is continuous. If $r = t = \frac{2N}{N+\alpha}$, then equality is reached in the previous inequality if and only if $g \equiv h$ (up to multiplicative constants), and $g(x) = (1+|x|^2)^{-\frac{N+\alpha}{2}}$ (up to translations and rescaling).

In the limiting case $g \in L^{\frac{N}{\alpha}}(\mathbb{R}^N)$ (i.e. $\frac{Nr}{N-\alpha r} \to \infty$) we have that $I_\alpha * g$ is a BMO function (see [304, Appendix A.2] and references therein). Anyway we have

- *If $g \in L^{\frac{N}{\alpha}-\varepsilon}(\mathbb{R}^N) \cap L^{\frac{N}{\alpha}+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon > 0$, then $I_\alpha * g \in C_0(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$.*

Proof. We show only the last claim, i.e. [275, Lemma 4.5(ii)]; we argue as in [115, Proposition 4.5] (see also Proposition 4.4.6 and Remark 1.5.8). Recall theat, by Young's Theorem, if two functions belong to two Lebesgue spaces with conjugate (finite) indexes, then their convolution belong to $C_0(\mathbb{R}^N)$. We first split

$$I_\alpha * g = (I_\alpha \chi_{B_1}) * g + (I_\alpha \chi_{B_1^c}) * g$$

where

$$I_\alpha \chi_{B_1} \in L^{r_1}(\mathbb{R}^N), \quad \text{for } r_1 \in [1, \tfrac{N}{N-\alpha}),$$
$$I_\alpha \chi_{B_1^c} \in L^{r_2}(\mathbb{R}^N), \quad \text{for } r_2 \in (\tfrac{N}{N-\alpha}, \infty].$$

We need that $g \in L^{q_1}(\mathbb{R}^N) \cap L^{q_2}(\mathbb{R}^N)$ for some q_i satisfying

$$\frac{1}{q_i} + \frac{1}{r_i} = 1, \quad i = 1, 2$$

that is

$$\frac{q_1}{q_1-1} \in \left[1, \frac{N}{N-\alpha}\right), \quad \frac{q_2}{q_2-1} \in \left(\frac{N}{N-\alpha}, \infty\right]$$

or equivalently $q_2 < \frac{N}{\alpha} < q_1$. Thus we have the claim. ∎

We emphasize the similarity of condition (1.3.33) and condition (1.2.1), when formally $\alpha = -2s$.

Positivity

We observe the following: if $g \in \mathcal{S}$ [351, Lemma 5.1.2] or if $\alpha \in (0, \frac{N}{2})$ and $g \in L^{\frac{2N}{N+2\alpha}}(\mathbb{R}^N)$ [266, Corollary 5.10] then we have

$$\mathcal{D}_\alpha(g,g) = \int_{\mathbb{R}^N} (I_\alpha * g) g dx = \int_{\mathbb{R}^N} \widehat{I_\alpha * g}\widehat{g} d\xi = \int_{\mathbb{R}^N} \widehat{I_\alpha} |\widehat{g}|^2 d\xi = \int_{\mathbb{R}^N} \frac{|\widehat{g}|^2}{|\xi|^\alpha} d\xi \geq 0$$

(see also [275, Lemma 4.5(v)], [75, Lemma 2.7], [248, Section 1.1], [297, Theorem 2.8 in Chapter 2], [334, Sections 2.1.1 and 2.3.3] and [266, Theorem 5.9]). This shows that

$$g \mapsto \mathcal{D}_\alpha(g,g)$$

is a *positive functional* (i.e. its sign does not depend on the sign of g). A more general result can be adapted from $\alpha = 2$ [266, Theorem 9.8] to a generic $\alpha \in (0, N)$ as follows.

Proposition 1.3.2 ([266]). *Let $g : \mathbb{R}^N \to \mathbb{R}$ measurable be such that*

$$\mathcal{D}_\alpha(|g|, |g|) < \infty.$$

Then

$$\mathcal{D}_\alpha(g,g) \geq 0$$

and the above quantity is zero if and only if $g \equiv 0$ almost everywhere. In particular the following representation holds

$$\mathcal{D}_\alpha(g,g) = \int_0^{+\infty} t^{2N-\alpha-1} \int_{\mathbb{R}^N} |h(t\cdot) * g|^2 dx dt \geq 0$$

for a whatever nonnegative, radially symmetric $h \in C_c^\infty(\mathbb{R}^N)$ normalized in such a way that $\int_0^{+\infty} t^{N-\alpha-1}(h*h)(t)dt = C_{N,\alpha}$.

Decay

We investigate now the decay of $I_\alpha * g$: indeed, if $g \in L^1_{loc}(\mathbb{R}^N)$, $g \geq 0$ and $g > 0$ on some ball, then

$$(I_\alpha * g)(x) \geq I_\alpha(2x) \int_{B_{2|x|}(x)} g \gtrsim I_\alpha(x) \simeq \frac{1}{|x|^{N-\alpha}} \quad \text{for } |x| \gg 0$$

which shows a polynomial bound from below on the Riesz potential, whatever the decay of g is (even with compact support). Moreover, if $g \geq 0$ has at least a polynomial decay

$$g(x) \lesssim \frac{1}{|x|^\theta} \quad \text{as } |x| \to +\infty$$

with $\theta > \alpha$, then the following estimates from above hold [301, Lemma A.1] (see also [204, Lemma 2.1] and [211, Lemma 4.6])

$$(I_\alpha * g)(x) \lesssim \begin{cases} \frac{1}{|x|^{\theta-\alpha}} & \text{if } \theta \in (\alpha, N), \\ \frac{\log(x)}{|x|^{N-\alpha}} & \text{if } \theta = N, \\ \frac{1}{|x|^{N-\alpha}} & \text{if } \theta \in (N, +\infty). \end{cases} \tag{1.3.34}$$

In particular, if $\theta > N$ the decay of $I_\alpha * g$ is exactly the same of I_α, as stated in the following result.

Lemma 1.3.3 ([300]). *Let $g \in L^\infty(\mathbb{R}^N)$ be continuous and $\theta > N$ be such that*

$$\sup_{x\in\mathbb{R}^N} |g(x)||x|^\theta < +\infty.$$

Then there exists $C = C(N,\alpha) > 0$ such that

$$\left|(I_\alpha * g)(x) - I_\alpha(x)\int_{\mathbb{R}^N} g(y)dy\right| \leq \frac{C\|g\|_{\infty,\theta}}{|x|^{N-\alpha}}\left(\frac{1}{1+|x|}+\frac{1}{1+|x|^{\theta-N}}\right)$$

for each $x \in \mathbb{R}^N$, $x \neq 0$, where we recall that $\|g\|_{\infty,\theta} = \|g(\cdot)(1+|\cdot|^\theta)\|_\infty$.

Proof. See [300, Lemma 6.2]. See also [190, Lemma C.3]. ∎

The rigidity of the previous result in particular highlights that it is not possible to implement a *bootstrap-type* argument in order to show fine results on the decay of a solutions. See Section 4.6.6.

1.3.1 The Riesz potential as the inverse of the fractional Laplacian

Since roughly

$$(-\Delta)^{\alpha/2} I_\alpha = \mathcal{F}^{-1}\left(|\xi|^\alpha \widehat{I_\alpha}(\xi)\right) = \mathcal{F}^{-1}\left(|\xi|^\alpha \frac{1}{|\xi|^{N-(N-\alpha)}}\right) = \mathcal{F}^{-1}(1) = \delta_0$$

then the Riesz kernel can be seen as the fundamental (distributional) solution for the fractional Laplacian [4, Theorem 5.10] (see also [201, Theorem 8.4] and [75, Theorem 2.3] for the case $\alpha \in (0,2)$)

$$(-\Delta)^{\alpha/2} I_\alpha = \delta_0 \quad \text{in } \mathbb{R}^N; \tag{1.3.35}$$

thus the Riesz potential generates the solutions of fractional equations in $\mathbb{R}^N$, that is

$$\phi = I_\alpha * g \iff (-\Delta)^{\alpha/2}\phi = g \quad \text{in } \mathbb{R}^N.$$

Therefore we may roughly say that ([351, Section 5.1], [201, equation (2.7)], [346, equation (2.3)] and [9, equation (1.2.7)])

$$I_\alpha * \equiv (-\Delta)^{-\alpha/2}.$$

More precisely

$$I_\alpha * ((-\Delta)^{\alpha/2} v) = v = (-\Delta)^{\alpha/2}(I_\alpha * v) \quad \text{for every } v \in C_c^\infty(\mathbb{R}^N);$$

indeed, when the fractional Laplacian is defined through hypersingular integrals, the first equality can be found in [297, proof of Theorem 2.9 in Chapter 2 and Section 4.5] for Schwartz functions, while the second equality for L^p functions in [334, Theorems 3.22 and 3.24]: anyway the hypersingular definition coincides with the Fourier transform one at least on C_c^∞ functions (see [334, Lemma 3.1] and [2, Theorem 1.8]). See also [248, equation (1.1.12')]. For the case $\alpha \in (0,2)$ see also [75, Theorem 2.8 and Corollary 2.9] and [352, Theorem 6], while for $\alpha = 2$ see also [275, Lemma 4.5(iii)].

Let us state this relation in a more general framework.

Proposition 1.3.4. *Let $\alpha \in (0,N)$ (i.e., set $s := \frac{\alpha}{2}$, we ask $N > 2s$).*

i) Assume $g \in L^p(\mathbb{R}^N)$ *for some* $p \in [1, \frac{N}{\alpha})$*. Then*

$$(-\Delta)^{\alpha/2}(I_\alpha * g) = g \quad \text{in } \mathbb{R}^N$$

in the strong sense; notice, in particular, that the fractional Laplacian of $\phi = I_\alpha * g$ *is well defined pointwise (i.e., finite) almost everywhere. Moreover, if* x *is a Lebesgue point for* g *(e.g.,* g *is continuous at* x*), then the previous relation holds at* x*.*

ii) If $g \in L^p(\mathbb{R}^N) \cap X$ *for some* $p \in [1, \frac{N}{\alpha})$ *and some function space* X*, then* $(-\Delta)^\alpha(I_\alpha * g) \in L^p(\mathbb{R}^N) \cap X$*; in particular if* $g \in L^p(\mathbb{R}^N) \cap L^{\frac{N}{N-\alpha p}}(\mathbb{R}^N)$*, then*

$$I_\alpha * g \in W^{\alpha, \frac{N}{N-\alpha p}}(\mathbb{R}^N).$$

iii) Let $g \in L^p(\mathbb{R}^N)$ *for some* $p \in [1, \frac{N}{\alpha})$*. Then* $\phi = I_\alpha * g$ *is the only (distributional) solution to*

$$(-\Delta)^{\alpha/2}\phi = g \quad \text{in } \mathbb{R}^N$$

belonging to $L^{\frac{Np}{N-\alpha p}}(\mathbb{R}^N)$*.*

iv) Let $\phi \in W^{\alpha, \frac{N}{N-\alpha p}}(\mathbb{R}^N)$ *for some* $p \in [1, \frac{N}{\alpha})$*; assume moreover that* $(-\Delta)^{\alpha/2}\phi \in L^p(\mathbb{R}^N)$*. Then*

$$I_\alpha * ((-\Delta)^{\alpha/2}\phi) = \phi \quad \text{in } \mathbb{R}^N$$

in the strong sense.

Proof. Point $i)$ is stated in [245, page 22, Definition 2.5 and Proposition 7.1] (see also [4, Corollary 5.16] for compactly supported $g \in L^1(\mathbb{R}^N)$ and [266, Corollary 5.10] for $\alpha \in (0, \frac{N}{2})$ and $g \in L^{\frac{2N}{N+2\alpha}}(\mathbb{R}^N)$); see instead [334, Theorems 3.22 and 3.24], and [297, Theorem 5.1 and Remark 5.1 in Chapter 4] for a hypersingular approach. Point $ii)$ is a direct consequence.

To show $iii)$, by linearity it is sufficient to prove the statement for $g = 0$; this can be done as in [100, Theorems 1.3 and Theorem 3.1]. See also [172, Corollary 1.4], [175, Corollary 1.3] and [159, Theorem 1.5] for $\alpha \in (0, 2)$, [4, Theorem 5.17] for $\alpha \notin 2\mathbb{N}$ and [225] for $\alpha \in 2\mathbb{N}$.

We give some details only on $iv)$ (see also [98]). Indeed, consider

$$(-\Delta)^{\alpha/2}\phi = 0 \quad \text{in } \mathbb{R}^N;$$

by $iii)$ we know that the only solution $\phi \in W^{\alpha, \frac{Np}{N-\alpha p}}(\mathbb{R}^N)$ is the null function. Thus the kernel of the linear operator

$$(-\Delta)^{\alpha/2} : W^{\alpha, \frac{Np}{N-\alpha p}}(\mathbb{R}^N) \to L^{\frac{Np}{N-\alpha p}}(\mathbb{R}^N)$$

is null, and hence the operator injective. In particular, considered the homogeneous space

$$\dot{W}^{\alpha,p}(\mathbb{R}^N) := \Big\{u \text{ measurable} \mid (-\Delta)^{\alpha/2}u \in L^p(\mathbb{R}^N)\Big\}$$

we have that

$$(-\Delta)^{\alpha/2} : \dot{W}^{\alpha,p}(\mathbb{R}^N) \cap W^{\alpha, \frac{Np}{N-\alpha p}}(\mathbb{R}^N) \to L^p(\mathbb{R}^N) \cap L^{\frac{Np}{N-\alpha p}}(\mathbb{R}^N)$$

is injective, and thus admits a left inverse. On the other hand, by $ii)$ we have

$$I_\alpha * : L^p(\mathbb{R}^N) \cap L^{\frac{Np}{N-\alpha p}}(\mathbb{R}^N) \to \dot{W}^{\alpha,p}(\mathbb{R}^N) \cap W^{\alpha, \frac{Np}{N-\alpha p}}(\mathbb{R}^N)$$

and moreover, by $i)$, it is a right inverse for $(-\Delta)^{\alpha/2}$. Therefore the left and right inverse must coincide, which means that $I_\alpha *$ is a right inverse for $(-\Delta)^{\alpha/2}$. This concludes the proof. ∎

Remark 1.3.5. *By the previous proof, we see that, if* $p \in [1, \frac{N}{\alpha})$*, then*

$$I_\alpha * \equiv (-\Delta)^{-\alpha/2}.$$

when looked on the spaces $\dot{W}^{\alpha,p}(\mathbb{R}^N) \cap W^{\alpha, \frac{Np}{N-\alpha p}}(\mathbb{R}^N)$ *and* $L^p(\mathbb{R}^N) \cap L^{\frac{Np}{N-\alpha p}}(\mathbb{R}^N)$*.*

Regularity

As already seen by point *ii*) of Proposition 1.3.4, the Riesz potential has a regularizing effect. We give more details in the following result.

Proposition 1.3.6. *Let $\alpha \in (0,N)$ and $p \in [1, \frac{N}{\alpha})$, and let $g \in L^p(\mathbb{R}^N)$.*

i) Assume $g \in L^q(\mathbb{R}^N)$ for some $q \in (\frac{N}{\alpha}, \infty)$ with $\alpha - \frac{N}{q} \in (0,1)$. Then $I_\alpha * g \in C^{\alpha - \frac{N}{q}}(\mathbb{R}^N)$.

ii) Assume $\alpha \in (0,2)$ and $g \in L^\infty(\mathbb{R}^N)$, and we assume a priori that $I_\alpha * g \in L^\infty(\mathbb{R}^N)$. Then

$$I_\alpha * g \in \begin{cases} C^{0,\gamma}(\mathbb{R}^N) \ \text{ for } \gamma < \alpha & \text{if } \alpha \in (0,1), \\ \Lambda_1(\mathbb{R}^N), \ \text{thus } C^{0,\gamma}(\mathbb{R}^N) \ \text{ for } \gamma < 1 & \text{if } \alpha = 1, \\ C^{1,\gamma-1}(\mathbb{R}^N) \ \text{ for } \gamma < \alpha & \text{if } \alpha \in (0,2). \end{cases}$$

iii) Assume $g \in C^{0,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $\sigma \in (0,1]$, and $g \geq 0$. Then $I_\alpha * g \in C^{0,\gamma}(\mathbb{R}^N)$ for each $\gamma < (1 - \frac{\alpha}{N}p)\sigma$.[2]

iv) Assume $\alpha \in (0,1)$ and $g \in C^{0,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (0,1)$ such that $\sigma + \alpha \in (0,1)$. Then $I_\alpha * g \in Lip(\sigma + \alpha)$.
In particular, if we assume a priori also $I_\alpha * g \in L^\infty(\mathbb{R}^N)$, then $I_\alpha * g \in C^{\sigma+\alpha}(\mathbb{R}^N)$.

v) Assume $\alpha \in (0,2)$ and $g \in C^{0,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (0,1]$, and we assume a priori that $I_\alpha * g \in L^\infty(\mathbb{R}^N)$. Then

$$I_\alpha * g \in \begin{cases} C^{0,\sigma+\alpha}(\mathbb{R}^N) & \text{if } \sigma + \alpha \in (0,1), \\ \Lambda_1, \ \text{thus } C^{0,\gamma}(\mathbb{R}^N) \ \text{ for } \gamma < 1 & \text{if } \sigma + \alpha = 1, \\ C^{1,\sigma+\alpha-1}(\mathbb{R}^N) & \text{if } \sigma + \alpha \in (1,2), \\ \Lambda_2, \ \text{thus } C^{1,\gamma}(\mathbb{R}^N) \ \text{ for } \gamma < 1 & \text{if } \sigma + \alpha = 2, \\ C^{2,\sigma+\alpha-2}(\mathbb{R}^N) & \text{if } \sigma + \alpha \in (2,3); \end{cases}$$

the previous relations holds also if we substitute global Hölder spaces with local spaces.

Proof. Point *i*) can be found in [297, Theorem 2.2 in Section 4.2] (see also [164, Theorem 2] and [328, Theorem 1.6]); point *iv*) can be found in [164, Theorem 1] and Remark 1.1.1. Points *ii*) and *v*) are consequences of Proposition 1.2.25 and Proposition 1.3.4.

We are left to prove *iii*). Indeed, let $r := \frac{\sigma}{\gamma} > \frac{N}{N - \alpha p} > 1$. We can find thus $\theta \in (1, \frac{N}{\alpha})$ such that $(1 - \frac{1}{r})\theta \geq p$. We thus have, for $x, y, z \in \mathbb{R}^N$, exploiting $|a^r - b^r| \lesssim |a-b||a^{r-1} - b^{r-1}|$

$$\begin{aligned} |g(x-z) - g(y-z)| &\lesssim |(g(x-z))^{\frac{1}{r}} - (g(y-z))^{\frac{1}{r}}||(g(x-z))^{\frac{r-1}{r}} - (g(y-z))^{\frac{r-1}{r}}| \\ &\lesssim |g(x-z) - g(y-z)|^{\frac{1}{r}}|(g(x-z))^{\frac{r-1}{r}} - (g(y-z))^{\frac{r-1}{r}}| \\ &\lesssim |x-y|^{\frac{\sigma}{r}}|(g(x-z))^{\frac{r-1}{r}} - (g(y-z))^{\frac{r-1}{r}}|; \end{aligned}$$

as a consequence

$$\begin{aligned} |(I_\alpha * g)(x) - (I_\alpha * g)(y)| &\lesssim |x-y|^\gamma \int_{\mathbb{R}^N} \frac{|(g(x-z))^{\frac{r-1}{r}} - (g(y-z))^{\frac{r-1}{r}}|}{|y|^{N-\alpha}} dz \\ &\leq |x-y|^\gamma \Big(\int_{B_1(0)} \frac{|g(x-z)|^{\frac{r-1}{r}} + |g(x-z)|^{\frac{r-1}{r}}}{|y|^{N-\alpha}} dz + \end{aligned}$$

[2] Actually, if in addition $\int_{\mathbb{R}^N} g^q < \infty$ for some $q \in (0,1)$, then we can take $\gamma < (1 - \frac{\alpha}{N}q)\sigma$. In particular, if $q = \frac{N}{N+\alpha}$, then $\gamma < \frac{N}{N+\alpha}\sigma$.

$$+\int_{B_1^c(0)} \frac{|g(x-z)|^{\frac{r-1}{r}} + |g(x-z)|^{\frac{r-1}{r}}}{|y|^{N-\alpha}} dz\Big)$$
$$\leq |x-y|^\gamma \Big(2\|g\|_\infty^{\frac{r-1}{r}} \int_{B_1(0)} \frac{1}{|y|^{N-\alpha}} dz + 2\|g\|_{\frac{r-1}{r}\theta}^{\frac{r-1}{r}} \int_{B_1^c(0)} \frac{1}{|y|^{(N-\alpha)\theta'}} dz\Big)$$

which gives the claim, being $(N-\alpha)\theta' > N$ and $\frac{r-1}{r}\theta \in [p, \infty)$. ∎

Limiting cases

We wonder what happens when $\alpha \to 0$ or $\alpha \to N$. In the first case, the Riesz potential collapses into a local operator (as one may expect from the representation $I_\alpha * g \equiv \mathcal{F}^{-1}(|\xi|^{-\alpha}\widehat{g})$), that is

$$I_\alpha * g \overset{\alpha\to 0^+}{\to} \delta_0 * g = g; \tag{1.3.36}$$

in the second case, as one may imagine looking at the Poisson equation (1.3.35) with $\alpha = N$ (for example, in the planar case $N = 2$ with the classical Laplacian), the Riesz kernel converges (up to constants) to a logarithm kernel

$$I_\alpha * g \overset{\alpha\to N^-}{\to} \log\left(\tfrac{1}{|\cdot|}\right) * g \tag{1.3.37}$$

whenever computed on a function with zero mean $\int_{\mathbb{R}^N} g = 0$. See [248, pages 46 and 50] for precise statements.

Definitions of solutions

The definitions of weak and viscosity solutions apply, mutatis mutandis, to nonlocal equations of the type

$$(-\Delta)^s u + \mu u = (I_\alpha * F(u)) f(u) \quad \text{on } \mathbb{R}^N \tag{1.3.38}$$

where we ask u to satisfy the equation in the classic/strong/weak/viscosity sense with nonlinearity $g(x) := (I_\alpha * F(u))(x) f(u(x))$. When dealing with weak solution, we need the term to be summable (see Remark 1.5.7); while, when dealing with classical and viscosity solutions, we need $I_\alpha * F(u)$ to be well defined pointwise (see Remark 1.5.8).

We notice that, under the assumptions of invertibility of the fractional Poisson equation (see Proposition 1.3.4), equation (1.3.38) can be rewritten as a fractional Schrödinger-Newton system

$$\begin{cases} (-\Delta)^s u + \mu u = \phi f(u) & \text{in } \mathbb{R}^N, \\ (-\Delta)^{\alpha/2}\phi = F(u) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3.39}$$

1.4 Some manipulations: absolute value and polarization

If one considers a function $u \in H^1(\mathbb{R}^N)$ and its absolute value, it is easy to see that

$$|\nabla |u|| = |\nabla u|.$$

Actually, the equality is not the case of the fractional Laplacian, generally. We show thus how the fractional Laplacian, and the Riesz potential, behave with respect to the absolute value.

Lemma 1.4.1. *Let $s \in (0,1)$ and $\alpha \in (0, N)$.*

- *Let $u \in H^s(\mathbb{R}^N)$. Then $|u| \in H^s(\mathbb{R}^N)$ and*

$$\|(-\Delta)^{s/2}|u|\|_2 \leq \|(-\Delta)^{s/2} u\|_2.$$

As a consequence, if $u = u_+ - u_-$, then $u_\pm = \frac{|u|\pm u}{2} \in H^s(\mathbb{R}^N)$.

- *Let $F : \mathbb{R} \to \mathbb{R}$ continuous and $u : \mathbb{R}^N \to \mathbb{R}$ measurable be such that $\mathcal{D}_\alpha(F(|u|), F(|u|)) < \infty$.*
  If F is even, then
  $$\mathcal{D}_\alpha(F(|u|), F(|u|)) = \mathcal{D}_\alpha(F(u), F(u));$$
  if F is odd and has constant sign on $(0, +\infty)$, then
  $$\mathcal{D}_\alpha(F(|u|), F(|u|)) \geq \mathcal{D}_\alpha(F(u), F(u)).$$

Proof. By (1.2.5) we have

$$\begin{aligned}
\|(-\Delta)^{s/2}|u|\|_2^2 &= C_{N,s} \int_{\mathbb{R}^{2N}} \frac{\big(|u(x)| - |u(y)|\big)^2}{|x-y|^{N+2s}} \, dxdy \\
&= C_{N,s} \int_{\mathbb{R}^{2N}} \frac{|u|^2(x) + |u|^2(y) - 2|u|(x)|u|(y)}{|x-y|^{N+2s}} \, dxdy \\
&\leq C_{N,s} \int_{\mathbb{R}^{2N}} \frac{u^2(x) + u^2(y) - 2u(x)u(y)}{|x-y|^{N+2s}} \, dxdy \\
&= C_{N,s} \int_{\mathbb{R}^{2N}} \frac{\big(u(x) - u(y)\big)^2}{|x-y|^{N+2s}} \, dxdy = \|(-\Delta)^{s/2} u\|_2^2,
\end{aligned}$$

thus the first claim. Focus on the second claim: if F is odd and with constant sign on $(0, +\infty)$, then, set for brevity $A^\pm := \{\pm u > 0\}$,

$$\begin{aligned}
&\mathcal{D}_\alpha(F(|u|), F(|u|)) \\
&= \int_{A^+ \times A^+} I_\alpha(x-y)F(u(x))F(u(y)) - \int_{A^- \times A^+} I_\alpha(x-y)F(u(x))F(u(y)) - \\
&\quad - \int_{A^+ \times A^-} I_\alpha(x-y)F(u(x))F(u(y)) + \int_{A^- \times A^-} I_\alpha(x-y)F(u(x))F(u(y)) \\
&\geq \int_{A^+ \times A^+} I_\alpha(x-y)F(u(x))F(u(y)) + \int_{A^- \times A^+} I_\alpha(x-y)F(u(x))F(u(y)) + \\
&\quad + \int_{A^+ \times A^-} I_\alpha(x-y)F(u(x))F(u(y)) + \int_{A^- \times A^-} I_\alpha(x-y)F(u(x))F(u(y)) \\
&= \mathcal{D}_\alpha(F(u), F(u)),
\end{aligned}$$

which concludes the proof, observing that equality holds if F is instead even. ∎

Remark 1.4.2. *We highlight that, in Sobolev spaces, the absolute value conserves the weak convergences. Indeed, assume $u_k \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. Since u_k is bounded and $\||u_k|\|_{H^s(\mathbb{R}^N)} \leq \|u_k\|_{H^s(\mathbb{R}^N)}$ we have that $|u_k|$ is bounded too. Therefore, $|u_k| \rightharpoonup v$ in $H^s(\mathbb{R}^N)$ up to a subsequence. As a consequence, up to a subsequence, $u_k \to u$ and $|u_k| \to v$ almost everywhere, which means that $|u| = v$ almost everywhere. This means that $|u_k| \rightharpoonup |u|$ in $H^s(\mathbb{R}^N)$.*

Notice that, in general, for weak convergences in L^p-spaces the implication is not true [389, Section 5].

We turn now to the study of symmetries. We exploit the tool of the polarization, useful in the presence of the Riesz potential. Let

$$\mathcal{H} := \{H \subset \mathbb{R}^N \text{closed half-space}, \ 0 \in H\}.$$

For any $H \in \mathcal{H}$ let σ_H be the reflection with respect to ∂H. The *polarization* (or *two-points symmetrization*) of a function $u : \mathbb{R}^N \to \mathbb{R}$ is defined as

$$u^H(x) := \begin{cases} \max\{u(x), u(\sigma_H(x))\} & \text{if } x \in H, \\ \min\{u(x), u(\sigma_H(x))\} & \text{if } x \notin H. \end{cases}$$

For example, if $u = \chi_\Omega$, with $\Omega \subset \mathbb{R}^N$ crossing ∂H, then

$$(\chi_\Omega)^H(x) = \begin{cases} \chi_{\Omega\cup\sigma_H(\Omega)}(x) & \text{if } x \in H, \\ \chi_{\Omega\cap\sigma_H(\Omega)}(x) & \text{if } x \notin H, \end{cases}$$

which roughly means that u^H brings mass from H^c to H. One can see [77], [380, Section 8.3] and [364] and references therein for an introduction on the topic and some relations with the symmetric decreasing rearrangement.

Clearly we have

$$u^H \equiv u \iff u \geq u \circ \sigma_H \quad \text{on } H,$$
$$u^H \equiv u \circ \sigma_H \iff u \leq u \circ \sigma_H \quad \text{on } H$$

which means, roughly, that there is more mass of u on H than on H^c. We expect that, if u coincide with u^H for all the hyperplanes, then some symmetry must hold. This is actually stated in the following result [300, Lemma 5.4] (see also [365, Proposition 3.15] and [72, Lemma 6.3]).

Proposition 1.4.3. *Let $u \in L^p(\mathbb{R}^N)$, for some $p \in [1, +\infty)$, be nonnegative. Then u is radially symmetric if and only if for every $H \in \mathcal{H}$ it results that $u^H = u$, while u is radially symmetric up to a translation if and only if for every $H \in \mathcal{H}$ it results that $u^H = u$ or $u^H = u \circ \sigma_H$.*

We state now a proposition which shows both how the Riesz potential behaves with respect to polarization, and why this tool is particularly effective in this framework [300, Lemma 5.3].

Proposition 1.4.4. *Let $\alpha \in (0, N)$ and $H \in \mathcal{H}$, and let $g \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ be nonnegative. Then*

$$\mathcal{D}_\alpha(g^H, g^H) \geq \mathcal{D}_\alpha(g, g)$$

and equality holds if and only if $u^H \equiv u$ or $u^H \equiv u \circ \sigma_H$.

We investigate now how the fractional Laplacian behaves with respect to polarization, see [47, equation (2.14)] and [72, Lemma 5.3] (see also [40, page 4818]).

Proposition 1.4.5. *Let $s \in (0, 1)$ and $H \in \mathcal{H}$, and let $u \in H^s(\mathbb{R}^N)$. Then $u^H \in H^s(\mathbb{R}^N)$ and*

$$\|(-\Delta)^{s/2} u^H\|_2 \leq \|(-\Delta)^{s/2} u\|_2.$$

When $s = 1$, the equality holds.

Finally, it is easy to verify that [380, Proposition 8.3.7], for every $p \in [1, +\infty)$,

$$\|u^H\|_p = \|u\|_p$$

and that, if $F : \mathbb{R} \to \mathbb{R}$ is nondecreasing, then

$$F(u^H) = (F(u))^H. \tag{1.4.40}$$

We refer to [259] for other interesting results about manipulations of nonlocal quantities.

1.5 Berestycki-Lions type assumptions: some convergences

The assumptions considered throughout the thesis are in the spirit of the ones proposed by Berestycki and Lions [50, 51], adapted then to the fractional framework in [79, 95] and to the Choquard-Hartree-Pekar framework by Moroz and Van Schaftingen [302]. These assumptions cover different models which arise in applications, see Examples 1.5.1.

In the case of the unconstrained problem (frequency fixed, mass free), as shown in the abovementioned papers (see also [138, 237, 300]), these assumptions are somehow *almost optimal*, in the sense that when the nonlinearity collapses to a power, the growth condition are optimal for the existence of a (sufficiently regular) variational solution. See also [90, 353, 392] for the case of combination of powers and [291] for some further generalizations to the so called infinity-mass regime.

We highlight that no pointwise condition of Ambrosetti-Rabinowitz type, nor of monotonicity type, is assumed, and this lack of additional assumptions obstructs some classical arguments related both to compactness and geometry of the problems.

In the L^2-constrained case (frequency free, mass prescribed), different qualitative phenomena are related to sub and super L^2-critical cases: for instance, the sub or super L^2-criticality of the exponent influences the boundedness of the functional on the L^2-sphere, as well as the lifespan and the stability of the solutions in some related equations (see [92]). In this thesis we restrict our analysis to the L^2-subcritical regime: we aim to extend our results to the L^2-critical and supercritical regime in the future.

In this Section, for the sake of clarity, we list all the assumptions on the nonlinearities that will come into play in the following Chapters, both in the fractional framework and in the Choquard framework; we let here $s \in (0, 1]$ and $\alpha \in (0, N)$. In particular, we show the role of the subcriticality growth in the convergence of nonlinear functionals.

We highlight that the labeling here introduced will be changed throughout different Chapters, in order to avoid cumbersome notations.

1.5.1 Local nonlinearities

For local nonlinearities of the type $g(u)$, $G(t) = \int_0^t g(\tau)d\tau$, we introduce the following notations:

- *Lower critical exponent:* $2^\# := 2$,
- *Upper critical exponent:* $2^*_s := 2 + \frac{4s}{N-2s} = \frac{2N}{N-2s} \in (2, +\infty)$,
- *L^2-critical exponent:* $2^m_s := 2 + \frac{4s}{N} = \frac{2N+4s}{N} \in (2, 2 + \frac{4}{N})$,

and notice that

$$2 = 2^\# < 2^m_s < 2^*_s < +\infty.$$

Moreover we introduce the following set of assumptions:

(h0) *Continuity:* $g \in C(\mathbb{R})$,

(h0') *Pohozaev regularity:* $s \in (\frac{1}{2}, 1)$ or $g \in C^\sigma_{loc}(\mathbb{R})$ for some $\sigma > 1 - 2s$,

(h1) *Nontriviality (frequency free):* there exists $t_0 > 0$ such that $G(t_0) > 0$,

(h1') *Nontriviality (frequency $\mu > 0$ fixed):* there exists $t_0 = t_0(\mu) > 0$ such that $G(t_0) \geq \frac{\mu}{2} t_0^2$,

(h2) *Supercriticality in 0:* $\lim_{t \to 0} \frac{g(t)}{t} = 0$,

(h2*) *L^2-subcriticality in 0:* $\lim_{t \to 0} \frac{G(t)}{|t|^{2^m_s}} = +\infty$,

(h3) *Subcriticality at ∞:* $\lim_{|t| \to +\infty} \frac{g(t)}{|t|^{2^*_s - 2} t} = 0$,

(h3') *Subcriticality (strict) at ∞:* $\lim_{|t| \to +\infty} \frac{g(t)}{|t|^{p-2} t} = 0$ for some $p \in (2, 2^*_s)$,

(h3") *L^2-subcriticality at ∞:* $\lim_{|t| \to +\infty} \frac{g(t)}{|t|^{2^m_s - 2} t} = 0$,

(h3*) *Criticality at ∞:* $\lim_{|t| \to +\infty} \frac{g(t)}{|t|^{2^*_s - 2} t} = a \neq 0$; if $a > 0$ we also assume $g(t) \geq a t^{2^*_s - 1} + C t^{p-1}$ for some $C > 0$ and $p \in (\max\{2^*_s - 2s, 2\}, 2^*_s)$ and every $t > 0$, (h4) *Symmetry:* g odd,

(h5) *Negative-cut (for positivity):* $g \equiv 0$ on $(-\infty, 0]$.

Notice that

$$\text{(h2*)} \vee \text{(h3*)} \implies \text{(h1')} \implies \text{(h1)}, \quad \text{(h3")} \implies \text{(h3')} \implies \text{(h3)}.$$

Example 1.5.1. *These general assumptions include different models arising in applications. For examples, they cover pure powers $g(t) = |t|^{q-2}t$, with $q \in (2, 2_s^*)$ (or $q \in (2, 2_s^m)$), and combined powers like $g(t) = |t|^{q-2}t + |t|^{r-2}t$ (cooperation models) and $g(t) = |t|^{q-2}t - |t|^{r-2}t$ (competion models). Other physical models can be found for example in asymptotically linear functions*

$$g(t) = \frac{t^3}{1+t^2}, \quad G(t) = \frac{1}{2}\left(t^2 - \log(1+t^2)\right),$$

which arise in the saturation effect in nonlinear optics for photorefractive media [161, 226, 281, 327, 383], or also

$$g(t) = \left(1 - \frac{1}{\sqrt{1+t^2}}\right)t, \quad G(t) = \frac{1}{2}\left(t^2 - 2\sqrt{1+t^2} + 2\right)$$

of square-root type, which describes narrow-gap semiconductors [317, 348].

Remark 1.5.2. *We trivially observe that assigning a condition on g is generally stronger than assigning a similar condition on G. Indeed, by De l'Hôpital theorem,*

$$\lim_{|t|\to+\infty} \frac{g(t)}{|t|^{q-2}t} = l \in \overline{\mathbb{R}} \implies \lim_{|t|\to+\infty} \frac{G(t)}{|t|^q} = l,$$

or more generally

$$\liminf_{|t|\to+\infty} \frac{g(t)}{|t|^{q-2}t} \leq \liminf_{|t|\to+\infty} \frac{G(t)}{|t|^q} \leq \limsup_{|t|\to+\infty} \frac{G(t)}{|t|^q} \leq \limsup_{|t|\to+\infty} \frac{g(t)}{|t|^{q-2}t}.$$

The viceversa is generally not true: consider for example $G(t) := t^q\left(\int_0^t \frac{\sin(\tau)}{\tau} - \frac{\pi}{2}\right)$ which verifies

$$\lim_{t\to+\infty} \frac{G(t)}{t^q} = 0, \quad \liminf_{t\to+\infty} \frac{g(t)}{t^{q-1}} = -1, \quad \limsup_{t\to+\infty} \frac{g(t)}{t^{q-1}} = 1;$$

notice that the $\limsup$ *is finite (consider $G(t) = t^{q-\frac{1}{2}}\cos(t)$ for an infinite* $\limsup$*). On the other hand, if one assume a priori that $\lim_{|t|\to+\infty} \frac{g(t)}{|t|^{q-2}t}$ exists, then the viceversa holds true.*

Moreover, if $\delta \in (0,1)$, by choosing $\varepsilon \in (0, 1-\delta)$ and setting $G(t) = t^{q-\varepsilon}\cos(t)$ we see that

$$\lim_{|t|\to+\infty} \frac{G(t)}{|t|^q} = 0 \quad \text{but} \quad \lim_{|t|\to+\infty} \frac{g(t)}{|t|^{q+\delta-1}} \neq 0;$$

in particular, since generally $2_s^* - 2_s^m \in (0, +\infty)$, we have

$$\lim_{|t|\to+\infty} \frac{G(t)}{|t|^{2_s^m}} = 0 \not\Rightarrow \lim_{|t|\to+\infty} \frac{g(t)}{|t|^{2_s^*-1}} = 0.$$

Similar considerations can be done for $t \to 0$ (consider $G(t) := t^q\left(\int_0^{1/t} \frac{\sin(\tau)}{\tau} - \frac{\pi}{2}\right)$ or $G(t) = t^{q+\varepsilon}\cos\left(\frac{1}{t}\right)$).

Remark 1.5.3. *Generally, when $u \in H^s(\mathbb{R}^N)$, $g(u)$ will not lie on a precise Lebesgue space, but on a summation of spaces. To handle these quantities we remark that the following properties are equivalent [27, Proposition 2.3], for any $p, q \in (1, +\infty)$:*

- $g \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$,
- $|g| \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$,
- $|g| \leq h$ *for some* $h \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$.

Remark 1.5.4. *We write here in which spaces lie the considered quantities. Let* $u \in H^s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$. *By assuming*

$$\limsup_{t\to 0} \frac{|g(t)|}{|t|} < \infty, \quad \limsup_{|t|\to+\infty} \frac{|g(t)|}{|t|^{2^*_s}} < \infty \tag{1.5.41}$$

(for instance given by (h2) and (h3)) we have (see Remark 1.5.3)

$$\begin{aligned} g(u) &\in L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2s}}(\mathbb{R}^N) + L^{2\frac{N-2s}{N+2s}} \cap L^{\frac{2N}{N+2s}}(\mathbb{R}^N) \\ &\subset L^2(\mathbb{R}^N) + L^{\frac{2N}{N+2s}}(\mathbb{R}^N), \\ G(u) &\in L^1(\mathbb{R}^N) \cap L^{\frac{N}{N-2s}}(\mathbb{R}^N) + L^{2\frac{N-2s}{N}}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \\ &\subset L^1(\mathbb{R}^N). \end{aligned}$$

If $\varphi \in H^s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$ *is a test function, we notice that the found summability is enough to have* $\int_{\mathbb{R}^N} g(u)\varphi \, dx$ *well defined.*

We state now the convergence properties of the nonlinear functionals, in the case of a subcritical growth [95] (see also [298, Theorem 2 and Corollary 2]).

Proposition 1.5.5. *Assume (h0) and (1.5.41).*

- *Let* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^N)$. *Then for any* $\varphi \in H^s(\mathbb{R}^N)$ *we have*
$$\int_{\mathbb{R}^N} g(u_n)\varphi \to \int_{\mathbb{R}^N} g(u)\varphi.$$
- *Assume in addition (h2) and (h3). Let* $u_n \rightharpoonup u$ *in* $H^s_r(\mathbb{R}^N)$. *Then*
$$\int_{\mathbb{R}^N} |G(u_n) - G(u)| \to 0, \quad \int_{\mathbb{R}^N} |g(u_n)u_n - g(u)u| \to 0$$
as well as $\int_{\mathbb{R}^N} |g(u_n)v - g(u)v| \to 0$ *for each* $v \in H^s(\mathbb{R}^N)$.
- *Assume in addition (h2) and (h3). Let* $u_n \rightharpoonup u$ *in* $H^s(\Omega)$ *with* $\Omega \subset \mathbb{R}^N$ *bounded. Then*
$$\int_\Omega |G(u_n) - G(u)| \to 0, \quad \int_\Omega |g(u_n)u_n - g(u)u| \to 0$$
as well as $\int_{\mathbb{R}^N} |g(u_n)v - g(u)v| \to 0$ *for each* $v \in H^s(\Omega)$.

Proof. We prove the first claim. Let $\varphi \in C^\infty_c(\mathbb{R}^N)$, and let $\Omega := \mathrm{supp}(\varphi)$. Since $u_n \to u$ in $L^r(\Omega)$ for each $r \in [2, 2^*_s)$, we have (by the L^r-dominated convergence theorem) $g(u_n) \to g(u)$ in $L^r(\Omega)$ for each $r \in [1, \frac{2N}{N+2s})$. For a whatever of such r, let q be its conjugate; since $\varphi \in L^q(\mathbb{R}^N)$ for such q, we have $g(u_n)\varphi \to g(u)\varphi$ in $L^1(\Omega)$. Thus

$$\int_{\mathbb{R}^N} g(u_n)\varphi \to \int_{\mathbb{R}^N} g(u)\varphi \quad \forall \varphi \in C^\infty_c(\mathbb{R}^N).$$

We want to extend the relation to $H^s(\mathbb{R}^N)$. Indeed, observe first that, for $\varphi \in H^s(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N} g(u_n)\varphi\right| \lesssim \int_{\mathbb{R}^N} \left(|u_n| + |u_n|^{2^*_s - 1}\right)|\varphi|$$

$$\leq \|u_n\|_2 \|\varphi\|_2 + \|u_n\|_{2^*_s}^{\frac{N+2s}{N-2s}} \|\varphi\|_{2^*_s} \leq C\|\varphi\|_{H^s}$$

uniform in $n \in \mathbb{N}$, since u_n are equibounded in $L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$. Let now $\varphi_\varepsilon \in C^\infty_c(\mathbb{R}^N)$ approximating a fixed φ in $H^s(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} g(u_n)\varphi - \int_{\mathbb{R}^N} g(u)\varphi = \int_{\mathbb{R}^N} g(u_n)(\varphi - \varphi_\varepsilon) + \int_{\mathbb{R}^N} (g(u_n) - g(u))\varphi_\varepsilon + \int_{\mathbb{R}^N} g(u)(\varphi_\varepsilon - \varphi);$$

thus the first and the third quantities are small in ε (uniformly in n), and the second is small for $n = n(\varepsilon) \gg 0$. Hence we have the first claim.

The second and the third claims are a consequence of [95, Lemma 2.4]. We exhibit here an easier proof of the second point, by assuming the stronger condition (h3').

Recall that $H^s_r(\mathbb{R}^N)$ is compactly embedded in $L^p(\mathbb{R}^N)$, being $p \in (2, 2^*_s)$ introduced in (h3'). Then by standard argument one has, up to a subsequence, that

- $u_n \to u$ almost everywhere,
- $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$, with $|u_n|, |u| \leq w \in L^p(\mathbb{R}^N)$.

By the assumption there exists an M such that

$$|g(t)t| \leq \begin{cases} C_\delta |t|^2 & \text{if } |t| \leq M, \\ \delta |t|^p & \text{if } |t| \geq M. \end{cases}$$

Fixed a whatever $R > 0$, set

$$M_n := \{|u_n| \leq M\} \cap B_R(0),$$

we have

$$\begin{aligned} |g(u_n)u_n| &= |g(u_n)u_n|\chi_{M_n} + |g(u_n)u_n|\chi_{\mathbb{R}^N \setminus M_n} \leq C_\delta |u_n|^2 \chi_{M_n} + \delta |u_n|^p \chi_{\mathbb{R}^N \setminus M_n} \\ &\leq C_\delta M^2 \chi_{M_n} + \delta |u_n|^p \leq C_\delta M^2 \chi_{B_R(0)} + \delta |w|^p \in L^1(\mathbb{R}^N) \end{aligned}$$

and similarly for $G(u_n)$ and $|g(u_n)v|$. Moreover, since g is continuous, we have $g(u_n) \to g(u)$ almost everywhere. By dominated convergence theorem, we obtain the claim. ∎

1.5.2 Nonlocal nonlinearities

For nonlocal nonlinearities of the type $(I_\alpha * F(u))f(u)$, $F(t) = \int_0^t f(\tau)d\tau$, we introduce the following notations:

- *Lower critical exponent:* $2^\#_\alpha := 1 + \frac{\alpha}{N} = \frac{N+\alpha}{N} \in (1, 2)$,
- *Upper critical exponent:* $2^*_{\alpha,s} := 1 + \frac{\alpha+2s}{N-2s} = \frac{N+\alpha}{N-2s} \in (1, +\infty)$,
- *L^2-critical exponent:* $2^m_{\alpha,s} := 1 + \frac{\alpha+2s}{N} = \frac{N+\alpha+2s}{N} \in (1, 2 + \frac{2}{N})$,

and notice that

$$1 < 2^\#_\alpha < 2^m_{\alpha,s} < 2^*_{\alpha,s} < +\infty;$$

if $s = 1$, if there is no ambiguity from the framework, we write $2^*_\alpha \equiv 2^*_{\alpha,1} = 1 + \frac{\alpha+2s}{N-2} = \frac{N+\alpha}{N-2}$ and $2^m_\alpha \equiv 2^m_{\alpha,1} = 1 + \frac{\alpha+2}{N} = \frac{N+\alpha+2}{N}$.

Remark 1.5.6. *We observe that, defining the Riesz potential by $x \mapsto \frac{A_{N,\beta}}{|x|^\beta}$, as some authors do, we have that the critical exponents become $\frac{2N-\beta}{N} < \frac{2N-\beta+2s}{N} < \frac{2N-\beta}{N-2s}$.*

We introduce the following set of assumptions:

(H0) *Continuity:* $f \in C(\mathbb{R})$ (i.e. $F \in C^1(\mathbb{R})$),

(H0') *Additional regularity:* $f \in C^{\sigma}_{loc}(\mathbb{R})$ (i.e. $F \in C^{1,\sigma}_{loc}(\mathbb{R})$) for some $\sigma \in (0,1]$,

(H1) *Nontriviality:* $F \not\equiv 0$, i.e. there exists $t_0 \in \mathbb{R}^*$ such that $F(t_0) \neq 0$,

(H2) *Well posedness:* $\limsup_{t\to 0} \frac{|f(t)|}{|t|^{2^{\#}_{\alpha}-1}} < \infty$, $\limsup_{|t|\to+\infty} \frac{|f(t)|}{|t|^{2^*_{\alpha,s}-1}} < \infty$, or equivalently $|tf(t)| \leq C\big(|t|^{2^{\#}_{\alpha}} + |t|^{2^*_{\alpha,s}}\big)$ for some $C<0$,

(H2') *L^2-well posedness:* $\limsup_{t\to 0} \frac{|f(t)|}{|t|^{2^{\#}_{\alpha}-1}} < \infty$, $\limsup_{|t|\to+\infty} \frac{|f(t)|}{|t|^{2^m_{\alpha,s}-1}} < \infty$, or equivalently $|tf(t)| \leq C\big(|t|^{2^{\#}_{\alpha}} + |t|^{2^m_{s,\alpha}}\big)$ for some $C<0$,

(H3) *Supercriticality in* 0*:* $\lim_{t\to 0} \frac{F(t)}{|t|^{2^{\#}_{\alpha}}} = 0$,

(H3') *(Super)linerarity in* 0*:* $\limsup_{t\to 0} \frac{|f(t)|}{|t|} < \infty$,

(H3*) *L^2-subcriticality in* 0*:* $\lim_{t\to 0} \frac{|F(t)|}{|t|^{2^m_{\alpha,s}}} = +\infty$,

(H3*') *Sublinearity in* 0*:* $\lim_{t\to 0} \frac{|f(t)|}{|t|} = +\infty$,

(H4) *Subcriticality at* ∞*:* $\lim_{|t|\to+\infty} \frac{F(t)}{|t|^{2^*_{\alpha,s}}} = 0$,

(H4') *L^2-subcriticality at* ∞*:* $\lim_{|t|\to+\infty} \frac{F(t)}{|t|^{2^m_{\alpha,s}}} = 0$,

(H5) *Symmetry:* f is odd or even,

(H6) *Sign:* f has constant sign on $(0,+\infty)$.

Notice that

$$\text{(H0')} \implies \text{(H0)}, \quad \text{(H2')} \implies \text{(H2)}, \quad \text{(H3')} \implies \text{(H3)}, \quad \text{(H2')} \implies \text{(H4)},$$

$$\text{(H3*)} \vee \text{(H3*')} \implies \text{(H1)}, \quad \text{(H3)} \wedge \text{(H4)} \implies \text{(H2)},$$

while generally (H3*) and (H3*') are not related (since 2 and $2^m_{\alpha,s}$ are not so).

When searching for multiple normalized solutions in Choquard equations, in addition to (H3*) and (H5) we will ask the following technical assumption (see also Remark 3.1.3):

(H7) *Almost monotonicity:* if F is odd, then F has a constant sign in $(0,\delta_0]$ and

$$\sup_{t\in(0,\delta_0],\, h\in[0,1]} \left| \frac{F(th)}{F(t)} \right| < \infty$$

for some $\delta_0 > 0$ (e.g., $|F|$ is non-decreasing in $[0,\delta_0]$).

Remark 1.5.7. *We write here in which spaces lie the considered quantities. Let $u \in H^s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$. By (H2) we have (see Remark 1.5.3)*

$$\begin{aligned} f(u) &\in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) \cap L^{\frac{N}{\alpha}\frac{2N}{N-2s}}(\mathbb{R}^N) + L^{2\frac{N-2s}{\alpha+2s}} \cap L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N) \\ &\subset L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N), \\ F(u), f(u)u &\in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \cap L^{\frac{N}{N+\alpha}\frac{2N}{N-2s}}(\mathbb{R}^N) + L^{2\frac{N-2s}{N+\alpha}}(\mathbb{R}^N) \cap L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \\ &\subset L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N). \end{aligned}$$

Thus by the Hardy-Littlewood-Sobolev inequality we obtain

$$\begin{aligned} I_\alpha * F(u) &\in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) \cap L^{\frac{2N^2}{N^2-(\alpha+2s)N-2s\alpha}}(\mathbb{R}^N) + L^{\frac{2N(N-2s)}{N^2-\alpha N+4s\alpha}}(\mathbb{R}^N) \cap L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N) \\ &\subset L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N). \end{aligned}$$

Finally, by the Hölder inequality, we have

$$(I_\alpha * F(u))F(u) \in L^1(\mathbb{R}^N)$$

and

$$\begin{aligned} (I_\alpha * F(u))f(u) &\in L^2(\mathbb{R}^N) \cap L^{\frac{2N^2}{N^2-2s\alpha}}(\mathbb{R}^N) + L^{\frac{2N(N-2s)}{N^2+2\alpha s}}(\mathbb{R}^N) \cap L^{\frac{2N}{N+2s}}(\mathbb{R}^N) \\ &\subset L^2(\mathbb{R}^N) + L^{\frac{2N}{N+2s}}(\mathbb{R}^N); \end{aligned}$$

we observe that $(I_\alpha * F(u))f(u)$ does not lie in $L^2(\mathbb{R}^N)$, generally. On the other hand, if $\varphi \in H^s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$ is a test function, we notice that the found summability of $(I_\alpha * F(u))f(u)$ is enough to have $\int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)\varphi\, dx$ well defined, since $f(u)\varphi \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$.

Remark 1.5.8. *By Propsition 1.3.1 we see that $I_\alpha * F(u) \in C_0(\mathbb{R}^N)$ (and thus it is well defined pointwise) if $F(u)$ lies in $L^{\frac{N}{\alpha}-\varepsilon}(\mathbb{R}^N) \cap L^{\frac{N}{\alpha}+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon > 0$. In particular, if $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, it is sufficient to assume that F grows at most polynomially (and at least superlinearly) in zero and at infinity. Moreover, assuming (H0) and (H2) on f, we need to assume that $u \in L^{\frac{N+\alpha}{\alpha}-\varepsilon}(\mathbb{R}^N) \cap L^{\frac{N}{\alpha}\frac{N+\alpha}{N-2s}+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon > 0$; in particular, the convolution is pointwise well defined if $u \in L^2(\mathbb{R}^N) \cap L^{\frac{N}{\alpha}\frac{2N}{N-2s}}(\mathbb{R}^N)$.*

We state now the convergences for the nonlinear Choquard terms in the case of a subcritical growth (see also [302, pages 6565 and 6577], [37, page 11] and [22, page 353]).

Proposition 1.5.9. *Assume (H0) and (H2).*

- *Let $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. Then for any $\varphi \in H^s(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)\varphi \to \int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)\varphi.$$

- *Assume in addition (H3) and (H4). Let $u_n \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n) \to \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)$$

  and

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n))f(u_n)u_n \to \int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)u.$$

Proof. Let $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, then u_n is bounded in $L^2(\mathbb{R}^N) \cap L^{2^*_s}(\mathbb{R}^N)$. By Remark 1.5.7 we have $F(u_n)$ bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Moreover we can assume $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ for $p \in [1, 2^*_s)$, and thus $F(u_n) \to F(u)$ in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [1, \frac{2N}{N+\alpha})$. This two information on $F(u_n)$ imply $F(u_n) \rightharpoonup F(u)$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$[3].[4] By some standard topological argument, the convergence holds

[3]We argue in this way. First, fix $\Omega \subset \mathbb{R}^N$ bounded and $q \in [1, \frac{2N}{N+\alpha})$, so that $L^{\frac{2N}{N+\alpha}}(\Omega) \subset L^q(\Omega)$ for every $q \in [1, \frac{2N}{N+\alpha})$. Since $F(u_n)$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \subset L^{\frac{2N}{N+\alpha}}(\Omega)$ and $\frac{2N}{N+\alpha} > 1$, it converges to some $v \in L^{\frac{2N}{N+\alpha}}(\Omega) \subset L^q(\Omega)$; on the other hand $F(u_n) \rightharpoonup F(u) \in L^q(\mathbb{R}^N) \subset L^q(\Omega)$, thus by uniqueness $v = F(u_n)$. Let now φ be in the dual $L^{\frac{N-\alpha}{2N}}(\mathbb{R}^N)$, and consider $\varphi_k \in C^\infty_c(\mathbb{R}^N)$ approximating φ. Thus

$$\int_{\mathbb{R}^N} \big(F(u_n) - F(u)\big)\varphi \leq \|F(u_n) - F(u)\|_{\frac{2N}{N+\alpha}} \|\varphi - \varphi_k\|_{\frac{N-\alpha}{2N}} + \int_{\text{supp}(\varphi_k)} \big(F(u_n) - F(u)\big)\varphi_k;$$

exploiting that $F(u_n)$ is bounded, the first piece is small for k large (uniform in n), while the second is small (fixed this k), for n large, by the previous argument with $\Omega = \text{supp}(\varphi_k)$.

[4]We can deduce the implication also in this way: since $u_n \to u$ a. e. pointwise and F is continuous, then

for the whole sequence. Moreover, by Proposition 1.3.1 we gain

$$I_\alpha * F(u_n) \rightharpoonup I_\alpha * F(u) \quad \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).$$

Let now $\varphi \in C_c^\infty(\mathbb{R}^N)$, and set $\Omega := \mathrm{supp}(\varphi)$. Since $u_n \to u$ in $L^p(\Omega)$ for each $p \in [2, 2_s^*)$, we have (by the L^p-dominated convergence theorem) $f(u_n) \to f(u)$ in $L^p(\Omega)$ for each $p \in [1, \frac{2N}{\alpha+2s})$. Let $p \in (\frac{2N}{N+\alpha}, \frac{2N}{\alpha+2s})$ be whatever and let q be such that $\frac{1}{p} + \frac{1}{q} = \frac{N+\alpha}{2N}$; since $\varphi \in L^q(\mathbb{R}^N)$ for such q, we have $f(u_n)\varphi \to f(u)\varphi$ in $L^{\frac{2N}{N+\alpha}}(\Omega)$. Thus

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n)\varphi \to \int_{\mathbb{R}^N} (I_\alpha * F(u)) f(u)\varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

To extend the relation to $\varphi \in H^s(\mathbb{R}^N)$ we argue as in Proposition 1.5.5, after having observed that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} (I_\alpha * F(u_n)) f(u_n)\varphi\right| &\lesssim \|F(u_n)\|_{\frac{2N}{N+\alpha}} \|f(u_n)\varphi\|_{\frac{2N}{N+\alpha}} \\ &\lesssim \||u_n|^{2_\alpha^\# -1}\varphi\|_{\frac{2N}{N+\alpha}} + \||u_n|^{2_{\alpha,s}^*-1}\varphi\|_{\frac{2N}{N+\alpha}} \\ &\leq \||u_n|^{\frac{\alpha}{N}}\|_{\frac{2N}{\alpha}} \|\varphi\|_2 + \||u_n|^{\frac{\alpha+2s}{N-2s}}\|_{\frac{2N}{\alpha+2s}} \|\varphi\|_{2_s^*} \\ &\leq \|u_n\|_2^{2_\alpha^\# -1} \|\varphi\|_{H^s} + \|u_n\|_{2_s^*}^{2_{\alpha,s}^*-1} \|\varphi\|_{H^s} \lesssim \|\varphi\|_{H^s}.\end{aligned}$$

Assume now (H3) and (H4). Let $G(t) := (F(t))^{\frac{N+\alpha}{2N}}$. By the assumptions we have

$$\lim_{t\to 0} \frac{G(t)}{|t|^2} = \lim_{t\to 0} \left(\frac{F(t)}{|t|^{2_\alpha^\#}}\right)^{\frac{N+\alpha}{2N}} = 0, \qquad \lim_{t\to \infty} \frac{G(t)}{|t|^{2_s^*}} = \lim_{t\to 0} \left(\frac{F(t)}{|t|^{2_{\alpha,s}^*}}\right)^{\frac{N+\alpha}{2N}} = 0.$$

Thus, by Proposition 1.5.5 we gain $G(u_n) \to G(u)$ in $L^1(\mathbb{R}^N)$, which means $F(u_n) \to F(u)$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. In particular, by Proposition 1.3.1 we obtain

$$I_\alpha * F(u_n) \to I_\alpha * F(u) \quad \text{in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N).$$

Thus we get the first claim. Moreover, arguing as before we get $f(u_n)u_n \rightharpoonup f(u)u$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, and this concludes the proof. ∎

Remark 1.5.10. *When $\alpha \to 0$, by (1.3.36), we know that, under suitable assumptions,*

$$(I_\alpha * F(u)) f(u) \overset{\alpha\to 0}{\to} F(u) f(u) =: g(u);$$

notice that (by integration by parts) $G(u) = \frac{1}{2}F^2(u)$. This relation is coherent with the definitions of the critical exponents of the local and nonlocal frameworks; indeed:

$$2_0^\# + (2_0^\# - 1) = 2^\# - 1, \quad 2_{0,s}^* + (2_{0,s}^* - 1) = 2_s^* - 1, \quad 2_{0,s}^m + (2_{0,s}^m - 1) = 2_s^m - 1.$$

This correspondence lacks when comparing the nontriviality assumptions $F(t_0) \neq 0$ and $G(t_0) \geq \frac{\mu}{2}t_0^2$: this is due to the fact that, for any $\alpha \neq 0$, the pieces μu and $(I_\alpha * F(u))f(u)$ scales differently. Moreover, we see that while the subcriticality assumptions for the local problem are made for g, for the nonlocal problem are made for F, since essentially the product Ff automatically becomes subcritical if F is so.

$F(u_n) \to F(u)$ a. e. pointwise; moreover, being bounded, then $F(u_n) \rightharpoonup v$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ for some v, where $\frac{2N}{N+\alpha} > 1$; hence by [298, Lemma 1] we have $v = F(u)$.

Chapter 2

Fractional Schrödinger equations: prescribed and free mass problems

In this Chapter we study the following fractional Schrödinger equation

$$(-\Delta)^s u + \mu u = g(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 2$, $s \in (0,1)$, $u \in H^s(\mathbb{R}^N)$, $\mu > 0$ is a frequency and $g \in C(\mathbb{R}, \mathbb{R})$ satisfies Berestycki-Lions type conditions. First, we recall some known facts about the *unconstrained problem*, i.e. when μ is fixed, which has been investigated in [79, 95]. Then we study the *constrained* problem

$$\begin{cases} (-\Delta)^s u + \mu u = g(u) & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2 dx = m, \end{cases}$$

where $m > 0$ is a prescribed mass, $u \in H^s_r(\mathbb{R}^N)$ and μ is a Lagrange multiplier, part of the unknowns. Using a Lagrangian formulation, we prove the existence of a weak solution with prescribed mass when g has an L^2-subcritical growth. The approach relies on the construction of a minimax structure, by means of a *Pohozaev mountain* in a product space and some deformation arguments under a weaker version of the Palais-Smale condition. A multiplicity result of infinitely many normalized solutions is also obtained if g is odd, and this is new even for g power.

The present Chapter is mainly based on the paper [113] (see also [114]).

2.1 The fractional Schrödinger equation: a long-range interaction

In 1948, following a suggestion by Dirac, Feynman [182] proposed a new suggestive description of the time evolution of the state of a non-relativistic quantum particle. According to Feynman, the wave function solution of the Schrödinger equation should be given by a heuristic integral over the space of paths: the classical notion of a single, unique classical trajectory for a system is replaced by a functional integral over an infinity of quantum-mechanically possible trajectories. Following Feynman's path integral approach to quantum mechanics, Laskin [249–252] generalized the path integral over Brownian motions (random motion seen in swirling gas molecules) to Lévy flights (a mix of long trajectories and short, random movements found in turbulent fluids) and derived the *fractional nonlinear Schrödinger* ((fNLS) for short) equation

$$i\hbar\partial_t \psi = \hbar^{2s}(-\Delta)^s \psi + V(x)\psi - g(\psi), \quad (t, x) \in (0, +\infty) \times \mathbb{R}^N \tag{2.1.1}$$

where $s \in (0,1)$, $N > 2s$, the symbol $(-\Delta)^s$ denotes the fractional power of the Laplace operator (defined via Fourier transform on the spatial variable), $\hbar$ designates the usual Planck constant, V is a real potential and g is a Gauge invariant nonlinearity, i.e. $g(e^{i\theta}\rho) = e^{i\theta}g(\rho)$ for any ρ, $\theta \in \mathbb{R}$. The complex wave function $\psi(x,t)$ represents the quantum mechanical probability amplitude for a given unit mass particle to have position x at time t, under the confinement due to the potential V, and $|\psi|^2$ is the corresponding probability density.

Fractional integrals and derivatives in the calculation methods have been used for the explanation of physical phenomena which do not comply with the laws of classical statistical physics, for instance in modeling Bose-Einstein condensates. It is known that Bose-Einstein condensation, theoretically discovered in 1924 and observed experimentally with alkali metals in 1995, represents a topical subject due to the explanation of quantum effects seen on a macroscopic scale, transmission of matter and the behaviour of superconductivity and superfluids. In this respect, not only experimental studies are important but theoretical studies too, which lead to the analysis of class of (fNLS) equations (also known as fractional Gross-Pitaievskii equations). Numerical simulations show existence of standing waves solutions, having a soliton behaviour and bound states [165, 394], including mass conservation, energy conservation and dispersion relation, in which the fractional order exponent influences the shape of the state.

In 2015 a first optical realization of the fractional Schrödinger equation, based on transverse light dynamics in aspherical optical cavities, was achieved by Longhi [274]; subsequently, the propagation dynamics of wave packets were reported in Kerr nonlinearities, with constant or double-barrier potential. Numerical results showed the existence of solitons for (fNLS) equations where the Lévy index s and the saturation parameter can significantly affect the stability of these solitons [243, 262, 383, 387]. Numerous other applications of the (fNLS) equation arise in the physical sciences, ranging from models of boson stars (see Section 4.1) to geo-hydrology [25], from charge transport in biopolymers, like DNA [242] to anomalous diffusion phenomena [76, 295, 367], from water wave dynamics [232] to jump processes in probability theory with applications to financial mathematics (see also [153] and the references therein). Applications for wide ranges of s appears, for example, also in the dynamics of populations [85]: here small values of $s \approx 0$ or large values of $s \approx 1$ better model specific behaviours, according to the environments. We refer also to [29, 30] for some recent applications to the analysis of the amount of bromsulphthalein in the human liver and to the study of thermostat systems, and others.

From a mathematical view point, when searching for standing waves to (2.1.1), i.e. factorized solutions

$$\psi(t,x) = e^{i\mu t}u(x), \quad \mu > 0,$$

two possible directions can be pursued. A first possibility is to study (2.1.1) with a prescribed frequency μ and free mass. This approach, which we call the *unconstrained* problem, has been deeply developed: the literature concerning the local version of the unconstrained problem starts from the seminal papers of Berestycki and Lions [50, 51] (see also [48, 70, 237, 290]) and it is so large that we do not even make an attempt to summarize it. Some fundamental contributions for the fractional case $s \in (0,1)$ instead can be found in [84, 86, 190]; in particular, the existence and qualitative properties of the solutions for more general classes of fractional NLS equations with local source were studied in [22, 79, 95, 177, 229, 230].

A second approach is to prescribe the mass of u, thus conserved by ψ in time

$$\int_{\mathbb{R}^N} |\psi(x,t)|^2\, dx = m, \quad \forall\, t \in (0,+\infty)$$

and let the frequency μ to be free, becoming an unknown. This second approach is of considerable significance in physics, not only for the quantum probability normalization and the information on the mass itself, but also because the mass may also have specific meaning, such as the power supply in nonlinear optics, or the total number of atoms in Bose-Einstein condensation. Moreover, it can give better insights into the dynamical properties, such as the orbital stability or instability of solutions of (2.1.1) (see [92]).

In the local framework ($s = 1$) the seminal contribution to the study of *constrained* problems is due to Stuart [356], Cazenave and Lions [92]; see [35, 36, 55, 58–60, 224, 235, 292, 335, 343] for more recent contributions in the local case.

In the fractional case, the existence of a mass-constrained solution was, instead, recently considered in [156, 179, 385] for pure powers and in [278] for combined powers. It remains an open problem anyway to derive analytically the existence of infinitely many bound states with higher energy, including mass conservation.

The present Chapter is dedicated to the study of standing waves solutions of (2.1.1) (when $V = const$ and we fix $\hbar = 1$) with prescribed mass, by means of a new variational method. Namely, we are interested to seek for radially symmetric solutions of the fractional problem

$$\begin{cases} (-\Delta)^s u + \mu u = g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = m, \end{cases} \tag{2.1.2}$$

where $N \geq 2$, $s \in (0,1)$, $m > 0$ and μ is a Lagrange multiplier. We assume that the function g satisfies the following Berestycki-Lions type conditions:

(g1) $g : \mathbb{R} \to \mathbb{R}$ continuous and $\lim_{t\to 0} \frac{g(t)}{t} = 0$,

(g2) $\lim_{|t|\to\infty} \frac{g(t)}{|t|^p} = 0$ where $p = 2_s^m = 1 + \frac{4s}{N}$ (see also Remark 2.1.4),

(g3) there exists $t_0 > 0$ such that $G(t_0) > 0$,

where $G(t) = \int_0^t g(\tau)d\tau$. We recall that (g2) means that g has an L^2-subcritical growth.

The solutions to (2.1.2) can be characterized as critical points of the C^1-functional $\mathcal{L} : H_r^s(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{L}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - \int_{\mathbb{R}^N} G(u)$$

constrained on the sphere

$$\mathcal{S}_m := \{u \in H_r^s(\mathbb{R}^N) \mid \|u\|_2^2 = m\};$$

here we consider thus, as in [224], a Lagrangian formulation of the problem (2.1.2). In order to avoid technical issues with the boundary of $\mathbb{R}_+$ (see Section 4.2.2 for a different approach), we write

$$\mu \equiv e^\lambda$$

with $\lambda \in \mathbb{R}$ and define the C^1-functional $\mathcal{I}^m : \mathbb{R} \times H_r^s(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{I}^m(\lambda, u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - \int_{\mathbb{R}^N} G(u) + \frac{e^\lambda}{2}(\|u\|_2^2 - m).$$

We seek for critical points $(\lambda, u) \in \mathbb{R} \times H_r^s(\mathbb{R}^N)$ of $\mathcal{I}^m$, namely weak solutions of $\partial_u \mathcal{I}^m(\lambda, u) = 0$ and $\partial_\lambda \mathcal{I}^m(\lambda, u) = 0$ or equivalently

$$\begin{cases} \int_{\mathbb{R}^N} \Big((-\Delta)^{s/2} u \, (-\Delta)^{s/2}\phi + e^\lambda u \phi \Big) = \int_{\mathbb{R}^N} g(u)\phi, \quad \forall \phi \in H_r^s(\mathbb{R}^N), \\ \int_{\mathbb{R}^N} u^2 dx = m. \end{cases}$$

We implement a minimax approach to detect normalized solutions in the nonlocal framework using a Pohozaev type function. More precisely, inspired by the Pohozaev (or Pohozaev-Derrick) identity [318]

$$\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + N \int_{\mathbb{R}^N} \Big(\frac{\mu}{2} u^2 - G(u) \Big) = 0, \tag{2.1.3}$$

for any $s \in (0,1)$ we introduce the Pohozaev function $\mathcal{P} : \mathbb{R} \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{P}(\lambda, u) := \frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + N \int_{\mathbb{R}^N} \left(\frac{e^\lambda}{2} u^2 - G(u) \right)$$

and the Pohozaev set

$$\Omega := \{(\lambda, u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) > 0\} \cup \{(\lambda, 0) \mid \lambda \in \mathbb{R}\}.$$

We note that, for each $\lambda \in \mathbb{R}$, the set $\{u \in H^s_r(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) > 0\} \cup \{0\}$ is a neighborhood of $u = 0$, and thus

$$\partial\Omega = \{(\lambda, u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) = 0, \ u \neq 0\}.$$

Therefore $(\lambda, u) \in \partial\Omega$ if and only if $u \neq 0$ and u satisfies the Pohozaev identity. However we emphasize that under assumptions (g1)–(g3), if $u \in H^s(\mathbb{R}^N)$ solves $\partial_u \mathcal{I}^m(\lambda, \cdot) = 0$ with $\lambda \in \mathbb{R}$ fixed, then $\mathcal{P}(\lambda, u) = 0$ when $s \in (\frac{1}{2}, 1)$. A similar result for $s \in (0, \frac{1}{2}]$ is not available since the weak solutions are not proved to be C^1, in general (see Section 2.2).

In spite of this lack of regularity, which is a special feature of the nonlocal framework, we recognize a Mountain Pass structure [18] for the functional $\mathcal{I}^m$, where the mountain is given by the subset $\partial\Omega$. We refer to it as the *Pohozaev mountain*. This approach can be useful to deal with different problems in other contexts.

Inspired by [224, 231], we need to use a new variant of the Palais-Smale condition which takes into account the Pohozaev identity, and we establish some deformation theorems which enable us to perform our minimax arguments in the *product space* $\mathbb{R} \times H^s_r(\mathbb{R}^N)$.

As a byproduct, our solutions satisfy the Pohozaev identity, even if we assume that f is only a continuous function (see Corollary 2.6.3). We also note that solutions with the Pohozaev identity are essential, in the following sense: our deformation argument shows that only critical points with the Pohozaev identity contribute to the topology; that is, solutions without the Pohozaev identity are deformable with a suitable deformation flow and have no topological relevance.

Firstly we prove the following existence results for (2.1.2).

Theorem 2.1.1. *Suppose $N \geq 2$ and (g1)–(g3). Then there exists $m_0 \geq 0$ such that for any $m > m_0$, the problem (2.1.2) has a solution, satisfying the Pohozaev identity (2.1.3).*

Theorem 2.1.2. *Suppose $N \geq 2$, (g1)–(g3) and*

(g4) $\lim_{t \to 0} \frac{G(t)}{|t|^{p+1}} = +\infty$, *where* $p = 2^m_s = 1 + \frac{4s}{N}$.

Then for any $m > 0$, the problem (2.1.2) has a solution, satisfying the Pohozaev identity (2.1.3).

We highlight that the found solution is actually a minimum for $\mathcal{L}$ constrained to the sphere (see Proposition 4.2.9), which furnishes a strong indication to its stability properties. The techniques employed in [343] for the local case $s = 1$, to get directly the existence of a minimum for $\mathcal{L}$, are not easily adaptable to the fractional framework, because of the need of a control on the tails in the Brezis-Lieb lemma and in the Concentration-Compactness techniques. Anyway, our method not only gets around these difficulties, but moreover it is also suitable to get multiple solutions.

Indeed, if we also suppose the oddness of g, namely

(g5) $g(-t) = -g(t)$ for all $t \in \mathbb{R}$,

we have $\mathcal{I}^m(\lambda, -u) = \mathcal{I}^m(\lambda, u)$ for all $(\lambda, u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N)$ and we can establish the existence of infinitely many L^2-constrained standing waves solutions for the (fNLS) equation.

We prove the following multiplicity result.

Theorem 2.1.3. *Suppose $N \geq 2$ and (g1)–(g3) and (g5). Then we have:*

(i) *For any $k \in \mathbb{N}$ there exists $m_k \geq 0$ such that for each $m > m_k$, the problem* (2.1.2) *has at least k nontrivial, distinct pairs of solutions, satisfying the Pohozaev identity* (2.1.3).

(ii) *In addition assume* (g4). *For any $m > 0$ the problem* (2.1.2) *has countably many solutions $(u_n)_n$ (satisfying the Pohozaev identity* (2.1.3)*), which verify*

$$\mathcal{L}(u_n) < 0 \quad \textit{for all } n \in \mathbb{N},$$

$$\mathcal{L}(u_n) \to 0 \quad \textit{as } n \to +\infty.$$

We remark that our subcritical multiplicity result seems new even in the case of the pure power $g(t) = |t|^{q-2}t$ and in the non-monotone case of competing powers $g(t) = |t|^{q-2}t - |t|^{r-2}t$, and it has a physical relevance since it describes the existence of multiple bound states with arbitrary high energies (see e.g. [165]). We stress that the analytical solutions for fractional differential equations are still limited, while there is a large amount of numerical methods in discretizing the fractional differential operators. In Theorem 2.1.3 we furnish an analytical rigorous approach to detect infinitely many symmetric solitons, which can be applied to the computation of ground and excited states to (fNLS) equations.

Remark 2.1.4. *We highilight that* (g2) *can be weakened, with no changes in the proofs, by asking, for some $q \in (2_s^m - 1, 2_s^* - 1)$*

$$\lim_{|t|\to+\infty} \frac{g(t)}{|t|^q} = 0 \quad \textit{and} \quad \limsup_{|t|\to+\infty} \frac{G(t)}{|t|^{2_s^m}} = \limsup_{|t|\to+\infty} \frac{g(t)}{|t|^{2_s^m-1}} = 0.$$

See also Remark 5.5.8 for some additional discussions.

Remark 2.1.5. *We highlight that we assume* a priori *the positivity of the Lagrange multiplier μ in* (2.1.2). *As a matter of fact, this condition seems to be quite natural: if u is a ground state on the sphere $\int_{\mathbb{R}^N} u^2\, dx = m$ and its energy is negative, then* a posteriori *the corresponding Lagrange multiplier μ is strictly positive (see Proposition 2.8.1). In addition, from a physical perspective, in the study of standing waves the multiplier μ describes the frequency of the particle, and thus it is positive; moreover, this prescribed sign is characteristic also of chemical potentials in the description of ideal gases, see [267, 320].*

The Chapter is organized as follows. In Section 2.2, we establish some preliminaries related to the unconstrained problem. In Section 2.3 we give the Lagrangian formulation of the problem (2.1.2) and a description of the geometry of a functional in a product space. Section 2.4 concerns with the Palais-Smale-Pohozaev ((PSP) for short) condition and Section 2.5 is devoted to the construction of the deformation argument under this (PSP) condition. Section 2.6 deals with our minimax procedure to detect the normalized solutions by means of the Pohozaev mountain. Finally in Section 2.7 we derive the multiplicity result of infinitely many normalized solutions when g is odd.

2.2 The unconstrained problem

In this Section we consider the unconstrained fractional equation

$$(-\Delta)^s u + \mu u = g(u) \quad \text{in } \mathbb{R}^N, \tag{2.2.4}$$

where $s \in (0,1)$, $N \geq 2$, $u \in H^s(\mathbb{R}^N)$, $\mu > 0$ is fixed and g satisfies (g1) together with the following assumptions

(g2') $\limsup_{|t|\to\infty} \frac{g(t)}{|t|^q} = 0$ where $q \in (1, 2_s^* - 1)$, where $2_s^* = \frac{2N}{N-2s}$;

(g3') there exists $t_0 > 0$ such that $G(t_0) > \frac{\mu}{2}t_0^2$, where $G(t) = \int_0^t g(\tau)d\tau$.

Under the assumptions (g1)-(g2'), it is standard to show that any weak solution of (2.2.4) is a critical point of the C^1-functional $\mathcal{J}_\mu : H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{J}_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx + \frac{\mu}{2}\int_{\mathbb{R}^N} u^2\,dx - \int_{\mathbb{R}^N} G(u)\,dx.$$

In the celebrated paper [50], for the local case $s = 1$, Berestycki and Lions proved the existence of a classical solution to (2.2.4), which is radially symmetric and has an exponentially decay, under the assumption (g1)-(g2')-(g3'); these conditions are almost optimal for the existence of (2.2.4). The found solution is of least energy among all nontrivial solutions, and a Mountain Pass (MP for short) solution as shown by Jeanjean and Tanaka [237]. Successively Byeon, Jeanjean, Maris [80] showed that every least energy solution of (2.2.4) has constant sign and is radially symmetric (and decreasing) up to translations.

For the nonlocal case $s \in (0, 1)$, we begin to recall that in the recent paper [79], Byeon, Kwon and Seok established the following results (see also [95]).

Proposition 2.2.1 (Regularity)**.** *Suppose* (g1)-(g2')*. Let $u \in H^s(\mathbb{R}^N)$ be a weak solution of the fractional equation* (2.2.4)*. Then $u \in C^1(\mathbb{R}^N)$ if one of the following assumptions holds:*

(i) $s \in (1/2, 1)$*;*

(ii) $s \in (0, 1/2]$ *and* $g \in C^{0,\sigma}_{loc}(\mathbb{R})$ *for some* $\sigma \in (1 - 2s, 1)$*.*

Proposition 2.2.2 (Fractional Pohozaev identity)**.** *Suppose* (g1)-(g2') *and*

(g4') *if* $s \in (0, 1/2]$, $g \in C^{0,\sigma}_{loc}(\mathbb{R})$ *for some* $\sigma \in (1 - 2s, 1)$.

Then every weak solution $u \in H^s(\mathbb{R}^N)$ of the fractional equation of (2.2.4) *satisfies the Pohozaev identity* (2.1.3)*, which can be rewritten as*

$$\frac{1}{2_s^*}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + \frac{\mu}{2^\#}\int_{\mathbb{R}^N} u^2 - \int_{\mathbb{R}^N} G(u) = 0,$$

where $2_s^* = \frac{2N}{N-2s}$ and $2^\# = 2$ are the upper and lower critical exponents.

Roughly, we see that the Pohozaev identity essentially means $\frac{d}{d\theta}\mathcal{J}_\mu(u(\cdot/e^\theta))_{|\theta=0} = 0$, thus it is strictly related to the scaling invariance of the problem (which will be exploited through an augmented functional, see (2.4.28)). Anyway the fact that u is a critical point for $\mathcal{J}'_\mu$ does not imply directly this relation, since $\frac{d}{d\theta}\mathcal{J}_\mu(u(\cdot/e^\theta))_{|\theta=0} = (\mathcal{J}'_\mu(u), \nabla u \cdot x) = 0$ requires some restriction on $\mathcal{J}_\mu$ and u.

Indeed, the C^1-regularity of the weak solution seems crucial for proving formally a Pohozaev type identity. Under (g1)-(g2')-(g3') we know [79] that each weak solution of (2.2.4) belongs to $H^s(\mathbb{R}^N) \cap C^\beta(\mathbb{R}^N)$ with $\beta \in (0, 2s)$ and thus it is not known if the Pohozaev identity holds when $s \in (0, 1/2]$, without additional regularity assumptions on the nonlinearity g.

In [79], the authors further investigated the existence of MP weak solutions of (2.2.4). We recall that a weak solution u is said of MP type if

$$\mathcal{J}_\mu(u) = a(\mu), \tag{2.2.5}$$

where

$$a(\mu) := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}_\mu(\gamma(t))$$

and

$$\Gamma_\mu := \{\gamma(t) \in C\big([0, 1], H^s_r(\mathbb{R}^N)\big) \mid \gamma(0) = 0,\ \mathcal{J}_\mu(\gamma(1)) < 0\}. \tag{2.2.6}$$

As for $s = 1$, the functional $\mathcal{J}_\mu$ does not satisfies the Palais-Smale condition at level $a(\mu)$ under the assumptions (g1)-(g2')-(g3'), thus one can not directly apply the MP theorem. For the local case $s = 1$, any weak solution is C^1 and it satisfies the Pohozaev identity, so that one can reduce the search of MP solutions to that of minimizers on the Pohozaev type constraint. For the fractional case, this approach seems to work for $s \in (1/2, 1)$, while requires additional regularity on the nonlinearity if $s \in (0, 1/2]$.

Conversely, in [79] the authors established that every minimizer of $\mathcal{J}_\mu$ on the Pohozaev constraint corresponds to a MP weak solution and derived some radially symmetric properties of the minimizer using a fractional version of the Polya-Szego inequality. Namely they introduce the Pohozaev functional $\mathcal{P} : H^s_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{P}_\mu(u) := \frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + N \int_{\mathbb{R}^N} \left(\frac{\mu}{2} u^2 - G(u) \right)$$

and

$$P_\mu := \{ u \in H^s_r(\mathbb{R}^N) \setminus \{0\} \mid \mathcal{P}_\mu(u) = 0 \},$$

$$p(\mu) := \min_{u \in P_\mu} \mathcal{J}_\mu(u).$$

In [79, Theorem 1.2] they established the following result.

Theorem 2.2.3. *Assume (g1)-(g2')-(g3'). Let $s \in (0, 1)$ and $\mu > 0$. Then*

(i) *there exists a minimizer of $\mathcal{J}_\mu$ subject to P_μ;*

(ii) *every minimizer of $\mathcal{J}_\mu$ subject to P_μ is a MP weak solution of (2.2.4);*

From Theorem 2.2.3 it follows that

$$a(\mu) = p(\mu).$$

While the equivalence between Mountain Pass solutions and least energy solutions is shown for $s \in (1/2, 1)$, it is yet an open problem for $s \in (0, 1/2]$ under the assumptions (g1)-(g2')-(g3'). In [79], this equivalence is established under the same regularity assumption of Proposition 2.1, namely $g \in C^{0,\sigma}(\mathbb{R}^N)$ for some $\sigma \in (1 - 2s, 1)$; see Section 4.3 for more comments on this relation. In the following Sections, in contrast, we will show that, under L^2-constraint, least energy solutions have Mountain Pass characterization. See Proposition 2.8.1.

Remark 2.2.4. *We highlight that in [79] they define $a(\mu)$ and $p(\mu)$ on the whole space $H^s(\mathbb{R}^N)$, by additionally assuming*

$$g(t) \equiv 0 \quad \textit{for } t \leq 0;$$

indeed, thanks to this assumptions, they can pass from a generic minimization sequence to a positive one, and thus to a radially symmetric one (and exploit then compactness). With this additional assumption they also show that

(iii) *every minimizer of $\mathcal{J}_\mu$ subject to P_μ is positive and radially symmetric up to a translation.*

Without this assumption we notice that their arguments show that every positive *minimizer of $\mathcal{J}_\mu$ subject to P_μ is radially symmetric up to a translation.*

On the other hand, without this additional assumption but by assuming (g4'), one may argue as follows: by a result similar to Proposition 5.5.3 (see [50, Theorem 3] and [80, Lemma 1] for details) the ground state problem can be seen as a minimization problem; thus we can apply [275, Theorem 4.1] to deduce the radial symmetry (up to a translation) of any minimizer. See also [255].

Remark 2.2.5. *The existence of a Pohozaev minimum (Mountain Pass solution) when*

$$\limsup_{|t|\to\infty} \frac{g(t)}{|t|^{2^*_s-1}} = 0$$

substitutes (g2') *can be found in [95], where it is assumed that $g \in C^1(\mathbb{R})$ (or, more specifically, it is sufficient that Pohozaev holds for every solution). A result involving this assumption together with $g \in C(\mathbb{R})$ seems to lack in literature, even though the proof by [79] can be easily adapted. Anyway, we can obtain this result as a byproduct of our argument, similarly to Section 3.7. See instead [230] for the existence of infinitely many solutions.*

Some further properties of this autonomous equation will be invetigated in Section 5.2 and in Section 5.5.1.

2.3 Lagrangian formulation and Pohozaev geometry

We come back to the constrained case; from now on in this Chapter we briefly denote

$$p := 2^m_s - 1 = 1 + \frac{4s}{N}.$$

We consider the Lagrangian formulation of the problem (2.1.2) in the space of radially symmetric functions $H^s_r(\mathbb{R}^N)$. Namely, we seek for critical points of the functional $\mathcal{I}^m : \mathbb{R} \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{I}^m(\lambda, u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 - \int_{\mathbb{R}^N} G(u) + \frac{e^\lambda}{2}(\|u\|_2^2 - m). \tag{2.3.7}$$

Under the assumption (g1)–(g3), it is standard to prove that $\mathcal{I}^m$ is C^1 in the product space $\mathbb{R} \times H^s_r(\mathbb{R}^N)$. It is immediate to recognize that for any $m > 0$

$$\mathcal{I}^m(\lambda, u) = \mathcal{J}(\lambda, u) - \frac{e^\lambda}{2} m$$

where $\mathcal{J} : \mathbb{R} \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$ is the C^1-functional defined by $\mathcal{J}(\lambda, u) := \mathcal{J}_{e^\lambda}(u)$, i.e.

$$\mathcal{J}(\lambda, u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 - \int_{\mathbb{R}^N} G(u) + \frac{e^\lambda}{2}\int_{\mathbb{R}^N} u^2.$$

For a fixed $\lambda \in \mathbb{R}$, u is critical point of $\mathcal{J}(\lambda, \cdot)$ means that $u \in H^s_r(\mathbb{R}^N)$ solves, in the weak sense,

$$(-\Delta)^s u + e^\lambda u = g(u) \qquad \text{in } \mathbb{R}^N. \tag{2.3.8}$$

Inspired by the Pohozaev identity (2.1.3), for any $s \in (0, 1)$ we also introduce the Pohozaev functional $\mathcal{P} : \mathbb{R} \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{P}(\lambda, u) := \frac{N - 2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + N\int_{\mathbb{R}^N} \left(\frac{e^\lambda}{2}u^2 - G(u)\right).$$

By Proposition 2.2.2, it follows that for any $\lambda \in \mathbb{R}$, if $u \in H^s_r(\mathbb{R}^N)$ solves (2.3.8), then $\mathcal{P}(\lambda, u) = 0$ when $s \in (\frac{1}{2}, 1)$. A similar result for $s \in (0, \frac{1}{2}]$ is not known under (g1)–(g3).

We introduce now the Pohozaev set

$$\Omega := \{(\lambda, u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) > 0\} \cup \{(\lambda, 0) \mid \lambda \in \mathbb{R}\}.$$

Since $\int_{\mathbb{R}^N} G(u) = o(\|u\|^2_{H^s})$ as $u \to 0$ we have the following.

Lemma 2.3.1. *We have*

$$\{(\lambda, 0) \mid \lambda \in \mathbb{R}\} \subset int(\Omega). \tag{2.3.9}$$

Proof. For any fixed $\delta > 0$ there exists a suitable $C_\delta > 0$ such that

$$G(t) \le \delta |t|^2 + C_\delta |t|^{p+1},$$

where $p + 1 < 2^*_s$. Thus

$$\begin{aligned} 0 = \frac{1}{N}\mathcal{P}(\lambda, u) &\ge \frac{1}{2^*_s}\|(-\Delta)^{s/2}u\|_2^2 + \left(\frac{e^\lambda}{2} - \delta\right)\|u\|_2^2 - C_\delta \|u\|_{p+1}^{p+1} \\ &\gtrsim \|u\|_{H^s}^2 - \|u\|_{H^s}^{p+1} > 0 \end{aligned}$$

for δ small and $\|u\|_{H^s}$ small, $u \neq 0$. ∎

This last result implies

$$\partial\Omega = \{(\lambda, u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) = 0, \ u \neq 0\};$$

we call this set the *Pohozaev mountain*. We remark that $(\lambda, u) \in \partial\Omega$ if and only if $u \neq 0$ and u satisfies the Pohozaev identity $\mathcal{P}(\lambda, u) = 0$.

Contrary to assumption (g3'), the arbitrariness of the frequency μ and the corresponding assumption (g3) lead to different interactions between the pieces μu and $g(u)$, which have to be taken into account; these interactions are described by the quantity

$$\mu_0 := 2 \sup_{t \in \mathbb{R}, t \neq 0} \frac{G(t)}{t^2}; \tag{2.3.10}$$

we deduce $\mu_0 \in (0, +\infty]$ under the assumptions (g1)–(g3). We also denote

$$\lambda_0 := \log(\mu_0), \quad \text{if } \mu_0 \in (0, \infty), \tag{2.3.11}$$

otherwise $\lambda_0 := +\infty$. Analysing the two cases ($\lambda_0 \in \mathbb{R}$ and $\lambda_0 = +\infty$) will be of key importance in the study of the geometry of the problem.

Taking into account that $2^m_s < 2^*_s$, we deduce by (i) Theorem 2.2.3 that for any $\lambda \in (-\infty, \lambda_0)$ the functional

$$u \in H^s_r(\mathbb{R}^N) \mapsto \mathcal{J}(\lambda, u) \in \mathbb{R}$$

has a minimizer u_λ subject to

$$(\partial\Omega)_\lambda := \{u \in H^s_r(\mathbb{R}^N) \setminus \{0\} \mid \mathcal{P}(\lambda, u) = 0\},$$

namely

$$\mathcal{J}(\lambda, u_\lambda) = \min_{u \in (\partial\Omega)_\lambda} \mathcal{J}(\lambda, u). \tag{2.3.12}$$

Furthermore by (ii) of Theorem 2.2.3 such u_λ is a Mountain Pass critical point of $\mathcal{J}(\lambda, \cdot)$ at level $a(\lambda)$, i.e.

$$\mathcal{J}(\lambda, u_\lambda) = a(\lambda)$$

where

$$a(\lambda) := \inf_{\gamma \in \Gamma(\lambda)} \max_{t \in [0,1]} \mathcal{J}(\lambda, \gamma(t)) \tag{2.3.13}$$

and

$$\Gamma(\lambda) := \{\gamma \in C([0,1], H^s_r(\mathbb{R}^N)) \mid \gamma(0) = 0, \ \mathcal{J}(\lambda, \gamma(1)) < 0\}. \tag{2.3.14}$$

We notice that $\lambda \in (-\infty, \lambda_0) \mapsto a(\lambda) \in \mathbb{R}$ is strictly monotone increasing: this can be shown, for example, by relying on the fact that $a(\lambda)$ coincides with the Pohozaev minimum and exploiting some scaling argument.[1]

Lemma 2.3.2. *Let $\lambda \in \mathbb{R}$. Then the following statements are equivalent:*

(a) $\lambda < \lambda_0$.

(b) *There exists a $t_0 = t_0(\lambda) > 0$ such that*

$$G(t_0) > \frac{e^\lambda}{2} t_0^2.$$

(c) *There exists $u \in H^s_r(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{P}(\lambda, u) = 0$; in particular $(\partial\Omega)_\lambda \neq \emptyset$.*

(d) *$\Gamma(\lambda) \neq \emptyset$, and thus $a(\lambda)$ is well defined.*

As further consequence, we see that $\partial\Omega \neq \emptyset$. Finally, $a(\lambda) > 0$.

Proof. (a) $\iff$ (b). This is a straightforward consequence of the definition of λ_0.

(b) $\implies$ (c) Let $u \in H^s_r(\mathbb{R}^N)$ to be fixed. We have, for $t > 0$,

$$\mathcal{P}(\lambda, u(\cdot/t)) = \frac{N-2s}{2} t^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - N t^N \int_{\mathbb{R}^N} \left(G(u) - \frac{e^\lambda}{2} u^2 \right).$$

We notice that $\mathcal{P}(\lambda, u(\cdot/t)) > 0$ for small $t > 0$. In order to get a $\bar{t}$ such that $\mathcal{P}(\lambda, u(\cdot/\bar{t})) = 0$ we need the quantity

$$\int_{\mathbb{R}^N} \left(G(u) - \frac{e^\lambda}{2} u^2 \right)$$

to be positive. For any $R > 0$ we choose a smooth $u = u_R \in C^\infty_c$ such that $u_R = t_0$ in $B_R(0)$ and $u_R = 0$ out of $B_{R+\frac{1}{R^N}}(0)$, $0 \leq u_R \leq t_0$. We set

$$C := \sup_{t \in [0,t_0]} \left| G(t) - \tfrac{e^\lambda}{2} |t|^2 \right| < +\infty.$$

Then

$$\begin{aligned} \int_{\mathbb{R}^N} \left(G(u_R) - \frac{e^\lambda}{2} u_R^2 \right) &= \int_{B_{R+\frac{1}{R^N}} \setminus B_R} \left(G(u_R) - \frac{e^\lambda}{2} u_R^2 \right) + \int_{B_R} \left(G(u_R) - \frac{e^\lambda}{2} u_R^2 \right) \\ &\geq -C |B_{R+\frac{1}{R^N}} \setminus B_R| + |B_R| \left(G(t_0) - \frac{e^\lambda}{2} |t_0|^2 \right) \to +\infty \end{aligned}$$

and in particular it is positive for a sufficiently large R.

[1] Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and v be a λ_2-Pohozaev minimum (i.e. $\mathcal{J}(\lambda_2, v) = a(\lambda_2)$ and $\mathcal{P}(\lambda_2, v) = 0$). Let rescale v in such a way it belongs to the λ_1-Pohozaev set, i.e. $u := v(\cdot/\theta)$ with $\mathcal{P}(\lambda_1, u) = 0$, for some explicit $\theta = \left(1 + \frac{2^*_s}{2}(\lambda_2 - \lambda_1) \frac{\|v\|_2^2}{\|(-\Delta)^{s/2} v\|_2^2}\right)^{-\frac{1}{2s}}$. Thus, by the Pohozaev identities,

$$a(\lambda_1) \leq \mathcal{J}(\lambda_1, u) = \frac{s}{N} \|(-\Delta)^{s/2} u\|_2^2 = \theta^{N-2s} \frac{s}{N} \|(-\Delta)^{s/2} v\|_2^2 = \theta^{N-2s} \mathcal{J}(\lambda_2, v) = \theta^{N-2s} a(\lambda_2).$$

If $\lambda_1 < \lambda_2$ then $\theta < 1$ and thus the have claim $a(\lambda_1) < a(\lambda_2)$.

As a further result, since $\theta \to 1$ as λ_1 and λ_2 approach, we obtain also $a(\lambda_1) \leq \liminf_{\lambda_2 \to \lambda_1} a(\lambda_2)$ and $\limsup_{\lambda_1 \to \lambda_2} a(\lambda_1) \leq a(\lambda_2)$. Swapping the role of λ_1 and λ_2 actually we obtain the (extra) continuity property: $\lim_{\lambda \to \lambda_0} a(\lambda) = a(\lambda_0)$.

(c) $\implies$ (d). Let $u \in H^s_r(\mathbb{R}^N)$, $u \not\equiv 0$ such that $\mathcal{P}(\lambda, u) = 0$. We define $\gamma(t) := u(\cdot/t)$ for $t \neq 0$ and $\gamma(0) = 0$, so that $\gamma : [0, \infty) \to H^s_r(\mathbb{R}^N)$ is continuous. We have

$$\mathcal{J}(\lambda, \gamma(t)) = \frac{1}{2} t^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - t^N \int_{\mathbb{R}^N} \left(G(u) - \frac{e^\lambda}{2} u^2 \right).$$

Noting $\int_{\mathbb{R}^N} \left(G(u) - \frac{e^\lambda}{2} u^2 \right) > 0$ by $\mathcal{P}(\lambda, u) = 0$, we have $\mathcal{J}(\lambda, \gamma(t)) \to -\infty$ as $t \to \infty$ and thus $\Gamma(\lambda) \neq \emptyset$.

(d) $\implies$ (b). If $\gamma \in \Gamma(\lambda)$, then $\mathcal{J}(\lambda, \gamma(1)) < 0$, thus

$$\int_{\mathbb{R}^N} \left(G(\gamma(1)) - \frac{e^\lambda}{2} \gamma(1)^2 \right) > 0,$$

which implies that there exists an $x_0 \in \mathbb{R}^N$ such that

$$G(\gamma(1)(x_0)) - \frac{e^\lambda}{2} \gamma(1)^2(x_0) > 0.$$

The claim comes by setting $t_0 := \gamma(1)(x_0)$.

Finally, by Theorem 2.2.3, there exists a Pohozaev minimum u_λ which is also a Mountain Pass solution, thus $\mathcal{J}(\lambda, u_\lambda) = a(\lambda)$, $D_u \mathcal{J}(\lambda, u_\lambda) = 0$ and $\mathcal{P}(\lambda, u_\lambda) = 0$, which imply

$$a(\lambda) = \frac{s}{N} \|(-\Delta)^{s/2} u_\lambda\|_2^2 > 0. \qquad \blacksquare$$

Remark 2.3.3. *Assume $\lambda_0 < +\infty$. We observe that, in this case, for $\lambda \geq \lambda_0$ we have $\mathcal{P}(\lambda, u) \geq 0$ and $\mathcal{J}(\lambda, u) \geq 0$ for each u, both strictly positive for $u \not\equiv 0$. This means that $[\lambda_0, +\infty) \times H^s_r(\mathbb{R}^N) \subset \Omega$.*

In the next result, we consider the case $\lambda_0 \in \mathbb{R}$ and we investigate the behaviour of $a(\lambda)$ as λ approach λ_0.

Proposition 2.3.4. *Assume (g1)–(g3) and $\lambda_0 \in \mathbb{R}$. We have*

(a) *if $(\lambda, u) \in \partial\Omega$ for some $u \in H^s_r(\mathbb{R}^N)$, then $\lambda < \lambda_0$.*

(b) $\lim_{\lambda \to \lambda_0^-} a(\lambda) = +\infty$.

Proof. Let $(\lambda, u) \in \partial\Omega$, namely $\mathcal{P}(\lambda, u) = 0$ and $u \neq 0$. This implies that for some $x_0 \in \mathbb{R}^N$

$$G(u(x_0)) - \frac{e^\lambda}{2} u(x_0)^2 > 0$$

and thus $\lambda < \lambda_0$ and (a) holds.

Now we show point (b). Let $\lambda < \lambda_0$; by contradiction, since by Lemma 2.3.2 $a(\lambda)$ is increasing and strictly positive, we assume that $a(\lambda) \to c \in (0, +\infty)$ as $\lambda \to \lambda_0^-$, from which we deduce that $\|(-\Delta)^{s/2} u_\lambda\|_2$ is bounded. Moreover, for any fixed $\delta > 0$ there exists a suitable $C_\delta > 0$ such that

$$G(t) \leq \delta |t|^2 + C_\delta |t|^{p+1},$$

where we recall that $p = 1 + \frac{4s}{N}$.

Thus we have by the fractional Gagliardo-Nirenberg inequality (1.2.8) and the fact that $\|(-\Delta)^{s/2} u_\lambda\|_2$ is bounded,

$$0 = \frac{1}{N} \mathcal{P}(\lambda, u_\lambda) \geq \frac{1}{2_s^*} \|(-\Delta)^{s/2} u_\lambda\|_2^2 + \left(\frac{e^\lambda}{2} - \delta \right) \|u_\lambda\|_2^2 - C_\delta \|u_\lambda\|_{p+1}^{p+1}$$

$$\geq \frac{1}{2_s^*}\|(-\Delta)^{s/2}u_\lambda\|_2^2 + \left(\frac{e^\lambda}{2} - \delta\right)\|u_\lambda\|_2^2 - C'C_\delta\|(-\Delta)^{s/2}u_\lambda\|_2^2\|u_\lambda\|_2^{p-1}$$
$$\geq \left(\frac{e^\lambda}{2} - \delta\right)\|u_\lambda\|_2^2 - C''C_\delta\|u_\lambda\|_2^{\frac{4s}{N}}$$

for some C', $C'' > 0$. By choosing $\delta < \frac{e^\lambda}{2}$, since $\frac{4s}{N} < 2$, also $\|u_\lambda\|_2$ must be bounded, which means that $(u_\lambda)_{\lambda<\lambda_0}$ is bounded in $H^s_r(\mathbb{R}^N)$. Hence, up to a subsequence, $u_\lambda \rightharpoonup u_0$ in $H^s_r(\mathbb{R}^N)$. By the immersion (1.2.14) and taking into account that $\partial_u \mathcal{J}(\lambda, u_\lambda) = 0$, we deduce that $u_\lambda \to u_0$ strongly in $H^s_r(\mathbb{R}^N)$ with $\mathcal{J}(\lambda_0, u_0) = c$, $\partial_u\mathcal{J}(\lambda_0, u_0) = 0$, $\mathcal{P}(\lambda_0, u_0) = 0$. Since $c > 0$, we have $u_0 \neq 0$. By $\mathcal{P}(\lambda_0, u_0) = 0$, we conclude

$$G(u_0(x)) - \frac{e^{\lambda_0}}{2}u_0(x)^2 > 0$$

for some $x \in \mathbb{R}^N$, which contradicts the definition of λ_0. ∎

We consider now the case $\lambda_0 = +\infty$ and we investigate the behaviour of $a(\lambda)$ for λ large.

Proposition 2.3.5. *Assume that $\lambda_0 = +\infty$. Then*

$$\lim_{\lambda\to+\infty} \frac{a(\lambda)}{e^\lambda} = +\infty.$$

Proof. By (g1)-(g2) we have that for any $\delta > 0$ there exists $C_\delta > 0$ such that for all $t \in \mathbb{R}$

$$G(t) \leq \frac{\delta}{p+1}|t|^{p+1} + \frac{C_\delta}{2}|t|^2. \tag{2.3.15}$$

We also denote by $b(\delta)$ the MP value of $\mathcal{H}_\delta : H^s_r(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{H}_\delta(v) := \frac{1}{2}\|(-\Delta)^{s/2}v\|_2^2 + \frac{1}{2}\|v\|_2^2 - \frac{\delta}{p+1}\|v\|_{p+1}^{p+1}.$$

It is easy to see that[2]

$$b(\delta) \to +\infty \quad \text{as } \delta \to 0^+.$$

For $v \in H^s_r(\mathbb{R}^N) \setminus \{0\}$, we set

$$u_\theta := \theta^{N/2}v(\theta\cdot),$$

and for simplicity we write $\mu \equiv e^\lambda$ and $\mathcal{J}(\mu, \cdot) = \mathcal{J}(\lambda, \cdot)$. By (2.3.15), we pass to evaluate

$$\mathcal{J}(\mu, u_\theta) \geq \theta^{2s}\left(\frac{1}{2}\|(-\Delta)^{s/2}v\|_2^2 + \frac{1}{2}(\mu - C_\delta)\theta^{-2s}\|v\|_2^2 - \frac{\delta}{p+1}\|v\|_{p+1}^{p+1}\right).$$

Setting $\theta := (\mu - C_\delta)^{1/2s}$ for $\mu = \mu_\delta > C_\delta$, we have

$$\mathcal{J}(\mu, u_{(\mu-C_\delta)^{1/2s}}) \geq (\mu - C_\delta)\mathcal{H}_\delta(v) \tag{2.3.16}$$

and hence

$$\frac{\mathcal{J}(\mu, u_{(\mu-C_\delta)^{1/2s}})}{\mu} \geq \frac{\mu - C_\delta}{\mu}\mathcal{H}_\delta(v). \tag{2.3.17}$$

Thus we have

$$\frac{a(\mu)}{\mu} \geq \frac{\mu - C_\delta}{\mu}b(\delta). \tag{2.3.18}$$

[2]Indeed, by scaling, $\mathcal{H}_\delta(\delta^{-\frac{1}{p-1}}\cdot) = \delta^{-\frac{2}{p-1}}\mathcal{H}_1$, which implies $\Gamma_\delta = \delta^{-\frac{1}{p-1}}\Gamma_1$; here Γ_δ is the set of paths related to $\mathcal{H}_\delta$. Using these two relations one obtains $b(\delta) = \delta^{-\frac{2}{p-1}}b(1) \to +\infty$.

Choosing $\mu = \mu_\delta > 2C_\delta$ we obtain

$$\frac{a(\mu)}{\mu} \geq \frac{b(\delta)}{2};$$

since $\delta > 0$ is arbitrary, we derive

$$\lim_{\mu\to+\infty} \frac{a(\mu)}{\mu} = +\infty. \qquad \blacksquare$$

Finally we investigate the behaviour of $a(\lambda)$ for $\lambda \to -\infty$, under some more restrictive assumption in the origin.

Proposition 2.3.6. *Assume (g4) in addition to (g1)–(g3). Then*

$$\lim_{\lambda\to-\infty} \frac{a(\lambda)}{e^\lambda} = 0. \tag{2.3.19}$$

Proof. We fix $u \in H^s_r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|u\|_\infty = 1$. Recalled $p = 1 + \frac{4s}{N}$, by (g4) there exists $M_r > 0$ such that for all $r \in (0, 1]$

$$G(ru(x)) \geq M_r r^{p+1}|u(x)|^{p+1}, \qquad \forall x \in \mathbb{R}^N$$

with

$$M_r \to +\infty \quad \text{as } r \to 0.$$

We write again $\mu \equiv e^\lambda$ for the sake of simplicity. Therefore for $t > 0$ we have

$$\begin{aligned}
\mathcal{J}(\mu, ru(\cdot/t)) &\leq \frac{1}{2}r^2 t^{N-2s}\|(-\Delta)^{s/2}u\|_2^2 + \frac{\mu}{2}r^2 t^N\|u\|_2^2 - M_r r^{p+1}t^N\|u\|_{p+1}^{p+1} \\
&= r^2\mu^{-\frac{N-2s}{2s}}\left(\frac{1}{2}t^{N-2s}\mu^{\frac{N-2s}{2s}}\|(-\Delta)^{s/2}u\|_2^2 + \frac{1}{2}\mu^{\frac{N}{2s}}t^N\|u\|_2^2 - M_r r^{\frac{4s}{N}}\mu^{\frac{N-2s}{2s}}t^N\|u\|_{p+1}^{p+1}\right) \\
&= r^2\mu^{-\frac{N-2s}{2s}}\left(\frac{1}{2}\tau^{N-2s}\|(-\Delta)^{s/2}u\|_2^2 + \frac{1}{2}\tau^N\|u\|_2^2 - M_r r^{\frac{4s}{N}}\mu^{-1}\tau^N\|u\|_{p+1}^{p+1}\right)
\end{aligned}$$

after setting $\tau := \mu^{\frac{1}{2s}}t$. Moreover choosing $r := \mu^{\frac{N}{4s}}$ we infer

$$\begin{aligned}
&\mathcal{J}\left(\mu, \mu^{\frac{N}{4s}}u(\cdot/(\mu^{-1/(2s)}\tau))\right) \\
&\quad\leq \mu\left(\frac{1}{2}\tau^{N-2s}\|(-\Delta)^{s/2}u\|_2^2 + \frac{1}{2}\tau^N\|u\|_2^2 - M_{\mu^{N/(4s)}}\tau^N\|u\|_{p+1}^{p+1}\right).
\end{aligned}$$

For $\mu \in (0, 1)$, the map

$$\tau \in (0, \infty) \mapsto \mu^{\frac{N}{4s}}u(\cdot/\mu^{-1/(2s)}\tau) \in H^s_r(\mathbb{R}^N)$$

can be regarded as a path in $\Gamma(\mu)$ after rescaling. Thus

$$\frac{a(\mu)}{\mu} \leq \max_{\tau\in[0,\infty)}\left(\frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2\tau^{N-2s} + \frac{1}{2}\|u\|_2^2\tau^N - M_{\mu^{N/(4s)}}\|u\|_{p+1}^{p+1}\tau^N\right).$$

Since $M_{\mu^{N/(4s)}} \to \infty$ as $\mu \to 0$, we derive the conclusion. $\blacksquare$

Proposition 2.3.7. *Assume (g1)–(g3). Then we have*

(a) $\mathcal{J}(\lambda, u) \geq 0$ *for all* $(\lambda, u) \in \Omega$;

(b) $\mathcal{J}(\lambda, u) \geq a(\lambda) > 0$ *for all* $(\lambda, u) \in \partial\Omega$.

Proof. We notice that for all $(\lambda, u) \in \Omega$

$$\mathcal{J}(\lambda, u) \geq \mathcal{J}(\lambda, u) - \frac{1}{N}\mathcal{P}(\lambda, u) = \frac{s}{N}\|(-\Delta)^{s/2}u\|_2^2 \geq 0$$

and thus (a) follows.

The proposition (b) follows from the fact that every minimizer of $\mathcal{J}(\lambda, \cdot)$ subject to $(\partial\Omega)_\lambda$ is a mountain pass weak solution of (2.2.4) at level $a(\lambda)$ (see Theorem 2.2.3). ∎

We are ready to show that for any $m > 0$ the functional $\mathcal{I}^m$ is bounded from below on the Pohozaev set $\partial\Omega$.

Proposition 2.3.8. *Assume (g1)–(g3). For any $m > 0$, we set*

$$B_m := \inf_{\lambda<\lambda_0}\left(a(\lambda) - \frac{e^\lambda}{2}m\right)$$

and

$$E_m := \inf_{(\lambda,u)\in\partial\Omega} \mathcal{I}^m(\lambda, u).$$

Then

$$E_m \geq B_m > -\infty. \tag{2.3.20}$$

Proof. Let $m > 0$. If $(\lambda, u) \in \partial\Omega$, by (b) of Proposition 2.3.7 we have

$$\mathcal{I}^m(\lambda, u) = \mathcal{J}(\lambda, u) - \frac{e^\lambda}{2}m \geq a(\lambda) - \frac{e^\lambda}{2}m;$$

since, by (a) of Proposition 2.3.4 it results that $\lambda < \lambda_0$, we have, passing to the infimum,

$$E_m \geq B_m.$$

We distinguish now two cases. Firstly we assume $\lambda_0 \in \mathbb{R}$. From (b) of Proposition 2.3.4 we have $a(\lambda) \to +\infty$ as $\lambda \to \lambda_0^-$, and thus we conclude

$$\inf_{\lambda<\lambda_0}\left(a(\lambda) - \frac{e^\lambda}{2}m\right) > -\infty.$$

Secondly, we suppose that $\lambda_0 = +\infty$. We have

$$a(\lambda) - \frac{e^\lambda}{2}m = e^\lambda\left(\frac{a(\lambda)}{e^\lambda} - \frac{m}{2}\right)$$

and thus, by Proposition 2.3.5

$$\inf_{\lambda\in\mathbb{R}}\left(a(\lambda) - \frac{e^\lambda}{2}m\right) > -\infty.$$ ∎

2.4 Compactness by scaling

Firstly we introduce the notations:

$$K_b := \{(\lambda, u) \in \mathbb{R} \times H_r^s(\mathbb{R}^N) \mid \mathcal{I}^m(\lambda, u) = b,\ \partial_\lambda\mathcal{I}^m(\lambda, u) = 0,\ \partial_u\mathcal{I}^m(\lambda, u) = 0\},$$
$$K_b^{PSP} := \{(\lambda, u) \in K_b \mid \mathcal{P}(\lambda, u) = 0\}.$$

Clearly, we have $K_b^{PSP} \subset K_b$. We note that for the definition of K_b^{PSP} we do not need additional regularity about g.

Under the assumptions (g1)–(g3), it seems difficult to verify the standard Palais-Smale condition for the functional $\mathcal{I}^m$. Therefore we cannot recognize that the set K_b is compact.

Inspired [224, 231], we introduce the Palais-Smale-Pohozaev (shortly (PSP)) condition, which is a weaker compactness condition than the standard Palais-Smale one. Such (PSP) condition takes into account the scaling properties of $\mathcal{I}^m$ through the Pohozaev functional $\mathcal{P}$. Using this new condition we will show that K_b^{PSP} is compact when $b < 0$.

2.4.1 A limiting Pohozaev identity

We give the definition of (PSP) condition in the radial setting.

Definition 2.4.1. *For $b \in \mathbb{R}$, we say that $\mathcal{I}^m$ satisfies the Palais-Smale-Pohozaev condition at level b (shortly the $(PSP)_b$ condition), if for any sequence $(\lambda_n, u_n)_n \subset \mathbb{R} \times H^s_r(\mathbb{R}^N)$ such that*

$$\mathcal{I}^m(\lambda_n, u_n) \to b, \tag{2.4.21}$$

$$\partial_\lambda \mathcal{I}^m(\lambda_n, u_n) \to 0, \tag{2.4.22}$$

$$\|\partial_u \mathcal{I}^m(\lambda_n, u_n)\|_{(H^s_r(\mathbb{R}^N))^*} \to 0, \tag{2.4.23}$$

$$\mathcal{P}(\lambda_n, u_n) \to 0, \tag{2.4.24}$$

it happens that $(\lambda_n, u_n)_n$ has a strongly convergent subsequence in $\mathbb{R} \times H^s_r(\mathbb{R}^N)$.

We will show the following result.

Proposition 2.4.2. *Assume (g1)–(g3). Let $b < 0$. Then $\mathcal{I}^m$ satisfies the $(PSP)_b$ condition on $\mathbb{R} \times H^s_r(\mathbb{R}^N)$.*

Proof. Let $b < 0$ and suppose that $(\lambda_n, u_n) \subset \mathbb{R} \times H^s_r(\mathbb{R}^N)$ satisfies (2.4.21)–(2.4.24). We will show that (λ_n, u_n) has a strongly convergent subsequence in several steps.
Step 1: *λ_n is bounded from below.*
Indeed

$$\begin{aligned}\frac{m}{2} e^{\lambda_n} &= \frac{1}{N}\mathcal{P}(\lambda_n, u_n) - \mathcal{I}^m(\lambda_n, u_n) + \frac{s}{N}\|(-\Delta)^{s/2} u_n\|_2^2 \\ &\geq \frac{1}{N}\mathcal{P}(\lambda_n, u_n) - \mathcal{I}^m(\lambda_n, u_n)\end{aligned}$$

hence

$$\frac{m}{2} \liminf_n e^{\lambda_n} \geq 0 - b > 0,$$

which implies (since $m > 0$) that λ_n is bounded from below.
Step 2: *$\|u_n\|_2^2 \to m$.*
Indeed, we have

$$\partial_\lambda \mathcal{I}^m(\lambda_n, u_n) = \frac{e^{\lambda_n}}{2}\left(\|u_n\|_2^2 - m\right) \to 0,$$

which implies the claim by Step 1.
Step 3: *$\|(-\Delta)^{s/2} u_n\|_2^2$ and λ_n are bounded (from above) as $n \to +\infty$.*
Indeed, by (g1)-(g2) we have that for any $\delta > 0$ there exists $C_\delta > 0$ such that for all $t \in \mathbb{R}$

$$g(t) \leq \delta |t|^p + C_\delta |t|. \tag{2.4.25}$$

By (2.4.25) and the fractional Gagliardo-Nirenberg inequality (1.2.8) we have

$$\begin{aligned}\partial_u \mathcal{I}^m(\lambda_n, u_n) u_n &= \|(-\Delta)^{s/2} u_n\|_2^2 + e^{\lambda_n}\|u_n\|_2^2 - \int_{\mathbb{R}^N} g(u_n) u_n \\ &\geq \|(-\Delta)^{s/2} u_n\|_2^2 + \left(e^{\lambda_n} - C_\delta\right)\|u_n\|_2^2 - \delta \|u_n\|_{p+1}^{p+1} \\ &\geq \|(-\Delta)^{s/2} u_n\|_2^2 + \left(e^{\lambda_n} - C_\delta\right)\|u_n\|_2^2 - \delta C \|(-\Delta)^{s/2} u_n\|_2^2 \|u_n\|_2^{p-1};\end{aligned}$$

moreover

$$\begin{aligned}|\partial_u \mathcal{I}^m(\lambda_n, u_n) u_n| &\leq \|\partial_u \mathcal{I}^m(\lambda_n, u_n)\|_{(H^s_r(\mathbb{R}^N))^*} \|u_n\|_{H^s_r(\mathbb{R}^N)} \\ &= \|\partial_u \mathcal{I}^m(\lambda_n, u_n)\|_{(H^s_r(\mathbb{R}^N))^*} \sqrt{\|(-\Delta)^{s/2} u_n\|_2^2 + \|u_n\|_2^2}.\end{aligned}$$

Set $\varepsilon_n := \|\partial_u \mathcal{I}^m(\lambda_n, u_n)\|_{(H^s_r(\mathbb{R}^N))^*}$ and (by Step 2) $\|u_n\|_2^2 = m + o(1)$, we finally have, joining the previous two inequalities, that

$$\|(-\Delta)^{s/2}u_n\|_2^2 \left(1 - \delta C(m+o(1))^{\frac{p-1}{2}}\right) + \left(e^{\lambda_n} - C_\delta\right)(m + o(1)) \\ \leq \varepsilon_n \sqrt{\|(-\Delta)^{s/2}u_n\|_2^2 + m + o(1)}.$$

Choosing $\delta > 0$ small so that $\delta C m^{\frac{p-1}{2}} < 1$, we obtain the claim.

Step 4: *Conclusion.*

By Steps 1-3, we have that (λ_n, u_n) is bounded in $\mathbb{R} \times H^s_r(\mathbb{R}^N)$. Hence, up to a subsequence, $\lambda_n \to \lambda$ and $u_n \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$. Therefore, we obtain (see Proposition 1.5.5)

$$\int_{\mathbb{R}^N} g(u_n)u_n \to \int_{\mathbb{R}^N} g(u)u \quad \text{and} \quad \int_{\mathbb{R}^N} g(u_n)u \to \int_{\mathbb{R}^N} g(u)u.$$

Again by the assumption $\partial_u \mathcal{I}^m(\lambda_n, u_n) \to 0$ we get

$$\begin{aligned} 0 &= \lim_n \partial_u \mathcal{I}^m(\lambda_n, u_n)u \\ &= \lim_n \left(\int_{\mathbb{R}^N} (-\Delta)^{s/2}u_n(-\Delta)^{s/2}u + e^{\lambda_n}\int_{\mathbb{R}^N} u_n u - \int_{\mathbb{R}^N} g(u_n)u\right) \\ &= \|(-\Delta)^{s/2}u\|_2^2 + e^{\lambda}\|u\|_2^2 - \int_{\mathbb{R}^N} g(u)u. \end{aligned} \tag{2.4.26}$$

Since $\partial_u \mathcal{I}^m(\lambda_n, u_n) \to 0$ and $u_n \rightharpoonup u$, we have $\partial_u \mathcal{I}^m(\lambda_n, u_n)u_n \to 0$; thus

$$\begin{aligned} 0 &= \lim_n \partial_u \mathcal{I}^m(\lambda_n, u_n)u_n \\ &= \lim_n \left(\|(-\Delta)^{s/2}u_n\|_2^2 + e^{\lambda_n}\|u_n\|_2^2 - \int_{\mathbb{R}^N} g(u_n)u_n\right) \\ &= \lim_n \left(\|(-\Delta)^{s/2}u_n\|_2^2 + e^{\lambda_n}\|u_n\|_2^2\right) - \int_{\mathbb{R}^N} g(u)u \end{aligned} \tag{2.4.27}$$

and hence, joining (2.4.26) and (2.4.27),

$$\|(-\Delta)^{s/2}u_n\|_2^2 + e^{\lambda_n}\|u_n\|_2^2 \to \|(-\Delta)^{s/2}u\|_2^2 + e^{\lambda}\|u\|_2^2,$$

which easily implies (since $e^{\lambda_n} \to e^{\lambda}$ and $\|u_n\|_2^2$ is bounded)

$$\|u_n\|_\lambda^2 \to \|u\|_\lambda^2,$$

where $\|\cdot\|_\lambda^2 := \|(-\Delta)^{s/2}\cdot\|_2 + e^{\lambda}\|\cdot\|_2^2$ is an equivalent norm on $H^s_r(\mathbb{R}^N)$. This, together with $u_n \rightharpoonup u$ in $H^s_r(\mathbb{R}^N)$ gives $u_n \to u$ strongly in $H^s_r(\mathbb{R}^N)$. ∎

Corollary 2.4.3. *Assume (g1)–(g3). Let $b \in \mathbb{R}$, $b < 0$. Then $K_b^{PSP} \cap (\mathbb{R} \times \{0\}) = \emptyset$ and K_b^{PSP} is compact.*

Proof. Since $\partial_\lambda \mathcal{I}^m(\lambda, 0) = -\frac{e^\lambda}{2m} \neq 0$, we have $K_b^{PSP} \cap (\mathbb{R} \times \{0\}) = \emptyset$. Proposition 2.4.2 implies that K_b^{PSP} is compact. ∎

Remark 2.4.4. *We emphasize that the $(PSP)_b$ condition does not hold at level $b = 0$. Indeed we can consider the unbounded sequence $(\lambda_j, 0)$ with $\lambda_j \to -\infty$ such that*

$$\mathcal{I}^m(\lambda_j, 0) = \partial_\lambda \mathcal{I}^m(\lambda_j, 0) = -\frac{e^{\lambda_j}}{2}m \to 0$$

and

$$\partial_u \mathcal{I}^m(\lambda_j, 0) = 0, \quad \mathcal{P}(\lambda_j, 0) = 0.$$

2.4.2 A functional in an augmented space

Following [223, 224, 235] we introduce the augmented functional $\mathcal{H}^m : \mathbb{R} \times \mathbb{R} \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{H}^m(\theta, \lambda, u) := \mathcal{I}^m(\lambda, u(e^{-\theta}\cdot)) \tag{2.4.28}$$

for $(\theta, \lambda, u) \in \mathbb{R} \times \mathbb{R} \times H^s_r(\mathbb{R}^N)$. By the scaling properties of $\mathcal{I}^m$ we can recognize that

$$\mathcal{H}^m(\theta, \lambda, u) = \frac{e^{(N-2s)\theta}}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - e^{N\theta} \int_{\mathbb{R}^N} G(u) + \frac{e^\lambda}{2}(e^{N\theta}\|u\|_2^2 - m) \tag{2.4.29}$$

for all $(\theta, \lambda, u) \in \mathbb{R} \times \mathbb{R} \times H^s_r(\mathbb{R}^N)$.

Moreover, by standard calculations we have the following proposition.

Proposition 2.4.5. *For all $(\theta, \lambda, u) \in \mathbb{R} \times \mathbb{R} \times H^s_r(\mathbb{R}^N)$, $h \in H^s_r(\mathbb{R}^N)$, $\beta \in \mathbb{R}$, we have*

(i) $\partial_\theta \mathcal{H}^m(\theta, \lambda, u) = \mathcal{P}(\lambda, u(\cdot/e^\theta))$,

(ii) $\partial_\lambda \mathcal{H}^m(\theta, \lambda, u) = \partial_\lambda \mathcal{I}^m(\lambda, u(\cdot/e^\theta))$,

(iii) $\partial_u \mathcal{H}^m(\theta, \lambda, u)h = \partial_u \mathcal{I}^m(\lambda, u(\cdot/e^\theta))h(\cdot/e^\theta)$,

(iv) $\mathcal{H}^m(\theta + \beta, \lambda, u(e^\beta\cdot)) = \mathcal{H}^m(\theta, \lambda, u)$.

Now we define a metric on the Hilbert manifold

$$M := \mathbb{R} \times \mathbb{R} \times H^s_r(\mathbb{R}^N)$$

by setting

$$\begin{aligned}\|(\alpha, \nu, h)\|^2_{(\theta,\lambda,u)} &:= \left|\Big(\alpha, \nu, \|h(e^{-\theta}\cdot)\|_{H^s_r(\mathbb{R}^N)}\Big)\right|^2 \\ &= \alpha^2 + \nu^2 + e^{N\theta}\|h\|_2^2 + e^{(N-2s)\theta}\|(-\Delta)^{s/2}h\|_2^2\end{aligned}$$

for any $(\alpha, \nu, h) \in T_{(\theta,\lambda,u)}M \equiv \mathbb{R} \times \mathbb{R} \times H^s_r(\mathbb{R}^N)$. We also denote the dual norm on $T^*_{(\theta,\lambda,u)}M$ by $\|\cdot\|_{(\theta,\lambda,u),*}$. We notice that $\|(\cdot,\cdot,\cdot)\|^2_{(\theta,\lambda,u)}$ depends only on θ and we can write $\|(\cdot,\cdot,\cdot)\|^2_{(\theta,\cdot,\cdot)}$. Moreover for any $(\alpha, \nu, h) \in T_{(\theta,\lambda,u)}M$ and $\beta \in \mathbb{R}$ we have

$$\|(\alpha, \nu, h(e^\beta x))\|^2_{(\theta+\beta,\cdot,\cdot)} = \|(\alpha, \nu, h)\|^2_{(\theta,\cdot,\cdot)}. \tag{2.4.30}$$

Furthermore we define the standard distance between two points as the infimum of length of curves connecting the two points, namely

$$\mathrm{dist}_M\big((\theta_0, \lambda_0, h_0), (\theta_1, \lambda_1, h_1)\big) := \inf_{\gamma \in \mathcal{G}} \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)} dt$$

where

$$\mathcal{G} := \left\{\gamma \in C^1([0,1], M) \,\Big|\, \gamma(0) = (\theta_0, \lambda_0, h_0), \gamma(1) = (\theta_1, \lambda_1, h_1)\right\}.$$

Observe that, if σ is a path connecting (α_0, ν_0, h_0) and (α_1, ν_1, h_1), then by (2.4.30) $\tilde\sigma(t) := (\sigma_1(t) + \beta, \sigma_2(t), (\sigma_3(t))(e^\beta\cdot))$ is a path connecting $(\alpha_0 + \beta, \nu_0, h_0(e^\beta\cdot))$ and $(\alpha_1 + \beta, \nu_1, h_1(e^\beta\cdot))$ with same length, and hence

$$\mathrm{dist}_M\big((\alpha_0, \nu_0, h_0), (\alpha_1, \nu_1, h_1)\big) = \mathrm{dist}_M\big((\alpha_0 + \beta, \nu_0, h_0(e^\beta\cdot)), (\alpha_1 + \beta, \nu_1, h_1(e^\beta\cdot))\big). \tag{2.4.31}$$

Denote now for simplicity $D := (\partial_\theta, \partial_\lambda, \partial_u)$ the gradient with respect to all the variables; a direct computation shows that

$$D\mathcal{H}^m(\theta, \lambda, u)(\alpha, \nu, h) = \mathcal{P}(\lambda, u(e^{-\theta}\cdot))\alpha + \partial_\lambda \mathcal{I}^m(\lambda, u(e^{-\theta}\cdot))\nu + \partial_u \mathcal{I}^m(\lambda, u(e^{-\theta}\cdot))h(e^{-\theta}\cdot)$$

and thus we obtain

$$\begin{aligned}&\|D\mathcal{H}^m(\theta,\lambda,u)\|^2_{(\theta,\lambda,u),*}\\ &= \left|\Big(\mathcal{P}(\lambda,u(e^{-\theta}\cdot)),\partial_\lambda\mathcal{I}^m(\lambda,u(e^{-\theta}\cdot)),\|\partial_u\mathcal{I}^m(\lambda,u(e^{-\theta}\cdot))\|_{(H^s_r(\mathbb{R}^N))^*}\Big)\right|^2\\ &= |\mathcal{P}(\lambda,u(e^{-\theta}\cdot))|^2+|\partial_\lambda\mathcal{I}^m(\lambda,u(e^{-\theta}\cdot))|^2+\|\partial_u\mathcal{I}^m(\lambda,u(e^{-\theta}\cdot))\|^2_{(H^s_r(\mathbb{R}^N))^*}.\end{aligned}$$

Now defined

$$\tilde{K}_b := \{(\theta,\lambda,u)\in M \mid \mathcal{H}^m(\theta,\lambda,u)=b,\ D\mathcal{H}^m(\theta,\lambda,u)=0\}$$

the set of critical points at level b of $\mathcal{H}^m$, we deduce

$$\tilde{K}_b = \{(\theta,\lambda,u(e^{\theta}\cdot)) \mid (\lambda,u)\in K_b^{PSP},\ \theta\in\mathbb{R}\}. \tag{2.4.32}$$

Proposition 2.4.6. *Assume (g1)–(g3). Let $b\in\mathbb{R}$, $b<0$. Then the functional $\mathcal{H}^m$ satisfies the following Palais-Smale type condition $(\widetilde{PSP})_b$: for each sequence $(\theta_n,\lambda_n,u_n)_n$ such that*

$$\mathcal{H}^m(\theta_n,\lambda_n,u_n)\to b,$$

$$\|D\mathcal{H}^m(\theta_n,\lambda_n,u_n)\|_{(\theta_n,\lambda_n,u_n),*}\to 0,$$

we have, up to a subsequence,

$$\mathrm{dist}_M((\theta_n,\lambda_n,u_n),\tilde{K}_b)\to 0.$$

We note that $(\widetilde{PSP})_b$ condition is different from the standard Palais-Smale condition and it ensures the compactness of the sequence $(\theta_n,\lambda_n,u_n)_n$ after a suitable scaling. By (2.4.32) we also highlight that, if $\tilde{K}_b\neq\emptyset$, then $\tilde{K}_b$ is not compact.

Proof. Let $(\theta_n,\lambda_n,u_n)_n$ be as in $(\widetilde{PSP})_b$. Then set $\tilde{u}_n := u_n(e^{-\theta_n}\cdot)$ we have

$$\mathcal{P}(\lambda_n,\tilde{u}_n)\to 0,$$

$$\partial_\lambda\mathcal{I}^m(\lambda_n,\tilde{u}_n)\to 0,$$

$$\|\partial_u\mathcal{I}^m(\lambda_n,\tilde{u}_n)\|_{(H^s_r(\mathbb{R}^N))^*}\to 0,$$

and thus by Proposition 2.4.2 the sequence $(\lambda_n,\tilde{u}_n)$ is convergent (up to subsequences) to a $(\lambda,\tilde{u})\in K_b^{PSP}$. Observe that, for each n, set $v_n := \tilde{u}(e^{\theta_n}\cdot)$, we have $(\theta_n,\lambda,v_n)\in\tilde{K}_b$. Therefore by (2.4.31)

$$\begin{aligned}\mathrm{dist}_M((\theta_n,\lambda_n,u_n),\tilde{K}_b) &\le \mathrm{dist}_M((\theta_n,\lambda_n,u_n),(\theta_n,\lambda,v_n))\\ &= \mathrm{dist}_M((0,\lambda_n,\tilde{u}_n),(0,\lambda,\tilde{u}))\\ &\le \sqrt{|\lambda_n-\lambda|^2+\|\tilde{u}_n-\tilde{u}\|^2_{H^s_r(\mathbb{R}^N)}}\to 0,\end{aligned}$$

which reaches the claim. ∎

Notation. We use the following notation: for $\tilde{A}\subset M$ and $\rho>0$ we set

$$\tilde{N}_\rho(\tilde{A}) := \{(\theta,\lambda,u)\in M \mid \mathrm{dist}_M((\theta,\lambda,u),\tilde{A})<\rho\},$$

while for $A\subset\mathbb{R}\times H^s_r(\mathbb{R}^N)$ and $R>0$ we set

$$N_R(A) := \{(\lambda,u)\in\mathbb{R}\times H^s_r(\mathbb{R}^N) \mid d((\lambda,u),A)<R\},$$

where

$$d((\lambda,u),(\lambda',u')) := (|\lambda-\lambda'|^2+\|u-u'\|^2_{H^s_r})^{1/2}.$$

We also write for $a < b$

$$
\begin{aligned}
[\mathcal{I}^m \le b] &:= \{(\lambda,u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N) \mid \mathcal{I}(\lambda,u) \le b\},\\
[a \le \mathcal{I}^m \le b] &:= \{(\lambda,u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N) \mid a \le \mathcal{I}(\lambda,u) \le b\},\\
[\mathcal{H}^m \le b]_M &:= \{(\theta,\lambda,u) \in M \mid \mathcal{H}(\theta,\lambda,u) \le b\},\\
[a \le \mathcal{H}^m \le b]_M &:= \{(\theta,\lambda,u) \in M \mid a \le \mathcal{H}(\theta,\lambda,u) \le b\}.
\end{aligned}
$$

Using these notations, as a corollary to Proposition 2.4.6, we have

Corollary 2.4.7. *For any $\rho > 0$ there exists a $\delta_\rho > 0$ such that*

$$
\forall\,(\theta,\lambda,u) \in [b-\delta_\rho \le \mathcal{H}^m \le b+\delta_\rho]_M \setminus \tilde{N}_\rho(\tilde{K}_b) \;:\; \|D\mathcal{H}(\theta,\lambda,u)\|_{(\theta,\lambda,u),*} > \delta_\rho. \tag{2.4.33}
$$

Here, if $\tilde{K}_b = \emptyset$, we regard $\tilde{N}_\rho(\tilde{K}_b) = \emptyset$.

2.5 A deformation flow by projections

Exploiting an idea in [224] (see also [231]), we aim to prove the following Deformation Theorem in the fractional framework.

Theorem 2.5.1. *Let $b < 0$, and assume $K_b^{PSP} = \emptyset$. Let $\bar{\varepsilon} > 0$, then there exist $\varepsilon \in (0,\bar{\varepsilon})$ and $\eta : [0,1] \times (\mathbb{R} \times H^s_r(\mathbb{R}^N)) \to \mathbb{R} \times H^s_r(\mathbb{R}^N)$ continuous such that*

1. *$\eta(0,\cdot,\cdot) = id_{\mathbb{R}\times H^s_r(\mathbb{R}^N)}$;*
2. *η fixes $[\mathcal{I}^m \le b-\bar{\varepsilon}]$, that is, $\eta(t,\cdot,\cdot) = id_{[\mathcal{I}^m \le b-\bar{\varepsilon}]}$ for all $t \in [0,1]$;*
3. *$\mathcal{I}^m$ is non-increasing along η, and in particular $\mathcal{I}^m(\eta(t,\cdot,\cdot)) \le \mathcal{I}^m(\cdot,\cdot)$ for all $t \in [0,1]$;*
4. *$\eta(1,[\mathcal{I}^m \le b+\varepsilon]) \subseteq [\mathcal{I}^m \le b-\varepsilon]$.*

We omit the proof of the Theorem since it will be very similar to the one made in the case of multiplicity (see Theorem 2.7.1). We remark that this deformation flow is not C^1 and it does not satisfy the two properties of the standard deformation flows, in general [224, Remark 3.2]:

(1) $\eta(s+t,\lambda,u) = \eta(t,\eta(s,\lambda,u))$ with $s+t \in [0,1], (\lambda,u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N)$;

(2) for $t \in [0,1]$, the map $(\lambda,u) \mapsto \eta(t,\lambda,u)$ is a homeomorphism.

This is due to the fact that this deformation will be built through a projection of another deformation, built for the augmented functional $\mathcal{H}^m$.

We also stress that the deformation argument in Theorem 2.5.1 works for K_b^{PSP} but not for K_b and thus, if $K_b^{PSP} = \emptyset$, then we have the statement (4) in Theorem 2.5.1 even if $K_b \neq \emptyset$. By classical arguments, we derive the following existence theorem (see also the proof of Corollary 2.6.3).

Corollary 2.5.2 (Existence)**.** *Let $\bar{b} < 0$ be a MP minimax value for $\mathcal{I}^m$. Then $K_{\bar{b}}^{PSP} \neq \emptyset$, that is, $\mathcal{I}^m$ has a critical point $(\bar{\lambda},\bar{u})$ satisfying the Pohozaev identity, namely $\mathcal{P}(\bar{\lambda},\bar{u}) = 0$.*

2.6 Minimax critical points in the product space

For any $m > 0$, let B_m and E_m be the constants defined in Proposition 2.3.8, namely

$$
B_m = \inf_{\lambda<\lambda_0}\left(a(\lambda) - \frac{e^\lambda}{2}m\right), \quad E_m = \inf_{(\lambda,u)\in\partial\Omega} \mathcal{I}^m(\lambda,u).
$$

As a minimax class for $\mathcal{I}^m$, we define the paths going from Ω to Ω^c, such that the energy of the ending points is below the minimal energy on the mountain $\partial\Omega$:

$$\Gamma^m := \{\xi \in C([0,1], \mathbb{R} \times H^s_r(\mathbb{R}^N)) \mid \xi(0) \in \mathbb{R} \times \{0\},\ \mathcal{I}^m(\xi(0)) \le B_m - 1, \quad \xi(1) \notin \Omega,\ \mathcal{I}^m(\xi(1)) \le B_m - 1\}.$$

We have the following result.

Proposition 2.6.1. *Assume (g1)–(g3).*

(i) *For any $m > 0$, we have $\Gamma^m \neq \emptyset$.*

(ii) *For sufficiently large $m > 0$ there exists $\xi \in \Gamma^m$ such that*

$$\max_{t\in[0,1]} \mathcal{I}^m(\xi(t)) < 0. \tag{2.6.34}$$

(iii) *Assume (g4). Then for any $m > 0$ there exists $\xi \in \Gamma^m$ with the property (2.6.34).*

Proof. Let $\lambda_0 \in (-\infty, \infty]$ be defined in (2.3.11). For any $\lambda < \lambda_0$ we show there exists a path $\psi_\lambda \in \Gamma^m$ such that

$$\max_{t\in[0,1]} \mathcal{I}^m(\psi_\lambda(t)) \le a(\lambda) - \frac{e^\lambda}{2} m. \tag{2.6.35}$$

Let u_λ be a MP solution of $\partial_u \mathcal{J}(\lambda, u) = 0$ (by Theorem 2.2.3). Set $\zeta_\lambda(t) := u_\lambda(\cdot/t)$ for $t > 0$ and $\zeta_\lambda(0) := 0$ and note that, since u_λ satisfies the Pohozaev identity, we have $\mathcal{I}^m(\lambda, \zeta_\lambda(t)) \to -\infty$ and $\mathcal{P}(\lambda, \zeta_\lambda(t)) \to -\infty$ as $t \to +\infty$. We can find $\gamma_\lambda := \zeta_\lambda(L\cdot)$ for $L \gg 1$ satisfying

$$a(\lambda) = \max_{t\in[0,1]} \mathcal{J}(\lambda, \gamma_\lambda(t)),$$

$$\mathcal{I}^m(\lambda, \gamma_\lambda(1)) \le B_m - 1, \quad (\lambda, \gamma_\lambda(1)) \notin \Omega.$$

We also note that $t \mapsto \mathcal{I}^m(t, 0) = -\frac{e^t}{2} m$ is decreasing and tending to $-\infty$ as $t \to +\infty$. Thus, joining γ_λ and $t \mapsto (\lambda + Lt, 0)$; $[0,1] \to \mathbb{R} \times H^s_r(\mathbb{R}^N)$ for $L \gg 1$, we find a path $\psi_\lambda \in \Gamma^m$, defined as

$$\psi_\lambda(t) := \begin{cases} (\lambda + L(1-2t),\, 0) & \text{if } t \in [0, 1/2], \\ (\lambda,\, \gamma_\lambda(2t-1)) & \text{if } t \in (1/2, 1] \end{cases}$$

with (2.6.35). Thus in particular we have (i).

Next we deal with (ii) and (iii). By (2.6.35), we have that (ii) follows easily; (iii) also follows from Proposition 2.3.6. ∎

We notice that each path in Γ^m passes through $\partial\Omega$, thus the minimax value

$$b_m := \inf_{\xi\in\Gamma^m} \max_{t\in[0,1]} \mathcal{I}^m(\xi(t)) \tag{2.6.36}$$

verifies $b_m \ge E_m$ and hence by Proposition 2.3.8 it is well defined and finite. Since the Palais-Smale-Pohozaev condition holds on $(-\infty, 0)$, it is important to estimate b_m. We have the following result.

Proposition 2.6.2. *Assume (g1)–(g3). We have*

$$b_m \le a(\lambda) - \frac{e^\lambda}{2} m \quad \textit{for all } \lambda < \lambda_0. \tag{2.6.37}$$

Moreover

(i) *Setting*

$$m_0 := 2 \inf_{\lambda<\lambda_0} \frac{a(\lambda)}{e^\lambda} \geq 0,$$

we have

$$b_m < 0 \quad \textit{for } m > m_0.$$

(ii) *Assume* (g4) *in addition, then* $m_0 = 0$*, that is,*

$$b_m < 0 \quad \textit{for all } m > 0.$$

(iii) *We have* $b_m = E_m = B_m$.

(iv) $\limsup_{m\to+\infty} \frac{b_m}{m} \leq -\frac{e^{\lambda_0}}{2}$. *If* $\lambda_0 = +\infty$*, then* $\lim_{m\to+\infty} \frac{b_m}{m} = -\infty$ *(see [114]).*

Proof. By (2.6.35) we have (2.6.37), and thus

$$b_m \leq e^\lambda \left(\frac{a(\lambda)}{e^\lambda} - \frac{m}{2} \right) \quad \text{for all } \lambda < \lambda_0.$$

By definition of m_0, we have $b_m < 0$ for $m > m_0$. Thus we have (i). By Proposition 2.3.6, we have $m_0 = 0$ under the assumption (g4) and thus we have (ii).

Furthermore, from (2.6.37) it follows $b_m \leq B_m$. As already observed $b_m \geq E_m \geq B_m$, from which we deduce (iii).

Finally for any $\lambda \in \mathbb{R}$ we have, again by (2.6.35),

$$\limsup_{m\to+\infty} \frac{b_m}{m} \leq \lim_{m\to+\infty} \left(\frac{a(\lambda)}{m} - \frac{e^\lambda}{2} \right) = -\frac{e^\lambda}{2}.$$

Since λ is arbitrary, we get (iv). ∎

By Proposition 2.6.2 and Corollary 2.5.2 we conclude that the level b_m, defined in (2.6.36), is a critical value of $\mathcal{I}^m$ in the product space $\mathbb{R} \times H^s_r(\mathbb{R}^N)$ and thus Theorem 2.1.1 and Theorem 2.1.2 hold.

Corollary 2.6.3. *Let* $m > m_0$*. Then there exists a solution of problem* (2.1.2) *which satisfies the Pohozaev identity* (2.1.3)*. If moreover* (g4) *holds, then there exists a solution of* (2.1.2) *for each* $m > 0$*.*

Proof. Let $\bar{\varepsilon} \in (0, 1)$. By Theorem 2.5.1, in correspondence to $b_m < 0$, there exists $\varepsilon \in (0, \bar{\varepsilon})$ and η satisfying $1) - 4)$. By definition of inf, there exists $\gamma \in \Gamma^m$ such that

$$\max_{t\in[0,1]} \mathcal{I}^m(\gamma(t)) < b_m + \varepsilon,$$

that is

$$\gamma([0, 1]) \subseteq [\mathcal{I}^m \leq b_m + \varepsilon]. \tag{2.6.38}$$

Set

$$\tilde{\gamma}(t) := \eta(1, \gamma(t)),$$

we show that $\tilde{\gamma} \in \Gamma^m$. Indeed for $i \in \{0, 1\}$, since $\mathcal{I}^m(\gamma(i)) \leq B_m - 1 \leq b_m - \bar{\varepsilon}$, Theorem 2.5.1 implies that $\tilde{\gamma}(i) = \eta(1, \gamma(i)) = \gamma(i) \in [\mathcal{I}^m \leq b_m + \bar{\varepsilon}]$, and thus $\tilde{\gamma}(0) = \gamma(0) \in \mathbb{R} \times \{0\}$, $\tilde{\gamma}(1) = \gamma(1) \notin \Omega$. Therefore

$$b_m \leq \max_{t\in[0,1]} \mathcal{I}^m(\tilde{\gamma}(t)). \tag{2.6.39}$$

By contradiction, assume $K^{PSP}_{b_m} = \emptyset$. By the properties of η and (2.6.38) we obtain that $\tilde{\gamma}([0, 1]) = \eta(1, \gamma([0, 1])) \subseteq [\mathcal{I}^m \leq b_m - \varepsilon]$, that is

$$\max_{t\in[0,1]} \mathcal{I}^m(\eta(1, \gamma(t))) \leq b_m - \varepsilon.$$

This is in contradiction with (2.6.39), and we conclude the proof. ∎

Remark 2.6.4. *We observe that, by Proposition 2.6.2 (iii), the found Mountain Pass solution* $(\overline{\mu}, \overline{u})$ *at level* b_m *is a Pohozaev minimum on the product space* $\mathbb{R} \times H^s_r(\mathbb{R}^N)$. *This additionally implies that the found solution is a Pohozaev minimum for the unconstrained case, once fixed* $\overline{\mu}$; *see also Remark 4.5.9.*

2.7 Multiple normalized solutions

We focus now on the existence of multiple solutions. In the whole Section we assume, in addition, (g5).

2.7.1 Symmetric deformation theorems

In what follows we will use the following terminology. Consider the action σ of $\mathbb{G} := \mathbb{Z}_2$ on the last components of $\mathbb{R} \times H^s_r(\mathbb{R}^N)$ and $M = \mathbb{R} \times \mathbb{R} \times H^s_r(\mathbb{R}^N)$, that is

$$\sigma : (\pm 1, \lambda, u) \in \mathbb{G} \times (\mathbb{R} \times H^s_r(\mathbb{R}^N)) \mapsto (\lambda, \pm u) \in \mathbb{R} \times H^s_r(\mathbb{R}^N),$$

$$\sigma : (\pm 1, \theta, \lambda, u) \in \mathbb{G} \times M \mapsto (\theta, \lambda, \pm u) \in M.$$

We notice that $\mathcal{I}^m$ and $\mathcal{H}^m$ are invariant under this action (i.e. they are even in u), as well as the set Ω (i.e. it is symmetric with respect the axis $\mathbb{R}$). In particular this means that, if u is a solution, then $-u$ is a solution as well. We highlight instead that the function $\eta = (\eta_1, \eta_2) : \mathbb{R} \times H^s_r(\mathbb{R}^N) \to \mathbb{R} \times H^s_r(\mathbb{R}^N)$ (resp. $\tilde{\eta} = (\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2) : M \to M$) is equivariant if η_1 is even in u and η_2 is odd in u (resp. $\tilde{\eta}_0$ and $\tilde{\eta}_1$ are even and $\tilde{\eta}_2$ is odd).

Theorem 2.7.1. *Let* $b < 0$, *and let* $\mathcal{O}$ *be a neighborhood of* K_b^{PSP}. *Then for each* $\bar{\varepsilon} > 0$ *there exist* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\eta : [0,1] \times (\mathbb{R} \times H^s_r(\mathbb{R}^N)) \to (\mathbb{R} \times H^s_r(\mathbb{R}^N))$ *continuous such that*

1. $\eta(0, \cdot, \cdot) = id_{\mathbb{R} \times H^s_r(\mathbb{R}^N)}$;
2. η *fixes* $[\mathcal{I}^m \leq b - \bar{\varepsilon}]$, *that is,* $\eta(t, \cdot, \cdot) = id_{[\mathcal{I}^m \leq b - \bar{\varepsilon}]}$ *for all* $t \in [0,1]$;
3. $\mathcal{I}^m$ *is non-increasing along* η, *and in particular* $\mathcal{I}^m(\eta(t, \cdot, \cdot)) \leq \mathcal{I}^m(\cdot, \cdot)$ *for all* $t \in [0,1]$;
4. *if* $K_b^{PSP} = \emptyset$, *then* $\eta(1, [\mathcal{I}^m \leq b + \varepsilon]) \subseteq [\mathcal{I}^m \leq b - \varepsilon]$;
5. *if* $K_b^{PSP} \neq \emptyset$, *then*
$$\eta(1, [\mathcal{I}^m \leq b + \varepsilon] \setminus \mathcal{O}) \subseteq [\mathcal{I}^m \leq b - \varepsilon]$$
and
$$\eta(1, [\mathcal{I}^m \leq b + \varepsilon]) \subseteq [\mathcal{I}^m \leq b - \varepsilon] \cup \mathcal{O};$$
6. $\eta(t, \cdot, \cdot)$ *is* $\mathbb{G}$*-equivariant, in the sense mentioned before.*

To prove this, we work first on the functional $\mathcal{H}^m$, for which we obtained the $(\widetilde{PSP})$ condition.

Theorem 2.7.2. *Let* $b < 0$, $\rho > 0$ *and write* $\tilde{\mathcal{O}} := \tilde{N}_\rho(\tilde{K}_b)$. *Then for each* $\bar{\varepsilon} > 0$ *there exist* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\tilde{\eta} : [0,1] \times M \to M$ *continuous such that*

1. $\tilde{\eta}(0, \cdot, \cdot) = id_M$;
2. $\tilde{\eta}$ *fixes* $[\mathcal{H}^m \leq b - \bar{\varepsilon}]_M$, *that is* $\tilde{\eta}(t, \cdot, \cdot) = id_{[\mathcal{H}^m \leq b - \bar{\varepsilon}]_M}$ *for all* $t \in [0,1]$;
3. $\mathcal{H}^m$ *is non-increasing along* $\tilde{\eta}$, *and in particular* $\mathcal{H}^m(\tilde{\eta}(t, \cdot, \cdot, \cdot)) \leq \mathcal{H}^m(\cdot, \cdot, \cdot)$ *for all* $t \in [0,1]$;
4. *if* $\tilde{K}_b = \emptyset$, *then* $\tilde{\eta}(1, [\mathcal{H}^m \leq b + \varepsilon]_M) \subseteq [\mathcal{H}^m \leq b - \varepsilon]_M$;

5. *if* $\tilde{K}_b \neq \emptyset$*, then*
$$\tilde{\eta}(1, [\mathcal{H}^m \leq b+\varepsilon]_M \setminus \tilde{\mathcal{O}}) \subseteq [\mathcal{H}^m \leq b-\varepsilon]_M$$
and
$$\tilde{\eta}(1, \mathcal{H}^{b+\varepsilon}) \subseteq \mathcal{H}^{b-\varepsilon} \cup \tilde{\mathcal{O}};$$

6. $\tilde{\eta}(t,\cdot,\cdot)$ *is* $\mathbb{G}$*-equivariant, in the sense mentioned before.*

We postpone the proof of Theorem 2.7.2 for $\mathcal{H}^m$ and see now how to use it to deduce the one for $\mathcal{I}^m$. Introduce first the following notation:

$$\pi : M \to \mathbb{R} \times H^s_r(\mathbb{R}^N),\ \pi(\theta,\lambda,u) := (\lambda, u(e^{-\theta}\cdot)),$$

$$\iota : \mathbb{R} \times H^s_r(\mathbb{R}^N) \to M,\ \iota(\lambda,u) := (0,\lambda,u),$$

which are a kind of rescaling projection and immersion. Observe that

$$\pi \circ \iota = id_{\mathbb{R}\times H^s_r(\mathbb{R}^N)}, \quad \text{(while } \iota\circ\pi \neq id_M),$$

$$\mathcal{H}^m \circ \iota = \mathcal{I}^m, \quad \mathcal{I}^m \circ \pi = \mathcal{H}^m,$$

$$\pi(\tilde{K}_b) = K_b^{PSP}.$$

For $\tilde{\eta}$ obtained in Theorem 2.7.2, define "$\eta = \pi\circ\tilde{\eta}\circ\iota$" up to the time; more precisely

$$\eta(t,\lambda,u) := \pi(\tilde{\eta}(t,\iota(\lambda,u))). \tag{2.7.40}$$

It is now a straightforward computation showing that η satisfies the requests of Theorem 2.7.1. A delicate issue, anyway, is to show the intuitive fact that neighborhoods of $\tilde{K}_b$ are brought to neighborhoods of K_b^{PSP}. More precisely we have the following result.

Lemma 2.7.3. *Assume that* K_b^{PSP} *is compact (for instance,* $b<0$*). Let* $\rho>0$*, then there exists* $R(\rho)>0$ *such that, set* $\tilde{\mathcal{O}} := \tilde{N}_\rho(\tilde{K}_b)$ *and* $\mathcal{O} := N_{R(\rho)}(K_b^{PSP})$*, we have*

$$\pi(\tilde{\mathcal{O}}) \subset \mathcal{O},$$

i.e.
$$\mathrm{dist}_M((\theta,\lambda,u),\tilde{K}_b) \leq \rho \implies d((\lambda,u(e^{-\theta}\cdot)),K_b^{PSP}) \leq R(\rho).$$
In particular, for $\theta=0$ *we have*

$$\mathrm{dist}_M((0,\lambda,u),\tilde{K}_b) \leq \rho \implies d((\lambda,u),K_b^{PSP}) \leq R(\rho), \tag{2.7.41}$$

that is
$$\iota(\complement\mathcal{O}) \subseteq \complement\tilde{\mathcal{O}}$$
where $\complement$ *denotes the complement of the set. Moreover*

$$\lim_{\rho\to 0} R(\rho) = 0.$$

Proof. We observe that is sufficient to prove (2.7.41) since by (2.4.31)

$$\mathrm{dist}_M((\theta,\lambda,u),\tilde{K}_b) = \mathrm{dist}_M((0,\lambda,u(e^{-\theta}\cdot)),\tilde{K}_b).$$

Let $\varepsilon>0$. By definition of $\mathrm{dist}_M((0,\lambda,u),\tilde{K}_b)$ there exists a $\sigma=\sigma(t)$, $\sigma=(\theta,\lambda,u)$, such that $\sigma(0)=(0,\lambda,u)$, $\sigma(1)\in\tilde{K}_b$ and

$$\int_0^1 \|\dot{\sigma}(t)\|_{\sigma(t)} dt \leq \rho+\varepsilon. \tag{2.7.42}$$

By (2.4.32) we have $(\lambda(1), u(1)(e^{-\theta(1)}\cdot)) \in K_b^{PSP}$ and thus

$$\begin{aligned}
&\mathrm{dist}((\lambda, u), K_b^{PSP}) \\
&\quad\leq \|(\lambda, u) - (\lambda(1), u(1)(e^{-\theta(1)}\cdot))\|_{\mathbb{R}\times H^s_r(\mathbb{R}^N)} \\
&\quad\leq \|(\lambda, u) - (\lambda(1), u(1))\|_{\mathbb{R}\times H^s_r(\mathbb{R}^N)} + \|(\lambda(1), u(1)) - (\lambda(1), u(1)(e^{-\theta(1)}\cdot))\|_{\mathbb{R}\times H^s_r(\mathbb{R}^N)} \\
&\quad= \|(\lambda(0), u(0)) - (\lambda(1), u(1))\|_{\mathbb{R}\times H^s_r(\mathbb{R}^N)} + \|u(1) - u(1)(e^{-\theta(1)}\cdot)\|_{H^s_r(\mathbb{R}^N)} \\
&\quad= I + II.
\end{aligned}$$

Focus on I. We have, by the fundamental theorem of calculus and Hölder inequality,

$$\begin{aligned}
I = \|(\lambda(0), u(0)) - (\lambda(1), u(1))\|_{\mathbb{R}\times H^s_r(\mathbb{R}^N)} &\leq \int_0^1 \left(|\dot\lambda(t)|^2 + \|\dot u(t)\|^2_{H^s_r(\mathbb{R}^N)}\right)^{1/2} dt \\
&= \int_0^1 \left(|\dot\lambda(t)|^2 + \|\dot u(t)\|_2^2 + \|(-\Delta)^{s/2}\dot u(t)\|_2^2\right)^{1/2} dt.
\end{aligned}$$

In order to use (2.7.42) it must appear the norm associated to M, which we recall is

$$\|\dot\sigma(t)\|^2_{\sigma(t)} = \dot\theta(t)^2 + \dot\lambda(t)^2 + e^{N\theta(t)}\|\dot u(t)\|_2^2 + e^{(N-2s)\theta(t)}\|(-\Delta)^{s/2}\dot u(t)\|_2^2.$$

Since we do not know the sign of $N\theta(t)$, we need an estimate on $\theta(t)$ and a corrective factor. Indeed, recalled that $\theta(0) = 0$, we have

$$|\theta(t)| = |\theta(t) - \theta(0)| \leq \int_0^1 |\dot\theta(t)| dt \leq \int_0^1 \|\dot\sigma(t)\|_{\sigma(t)} dt \leq \rho + \varepsilon.$$

Thus $\theta(t) \geq -(\rho+\varepsilon) \geq -\frac{N}{N-2s}(\rho+\varepsilon)$ which imply

$$e^{N(\rho+\varepsilon)} \geq 1, \quad e^{N(\rho+\varepsilon)} e^{N\theta(t)} \geq 1, \quad e^{N(\rho+\varepsilon)} e^{(N-2s)\theta(t)} \geq 1$$

and hence we obtain

$$\begin{aligned}
I &\leq e^{\frac{N(\rho+\varepsilon)}{2}} \int_0^1 \left(|\dot\lambda(t)|^2 + e^{N\theta(t)}\|\dot u(t)\|_2^2 + e^{(N-2s)\theta(t)}\|(-\Delta)^{s/2}\dot u(t)\|_2^2\right)^{1/2} dt \\
&\leq e^{\frac{N(\rho+\varepsilon)}{2}} \int_0^1 \left(|\dot\theta(t)^2| + |\dot\lambda(t)|^2 + e^{N\theta(t)}\|\dot u(t)\|_2^2 + e^{(N-2s)\theta(t)}\|(-\Delta)^{s/2}\dot u(t)\|_2^2\right)^{1/2} dt \\
&= e^{\frac{N(\rho+\varepsilon)}{2}} \int_0^1 \|\dot\sigma(t)\|_{\sigma(t)} dt \leq e^{\frac{N(\rho+\varepsilon)}{2}}(\rho+\varepsilon) \overset{\varepsilon\to 0}{\to} e^{\frac{N\rho}{2}}\rho.
\end{aligned}$$

Focus now on II. Set $\bar\omega := u(1)(e^{-\theta(1)}\cdot)$ we have $\bar\omega \in P_2(K_b^{PSP})$ (where P_2 is the projection on the second component) with $|\theta(1)| \leq \rho+\varepsilon$, and thus

$$\begin{aligned}
II = \|u(1) - u(1)(e^{-\theta(1)}\cdot)\|_{H^s_r(\mathbb{R}^N)} &= \|\bar\omega(e^{\theta(1)}\cdot) - \bar\omega\|_{H^s_r(\mathbb{R}^N)} \\
&\leq \sup\left\{\|\omega(e^\alpha\cdot) - \omega\|_{H^s_r(\mathbb{R}^N)} \mid |\alpha| \leq \rho+\varepsilon,\ \omega \in P_2(K_b^{PSP})\right\}.
\end{aligned}$$

Since $P_2(K_b^{PSP})$ is compact, it is simple to show that, as $\varepsilon \to 0$,

$$II \leq \sup\left\{\|\omega(e^\alpha\cdot) - \omega\|_{H^s_r(\mathbb{R}^N)} \mid |\alpha| \leq \rho,\ \omega \in P_2(K_b^{PSP})\right\}.$$

Summing up, we have

$$\begin{aligned}
\mathrm{dist}((\lambda, u), K_b^{PSP}) &\leq e^{\frac{N\rho}{2}}\rho + \sup\left\{\|\omega(e^\alpha\cdot) - \omega\|_{H^s_r(\mathbb{R}^N)} \mid |\alpha| \leq \rho,\ \omega \in P_2(K_b^{PSP})\right\} \\
&=: R(\rho) < \infty.
\end{aligned}$$

Here we have

$$\lim_{\rho\to 0} R(\rho) = 0,$$

which concludes the proof. ∎

We are now ready to show that η satisfies the desired properties.

Proof of Theorem 2.7.1. Let $\mathcal{O}$ be a neighborhood of K_b^{PSP}, and choose R such that $N_R(K_b^{PSP}) \subset \mathcal{O}$. By Lemma 2.7.3 choose $\rho \ll 1$ satisfying $R(\rho) < R$ and thus $N_{R(\rho)}(K_b^{PSP}) \subset \mathcal{O}$. Consequently, by Theorem 2.7.2, there exists a deformation $\tilde\eta$ corresponding to the neighborhood $\tilde{\mathcal{O}} := \tilde N_\rho(\tilde K_b)$. We thus define η by (2.7.40) and prove the properties. Start observing that

$$\begin{aligned}(\lambda,u)\in[\mathcal{I}^m \le b\pm\delta] &\implies b\pm\delta > \mathcal{I}^m(\lambda,u) = \mathcal{H}^m(\iota(\lambda,u))\\ &\implies \iota(\lambda,u)\in[\mathcal{H}^m\le b\pm\delta]_M,\end{aligned}$$

i.e. $\iota([\mathcal{I}^m \le b\pm\delta]) \subset [\mathcal{H}^m \le b\pm\delta]_M$; similarly, $\pi([\mathcal{H}^m \le b\pm\delta]_M) \subset [\mathcal{I}^m \le b\pm\delta]$.

1. $\eta(0,\lambda,u) = \pi(\tilde\eta(0,\iota(\lambda,u))) = \pi(\iota(\lambda,u)) = (\lambda,u)$.
2. If $(\lambda,u)\in[\mathcal{I}^m \le b-\bar\varepsilon]$, then $\iota(\lambda,u)\in[\mathcal{H}^m\le b-\bar\varepsilon]_M$. Thus $\eta(t,\lambda,u) = \pi(\tilde\eta(t,\iota(\lambda,u))) = \pi(\iota(\lambda,u)) = (\lambda,u)$.
3. $\mathcal{I}^m(\eta(t,\lambda,u)) = \mathcal{I}^m(\pi(\tilde\eta(t,\iota(\lambda,u)))) = \mathcal{H}^m(\tilde\eta(t,\iota(\lambda,u))) \le \mathcal{H}^m(\iota(\lambda,u)) = \mathcal{I}^m(\lambda,u)$.
4. If $K_b^{PSP} = \emptyset$, then $\tilde K_b = \emptyset$. Thus for $(\lambda,u)\in[\mathcal{I}^m \le b+\varepsilon]$, we have $\mathcal{I}^m(\eta(1,\lambda,u)) = \mathcal{I}^m(\pi(\tilde\eta(1,\iota(\lambda,u)))) = \mathcal{H}^m(\tilde\eta(1,\iota(\lambda,u))) \le b-\varepsilon$.
5. We have, by previous arguments and Lemma 2.7.3, that $\iota([\mathcal{I}^m \le b+\varepsilon]\setminus\mathcal{O}) = \iota([\mathcal{I}^m \le b+\varepsilon]\cap \complement\mathcal{O}) \subseteq \iota([\mathcal{I}^m\le b+\varepsilon])\cap\iota(\complement\mathcal{O}) \subseteq [\mathcal{H}^m\le b+\varepsilon]_M \cap(\complement\tilde{\mathcal{O}}) = [\mathcal{H}^m \le b+\varepsilon]_M\setminus\tilde{\mathcal{O}}$ and thus

   $$\begin{aligned}&\eta(1,[\mathcal{I}^m\le b+\varepsilon]\setminus\mathcal{O})\\ &\quad = \pi(\tilde\eta(1,\iota([\mathcal{I}^m\le b+\varepsilon]\setminus\mathcal{O}))) \subset \pi(\tilde\eta(1,[\mathcal{H}^m\le b+\varepsilon]_M\setminus\tilde{\mathcal{O}}))\\ &\quad \subset \pi([\mathcal{H}^m\le b-\varepsilon]_M) \subset [\mathcal{I}^m \le b-\varepsilon].\end{aligned}$$

   The other inclusion is similar and easier.
6. We write $\tilde\eta(t,\theta,\lambda,u) = \big(\tilde\eta_0(t,\theta,\lambda,u),\tilde\eta_1(t,\theta,\lambda,u),\tilde\eta_2(t,\theta,\lambda,u)\big)$. Then by definition

   $$(\eta_1(t,\lambda,u),\eta_2(t,\lambda,u)) = \Big(\tilde\eta_1(t,0,\lambda,u),\tilde\eta_2\big(t,0,\lambda,u(e^{-\tilde\eta_0(t,0,\lambda,u)}\cdot)\big)\Big)$$

   thus by the property 6 of Theorem 2.7.2,

   $$\begin{aligned}(\eta_1(t,\lambda,-u),\eta_2(t,\lambda,-u)) &= \Big(\tilde\eta_1(t,0,\lambda,-u),\tilde\eta_2\big(t,0,\lambda,-u(e^{-\tilde\eta_0(t,0,\lambda,-u)}\cdot)\big)\Big)\\ &= \Big(\tilde\eta_1(t,0,\lambda,u),-\tilde\eta_2\big(t,0,\lambda,u(e^{-\tilde\eta_0(t,0,\lambda,u)}\cdot)\big)\Big)\\ &= (\eta_1(t,\lambda,u),-\eta_2(t,\lambda,u)).\end{aligned}$$

The theorem is hence proved. ∎

Now we are ready to prove the main theorem for $\mathcal{H}^m$.

Proof of Theorem 2.7.2. To avoid cumbersome notation, we write $\xi = (\theta,\lambda,u)\in M$. Set

$$M' := \{D\mathcal{H}^m(\xi)\neq 0\}.$$

It is known [14] that there exists a pseudo-gradient on the Hilbert manifold M associated to $\mathcal{H}^m$, namely a locally Lipschitz vector field $\mathcal{V}: M'\to TM$ such that

(a) $\|\mathcal{V}(\xi)\|_\xi \leq 2\|D\mathcal{H}^m(\xi)\|_{\xi,*}$,

(b) $D\mathcal{H}^m(\xi)\cdot\mathcal{V}(\xi) \geq \|D\mathcal{H}^m(\xi)\|^2_{\xi,*}$;

in particular,

$$\frac{1}{2}\|\mathcal{V}(\xi)\|_\xi \leq \|D\mathcal{H}^m(\xi)\|_{\xi,*} \leq \|\mathcal{V}(\xi)\|_\xi. \tag{2.7.43}$$

Moreover, we can ask, in the construction of the pseudo-gradient, that $\mathcal{V}$ is $\mathbb{G}$-equivariant, since $\mathcal{H}^m$ is $\mathbb{G}$-invariant. Namely, set $\mathcal{V}=(\mathcal{V}_0,\mathcal{V}_1,\mathcal{V}_2)$, then $\mathcal{V}_0$ and $\mathcal{V}_1$ are even in u, while $\mathcal{V}_2$ is odd in u.

By Corollary 2.4.7, there exists $\delta=\delta_{\frac{\rho}{3}}>0$ such that

$$\forall\,\xi\in[b-\delta\leq\mathcal{H}^m\leq b+\delta]_M \ \ s.t.\ \ \mathrm{dist}_M(\xi,\tilde{K}_b)>\frac{\rho}{3}\ :\ \|D\mathcal{H}^m(\xi)\|_{\xi,*}>\delta. \tag{2.7.44}$$

We assume

$$\varepsilon<\min\Big\{\frac{1}{2}\bar{\varepsilon},\frac{1}{4}\delta,\frac{1}{6}\rho\delta\Big\}. \tag{2.7.45}$$

Set the following

$$A:=[b-\varepsilon\leq\mathcal{H}^m\leq b+\varepsilon]_M,\quad B:=[b-2\varepsilon\leq\mathcal{H}^m\leq b+2\varepsilon]_M$$

and choose a locally Lipschitz function $g\in C(M,[0,1])$ such that

$$g=1\ \text{ on } A,\quad g=0\ \text{ on } \complement B,$$

for instance $g(\xi):=\frac{d(\xi,\complement B)}{d(\xi,\complement B)+d(\xi,A)}$.

When $\tilde{K}_b\neq\emptyset$, we choose a locally Lipschitz function $\tilde{g}\in C(M,[0,1])$ satisfying

$$\tilde{g}=0\ \text{ on } \tilde{N}_{\frac{\rho}{3}}(\tilde{K}_b),\quad \tilde{g}=1\ \text{ on } \complement\tilde{N}_{\frac{2}{3}\rho}(\tilde{K}_b).$$

When $\tilde{K}_b=\emptyset$, we set $\tilde{g}\equiv 1$. Moreover we introduce, for any $r\geq 0$,

$$b(r):=\begin{cases}\dfrac{1}{r} & \text{if } r\geq 1\\ 1 & \text{if } 0\leq r<1.\end{cases}$$

Finally define, for $\xi\in M$,

$$W(\xi):=-g(\xi)\tilde{g}(\xi)b\left(\|\mathcal{V}(\xi)\|_\xi\right)\mathcal{V}(\xi)$$

and, fixed $\xi\in M$, consider the Cauchy problem

$$\begin{cases}\tilde{\eta}'=W(\tilde{\eta}),\\ \tilde{\eta}(0)=\xi.\end{cases}$$

We have that W is well defined on M and

$$\|W(\xi)\|_\xi\leq\|\mathcal{V}(\xi)\|_\xi\, b\left(\|\mathcal{V}(\xi)\|_\xi\right)\leq 1,$$

where we have used that $|g|$, $|\tilde{g}|\leq 1$. Therefore we have the global existence of a flow $\tilde{\eta}=\tilde{\eta}(t,\xi)$; we are interested in $\tilde{\eta}$ restricted to $[0,1]$. We now verify the desired properties.

1) $\tilde{\eta}(0,\xi)=\xi$ by construction of the flow.

2) If $\xi\in[\mathcal{H}^m\leq b-\bar{\varepsilon}]_M$, then $g(\xi)=0$, and thus $W(\xi)=0$. This means that $\tilde{\eta}(t,\xi)\equiv\xi$ is an equilibrium solution. Since $W\in Lip_{loc}(M)$ we have uniqueness of the solution, hence actually $\tilde{\eta}(t,\xi)\equiv\xi$.

3) We have

$$\begin{aligned}\frac{d}{dt}\mathcal{H}^m(\tilde{\eta}(t,\xi)) &= D\mathcal{H}^m(\tilde{\eta}(t,\xi))\tilde{\eta}'(t,\xi)\\ &= -D\mathcal{H}^m(\tilde{\eta}(t,\xi))\mathcal{V}(\tilde{\eta}(t,\xi))g(\tilde{\eta}(t,\xi))\tilde{g}(\tilde{\eta}(t,\xi))b\left(\|\mathcal{V}(\tilde{\eta}(t,\xi))\|_{\tilde{\eta}(t,\xi)}\right)\\ &\leq -\|D\mathcal{H}^m(\tilde{\eta}(t,\xi))\|^2_{\tilde{\eta}(t,\xi),*}g(\tilde{\eta}(t,\xi))\tilde{g}(\tilde{\eta}(t,\xi))b\left(\|\mathcal{V}(\tilde{\eta}(t,\xi))\|_{\tilde{\eta}(t,\xi)}\right)\\ &\leq 0\end{aligned}$$

that is the claim; we have used that g, $\tilde{g}$, b are positive and the property (b).

4) We assume here $\tilde{K}_b = \emptyset$. By using the fundamental theorem of calculus and previous arguments, we obtain

$$\begin{aligned}\mathcal{H}^m(\tilde{\eta}(1,\xi)) - \mathcal{H}^m(\tilde{\eta}(0,\xi)) &= \int_0^1 \frac{d}{ds}\mathcal{H}^m(\tilde{\eta}(s,\xi))ds\\ &= -\int_0^1 D\mathcal{H}^m(\tilde{\eta}(s,\xi))\mathcal{V}(\tilde{\eta}(s,\xi))g(\tilde{\eta}(s,\xi))b\left(\|\mathcal{V}(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi)}\right)ds\\ &\leq -\int_0^1 \|D\mathcal{H}^m(\tilde{\eta}(s,\xi))\|^2_{\tilde{\eta}(s,\xi),*}g(\tilde{\eta}(s,\xi))b\left(\|\mathcal{V}(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi)}\right)ds.\end{aligned}$$

Let now $\xi \in [\mathcal{H}^m \leq b+\varepsilon]_M$. This means, by point 3), that for $s \in [0,1]$

$$\mathcal{H}^m(\tilde{\eta}(s,\xi)) \leq \mathcal{H}^m(\tilde{\eta}(0,\xi)) = \mathcal{H}^m(\xi) \leq b+\varepsilon,$$

thus $\tilde{\eta}(s,\xi) \in [\mathcal{H}^m \leq b+\varepsilon]_M$ and

$$\mathcal{H}^m(\tilde{\eta}(1,\xi)) \leq b+\varepsilon - \int_0^1 \|D\mathcal{H}^m(\tilde{\eta}(s,\xi))\|^2_{\tilde{\eta}(s,\xi),*}g(\tilde{\eta}(s,\xi))b\left(\|\mathcal{V}(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi)}\right)ds.$$

Assume now by contradiction that $\mathcal{H}^m(\tilde{\eta}(1,\xi)) > b-\varepsilon$, which implies (again by point 3)) $\mathcal{H}^m(\tilde{\eta}(s,\xi)) > b-\varepsilon$, for all $s \in [0,1]$. Thus for all $s \in [0,1]$ we have $\tilde{\eta}(s,\xi) \in [b-\varepsilon \leq \mathcal{H}^m \leq b+\varepsilon]_M$ and in particular, since $\varepsilon < \frac{1}{2}\bar{\varepsilon}$, that $g(\tilde{\eta}(s,\xi)) = 1$; hence

$$\mathcal{H}^m(\tilde{\eta}(1,\xi)) \leq b+\varepsilon - \int_0^1 \|D\mathcal{H}^m(\tilde{\eta}(s,\xi))\|^2_{\tilde{\eta}(s,\xi),*}b\left(\|\mathcal{V}(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi)}\right)ds.$$

By (2.7.43), by the fact that $\tilde{\eta}(s,\xi) \in [b-\varepsilon \leq \mathcal{H}^m \leq b+\varepsilon]_M \subset [b-\delta \leq \mathcal{H}^m \leq b+\delta]_M$ and by (2.4.33), we have

$$\|\mathcal{V}(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi)} \geq \|D\mathcal{H}^m(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi),*} \geq \delta \geq 4\varepsilon; \tag{2.7.46}$$

in particular,

$$b\left(\|\mathcal{V}(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi)}\right) = \frac{1}{\|\mathcal{V}(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi)}}.$$

Thus, exploiting again (2.7.43) and (2.7.46) we obtain

$$\begin{aligned}\mathcal{H}^m(\tilde{\eta}(1,\xi)) &\leq b+\varepsilon - \frac{1}{2}\int_0^1 \|D\mathcal{H}^m(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi),*}ds\\ &\leq b+\varepsilon - 2\int_0^1 \varepsilon ds = b-\varepsilon,\end{aligned}$$

which is an absurd.

5) We assume now $\tilde{K}_b \neq \emptyset$. Let now $\xi \in [\mathcal{H}^m \leq b+\varepsilon]_M \setminus \tilde{\mathcal{O}}$. Assume again by contradiction that $\mathcal{H}^m(\tilde{\eta}(1,\xi)) > b-\varepsilon$, which implies again $\tilde{\eta}(s,\xi) \in [b-\varepsilon \leq \mathcal{H}^m \leq b+\varepsilon]_M$. We distinguish two cases.
Case 1: $\tilde{\eta}(t,\xi) \notin \tilde{N}_{\frac{2}{3}\rho}(\tilde{K}_b)$ for all $t \in [0,1]$. In this case we proceed as in the proof of 4). Indeed since $\varepsilon < \delta_{\frac{\rho}{3}}$, we are in the assumptions of (2.4.33) and thus

$$\|D\mathcal{H}^m(\tilde{\eta}(s,\xi))\|_{\tilde{\eta}(s,\xi),*} > \delta > 4\varepsilon.$$

We argue as before and conclude.
Case 2: $\tilde{\eta}(t^*,\xi) \in \tilde{N}_{\frac{2}{3}\rho}(\tilde{K}_b)$ for some $t^* \in [0,1]$. In this case ε has to be better specified. We make a finer argument by choosing suitable $[\alpha,\beta] \subset [0,1]$ and observing that

$$\begin{aligned}\mathcal{H}^m(\tilde{\eta}(1,\xi)) \leq \mathcal{H}^m(\tilde{\eta}(\beta,\xi)) &= \mathcal{H}^m(\tilde{\eta}(\alpha,\xi)) + \int_\alpha^\beta \frac{d}{ds}\mathcal{H}^m(\tilde{\eta}(s,\xi))ds \\ &\leq \mathcal{H}^m(\tilde{\eta}(0,\xi)) + \int_\alpha^\beta \frac{d}{ds}\mathcal{H}^m(\tilde{\eta}(s,\xi))ds \\ &\leq b+\varepsilon + \int_\alpha^\beta \frac{d}{ds}\mathcal{H}^m(\tilde{\eta}(s,\xi))ds.\end{aligned}$$

Noting that $\tilde{\eta}(0,\xi) = \xi \notin \tilde{\mathcal{O}} = \tilde{N}_\rho(\tilde{K}_b)$ and $\tilde{\eta}(t^*,\xi) \in \tilde{N}_{\frac{2}{3}\rho}(\tilde{K}_b)$, we can find α and β such that

$$\tilde{\eta}(\alpha) \in \partial\tilde{N}_\rho(\tilde{K}_b), \quad \tilde{\eta}(\beta) \in \partial\tilde{N}_{\frac{2}{3}\rho}(\tilde{K}_b),$$

and

$$\tilde{\eta}(s) \in \tilde{N}_\rho(\tilde{K}_b) \setminus \tilde{N}_{\frac{2}{3}\rho}(\tilde{K}_b) \quad \forall\, s \in (\alpha,\beta).$$

Hence we obtain by (2.7.44)

$$\mathcal{H}^m(\tilde{\eta}(1,\xi)) \leq b+\varepsilon-\delta(\beta-\alpha).$$

We need an estimate from below of $\beta-\alpha$, which is obtained by observing that $\tilde{\eta}(\cdot,\xi)$ is a path connecting $\tilde{\eta}(\alpha,\xi)$ and $\tilde{\eta}(\beta,\xi)$, thus (recall that $1 \geq \|W(\xi)\|_\xi$)

$$\begin{aligned}\beta-\alpha = \int_\alpha^\beta dt &\geq \int_\alpha^\beta \|W(\tilde{\eta}(t,\xi))\|_{\tilde{\eta}(t,\xi)}dt \\ &= \int_\alpha^\beta \|\tilde{\eta}'(t,\xi)\|_{\tilde{\eta}(t,\xi)}dt \geq \mathrm{dist}_M(\tilde{\eta}(\alpha,\xi),\tilde{\eta}(\beta,\xi)) \\ &\geq \mathrm{dist}_M\left(\tilde{N}_\rho(\tilde{K}_b),\tilde{N}_{\frac{2}{3}\rho}(\tilde{K}_b)\right) \geq \frac{1}{3}\rho.\end{aligned}$$

Finally

$$\mathcal{H}^m(\tilde{\eta}(1,\xi)) \leq b+\varepsilon-\tfrac{1}{3}\rho\delta \leq b-\varepsilon$$

by our choice (2.7.45) of ε.

As regards the second inclusion, we argue in a similar way. Let $\xi \in [\mathcal{H}^m \leq b+\varepsilon]_M$. Case 1 can be done verbatim. In Case 2, if $\tilde{\eta}(1,\xi) \in \tilde{\mathcal{O}}$ we are done; if not, then we repeat the argument but with the path built thanks to $\tilde{\eta}(1,\xi) \notin \tilde{N}_\rho(\tilde{K}_b)$ and $\tilde{\eta}(t^*,\xi) \in \tilde{N}_{\frac{2}{3}\rho}(\tilde{K}_b)$.

6) Notice that, written $W = (W_0,W_1,W_2)$, we have that W_0 and W_1 are even in u while W_2 is odd in u, since $\mathcal{V}$ is so and g, $b(\|D\mathcal{H}^m(\cdot)\|_{\cdot,*})$ are even in u. Thus, by uniqueness of the solution, we have that $\tilde{\eta}$ satisfies the required symmetry properties.

The proof is thus concluded. ∎

2.7.2 Minimax values

Minimax values $a_j(\lambda)$

We write for $j \in \mathbb{N}$, $D_j := \{\xi \in \mathbb{R}^j \mid |\xi| \le 1\}$ and we introduce the set of paths

$$\Gamma_j(\lambda) := \{\gamma \in C(D_j, H^s_r(\mathbb{R}^N)) \mid \gamma \text{ odd}, \mathcal{J}(\lambda, \gamma(\xi)) < 0 \,\forall \xi \in \partial D_j\}$$

and

$$a_j(\lambda) := \inf_{\gamma \in \Gamma_j(\lambda)} \sup_{\xi \in D_j} \mathcal{J}(\lambda, \gamma(\xi)).$$

By an odd extension from $[0,1]$ to $[-1,1] = D_1$, we may regard $\Gamma_1(\lambda) \equiv \Gamma(\lambda)$ and $a_1(\lambda) \equiv a(\lambda)$. Thus these quantities can be seen as generalizations. As for $j = 1$, we prove the following properties.

Proposition 2.7.4. *Let $\lambda_0 \in \mathbb{R} \cup \{+\infty\}$ be given in (2.3.11), $\lambda < \lambda_0$ and $j \in \mathbb{N}$.*

1. *$\Gamma_j(\lambda) \neq \emptyset$, thus $a_j(\lambda)$ is well defined. Moreover, it is increasing with respect to λ;*
2. *$a_j(\lambda) \le a_{j+1}(\lambda)$;*
3. *$a_j(\lambda) > 0$;*
4. *$\lim_{\lambda \to \lambda_0^-} \frac{a_j(\lambda)}{e^\lambda} = +\infty$;*
5. *if (g4) holds, then $\lim_{\lambda \to -\infty} \frac{a_j(\lambda)}{e^\lambda} = 0$.*

Proof. The proofs are quite the same of Propositions 2.3.2–2.3.6. We point out just some slight differences.

1. For $\lambda < \lambda_0$, there exists $t_0 > 0$ such that
$$G(t_0) - \frac{e^\lambda}{2} t_0^2 > 0.$$
As in [51], we find that there exists a continuous odd map $\tilde{\gamma} : \partial D_j \to H^1_r(\mathbb{R}^N) \hookrightarrow H^s_r(\mathbb{R}^N)$ with $\mathcal{J}(\lambda, \tilde{\gamma}(\xi)) < 0$. Extending $\tilde{\gamma}$ onto D_j we find $\Gamma_j(\lambda) \neq \emptyset$.
2. Since $D_j \subset D_{j+1}$, we observe $\gamma_{|D_j} \in \Gamma_j(\lambda)$ for $\gamma \in \Gamma_{j+1}(\lambda)$. Thus we regard $\Gamma_{j+1}(\lambda) \subset \Gamma_j(\lambda)$ and obtain 2).
3. Clear by $a_1(\lambda) = a(\lambda) > 0$ and point 2).
4. Again by $\lim_{\lambda \to \lambda_0^-} \frac{a(\lambda)}{e^\lambda} = +\infty$ and point 2).
5. We consider the path $\tilde{\gamma} : \partial D_j \to H^s_r(\mathbb{R}^N)$ obtained in 1) and introduce a path
$$\xi \in D_j \mapsto \mu^{N/4} \tilde{\gamma}\left(\frac{\xi}{|\xi|}\right)(\,\cdot\,/\mu^{-\frac{1}{2s}}|\xi|) \in H^s_r(\mathbb{R}^N).$$
Arguing as in Proposition 2.3.6, we have 5). ∎

Minimax values b_j^m

We set

$$\Gamma_j^m := \{\Theta \in C(D_j, \mathbb{R} \times H_r^s(\mathbb{R}^N)) \mid \Theta \text{ is } \mathbb{G}\text{-equivariant}; \quad \mathcal{I}^m(\Theta(0)) \le B_m - 1; \quad \Theta(\xi) \notin \Omega,\ \mathcal{I}^m(\Theta(\xi)) \le B_m - 1 \text{ for all } \ \xi \in \partial D_j\}$$

and

$$b_j^m := \inf_{\Theta \in \Gamma_j^m} \sup_{\xi \in D_j} \mathcal{I}^m(\Theta(\xi)).$$

We notice that for $j = 1$ we obtain $\Gamma_1^m \equiv \Gamma^m$ (up to an even/odd extension from $[0, 1]$ to $[-1, 1] = D_1$) and $b_1^m \equiv b_m$. So Γ_j^m is a natural extension to build multiple solutions.

As in the case of Γ^m and b_m, we want to prove that $\Gamma_j^m \neq \emptyset$ and that, for a fixed $k \in \mathbb{N}$, there exists an $m_k \gg 0$ (possibly equal to 0) such that, if $m > m_k$, then $b_j^m < 0$ for $j = 1 \dots k$.

Proposition 2.7.5.

(i) *For any $\lambda < \lambda_0$, $m > 0$, $j \in \mathbb{N}$, we have $\Gamma_j^m \neq \emptyset$ and $b_j^m \le a_j(\lambda) - e^{\lambda} \frac{m}{2}$.*

(ii) *For any $k \in \mathbb{N}$, set*

$$m_k := 2 \inf_{\lambda < \lambda_0} \frac{a_k(\lambda)}{e^{\lambda}} \ge 0 \tag{2.7.47}$$

we have, for any $m > m_k$

$$b_j^m < 0 \quad \text{for } j = 1, 2, \dots, k.$$

(iii) *$m_k = 0$ for all $k \in \mathbb{N}$ if (g4) holds. That is,*

$$b_j^m < 0 \quad \text{for all } j \in \mathbb{N}.$$

Proof. For (i), the proof is similar to Proposition 2.6.1. We just need to set, for $\zeta_\lambda \in \Gamma_j(\lambda)$,

$$\psi_\lambda(\xi) := \begin{cases} (\lambda + L(1 - 2|\xi|),\, 0) & \text{if } |\xi| \in [0, 1/2], \\ \left(\lambda,\, \zeta_\lambda\left(\frac{\xi}{|\xi|}(2|\xi| - 1)\right)\right) & \text{if } |\xi| \in (1/2, 1] \end{cases}$$

and we come up again to the same proof.

For (ii), (iii), we come up with a proof similar to Proposition 2.6.2, observing in addition that $m_k \le m_{k+1}$ since $a_k(\lambda)$ are increasing in k. ∎

By Proposition 2.4.2 and Theorem 2.5.1 $\mathcal{I}^m$ satisfies the $(PSP)_b$ condition for $b < 0$ and the deformation lemma holds. Let $m_k \ge 0$ be a number given in Proposition 2.7.5. For $m > m_k$ we can see that $b_j^m < 0$ for $j = 1, 2, \dots, k$ are critical values of $\mathcal{I}^m$. If b_j^m are different, we directly have multiplicity of solutions. To deal with the case $b_j^m = b_{j'}^m$ for some $j \neq j'$, we need another family of minimax methods, which exploits the topological information hidden in this equality.

Minimax values c_j^m

Let us define minimax families Λ_j^m which allow to find multiple solutions. We use an idea from [325]. In what follows, we denote by genus(A) the genus of closed symmetric sets A with $0 \notin A$ (see Appendix A.6).

Define, for each $j \in \mathbb{N}$,

$$\Lambda_j^m := \{A = \Theta(\overline{D_{j+l} \setminus Y}) \quad | \quad l \ge 0,\ \Theta \in \Gamma_{j+l}^m, \quad Y \subseteq D_{j+l} \setminus \{0\} \ \text{ is closed, symmetric in } 0,$$

and genus$(Y) \leq l\}$

and

$$c_j^m := \inf_{A \in \Lambda_j^m} \sup_A \mathcal{I}^m.$$

In the following lemma, we observe that Λ_j^m includes, in some way, Γ_j^m and that it inherits the property that the paths intersect $\partial\Omega$.

Lemma 2.7.6.

(i) $\Lambda_j^m \neq \emptyset$;

(ii) $c_j^m \leq b_j^m$;

(iii) *for any $A \in \Lambda_j^m$, we have $A \cap \partial\Omega \neq \emptyset$. As a consequence, we obtain*

$$b_m = B_m = E_m \leq c_j^m.$$

Proof. Indeed, we see that, by choosing $l = 0$ and $Y = \emptyset$ we have

$$\{A = \Theta(D_j) \mid \Theta \in \Gamma_j^m\} \subset \Lambda_j^m$$

from which come the first two claims.

Focus on the third claim. Let $A = \Theta(\overline{D_{j+l} \setminus Y})$ and set $U := \Theta^{-1}(\Omega)$. By the symmetry in (λ, u) of Θ and the symmetry in u of Ω we have that U is symmetric. Moreover, since $\Theta(0) \in \Omega$, we have that $U \subset D_{j+l} \subset \mathbb{R}^{j+l}$ is a symmetric neighborhood of the origin. By Proposition A.14 we have

$$\text{genus}(\partial U) = j + l. \tag{2.7.48}$$

Observe in addition the following chain of inclusions

$$\overline{\partial U \setminus Y} = \overline{(\partial U \cap D_{j+l}) \setminus Y} = \overline{(D_{j+l} \setminus Y) \cap \partial U} \subseteq \overline{D_{j+l} \setminus Y} \cap \overline{\partial U} = \overline{D_{j+l} \setminus Y} \cap \partial U$$

thus

$$\Theta\left(\overline{\partial U \setminus Y}\right) \subseteq \Theta\left(\overline{D_{j+l} \setminus Y} \cap \partial U\right) \subseteq \Theta\left(\overline{D_{j+l} \setminus Y}\right) \cap \Theta\left(\partial U\right) = A \cap \Theta\left(\partial U\right).$$

Assume for the moment that it holds

$$\Theta(\partial U) \subset \partial\Omega. \tag{2.7.49}$$

Then by the previous computation we have

$$\Theta\left(\overline{\partial U \setminus Y}\right) \subseteq A \cap \partial\Omega.$$

Thus, to reach the claim, we need to show that $\overline{\partial U \setminus Y} \neq \emptyset$. But is an immediate consequence of (2.7.48) and Proposition A.14 that

$$\text{genus}(\overline{\partial U \setminus Y}) \geq \text{genus}(\partial U) - \text{genus}(Y) \geq (j + l) - l = j \geq 1$$

which directly excludes the possibility that $\overline{\partial U \setminus Y}$ is empty.

Focus now on (2.7.49); we first observe that, by continuity, we have

$$\partial_{D_{j+l}} U = \partial_{D_{j+l}}(\Theta^{-1}(\Omega)) \subset \Theta^{-1}(\partial\Omega),$$

where $\partial_{D_{j+l}}$ is the boundary with respect to the topology restricted to D_{j+l}, but this is not enough, since $\partial_{D_{j+l}} U$ is generally smaller than ∂U (the boundary made with respect to the whole space $\mathbb{R}^{j+l}$), which is the one appearing in (2.7.48). Let thus $\xi \in \partial U$; we need to show that $\Theta(\xi) \in \partial\Omega$. By definition of U, we have $\Theta(\xi) \in \Theta(\overline{\Theta^{-1}(\Omega)}) \subset \overline{\Omega}$; assume by contradiction

$\Theta(\xi) \in \Omega$. We first observe that $\xi \notin \partial D_{j+l}$, by definition of $\Theta \in \Gamma^m_{j+l}$, thus ξ is in the interior of D_{j+l}. We then can find a neighborhood N_1 of ξ (with respect to $\mathbb{R}^{j+l}$) contained in D_{j+l}, and a neighborhood M_2 of $\Theta(\xi)$ contained in Ω; set $N := N_1 \cap \Theta^{-1}(M_2)$, we have that N is a neighborhood of ξ (with respect to $\mathbb{R}^{j+l}$) contained in U, which implies that ξ is in the interior of U, absurd. This concludes the proof of the first part.

We prove now the consequence. Indeed, for each $A \in \Lambda^m_j$ we have

$$E_m = \inf_{\partial\Omega} \mathcal{I}^m \leq \inf_{\partial\Omega\cap A} \mathcal{I}^m \leq \sup_{\partial\Omega\cap A} \mathcal{I}^m \leq \sup_A \mathcal{I}^m$$

and thus the claim passing to the infimum over Λ^m_j. ∎

Let us now show the main properties of Λ^m_j and c^m_j, which will actually be the only ones used in the multiplicity result.

Proposition 2.7.7. *Let $j \in \mathbb{N}$.*

1. *$\Lambda^m_j \neq \emptyset$;*
2. *$\Lambda^m_{j+1} \subseteq \Lambda^m_j$, and thus $c^m_j \leq c^m_{j+1}$;*
3. *let $A \in \Lambda^m_j$ and $Z \subset \mathbb{R} \times H^s_r(\mathbb{R}^N)$ be $\mathbb{G}$-invariant, closed, and such that $0 \notin \overline{P_2(Z)}$ and $\mathrm{genus}(\overline{P_2(Z)}) \leq i$. Then $\overline{A \setminus Z} \in \Lambda^m_{j-i}$.*

Fix now $k \in \mathbb{N}$, and let $m > m_k$, where m_k has been introduced in (2.7.47). Then

4. *$c^m_j < 0$ and $\mathcal{I}^m$ satisfies $(PSP)_{c^m_j}$;*
5. *if $A \in \Lambda^m_j$ and η is a deformation as in Theorem 2.7.1 for $b = c^m_j$, then $\eta(1, A) \in \Lambda^m_j$.*

Proof. Properties 1) and 4) have already been shown in the Lemma 2.7.6, while property 2) is a consequence of the definition. Let us see properties 3) and 5).

3) Let $A = \Theta(\overline{D_{j+l} \setminus Y}) \in \Lambda^m_j$ and let Z be $\mathbb{G}$-invariant, closed and such that $0 \notin \overline{P_2(Z)}$ and $\mathrm{genus}(\overline{P_2(Z)}) \leq i$. Assume it holds

$$\begin{aligned} \overline{A \setminus Z} &= \Theta(\overline{(D_{j+l} \setminus Y) \setminus \Theta^{-1}(Z)}) \\ &= \Theta(\overline{D_{(j-i)+(l+i)} \setminus (Y \cup \Theta^{-1}(Z))}); \end{aligned} \tag{2.7.50}$$

if $\mathrm{genus}(Y \cup \Theta^{-1}(Z)) \leq l + i$ we have the claim. But this is a direct consequence of the assumptions and Proposition A.14, since

$$\begin{aligned} \mathrm{genus}(Y \cup \Theta^{-1}(Z)) &\leq \mathrm{genus}(Y) + \mathrm{genus}(\Theta^{-1}(Z)) \\ &\leq l + \mathrm{genus}(\overline{h(\Theta^{-1}(Z))}) \\ &= l + \mathrm{genus}(\overline{P_2(Z)}) \leq l + i \end{aligned}$$

where we have set $h := P_2 \circ \Theta$, which is an odd map and thus admissible for the genus.

Turn now to (2.7.50). Set $B := D_{j+l} \setminus Y$ and $W := \Theta^{-1}(Z)$ we have to prove

$$\overline{\Theta(\overline{B}) \setminus \Theta(W)} = \Theta(\overline{B \setminus W}).$$

We have

$$\overline{\Theta(\overline{B}) \setminus \Theta(W)} \subseteq \overline{\Theta(\overline{B} \setminus W)} \overset{(i)}{\subseteq} \overline{\Theta(\overline{B \setminus W})} \overset{(ii)}{=} \Theta(\overline{B \setminus W})$$

and

$$\Theta(\overline{B \setminus W}) \overset{(iii)}{\subseteq} \overline{\Theta(B \setminus W)} \overset{(iv)}{=} \overline{\Theta(B) \setminus \Theta(W)} \subseteq \overline{\Theta(\overline{B}) \setminus \Theta(W)}$$

where

(i) is due to the fact that W is closed;

(ii) $\overline{B \setminus W} \subseteq D_{j+l}$ is compact, thus $\Theta(\overline{B \setminus W})$ is closed;

(iii) derives from the continuity of Θ;

(iv) is due to the fact that W is a preimage.

5) Consider $0 < \bar{\varepsilon} < 1$, $b = c_j^m \geq B_m$ and η as in the deformation lemma, and fix $A = \Theta(\overline{D_{j+l} \setminus Y}) \in \Lambda_j^m$ with $\Theta \in \Gamma_{j+l}^m$. To show that $\eta(1, A) \in \Lambda_j^m$ and conclude the proof, it is sufficient to show that $\tilde{\Theta} := \eta(1, \Theta) \in \Gamma_{j+l}^m$ as well.

- $\tilde{\Theta}(-\xi) = \eta(1, \Theta(-\xi)) = \eta(1, \Theta_1(-\xi), \Theta_2(-\xi)) = \eta(1, \Theta_1(\xi), -\Theta_2(\xi))$ and thus

$$\begin{aligned}(\tilde{\Theta}_1(-\xi), \tilde{\Theta}_2(-\xi)) &= \Big(\eta_1(1, \Theta_1(\xi), -\Theta_2(\xi)), \eta_2(1, \Theta_1(\xi), -\Theta_2(\xi))\Big) \\ &= \Big(\eta_1(1, \Theta_1(\xi), \Theta_2(\xi)), -\eta_2(1, \Theta_1(\xi), \Theta_2(\xi))\Big) = (\tilde{\Theta}_1(\xi), -\tilde{\Theta}_2(\xi))\end{aligned}$$

  which shows that $\tilde{\Theta}_1$ is even and $\tilde{\Theta}_2$ is odd.
- By Lemma 2.7.6, for $\xi = 0$ and $\xi \in \partial D_{j+l}$ we have $\mathcal{I}^m(\Theta(\xi)) \leq B_m - 1 = E_m - 1 \leq c_j^m - \bar{\varepsilon}$, thus $\Theta(\xi) \in [\mathcal{I}^m \leq c_j^m - \bar{\varepsilon}]$. Therefore $\tilde{\Theta}(\xi) = \eta(1, \Theta(\xi)) = \Theta(\xi)$ for $\xi = 0$ and $\xi \in \partial D_{j+l}$, and the same properties are satisfied. ∎

2.7.3 Multiplicity theorem

Fix $k \in \mathbb{N}^*$, and let Λ_j^m and c_j^m be given in the previous Section for $j = 1 \ldots k$. Exploiting the properties given in Proposition 2.7.7, we can find multiple solutions.

Theorem 2.7.8. *Fix $k \in \mathbb{N}^*$, and assume $m > m_k$. We have that*

$$c_1^m \leq c_2^m \leq \cdots \leq c_k^m < 0$$

are critical values of $\mathcal{I}^m$. Moreover

(i) *if, for some $q \geq 1$,*

$$c_j^m < c_{j+1}^m < \cdots < c_{j+q}^m$$

then we have $q+1$ different nonzero critical values, and thus $q+1$ different (pairs of) nontrivial solutions of (2.1.2)*;*

(ii) *if instead, for some $q \geq 1$,*

$$c_j^m = c_{j+1}^m = \cdots = c_{j+q}^m \equiv b \tag{2.7.51}$$

then

$$\text{genus}(P_2(K_b^{PSP})) \geq q + 1 \tag{2.7.52}$$

and thus (by Proposition A.14) $\#P_2(K_b^{PSP}) = +\infty$, which means that we have infinite different solutions of (2.1.2)*.*

Summing up, we have at least k different (pairs of) solutions of (2.1.2) *which satisfy the Pohozaev identity* (2.1.3)*.*

Proof. It is sufficient to show only the property (2.7.52) on the genus: indeed by choosing $q = 0$ we have that, for each j, $\#(K_{c_j^m}^{PSP}) \geq 1$ and thus c_j^m is a nontrivial critical value.

By the $(PSP)_b$ we have that K_b^{PSP} is compact, thus $P_2(K_b^{PSP})$ is compact; moreover it is symmetric with respect to 0 and does not contain 0 (see Corollary 2.4.3).

By Proposition A.14 we can find a (closed, symmetric with respect to origin, not containing the zero) neighborhood N of $P_2(K_b^{PSP})$ which preserves the genus, i.e. $\text{genus}(N) = \text{genus}(P_2(K_b^{PSP}))$.

We can easily think N as a projection of a neighborhood Z of K_b^{PSP} (i.e. $N = P_2(Z)$) satisfying the properties of Proposition 2.7.7.

By Theorem 2.7.1, there exist a sufficiently small ε and an η such that $\eta([\mathcal{I}^m \le b+\varepsilon] \setminus Z) \subseteq [\mathcal{I}^m \le b-\varepsilon]$. Corresponding to ε, by definition of c_j^m, there exists an $A \in \Lambda_{j+q}^m$ such that $\sup_A \mathcal{I}^m < b+\varepsilon$, that is $A \subseteq [\mathcal{I}^m \le b+\varepsilon]$. Thus, being $\eta(1,\cdot)$ continuous

$$\begin{aligned}\eta(1,\overline{A\setminus Z}) &\subseteq \eta(1,\overline{[\mathcal{I}^m \le b+\varepsilon]\setminus Z}) \subseteq \overline{\eta(1,[\mathcal{I}^m \le b+\varepsilon]\setminus Z)}\\ &\subseteq \overline{[\mathcal{I}^m \le b-\varepsilon]} = [\mathcal{I}^m \le b-\varepsilon],\end{aligned}$$

and hence

$$\sup_{\eta(1,\overline{A\setminus Z})} \mathcal{I}^m \le b-\varepsilon. \tag{2.7.53}$$

On the other hand, assume by contradiction that $\text{genus}(P_2(K_b^{PSP})) \le q$, i.e. $\text{genus}(P_2(Z)) \le q$. We use now the properties on c_j^m and Λ_j^m.

Replacing j with $j+q$ and i with q and applying Proposition 2.7.7, we have $\overline{A\setminus Z} \in \Lambda_j^m$; by property 5) of Proposition 2.7.7 we obtain $\eta(1,\overline{A\setminus Z}) \in \Lambda_j^m$, which implies (by definition of c_j^m)

$$\sup_{\eta(1,\overline{A\setminus Z})} \mathcal{I}^m \ge c_j^m = b.$$

This is a contradiction with (2.7.53), and thus concludes the proof. ∎

Proof of Theorem 2.1.3. As consequence of Theorem 2.7.8, we derive (i). We pass to prove (ii). Under condition (g4), we have $m_k = 0$ for all $k \in \mathbb{N}$. Thus for any $j \in \mathbb{N}$, c_j^m is a critical value of $\mathcal{I}^m$ and $c_j^m \le b_j^m < 0$. Since c_j^m is an increasing sequence, we have $c_j^m \to \bar{c} \le 0$ as $j \to \infty$. We need to show that $\bar{c} = 0$.

By contradiction we assume $\bar{c} < 0$. Then $K_{\bar{c}}^{PSP}$ is compact and $K_{\bar{c}}^{PSP} \cap (\mathbb{R} \times \{0\}) = \emptyset$. It follows that $q = \text{genus}(P_2(K_{\bar{c}}^{PSP})) < \infty$. Arguing as in the proof of Theorem 2.7.8, let $\delta > 0$ be such that $q = \text{genus}(P_2(N_\delta(K_{\bar{c}}^{PSP}))) < \infty$. By Theorem 2.7.1, there exist $\varepsilon \in (0,1)$ small and $\eta : [0,1] \times \mathbb{R} \times H_r^s(\mathbb{R}^N) \to \mathbb{R} \times H_r^s(\mathbb{R}^N)$ satisfying

$$\eta(1,[\mathcal{I}^m \le \bar{c}+\varepsilon] \setminus N_\delta(K_{\bar{c}}^{PSP})) \subseteq [\mathcal{I}^m \le \bar{c}-\varepsilon] \tag{2.7.54}$$

and

$$\eta(t,\lambda,u) = (\lambda,u) \quad \text{if } \mathcal{I}^m(\lambda,u) \le B_m - 1. \tag{2.7.55}$$

We can choose $j \in \mathbb{N}$ sufficiently large such that $c_j^m > \bar{c} - \varepsilon$ and take $B \in \Lambda_{j+q}^m$ such that $B \subset [\mathcal{I}^m \le \bar{c}+\varepsilon]$. Then we have

$$\overline{B \setminus N_\delta(K_{\bar{c}}^{PSP})} \in \Lambda_j^m.$$

From equations (2.7.54), (2.7.55) we derive $c_j^m \le \bar{c} - \varepsilon$, which gives a contradiction. ∎

Remark 2.7.9. *We observe that, even if the problem is invariant under translations, the found solutions are not translations of a same solution since they all are radially symmetric. Moreover, assuming* (g4)*, since* $0 > c_j^m \to 0$ *we easily find a sequence of solutions with distinct energy levels.*

2.8 L^2-minimum

In Theorems 2.1.1 and 2.1.2 we find a solution via mountain pass minimax methods. We remark that this solution is characterized as minimizer of the functional $\mathcal{L}$ on $\mathcal{S}_m$, where $\mathcal{L} : H_r^s(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\mathcal{L}(u) := \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 - \int_{\mathbb{R}^N} G(u)$$

and $\mathcal{S}_m$ is the L^2-sphere in $H^S_r(\mathbb{R}^N)$, i.e.

$$\mathcal{S}_m := \{u \in H^s_r(\mathbb{R}^N) \mid \|u\|_2^2 = m\}.$$

Set

$$\kappa_m := \inf_{u\in\mathcal{S}_m} \mathcal{L}(u).$$

Proposition 2.8.1. *Assume (g1)–(g3), and let $m \geq m_0$, where m_0 is introduced in Proposition 2.6.2. We have that the following statements hold.*

(i) The Mountain Pass level and the ground state level coincide, i.e.

$$\kappa_m = b_m. \tag{2.8.56}$$

In particular, thanks to Corollary 2.6.3, there exists a ground state of $\mathcal{L}_{|\mathcal{S}_m}$.

(ii) Every ground state of $\mathcal{L}_{|\mathcal{S}_m}$ satisfies the Pohozaev identity (2.1.3) with μ the associated Lagrange multiplier. Thanks to (2.8.56), the same conclusion holds for every Mountain Pass solution at level b_m.

(iii) Every ground state of $\mathcal{L}_{|\mathcal{S}_m}$ has a positive associated Lagrange multiplier. This means that every ground state of $\mathcal{L}_{|\mathcal{S}_m}$ is a solution of problem (2.1.2).

Moreover, if (g4) holds, then $m_0 = 0$.

Proof. (i) Let u_* be the Mountain Pass solution obtained in Corollary 2.6.3, which verifies $\|u_*\|_2^2 = m$. Thus,

$$\kappa_m \leq \mathcal{L}(u_*) = b_m < 0. \tag{2.8.57}$$

In particular, by (2.8.57) we can find a minimizing sequence $(u_n)_n \subset \mathcal{S}_m$ for κ_m satisfying $\mathcal{L}(u_n) < 0$, and thus we can set

$$e^{\lambda_n} := \frac{2}{Nm}\Big(s\|(-\Delta)^{s/2}u_n\|_2^2 - N\mathcal{L}(u_n)\Big) > 0$$

so that $\mathcal{P}(\lambda_n, u_n) = 0$, i.e., $(\lambda_n, u_n) \in \partial\Omega$. At this point Proposition 2.6.2 implies

$$\kappa_m + o(1) = \mathcal{L}(u_n) = \mathcal{I}^m(\lambda_n, u_n) \geq E_m = b_m.$$

Passing to the limit, together with (2.8.57), we have (2.8.56).

(ii) Let u_0 be a minimizer of $\mathcal{L}$ on $\mathcal{S}_m$. Corresponding to u_0, there exists a Lagrange multiplier $\mu_0 \in \mathbb{R}$ such that

$$(-\Delta)^{s/2}u_0 + \mu_0 u_0 = g(u_0),$$

and thus, in particular,

$$\|(-\Delta)^{s/2}u_0\|_2^2 + \mu_0\|u_0\|_2^2 - \int_{\mathbb{R}^N} g(u_0)u_0\,dx = 0. \tag{2.8.58}$$

We show first that u_0 satisfies the Pohozaev identity. In fact, we consider the $\mathbb{R}$-action Φ : $\mathbb{R} \times \mathcal{S}_m \to \mathcal{S}_m$ defined by

$$(\Phi_\theta v)(x) := e^{\frac{N}{2}\theta}v(e^\theta x), \tag{2.8.59}$$

since $\|\Phi_\theta v\|_2^2 = \|v\|_2^2$. Then we have

$$\mathcal{L}(\Phi_\theta u_0) = \frac{1}{2}e^{2s\theta}\|(-\Delta)^{s/2}u_0\|_2^2 - e^{-N\theta}\int_{\mathbb{R}^N} G(e^{\frac{N}{2}\theta}u_0).$$

Since u_0 is a minimizer, we have $\frac{d}{d\theta}\big|_{\theta=0}\mathcal{L}(\Phi_\theta u_0) = 0$, that is,

$$s\|(-\Delta)^{s/2}u_0\|_2^2 + N\int_{\mathbb{R}^N} G(u_0) - \frac{N}{2}\int_{\mathbb{R}^N} g(u_0)u_0\,dx = 0. \tag{2.8.60}$$

From (2.8.58) and (2.8.60), the Pohozaev identity follows

$$\frac{N-2s}{2}\|(-\Delta)^{s/2}u_0\|_2^2 + \frac{N}{2}\mu_0\|u_0\|_2^2 - N\int_{\mathbb{R}^N} G(u_0) = 0. \tag{2.8.61}$$

(iii) Finally, from (2.8.57) we have $\mathcal{L}(u_0) = \kappa_m < 0$, that is

$$\frac{1}{2}\|(-\Delta)^{s/2}u_0\|_2^2 - \int_{\mathbb{R}^N} G(u_0) = \kappa_m < 0, \tag{2.8.62}$$

which joined to (2.8.61) gives $\mu_0 > 0$. This concludes the proof. ∎

Remark 2.8.2. *By [275, Theorem 4.1], we have that actually every L^2-minimum is radially symmetric (up to a translation). Thus κ_m coincide with the infimum made on the L^2-ball of the whole space $H^s(\mathbb{R}^N)$.*

2.9 Relation between constrained and unconstrained problems

Let $0 < \mu < \mu_0$ and $m > 0$. By joining the results of Proposition 2.6.2 and Proposition 4.2.9, we proved the following relation.

$$\kappa(m) = \inf_{\mu\in(0,\mu_0)} \big(p(\mu) - \mu\, m\big) \tag{2.9.63}$$

where we slightly changed the definition of L^2-minimum

$$\kappa(m) := \inf_{\substack{u\in H^s_r(\mathbb{R}^N)\\ \frac{1}{2}\|u\|_2^2=m}} \left(\frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 - \int_{\mathbb{R}^N} G(u)\right)$$

and of Pohozaev minimum

$$p(\mu) := \inf_{\substack{u\in H^s_r(\mathbb{R}^N)\setminus\{0\}\\ \|(-\Delta)^{s/2}u\|_2^2 + 2^*_s\left(\frac{\mu}{2}\|u\|_2^2 - \int_{\mathbb{R}^N} G(u)\right)=0}} \left(\frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 - \int_{\mathbb{R}^N} G(u) + \frac{\mu}{2}\|u\|_2^2\right);$$

we recall that, when $s \in (\frac{1}{2}, 1)$ or $g \in C^\sigma_{loc}(\mathbb{R})$ for some $\sigma > 1-2s$, $p(\mu)$ is actually a ground state level.

The relation between the unconstrained and the constrained problem is an old-fashioned problem, which has been deeply investigated in a recent paper by Jeanjean and Lu [236] in the case $s = 1$. We see that equation (2.9.63) gives an interesting relation between the two energy levels: this relation may be also reformulated by saying that

$$\kappa(m) = -p^*(m) \tag{2.9.64}$$

where p^* is the *Legendre transform* of a. A relation of this type, but in a different framework, has been also obtained by Dovetta, Serra and Tilli in a very recent paper [162]. Here, relying on the convexity of the energy functions (due to the polynomial shape of g), they exploit (2.9.64) in order to achieve interesting results.

We believe thus that (2.9.64) could give more insights in the study of the relation between these two problems.

CHAPTER 3

Choquard-Hartree-Pekar equations: multiplicity of solutions

In this Chapter we study the following nonlinear *Choquard-Hartree-Pekar equation*

$$-\Delta u + \mu u = (I_\alpha * F(u))F'(u) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in (0, N)$, I_α is the Riesz potential, and F is an almost optimal subcritical nonlinearity. The goal is to prove existence of infinitely many solutions $u \in H^1_r(\mathbb{R}^N)$, by assuming F odd or even.

We analyze the two cases: μ is a fixed positive constant or μ is unknown and the L^2-norm of the solution is prescribed, i.e. $\int_{\mathbb{R}^N} u^2 = m > 0$. Since the presence of the nonlocality prevents to apply the classical approach introduced by Berestycki and Lions in [51], we implement a new construction of multidimensional odd paths, and we find a nonlocal counterpart of their multiplicity result. In particular we extend the existence result in [302], due to Moroz and Van Schaftingen.

This Chapter is mainly based on the paper [116].

3.1 Convolution with Riesz potential: a self-interaction

Given a nonlinearity $F \in C^1(\mathbb{R}, \mathbb{R})$ and set $f := F'$, we are interested to seek for multiple solutions $u \in H^1_r(\mathbb{R}^N)$ of the nonlocal equation

$$-\Delta u + \mu u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{3.1.1}$$

where $N \geq 3$ and $\alpha \in (0, N)$. In literature the semilinear equation (3.1.1) with nonlocal source has several physical motivations and it is usually called nonlinear *Choquard* (or *Hartree*, or *Pekar*) *equation*.

In 1954 the equation (3.1.1) with $N = 3$, $\alpha = 2$ and $F(s) = \frac{1}{2}|s|^2$, that is

$$-\Delta u + \mu u = \left(\frac{1}{4\pi|x|} * |u|^2\right) u \quad \text{in } \mathbb{R}^3, \tag{3.1.2}$$

was elaborated by Pekar in [313] (see also [260]) to describe the quantum theory of a polaron at rest, through the use of the Newton potential $\frac{1}{4\pi|x|}$. The idea of the convolution as a feature of interaction of a body with itself was exploited also by other authors: in 1976 it was arisen in the work [264] suggested by Choquard [106] on the modeling of an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one-component plasma (see also [194, 196, 355]). In 1996 the same equation was derived by Penrose in his discussion on the self-gravitational collapse of a quantum mechanical wave-function [299, 314–316] (see also [196, 361, 362]) and in that context it is referred as *Schrödinger-Newton system* (see (1.3.39)). See also Section 3.1 for a derivation concerning exotic stars.

If u is a solution of (3.1.2), then we notice that the wave function

$$\psi(x,t) = e^{i\mu t}u(x), \quad (x,t) \in \mathbb{R}^3 \times [0,+\infty)$$

is a solitary wave of the time-dependent *Hartree equation* [216]

$$i\psi_t = -\Delta\psi - \left(\frac{1}{4\pi|x|} * |\psi|^2\right)\psi \quad \text{in } \mathbb{R}^3 \times (0,+\infty); \tag{3.1.3}$$

thus (3.1.2) represents the stationary nonlinear Hartree equation.

As already pointed out in Chapter 2, the study of standing waves of (3.1.3) has been pursed in two main directions, which opened two different challenging research fields.

A first topic regards the search for solutions of (3.1.2) with a prescribed frequency μ and free mass, the so-called *unconstrained* problem. The second line of investigation of the problem (3.1.3) consists of prescribing the mass $m > 0$ of u, thus conserved by ψ in time

$$\int_{\mathbb{R}^3} |\psi(x,t)|^2\, dx = m \quad \forall\, t \in [0,+\infty),$$

and letting the frequency μ to be free. Such problem is usually said *constrained*.

For the unconstrained problem, the first investigations for existence and symmetry of the solutions to (3.1.2) go back to the works of Lieb [265] and Menzala [293], and also to [108, 299, 355] by means of ordinary differential equations techniques. We mention also the recent papers by Lenzmann [257] and by Winter and Wei [375] about the nondegeneracy of the unique radial solution of (3.1.2).

Variational methods were also employed to derive existence and qualitative results of standing wave solutions for more generic values of $\alpha \in (0,N)$ and of power type nonlinearities $F(t) = \frac{1}{p}|t|^p$: in particular Moroz and Van Schaftingen [300] (see also [304]) considered the special model

$$-\Delta u + \mu u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{3.1.4}$$

and they proved that (3.1.4) has solutions if

$$2^\#_\alpha = \frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2} = 2^*_\alpha. \tag{3.1.5}$$

When dealing with variational (and regular) solutions, they proved that range (3.1.5) is optimal. Moreover in [300] they showed that all positive ground states of (3.1.4) are radially symmetric and monotone decreasing about some point and derived the decay asymptotics at infinity of such ground states (see [109] for $p \geq 2$, and also [279]). Furthermore, in [205, 206, 332] the authors study, for some values of p and α, least energy nodal solutions, odd with respect to a hyperplane; see also [109, 128, 372, 378, 384] for other results on sign-changing solutions with various symmetries and saddle type solutions.

Recently in [302] Moroz and Van Schaftingen considered the problem (3.1.1) when F is a Berestycki-Lions type function under the following general assumptions:

(F1) $F \in C^1(\mathbb{R},\mathbb{R})$;

(F2) there exists $C > 0$ such that, for every $s \in \mathbb{R}$,

$$|sf(s)| \leq C\big(|s|^{2^\#_\alpha} + |s|^{2^*_\alpha}\big);$$

(F3)
$$\lim_{s\to 0}\frac{F(s)}{|s|^{2^\#_\alpha}}=0,\quad \lim_{s\to+\infty}\frac{F(s)}{|s|^{2^*_\alpha}}=0;$$

(F4) $F(s)\not\equiv 0$, that is, there exists $s_0\in\mathbb{R}$, $s_0\neq 0$ such that $F(s_0)\neq 0$.

In particular they prove the following theorem (see [302, Theorems 1 and 4]).

Theorem 3.1.1 ([302])**.** *We have the following results.*

- *Assume (F1)–(F4). Then there exists a ground state solution $u\in H^1(\mathbb{R}^N)$. Moreover $u\in W^{2,q}_{loc}(\mathbb{R}^N)$ for each $q\geq 1$ (in particular, u is Hölder continuous);*
- *Assume (F1)-(F2), f odd and with constant sign on $(0,+\infty)$. Then every ground state has strict constant sign (strictly positive or negative) and it is radially symmetric with respect to some point in $\mathbb{R}^N$.*

The qualitative result contained in Theorem 3.1.1 will be extended in this thesis to the case f even, see Theorem 4.5.3. The existence of an infinite number of standing wave solutions to (3.1.2) was instead faced by Lions in [271] (see also [128]); here the homogeneity of the source plays a crucial role in order to work on finite dimensional subspaces. Similar ideas have been applied in [8, 323] in presence of more general sources satisfying Ambrosetti-Rabinowitz type conditions. We remark that all these multiplicity results deal with odd power nonlinearities f.

To our knowledge it is still an open problem the existence of infinitely many radially symmetric solutions for the nonlinear Choquard equation (3.1.1) under the optimal assumptions (F1)–(F4) and symmetric conditions on the nonlocal source term $(I_\alpha * F(u))f(u)$, and this is the aim of this Chapter. We note that this nonlinear term is odd both if f is even or odd.

Existence of a solution for the nonlinear Choquard equation (3.1.4) under mass constraint has been obtained by Ye [388]; see also [261] for odd powers-sum type functions. More recently, Cingolani and Tanaka in [124] obtained existence of a solution $u\in H^1_r(\mathbb{R}^N)$ to

$$\begin{cases}-\Delta u+\mu u=(I_\alpha * F(u))f(u) & \text{in } \mathbb{R}^N,\\ \displaystyle\int_{\mathbb{R}^N} u^2dx=m,\end{cases}\tag{3.1.6}$$

assuming that F satisfies (F1), (F4) and it is L^2-subcritical, namely

(CF2) there exists $C>0$ such that, set $2^m_\alpha=\frac{N+\alpha+2}{N}$, for every $s\in\mathbb{R}$,
$$|sf(s)|\leq C\big(|s|^{2^\#_\alpha}+|s|^{2^m_\alpha}\big);$$

(CF3)
$$\lim_{s\to 0}\frac{F(s)}{|s|^{2^\#_\alpha}}=0,\quad \lim_{s\to+\infty}\frac{F(s)}{|s|^{2^m_\alpha}}=0.$$

The existence result in [124] relies on a Lagrangian formulation of the problem, in the spirit of Chapter 2.

Multiplicity of radial standing wave solutions to (3.1.3) with prescribed L^2-norm has been instead faced again by Lions in [271] (see also [118] for the planar logarithmic Choquard equation); as regards instead the case of general nonlinearities f, recently Bartsch et al. [37] obtained the existence of infinitely many solutions of (3.1.6) by assuming that f is an odd function which satisfies monotonicity and Ambrosetti-Rabinowitz conditions. We highlight that the restriction on odd functions is not just a matter of symmetry of the functional, but it is related also to some sign restriction on the function f. The authors in [37] rely on mountain pass and Concentration-Compactness arguments, together with the use of a stretched functional, i.e. a functional in an augmented space which takes into consideration scaling properties and the Pohozaev identity.

It remains open the challenging problem of the existence of infinitely many solutions for the constrained nonlinear Choquard equation (3.1.6) under optimal assumptions on the nonlinearity f, when monotonicity and Ambrosetti-Rabinowitz type conditions do not hold or f is not odd.

In the present Chapter we will give an affirmative answer to both the *unconstrained* and *constrained* problems when F satisfies the general Berestycki-Lions type assumptions (F1)–(F4) and (F1)-(CF2)-(CF3)-(F4) respectively, together with the symmetric condition

(F5) F is odd or even.

We begin to notice that despite [124], where existence is investigated, to gain multiplicity the symmetry of the function F plays a crucial role. In particular, we assume F to be *odd* or *even*, which guarantees the evenness of the energy functional associated to (3.1.1). We emphasize that the possibility to assume both the symmetries on F is a particular feature of the nonlocal source: indeed, in the source-local case [51,224] (see also Chapter 2), the nonlinear term is usually assumed odd in order to get the symmetry of the functional. We mention the recent paper [137] where the existence of a single nonradial solution to (3.1.1) is obtained under the condition (F5).

We start to analyze the constrained case, which appears, as usual, more delicate. By virtue of [310], radially symmetric solutions to (3.1.6) can be characterized as critical points of the C^1-functional $\mathcal{L} : H^1_r(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{L}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\, dx,$$

constrained on the sphere

$$\mathcal{S}_m := \left\{ u \in H^1_r(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} u^2\, dx = m \right\}.$$

A possible approach to problem (3.1.6) is to minimize $\mathcal{L}$ on the sphere $\mathcal{S}_m$, whenever the functional is here bounded. Nevertheless, in the spirit of Chapter 2, for the general class of nonlinearities related to [50, 302], considered in this thesis, we introduce a Lagrangian formulation of the nonlocal problem (3.1.6), extending a new approach introduced by Hirata and Tanaka [224] for the local case. We highlight again the advantage of this method, that can be suitably adapted to derive multiplicity results of normalized solutions in several different frameworks.

We recall here briefly the ideas of Chapter 2. Writing $\mathbb{R}_+ := (0, +\infty)$, a solution $(\mu, u) \in \mathbb{R}_+ \times H^1_r(\mathbb{R}^N)$ of (3.1.6) corresponds to a critical point of the functional $\mathcal{I}^m : \mathbb{R}_+ \times H^1_r(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{I}^m(\mu, u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\, dx + \frac{\mu}{2}\left(\int_{\mathbb{R}^N} u^2\, dx - m\right).$$

We seek for critical points $(\mu, u) \in \mathbb{R}_+ \times H^1_r(\mathbb{R}^N)$ of $\mathcal{I}^m$, namely weak solutions of $\partial_u \mathcal{I}^m(\mu, u) = 0$ and $\partial_\mu \mathcal{I}^m(\mu, u) = 0$.

Inspired by the Pohozaev identity, we introduce the Pohozaev functional $\mathcal{P} : \mathbb{R}_+ \times H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{P}(\mu, u) := \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + N\frac{\mu}{2}\int_{\mathbb{R}^N} u^2\, dx - \frac{N+\alpha}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\, dx$$

and the *Pohozaev set*

$$\Omega := \{(\mu, u) \in \mathbb{R}_+ \times H^1_r(\mathbb{R}^N) \mid \mathcal{P}(\mu, u) > 0\} \cup \{(\mu, 0) \mid \mu \in \mathbb{R}_+\}.$$

We note that $\{(\mu,0) \mid \mu \in \mathbb{R}_+\} \subset int(\Omega)$ and thus

$$\partial\Omega = \{(\mu,u) \in \mathbb{R}_+ \times H^1_r(\mathbb{R}^N) \mid \mathcal{P}(\mu,u)=0,\ u \not\equiv 0\},$$

where the interior and the boundary are taken with respect to the topology of $\mathbb{R}_+ \times H^1_r(\mathbb{R}^N)$. Therefore $(\mu,u) \in \partial\Omega$ if and only if $u \not\equiv 0$ satisfies the Pohozaev identity $\mathcal{P}(\mu,u)=0$. We recognize a Mountain Pass structure for the functional $\mathcal{I}^m$ in $\mathbb{R}_+ \times H^1_r(\mathbb{R}^N)$, where the mountain is given by $\partial\Omega$. We call $\partial\Omega$ a *Pohozaev mountain* for $\mathcal{I}^m$. We emphasize that under assumptions (F1)-(F2), if $u \in H^1_r(\mathbb{R}^N)$ solves $\partial_u \mathcal{I}^m(\mu,u)=0$ with $\mu \in \mathbb{R}_+$ fixed, then $\mathcal{P}(\mu,u)=0$.

Using a variant of the Palais-Smale condition [224,231], which takes into account the Pohozaev identity, we will prove a deformation theorem which enables us to apply minimax arguments in the product space $\mathbb{R}_+ \times H^1_r(\mathbb{R}^N)$. We will prove the existence of multiple L^2-normalized solutions detecting minimax structures in such product space.

We state our main results.

Theorem 3.1.2. *Suppose $N \geq 3$, $\alpha \in (0,N)$ and (F1)-(CF2)-(CF3)-(F4)-(F5).*

(i) For any $k \in \mathbb{N}$ there exists $m_k \geq 0$ such that for every $m > m_k$, the problem (3.1.6) has at least k pairs of nontrivial, distinct, radially symmetric solutions.

(ii) Assume in addition an L^2-subcritical growth also at zero, i.e.

(CF4)

$$\lim_{s\to 0} \frac{|F(s)|}{|s|^{2^m_\alpha}} = +\infty;$$

additionally, if F is odd, assume that there exists $\delta_0 > 0$ such that F has a constant sign in $(0,\delta_0]$ and

$$\sup_{s\in(0,\delta_0],\, h\in[0,1]} \frac{F(sh)}{F(s)} < +\infty; \tag{3.1.7}$$

for example, this is satisfied if $|F(s)|$ is assumed non-decreasing in $[0,\delta_0]$.

Then $m_k = 0$ for each $k \in \mathbb{N}$, that is for any $m > 0$ the problem (3.1.6) has countably many pairs of solutions $(\mu_n,u_n)_n$ satisfying $\mathcal{L}(u_n) < 0$, $n \in \mathbb{N}$. Moreover we have

$$\mathcal{L}(u_n) \to 0 \quad \text{as } n \to +\infty.$$

Remark 3.1.3. *We comment condition (3.1.7). Set*

$$M := \sup_{s\in(0,\delta_0],\, h\in[0,1]} \frac{F(sh)}{F(s)} < +\infty$$

we have, when $|F(s)|$ is non-decreasing, $M=1$. As a nontrivial example one can consider $\beta \in (2^\#_\alpha, 2^m_\alpha)$ and F oscillating near zero between $|s|^\beta$ and $2|s|^\beta$, so that $M \leq 2$; for instance the odd extension of

$$F(s) := s^\beta\big(2+\sin(\tfrac{1}{s})\big) \quad \text{as } s \to 0^+.$$

If instead F oscillates (not strictly) between $|s|^{\beta_1}$ and $|s|^{\beta_2}$, with $2^\#_\alpha < \beta_1 < \beta_2 < 2^m_\alpha$, then $M=+\infty$; thus for instance the odd extension of

$$F(s) := s^{\beta_1}\big(1+\sin(\tfrac{1}{s})\big) + s^{\beta_2}\big(1-\sin(\tfrac{1}{s})\big) \quad \text{as } s \to 0^+$$

is not covered by (3.1.7).

Remark 3.1.4. *We observe that, by substituting F with $-F$, there is no loss of generality in assuming*

$$F(s_0) > 0 \quad \textit{for some } s_0 \neq 0$$

in (F4) *(s_0 can be chosen positive if, for example,* (F5) *holds) and*

$$\lim_{s \to 0^+} \frac{F(s)}{|s|^{2_\alpha^m}} = +\infty$$

in (CF4). *Thus, for the remaining part of the Chapter, we assume this positivity on the right-hand side of zero.*

A key point of the argument is the construction of multidimensional odd paths. When f satisfies some Ambrosetti-Rabinowitz condition (i.e., F can be estimated from below by an homogeneous function $|t|^p$), the construction of such a path classically relies on the equivalence of the H^1-norm and the L^p-norm on finite dimensional subspaces of $H^1(\mathbb{R}^N)$. When such condition is no longer available, a finer construction is needed: in the celebrated paper [51] Berestycki and Lions build this path for a local problem by exploiting an inductive process based on piecewise affine functions.

In our nonlocal case, in order to prove the existence of multiple solutions for $m \gg 0$ (point (i) of Theorem 3.1.2), unlike the elaborated approach of [51] we can obtain the existence of a multidimensional odd path by exploiting the positivity of the Riesz potential functional. A similar approach can be implemented to gain existence of infinitely many solutions for any $m > 0$ when F is even (first part of point (ii) of Theorem 3.1.2), since in this case F can be assumed positive in a neighborhood of the origin. See anyway Remark 3.1.6 below.

A quite delicate issue, instead, comes up when F is odd. Differently from [224] and Chapter 2, the classical argument given by [51] cannot be applied directly in the context of nonlinear Choquard equations because of the presence of a nonlocal source, and we need to implement a new approach to gain the existence of an admissible odd path.

To this aim we proceed by finding suitable annuli: using characteristic functions corresponding to the annuli, we construct our multidimensional odd paths. Here interactions between these characteristic functions produced by the Riesz potential play a crucial role, in particular the index α is related to the strength of interaction and the case $\alpha \in (0, 1]$ reveals to be more delicate. To this aim we use sharp estimates for the Riesz potential obtained by Thim [359].

As a further byproduct of the previous approach we gain the existence of infinitely many solutions for the unconstrained problem. More precisely, defined the C^1-functional $\mathcal{J}_\mu : H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{J}_\mu(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{\mu}{2} \int_{\mathbb{R}^N} u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) \, dx,$$

we establish the following result.

Theorem 3.1.5. *Suppose $N \geq 3$, $\alpha \in (0, N)$ and $\mu > 0$ fixed. Assume that* (F1)–(F5) *hold. Then there exist countably many radial solutions $(u_n)_n$ of the nonlinear Choquard equation* (3.1.1). *Moreover we have*

$$\mathcal{J}_\mu(u_n) \to +\infty \quad \textit{as } n \to +\infty.$$

Our multiplicity result is the counterpart of what done in [51] for the local case with odd nonlinearities and extend the existence result in [302] due to Moroz and Van Schaftingen.

Remark 3.1.6. *We highlight that the easier approach for building a multidimensional path, based on the positivity of the Riesz kernel (Proposition* 1.3.2*), cannot generally be applied to more* *generally frameworks (also if F is even); for examples, when dealing with kernels $K = K(x, y)$ which do not makes the functional*

$$g \mapsto \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) g(x) g(y)$$

positive (for example, $K(x, y)$ sign-changing). In this case, the approach here developed, based on suitable annuli, might instead be adapted. This is an interesting line of research for the future.

The Chapter is organized as follows. In Section 3.2 we focus on the construction of multidimensional paths, by dealing first with an easier version based on the positivity of the Riesz potential, and then a refined version based on some suitable annuli and essential interaction estimates for non-local terms. Section 3.3 is then dedicated to the study of the asymptotic behaviour of the mountain pass values, according to variable values of μ. Afterwards, in Section 3.4, we detect a mountain pass structure, built on the Pohozaev mountain, for the constrained case, and in Section 3.5 we derive a Palais-Smale-Pohozaev condition. In Section 3.6 we introduce an augmented functional which will be used to gain a deformation lemma, and we further study suitable minimax values defined through the tool of the genus which allows to prove the main Theorem 3.1.2. Finally in Section 3.7 we deal with the unconstrained case by proving Theorem 3.1.5.

3.2 Multidimensional annuli-shaped paths: even and odd nonlinearities

In this Chapter we briefly denote by q the lower-critical exponent $2_\alpha^\#$ and by p the L^2-critical exponent 2_α^m, i.e.

$$q := 2_\alpha^\# = \frac{N+\alpha}{N}, \quad p := 2_\alpha^m = \frac{N+\alpha+2}{N}.$$

Again, to avoid problems with the boundary of $\mathbb{R}_+$, we write from now on (see Section 4.2.2 for a different approach)

$$\mu \equiv e^\lambda \in (0, +\infty), \quad \lambda \in \mathbb{R}.$$

We also set

$$\mathcal{D}(u) := \mathcal{D}_\alpha(F(u), F(u)) = \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u)\, dx.$$

Using Proposition 1.3.1 and (F1)-(F2), we notice that $\mathcal{D}$ is continuous on $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, where $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent, and thus continuous on $H^1_r(\mathbb{R}^N)$; notice that if we assume (CF2), then $\mathcal{D}$ is continuous also on $L^2(\mathbb{R}^N) \cap L^{2+\frac{4}{N+\alpha}}(\mathbb{R}^N)$.

To deal with the unconstrained problem, we further define the C^1-functional $\mathcal{J} : \mathbb{R} \times H^1_r(\mathbb{R}^N) \to \mathbb{R}$ by setting

$$\mathcal{J}(\lambda, u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\mathcal{D}(u) + \frac{e^\lambda}{2}\|u\|_2^2, \quad (\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N). \tag{3.2.8}$$

For a fixed $\lambda \in \mathbb{R}$, $u \in H^1_r(\mathbb{R}^N)$ is critical point of $\mathcal{J}(\lambda, \cdot)$ if and only if u solves (weakly)

$$-\Delta u + e^\lambda u = (I_\alpha * F(u)) f(u) \quad \text{in } \mathbb{R}^N. \tag{3.2.9}$$

In this Section we study the geometry of

$$u \in H^1_r(\mathbb{R}^N) \mapsto \mathcal{J}(\lambda, u) \in \mathbb{R},$$

for a fixed $\lambda \in \mathbb{R}$. We introduce a sequence of minimax values $a_n(\lambda)$, $n \in \mathbb{N}^*$: these values play important roles to find multiple solutions for the constrained problem (Theorem 3.1.2) as well as for the unconstrained problem (Theorem 3.1.5).

For $n \in \mathbb{N}^*$ and $\lambda \in \mathbb{R}$ we introduce the set of paths

$$\Gamma_n(\lambda) := \{\gamma \in C(D_n, H^1_r(\mathbb{R}^N)) \mid \gamma \text{ odd}, \mathcal{J}(\lambda, \gamma_{|\partial D_n}) < 0\}$$

and the minimax values

$$a_n(\lambda) := \inf_{\gamma \in \Gamma_n(\lambda)} \sup_{\xi \in D_n} \mathcal{J}(\lambda, \gamma(\xi)).$$

For $n \geq 2$ the nonemptiness of $\Gamma_n(\lambda)$ has to be checked; for $n = 1$ we refer to [302, claim 1 of Proposition 2.1]. Classically, in the local framework this fact was proved in [51] by constructing inductively piecewise affine paths. This construction does not fit the nonlocality interaction given by the Choquard term, thus we need another approach.

Proposition 3.2.1. *Assume (F1)–(F4) and $F(\pm s_0) \neq 0$. Let $n \in \mathbb{N}^*$ and $\lambda \in \mathbb{R}$. Then $\Gamma_n(\lambda) \neq \emptyset$, thus $a_n(\lambda)$ is well defined. Moreover, $a_n(\lambda) > 0$ and it is increasing with respect to λ and n.*

Proof. Start observing that the polyhedron

$$\Sigma := \left\{ t = (t_1, \ldots, t_n) \mid \max_{i=1,\ldots,n} |t_i| = 1 \right\}$$

is homeomorphic to ∂D_n (we passed from the L^2 to the L^∞ norm). Let us fix $e_1, \ldots, e_n \in C^\infty_c(\mathbb{R}^N)$, each of them between 0 and 1, radially symmetric, equal to one in some annulus A_i, and such that their supports are mutually disjoint. Then set $\gamma : \Sigma \to H^1_r(\mathbb{R}^N)$ by

$$\gamma(t)(x) := s_0 \sum_{i=1}^n t_i e_i(x) \tag{3.2.10}$$

for every $t = (t_1, \ldots, t_n) \in \Sigma$ and $x \in \mathbb{R}^N$. The map γ is clearly odd and continuous. Moreover every $t \in \Sigma$ has at least a nontrivial component $|t_i| = 1$, thus we have $F(\gamma(t)(x)) = F(s_0 t_i e_i(x)) = F(\pm s_0) \neq 0$ on A_i, hence $F(\gamma(t)) \not\equiv 0$. By Proposition 1.3.2 we have

$$\mathcal{D}(\gamma(t)) > 0 \quad \text{for each } t \in \Sigma.$$

Since

$$\mathcal{D} \circ \gamma : \Sigma \to \mathbb{R}$$

is continuous, and Σ is compact, we obtain

$$\min_{t \in \Sigma} \mathcal{D}(\gamma(t)) =: C > 0,$$

i.e.

$$\mathcal{D}(\gamma(t)) \geq C > 0 \quad \text{for each } t \in \Sigma.$$

Set moreover $M := \max_\Sigma \|\gamma\|^2_{H^1} \in \mathbb{R}$. By scaling, we obtain

$$\begin{aligned} \mathcal{J}_\lambda(\gamma(t)(\cdot/\theta)) &= \frac{\theta^{N-2}}{2} \|\nabla \gamma(t)\|_2^2 + \frac{\theta^N e^\lambda}{2} \|\gamma(t)\|_2^2 - \frac{\theta^{N+\alpha}}{2} \mathcal{D}(\gamma(t)) \\ &\leq \frac{\theta^{N-2}}{2} M + \frac{\theta^N e^\lambda}{2} M - \frac{\theta^{N+\alpha}}{2} C < 0 \end{aligned}$$

for some $\theta = \theta^* \gg 0$. Thus we consider $\tilde{\gamma} := \gamma(\cdot)(\cdot/\theta^*) : \partial D_n \to H^1_r(\mathbb{R}^N)$. Finally we extend $\tilde{\gamma}$ to D_n by

$$\tilde{\gamma}(\xi) := |\xi| \gamma \left(\frac{\xi}{|\xi|} \right)$$

for every $\xi \in D_n \setminus \{0\}$, and $\tilde{\gamma}(0) := 0$. Therefore $\tilde{\gamma} \in \Gamma_n(\lambda) \neq \emptyset$.

What remains to prove is the monotonicity and positivity of $a_n(\lambda)$. Since $D_n \subset D_{n+1}$, we may regard for $\gamma \in \Gamma_{n+1}(\lambda)$,

$$\gamma_{|D_n} \in \Gamma_n(\lambda).$$

Thus we have $a_n(\lambda) \le a_{n+1}(\lambda)$. Since $\mathcal{J}(\lambda, u)$ is monotone in λ, we also have the monotonicity with respect to λ.

The positivity of $a_1(\lambda)$ is essentially obtained in [302] (see also [124]). Thus

$$a_n(\lambda) \ge a_1(\lambda) > 0. \qquad \blacksquare$$

In the proof of Proposition 3.2.1 we hardly relied on the positivity of the Riesz potential functional given in Proposition 1.3.2, to obtain the existence of path $\gamma : D_n \to H^1_r(\mathbb{R}^N)$ and a $C > 0$ such that

$$\mathcal{D}(\gamma(\xi)) \ge C > 0 \quad \text{for each } \xi \in \partial D_n. \tag{3.2.11}$$

Notice moreover that this γ satisfies $\gamma(\theta\xi) = \theta\gamma(\xi)$ for any $\xi \in D_n$ and $\theta \in \mathbb{R}$. Anyway, no good information on the constant C appearing in (3.2.11) are given by this result.

A useful estimate in order to get infinitely many solutions for any $m > 0$, when (CF4) holds, is the one which relates $\mathcal{D}(\theta\gamma)$ to $F(\theta s_0)$ (see Lemma 3.2.8 and Section 3.3), that is

$$\mathcal{D}(\theta\gamma(\xi)) \ge C(F(\pm\theta s_0))^2 \quad \text{for each } \xi \in \partial D_n \text{ and } \theta \in [0,1] \tag{3.2.12}$$

for some uniform $C > 0$. When F is positive in a neighborhood of the origin (which is the case of F even and (CF4)), then one can build a suitable γ which satisfies (3.2.12).

Proposition 3.2.2. *Assume (F1)–(F4). Assume moreover that F is positive in some $[-s_0, s_0]$, $F(\pm s_0) \neq 0$. Let $n \in \mathbb{N}^*$ and $\lambda \in \mathbb{R}$. Then the path $\gamma \in \Gamma_n(\lambda)$ defined in (3.2.10) satisfies (3.2.12).*

Proof. Assume the notation of the proof of Proposition 3.2.1. For each $t \in \Sigma$, there exists $|t_k| = 1$, thus, by exploiting that $\theta s_0 t_i e_i \in [-s_0, s_0]$ we obtain

$$\begin{aligned}
\mathcal{D}(\theta\gamma(t)) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y) F\left(\theta s_0 \sum_{i=1}^n t_i e_i(x)\right) F\left(\theta s_0 \sum_{j=1}^n t_j e_j(x)\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y) F(\theta s_0 t_i e_i(x)) F(\theta s_0 t_j e_j(x)) \\
&\ge \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y) F(\theta s_0 t_k e_k(x)) F(\theta s_0 t_k e_k(y)) \\
&\ge \big(F(\pm\theta s_0)\big)^2 \int_{A_k}\int_{A_k} I_\alpha(x-y)
\end{aligned}$$

which is the claim. $\blacksquare$

When F is odd (and thus it cannot be positive around the origin) it seems not an easy task to build a $\gamma \in \Gamma_n(\lambda)$ satisfying (3.2.12); indeed some estimate from below on

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y) \frac{F(\theta\gamma(\xi)(x)) F(\theta\gamma(\xi)(y))}{\big(F(\theta s_0)\big)^2}$$

uniform for $\theta \to 0$ seems required; this is related to quotients of the type $\frac{F(sh)}{F(s)}$ with $s \in (0, s_0]$ and $h \in [0,1]$. This is essentially the meaning of condition (3.1.7).

To deal with this case we need a deep understanding of the Riesz potential on radial functions. We thus give now a different construction for a $\gamma \in \Gamma_n(\lambda)$: this procedure might be investigated also for more general Choquard-type equations, where different kernels (possibly sign-changing) appear.

We start by recalling a result contained in [359, Theorem 1] (see also [294, Lemma 6.3] and references therein).

Theorem 3.2.3 ([359]). *Let $\alpha \in (0,N)$ and $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be radial. Then $I_\alpha * u$ is radial and*

$$(I_\alpha * u)(r) = r^\alpha \int_0^\infty F_\alpha\left(\frac{r}{\rho}\right)\left(\frac{\rho}{r}\right)^\alpha u(\rho)\,\frac{d\rho}{\rho}, \tag{3.2.13}$$

where F_α is positive and it satisfies, for some constants $C_{N,0}$, $C_{N,\infty}$, $C_{N,\alpha} > 0$,

$$F_\alpha(s) \to C_{N,0} \quad \text{as } s \to 0, \qquad \frac{F_\alpha(s)}{s^{\alpha-N}} \to C_{N,\infty} \quad \text{as } s \to +\infty$$

and

$$\frac{F_\alpha(s)}{G_\alpha(s)} \to 1 \quad \text{as } s \to 1, \tag{3.2.14}$$

with

$$G_\alpha(s) := \begin{cases} C_{N,\alpha} & \text{if } \alpha \in (1,N), \\ C_{N,\alpha}|\log|s-1|| & \text{if } \alpha = 1, \\ C_{N,\alpha}|s-1|^{\alpha-1} & \text{if } \alpha \in (0,1). \end{cases} \tag{3.2.15}$$

For a proof of Proposition 3.2.1, we prepare some notation and some estimates. We introduce the annuli

$$A(R,h) := \{x \in \mathbb{R}^N \mid |x| \in [R-h, R+h]\}, \quad \chi(R,h;\cdot) := \chi_{A(R,h)}$$

for any $R \gg h > 0$. We have the following key estimates.

Lemma 3.2.4. *It results as $h \to 0$*

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x-y)\chi(1,h;x)\chi(1,h;y)\,dxdy \sim \begin{cases} h^2 & \text{if } \alpha \in (1,N), \\ h^2|\log h| & \text{if } \alpha = 1, \\ h^{1+\alpha} & \text{if } \alpha \in (0,1). \end{cases}$$

Proof. We apply Theorem 3.2.3 to $u(|x|) = \chi(1,h;|x|)$. In particular, by (3.2.13) we have

$$\begin{aligned} S_h &:= \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x-y)u(x)u(y)\,dxdy = C\int_0^\infty (I_\alpha * u)(r)u(r)r^{N-1}\,dr \\ &= C\int_0^\infty \int_0^\infty F_\alpha\left(\frac{r}{\rho}\right)\rho^{\alpha-1}r^{N-1}u(\rho)u(r)\,d\rho dr = C\iint_{[1-h,1+h]^2} F_\alpha\left(\frac{r}{\rho}\right)\rho^{\alpha-1}r^{N-1}. \end{aligned}$$

First we note that

$$\sup_{\rho,r \in [1-h,1+h]} \left|\frac{r}{\rho} - 1\right| \to 0 \quad \text{as } h \to 0.$$

We consider the following three cases separately:

$$\text{(i) } \alpha \in (1,N), \quad \text{(ii) } \alpha = 1, \quad \text{(iii) } \alpha \in (0,1).$$

(i) When $\alpha \in (1,N)$ we may assume $F(\frac{r}{\rho}) \sim C_{N,\alpha} > 0$. Thus

$$S_h \sim \iint_{[1-h,1+h]^2} \rho^{\alpha-1}r^{N-1}\,d\rho dr \sim h^2.$$

(ii) When $\alpha = 1$

$$F_\alpha\left(\frac{r}{\rho}\right) \sim G_1\left(\frac{r}{\rho}\right) = C_{N,1}\left|\log\left|\frac{r}{\rho} - 1\right|\right|$$

$$\sim |\log|r-\rho| - \log\rho| = -\log|r-\rho| + \log\rho.$$

Thus

$$S_h \sim \iint_{[1-h,1+h]^2} (-\log|r-\rho| + \log\rho) r^{N-1}\, d\rho dr.$$

Set

$$A_h := \{(\rho,r) \,|\, |\rho - r| \le \tfrac{1}{2}h,\ |r-1| \le \tfrac{1}{2}h\},$$
$$B_h := \{(\rho,r) \,|\, |\rho - r| \le 2h,\ |r-1| \le h\},$$

we have

$$A_h \subset [1-h,1+h]^2 \subset B_h.$$

Hence for some C, $C' > 0$

$$C\iint_{A_h} (-\log|r-\rho| + \log\rho) r^{N-1}\, d\rho dr \le S_h \le C' \iint_{B_h} (-\log|r-\rho| + \log\rho) r^{N-1}\, d\rho dr. \quad (3.2.16)$$

We compute

$$\begin{aligned}
&\iint_{B_h} (-\log|r-\rho| + \log\rho) r^{N-1}\, d\rho dr \\
&\le \iint_{B_h} (-\log|r-\rho| + \log(1+h))(1+h)^{N-1}\, d\rho dr \\
&= \iint_{[-2h,2h]\times[1-h,1+h]} (-\log|\tau| + \log(1+h))(1+h)^{N-1}\, d\tau dr \\
&= 4h(1+h)^{N-1}\int_0^{2h} (-\log\tau)\, d\tau + 8h^2(1+h)^{N-1}\log(1+h) \\
&= 4h(1+h)^{N-1}\big(-2h\log(2h) + 2h\big) + 8h^2(1+h)^{N-1}\log(1+h) \\
&\le C'' h^2 |\log h| \quad \text{as } h \to 0.
\end{aligned}$$

Similarly we have

$$\iint_{A_h} (-\log|r-\rho| + \log\rho) r^{N-1}\, d\rho dr \ge C''' h^2 |\log h|,$$

from which we obtain

$$S_h \sim h^2 |\log h| \quad \text{as } h \to 0.$$

(iii) When $\alpha \in (0,1)$

$$F_\alpha\left(\frac{r}{\rho}\right) \sim G_\alpha\left(\frac{r}{\rho}\right) = C_{N,\alpha}\left|\frac{r}{\rho} - 1\right|^{\alpha-1}.$$

Thus

$$S_h \sim \iint_{[1-h,1+h]^2} \left|\frac{r}{\rho} - 1\right|^{\alpha-1} \rho^{\alpha-1} r^{N-1}\, d\rho dr = \iint_{[1-h,1+h]^2} |r-\rho|^{\alpha-1} r^{N-1}\, d\rho dr.$$

Since

$$C\iint_{A_h} |r-\rho|^{\alpha-1}(1-h)^{N-1}\, d\rho dr \le S_h \le C' \iint_{B_h} |r-\rho|^{\alpha-1}(1+h)^{N-1}\, d\rho dr,$$

we have as in (3.2.16)

$$S_h \sim h^{1+\alpha} \quad \text{as } h \to 0.$$

This completes the proof. ∎

We show how to use it to build a continuous odd map in $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$. By a regularization argument, we will obtain a map in $\Gamma_n(\lambda)$.

By scaling, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)\chi(R,h;x)\chi(R,h;y)\,dxdy\\
&= R^{N+\alpha}\iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)\chi\Big(1,\frac{h}{R};x\Big)\chi\Big(1,\frac{h}{R};y\Big)\,dxdy\\
&\sim \begin{cases} R^{N+\alpha}(\frac{h}{R})^2 & \text{if } \alpha\in(1,N),\\ R^{N+1}(\frac{h}{R})^2|\log\frac{h}{R}| & \text{if } \alpha=1,\\ R^{N+\alpha}(\frac{h}{R})^{1+\alpha} & \text{if } \alpha\in(0,1).\end{cases}
\end{aligned}$$

For $R \geq 2$, we set the thickness of the annuli as

$$h_R := \begin{cases} R^{-\frac{N-2+\alpha}{2}} & \text{if } \alpha\in(1,N),\\ R^{-\frac{N-1}{2}}(\log R)^{-1/2} & \text{if } \alpha=1,\\ R^{-\frac{N-1}{1+\alpha}} & \text{if } \alpha\in(0,1),\end{cases}$$

so that a uniform bound is gained.

Corollary 3.2.5. *We have*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)\chi(R,h_R;x)\chi(R,h_R;y)\,dxdy \in [C_{01},C_{02}] \quad \textit{for } R\geq 2, \tag{3.2.17}$$

where C_{01}, $C_{02} > 0$ are independent of $R \geq 2$.

Proof. We check (3.2.17) only for $\alpha = 1$. We have

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)\chi\Big(1,\frac{h_R}{R};x\Big)\chi\Big(1,\frac{h_R}{R};y\Big)\,dxdy\\
&\sim R^{N+1}\Big(\frac{h_R}{R}\Big)^2\Big|\log\Big(\frac{h_R}{R}\Big)\Big|\\
&= R^{N+1}\left(\frac{R^{-\frac{N-1}{2}}|\log R|^{-1/2}}{R}\right)^2\left|\log\left(\frac{R^{-\frac{N-1}{2}}|\log R|^{-1/2}}{R}\right)\right|\\
&= (\log R)^{-1}\left|\log\left(R^{-\frac{N+1}{2}}(\log R)^{-1/2}\right)\right|\\
&= (\log R)^{-1}\left(\frac{N+1}{2}\log R+\frac{1}{2}\log(\log R)\right)\\
&\to \frac{N+1}{2} \quad \text{as } R\to\infty.
\end{aligned}$$

whic shows the claim. ∎

Next we compute the interaction effect between $\chi(R^i, h_{R^i};\cdot)$ and $\chi(R^j, h_{R^j};\cdot)$ with $i, j \in \mathbb{N}$, $i \neq j$ and $R \gg 1$.

Lemma 3.2.6. *For $i < j$ we have*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)\chi(R^i,h_{R^i};x)\chi(R^j,h_{R^j};y)\,dxdy \to 0 \quad \textit{as } R\to\infty.$$

Proof. Since $\operatorname{supp}\chi(R,h_R;\cdot) = A(R,h_R)$ we get

$$\begin{aligned}
\operatorname{dist}\big(\operatorname{supp}\chi(R^i,h_{R^i};\cdot),\operatorname{supp}\chi(R^j,h_{R^j};\cdot)\big) &= (R^j-h_{R^j})-(R^i+h_{R^i})\\
&= R^j - O(R^i).
\end{aligned}$$

Thus

$$\begin{aligned} I_R &:= \iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)\chi(R^i,h_{R^i};x)\chi(R^j,h_{R^j};y)\,dxdy \\ &\le C(R^j+O(R^i))^{-(N-\alpha)}\|\chi(R^i,h_{R^i};\cdot)\|_1\|\chi(R^j,h_{R^j};\cdot)\|_1. \end{aligned}$$

Here

$$\|\chi(R,h_R;\cdot)\|_1 = \mathrm{meas}(A(R,h_R)) \sim CR^{N-1}h_R$$

hence

$$\begin{aligned} I_R &\le qC(R^j-O(R^i))^{-(N-\alpha)}R^{(N-1)i}h_{R^i}R^{(N-1)j}h_{R^j} \\ &\le C'R^{(\alpha-1)j+(N-1)i}h_{R^i}h_{R^j}. \end{aligned}$$

When $\alpha\in(1,N)$, we have by the definition of h_R

$$\begin{aligned} I_R &\le CR^{(\alpha-1)j+(N-1)i}R^{-\frac{1}{2}(N-2+\alpha)(i+j)} \\ &= C'R^{-\frac{1}{2}(N-\alpha)(j-i)} \to 0 \quad \text{as } R\to\infty; \end{aligned}$$

when $\alpha=1$, we obtain

$$\begin{aligned} I_R &\le C'R^{(N-1)i}R^{-\frac{1}{2}(N-1)(i+j)}(\log R^i)^{-\frac{1}{2}}(\log R^j)^{-\frac{1}{2}} \\ &= C'R^{-\frac{1}{2}(N-1)(j-i)}(ij)^{-\frac{1}{2}}(\log R)^{-1} \to 0 \quad \text{as } R\to\infty; \end{aligned}$$

when $\alpha\in(0,1)$,

$$\begin{aligned} I_R &\le C'R^{(\alpha-1)j+(N-1)i}R^{-\frac{N-1}{1+\alpha}(i+j)} \\ &= C'R^{-\frac{1}{1+\alpha}((N-\alpha^2)j-\alpha(N-1)i)} \to 0 \quad \text{as } R\to\infty. \end{aligned}$$

This concludes the proof. ∎

We have now the tools to build a refined path $\gamma\in\Gamma_n(\lambda)$.

Proof of Proposition 3.2.1 (refined). We construct now a path $\gamma\in\Gamma_n(\lambda)$; this path will moreover satisfy

$$\max_{\xi\in D_n,\,x\in\mathbb{R}^N}|\gamma(\xi)(x)|\le s_0.$$

Step 1: *Construction of an odd path in L^r.*
For $n\ge 2$ we consider again the polyhedron

$$\Sigma = \{t=(t_1,\dots,t_n) \mid \max_{i=1,\dots,n}|t_i|=1\}.$$

For a large $R\gg 1$, which we will choose later, we define

$$\gamma_R(t)(x) := \sum_{i=1}^n \mathrm{sgn}(t_i)\chi(R^i,|t_i|h_{R^i};x) : \Sigma\to L^r(\mathbb{R}^N)$$

where $r\in[1,+\infty]$. Here we regard $\chi(R^i,0;x)\equiv 0$, and we notice that $\gamma_R(t)$ is radial for each $t\in\Sigma$. Considered s_0, we have

$$\begin{aligned} \mathcal{D}(s_0\gamma_R(t)) = \sum_{i,j} & F(\mathrm{sgn}(t_i)s_0)F(\mathrm{sgn}(t_j)s_0)\cdot \\ & \cdot\iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)\chi(R^i,|t_i|h_{R^i};x)\chi(R^i,|t_j|h_{R^i};y)\,dxdy. \end{aligned}$$

We note that

(i) For any $t = (t_1, \dots, t_n) \in \Sigma$, there exists at least one t_k such that $|t_k| = 1$.

(ii) By Lemma 3.2.4,

$$(F(\pm s_0))^2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x-y)\chi(R^k, h_{R^k}; x)\chi(R^k, h_{R^k}; y)\, dxdy \geq C_0.$$

(iii) By (i) and (ii),

$$\sum_{i=1}^n (F(\pm s_0))^2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x-y)\chi(R^i, h_{R^i}; x)\chi(R^i, h_{R^i}; y)\, dxdy \geq C_0.$$

(iv) If $i \neq j$, by Lemma 3.2.6,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha(x-y)\chi(R^i, h_{R^i}; x)\chi(R^j, h_{R^j}; x)\, dxdy \to 0 \quad \text{as } R \to \infty.$$

By (i)–(iv), we have for sufficiently large $R \gg 1$,

$$\mathcal{D}(s_0\gamma_R(t)) \geq C > 0 \quad \text{for all } t \in \Sigma. \tag{3.2.18}$$

In what follows we fix $R \gg 1$ so that (3.2.18) holds.

Step 2: *Construction of an odd path in H^1_r.*

For $0 \leq h \ll R$ and $\varepsilon > 0$, we set

$$\chi_\varepsilon(R, h; x) := \begin{cases} 1 & \text{if } x \in A(R,h), \\ 1 - \frac{1}{\varepsilon}\mathrm{dist}(x, A(R,h)), & \text{if } \mathrm{dist}(x, A(R,h)) \in (0, \varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

Here we regard $A(R, 0) = \{x \in \mathbb{R}^N \mid |x| = R\}$. We note that

$$\begin{aligned} &\chi_\varepsilon(R, h; \cdot) \in H^1_r(\mathbb{R}^N) \text{ for } \varepsilon > 0, \\ &\chi_\varepsilon(R, h; \cdot) \to \chi(R, h; \cdot) \text{ in } L^r(\mathbb{R}^N) \text{ as } \varepsilon \to 0 \text{ for all } r \in [1, \infty), \\ &\mathrm{supp}\, \chi_\varepsilon(R^i, h_{R^i}; \cdot) \cap \mathrm{supp}\, \chi_\varepsilon(R^j, h_{R^j}; \cdot) = \emptyset \text{ for } i \neq j \text{ for } \varepsilon \text{ small.} \end{aligned}$$

We set

$$\gamma_{\varepsilon,R}(t) := \sum_{i=1}^n \mathrm{sgn}(t_i)\chi_\varepsilon(R^i, |t_i| h_{R^i}; \cdot), \quad t \in \Sigma, \tag{3.2.19}$$

$\gamma_{\varepsilon,R} : \Sigma \to H^1_r(\mathbb{R}^N)$, continuous. By (3.2.18) and the continuity of $\mathcal{D}$ on $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, we have for $\varepsilon > 0$ small

$$\mathcal{D}(s_0\gamma_{\varepsilon,R}(t)) \geq C > 0 \quad \text{for all } t \in \Sigma.$$

Since

$$\mathcal{J}(\lambda, u(\cdot/\theta)) = \frac{1}{2}\theta^{N-2}\|\nabla u\|_2^2 + \frac{e^\lambda}{2}\theta^N\|u\|_2^2 - \frac{1}{2}\theta^{N+\alpha}\mathcal{D}(u),$$

we have for large $\theta \gg 1$

$$\mathcal{J}(\lambda, s_0\gamma_{\varepsilon,R}(t)(\cdot/\theta)) < 0 \quad \text{for all } t \in \Sigma \approx \partial D_n.$$

Considering $D_n = \{st \mid s \in [0,1],\, t \in \Sigma\}$ and extending $s_0\gamma_{\varepsilon,R}(t)(\cdot/\theta)$ to D_n by

$$\widetilde{\gamma}(st) := s s_0 \gamma_{\varepsilon,R}(t)(\cdot/\theta),$$

finally we obtain a path $\widetilde{\gamma} \in \Gamma_n(\lambda)$. ∎

Remark 3.2.7. *Even without assuming the positivity of F (see Proposition 3.2.2), we notice that the construction of an odd map in L^r gets easier when F is an even function. Indeed there is no negative contribution given by the mixed interactions. We give only an outline of the proof, highlighting that in this case we do not need to use the fine Theorem 3.2.3 given by [359].*

Define for every $i = 1, \dots n$ and $s \in [-1, 1]$ the annuli

$$A_i(s) := \{x \in \mathbb{R}^N \mid |x| \in [2ni - |s|, 2ni + |s|]\}.$$

For every $t = (t_1, \dots, t_n) \in \Sigma$ we have that $A_1(t_1), \dots, A_n(t_n)$ are disjoint. Moreover, if $t_i = 0$, then $\mathrm{meas}(A_i(t_i)) = 0$. Thus we define a continuous, odd map by

$$\gamma(t)(x) := \sum_{i=1}^{n} \mathrm{sgn}(t_i)\chi_{A_i(t_i)}(x) : \Sigma \to L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N).$$

Since F is even, we obtain

$$\begin{aligned}
&\mathcal{D}(s_0\gamma(t)) \\
&= \sum_{i,j} \iint_{A_i(t_i)\times A_j(t_j)} I_\alpha(x-y)F(s_0\mathrm{sgn}(t_i)\chi_{A_i(t_i)}(x))F(s_0\mathrm{sgn}(t_j)\chi_{A_j(t_j)}(y))\,dxdy \\
&= (F(s_0))^2 \sum_{i,j} \iint_{A_i(t_i)\times A_j(t_j)} I_\alpha(x-y)\,dxdy (F(s_0))^2 \ge C > 0,
\end{aligned}$$

where C does not depend on the specific t. The regularization to a H^1_r-path can be done as in the general case (or by mollification), as well as the extension to D_n.

We highlight that this construction can be adapted also to the local case, and thus it gives a simplified construction of a multidimensional path in the setting of Berestycki and Lions [51].

We are ready now to show that $\gamma_{R,\varepsilon} : \Sigma \to H^1_r(\mathbb{R}^N)$, defined in (3.2.19), has the desired property (3.2.12).

Lemma 3.2.8. *Assume (F1)–(F5), and $F > 0$ in some $(0, \delta_0]$. If F is odd, additionally assume (3.1.7). Then there exists a constant $A > 0$ independent of $s \in (0, \delta_0]$ and $t \in \Sigma$ such that*

$$\mathcal{D}(s\gamma_{R,\varepsilon}(t)) \ge \frac{1}{2}(F(s))^2(A + o_\varepsilon(1)) \quad \text{as } \varepsilon \to 0;$$

here $o_\varepsilon(1)$ is a quantity which goes to 0 as $\varepsilon \to 0$ uniformly in $t \in \Sigma$ and $s \in (0, \delta_0]$.

Proof. We prove Lemma 3.2.8 in two steps.

Step 1: For $t \in \Sigma$, set

$$a_{ij}(t) := \iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)\chi(R^i, |t_i|h_{R^i}; x)\chi(R^j, |t_j|h_{R^j}; y)\,dxdy.$$

Then for sufficiently large $R > 0$, we have

$$A := \inf_{t\in\Sigma}\left(\sum_{i=1}^{n} a_{ii}(t) - \sum_{i\neq j}^{n} a_{ij}(t)\right) > 0. \tag{3.2.20}$$

This fact follows from (3.2.17) and Lemma 3.2.6. We fix $R \gg 1$ so that (3.2.20) holds.

Step 2: $\mathcal{D}(s\gamma_{R,\varepsilon}(t)) \ge \frac{1}{2}F(s)^2 A$ *as* $\varepsilon \to 0$.

We note that for $\varepsilon > 0$ small

$$\mathrm{supp}\,\chi_\varepsilon(R^i, |t_i|h_{R^i}; \cdot) \cap \mathrm{supp}\,\chi_\varepsilon(R^j, |t_j|h_{R^j}; \cdot) = \emptyset \quad \text{for } i \neq j.$$

Thus we have

$$\begin{aligned}
&\mathcal{D}(s\gamma_{R,\varepsilon}(t)) \\
&= \sum_{i,j} \iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)F(s\,\mathrm{sgn}(t_i)\chi_\varepsilon(R^i,|t_i|h_{R^i};x))F(s\,\mathrm{sgn}(t_j)\chi_\varepsilon(R^j,|t_j|h_{R^j};y)) \quad (3.2.21)\\
&=: \sum_{i,j} B_{ij}(s,t). \quad (3.2.22)
\end{aligned}$$

We consider cases $i=j$ and $i\neq j$ separately.

First we focus on the case $i=j$. For both even and odd F we have

$$\begin{aligned}
&B_{ii}(s,t)\\
&= \iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)F(s\,\mathrm{sgn}(t_i)\chi_\varepsilon(R^i,|t_i|h_{R^i};x))F(s\,\mathrm{sgn}(t_i)\chi_\varepsilon(R^j,|t_i|h_{R^i};y))\\
&= \iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)F(s\chi_\varepsilon(R^i,|t_i|h_{R^i};x))F(s\chi_\varepsilon(R^j,|t_i|h_{R^i};y))\\
&\geq \iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)F(s\chi(R^i,|t_i|h_{R^i};x))F(s\chi(R^j,|t_i|h_{R^i};y))\\
&= (F(s))^2 a_{ii}(t), \quad (3.2.23)
\end{aligned}$$

where we used the positivity of F and the monotonicity of the integral. Next we consider the case $i\neq j$ for even F. Since $F(s)\geq 0$ for $s\in[-\delta_0,\delta_0]$ we obtain

$$B_{ij}(s,t)\geq 0 \quad \text{for all } t\in\Sigma. \quad (3.2.24)$$

Finally we consider the case $i\neq j$ for odd F. Since $|F(s)|=F(|s|)$ for $s\in[-\delta_0,\delta_0]$

$$\begin{aligned}
&B_{ij}(s,t)\\
&= \iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)F(s\,\mathrm{sgn}(t_i)\chi_\varepsilon(R^i,|t_i|h_{R^i};x))F(s\,\mathrm{sgn}(t_j)\chi_\varepsilon(R^j,|t_j|h_{R^j};y)) \quad (3.2.25)\\
&\geq -\iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)F(s\chi_\varepsilon(R^i,|t_i|h_{R^i};x))F(s\chi_\varepsilon(R^j,|t_j|h_{R^j};y)). \quad (3.2.26)
\end{aligned}$$

Setting

$$C_i(t,\varepsilon) := \{x \mid \mathrm{dist}(x, A(R^i,|t_i|h_{R^i}))\in(0,\varepsilon)\}$$

we have

$$\begin{aligned}
&\chi_\varepsilon(R^i,|t_i|h_{R^i};x)\in(0,1) \quad \text{for } x\in C_i(t_i,\varepsilon),\\
&\chi_\varepsilon(R^i,|t_i|h_{R^i};x)=\chi(R^i,|t_i|h_{R^i};x) \quad \text{for } x\notin C_i(t_i,\varepsilon),\\
&\mathrm{meas}(C_i(t_i,\varepsilon))\to 0 \quad \text{as } \varepsilon\to 0, \text{ uniformly in } t\in\Sigma.
\end{aligned}$$

Thus for $r\in[1,\infty)$ and $s\in(0,\delta]$

$$\begin{aligned}
&\left\|\frac{1}{F(s)}F(s\chi_\varepsilon(R^i,|t_i|h_{R^i};\cdot))-\chi(R^i,|t_i|h_{R^i};\cdot)\right\|_r^r\\
&\leq \int_{C_i(t_i,\varepsilon)}\left|\frac{1}{F(s)}F(s\chi_\varepsilon(R^i,|t_i|h_{R^i};x))\right|^r dx\\
&= \left(\max_{h\in[0,1]}\frac{|F(hs)|}{|F(s)|}\right)^r \mathrm{meas}(C_i(t_i,\varepsilon))\\
&\to 0 \qquad \text{as } \varepsilon\to 0 \text{ uniformly in } t\in\Sigma.
\end{aligned}$$

Here we use the fact that $\max_{h\in[0,1]} \frac{F(sh)}{F(s)} \le C$, which follows from the local almost-monotonicity assumption in (CF4). We note that (3.2.27) implies, exploiting again (CF4)

$$\left| \frac{1}{(F(s))^2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha(x-y)F(s\chi_\varepsilon(R^i,|t_i|h_{R^i};x))F(s\chi_\varepsilon(R^j,|t_j|h_{R^j};y)) - a_{ij}(t) \right| \\ \to 0 \quad \text{as } \varepsilon\to 0. \tag{3.2.27}$$

By (3.2.26) and (3.2.27),

$$B_{ij}(s,t) \ge -(F(s))^2(a_{ij}(t)+o(1)) \quad \text{as } \varepsilon\to 0. \tag{3.2.28}$$

Thus, it follows from (3.2.22)–(3.2.24) and (3.2.28) that

$$\begin{aligned} \mathcal{D}(s\gamma_{R,\varepsilon}(t)) &\ge (F(s))^2\left(\sum_{i=1}^n a_{ii}(t) - \sum_{i\ne j} a_{ij} + o(1)\right) \\ &\ge \frac{1}{2}(F(s))^2(A+o(1)) \end{aligned}$$

This concludes the proof. ∎

3.3 Asymptotic analysis of mountain pass values

We end this Section with some key estimates on the asymptotic behaviour of $a_n(\lambda)$ as $\lambda\to\pm\infty$.

Proposition 3.3.1. *Assume (F1)–(F4) and let $n\in\mathbb{N}^*$.*

(i) If (CF3) holds, then $\lim_{\lambda\to+\infty}\frac{a_n(\lambda)}{e^\lambda}=+\infty$.

(ii) If (CF4) holds, then $\lim_{\lambda\to-\infty}\frac{a_n(\lambda)}{e^\lambda}=0$.

Proof of (i) of Proposition 3.3.1. Recall $q=\frac{N+\alpha}{N}$, $p=\frac{N+\alpha+2}{N}$, and write $\mu=e^\lambda$ (and consequently adapt the notations) for the sake of simplicity.

Since $a_n(\mu)\ge a_1(\mu)$ for each $n\in\mathbb{N}^*$, it is sufficient to show the claim for $n=1$. By (CF3), for any $\delta>0$ there exists $C_\delta>0$ such that

$$|F(s)| \le \delta|s|^p + C_\delta|s|^q \quad \text{for all } s\in\mathbb{R}.$$

For $v\in H^1_r(\mathbb{R}^N)$, setting $u_s := s^{N/2}v(s\cdot)$, we have

$$\begin{aligned} \mathcal{D}(u_s) &= s^{-N-\alpha}\mathcal{D}(s^{N/2}v) \\ &\le s^{-N-\alpha}\int_{\mathbb{R}^N}\Big(I_\alpha*(\delta s^{\frac{N}{2}p}|v|^p + C_\delta s^{\frac{N}{2}q}|v|^q)\Big)(\delta s^{\frac{N}{2}p}|v|^p + C_\delta s^{\frac{N}{2}q}|v|^q)\,dx \\ &= s^2\int_{\mathbb{R}^N}\Big(I_\alpha*(\delta|v|^p + C_\delta s^{-1}|v|^q)\Big)(\delta|v|^p + C_\delta s^{-1}|v|^q)\,dx \\ &=: s^2 D_{\delta,C_\delta s^{-1}}(v), \end{aligned} \tag{3.3.29}$$

where we write for $\delta>0$ and $A\ge 0$,

$$\begin{aligned} D_{\delta,A}(v) &:= \int_{\mathbb{R}^N}\Big(I_\alpha*(\delta|v|^p + A|v|^q)\Big)(\delta|v|^p + A|v|^q)\,dx, \\ \mathcal{J}_{\delta,A}(v) &:= \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 - \frac{1}{2}D_{\delta,A}(v). \end{aligned}$$

We also denote by $b(\delta, A)$ the MP value of $\mathcal{J}_{\delta,A}$. Taking into account the continuity and monotonicity property of $b(\delta, A)$ with respect of each variable δ and A and observing that $\mathcal{J}_{\delta,A}$ satisfies the (PS) condition, we have

$$b(\delta, A) \to b(\delta, 0) \quad \text{as } A \to 0^+,$$

$$b(\delta, 0) \to +\infty \quad \text{as } \delta \to 0^+.$$

Thus, from (3.3.29) we have that

$$\mathcal{J}(\mu, u_s) \geq s^2 \left(\frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\mu s^{-2}\|v\|_2^2 - \frac{1}{2}D_{\delta, C_\delta s^{-1}}(v) \right).$$

Setting $s := \sqrt{\mu}$, we obtain

$$\mathcal{J}(\mu, u_{\sqrt{\mu}}) \geq \mu \mathcal{J}_{\delta, C_\delta \mu^{-1/2}}(v)$$

and thus $\frac{a_1(\mu)}{\mu} \geq b(\delta, C_\delta \mu^{-1/2})$, which implies

$$\liminf_{\mu\to\infty} \frac{a_1(\mu)}{\mu} \geq \lim_{A\to 0} b(\delta, A) = b(\delta, 0).$$

Since $\delta > 0$ is arbitrary, we gain

$$\lim_{\mu\to+\infty} \frac{a_1(\mu)}{\mu} = +\infty. \qquad \blacksquare$$

We deal now with the proof of (ii) of Proposition 3.3.1. We highlight that, when F is even, the proof can be simplified (see [124], Proposition 2.3.6, and Proposition 3.2.2).

The proof will be based on the key Lemma 3.2.8. We start noticing that, by (CF4) and Remark 3.1.4, for some $\delta_0 > 0$

$$F(s) > 0 \quad \text{for } s \in (0, \delta_0],$$

which implies

(i) when F is even, $F(s) > 0$ for all $s \in [-\delta_0, \delta_0] \setminus \{0\}$;

(ii) when F is odd, $F(s) < 0$ for all $s \in [-\delta_0, 0)$.

By (CF4), we also note that there exists $L_s > 0$ with $L_s \to +\infty$ as $s \to 0^+$ such that

$$F(\sigma) \geq L_s \sigma^p \quad \text{for all } \sigma \in [0, s]. \tag{3.3.30}$$

Proof of (ii) of Proposition 3.3.1. Let $\gamma_{R,\varepsilon}$ defined in (3.2.19). For $s_0 \in (0, \delta_0]$ and $\mu > 0$, we consider the map

$$st \in D_n \mapsto ss_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}}) \in H_r^1(\mathbb{R}^N).$$

We have by Lemma 3.2.8 (since $\varepsilon > 0$ is here fixed small, we write A instead of $A + o_\varepsilon(1)$)

$$\begin{aligned}
&\mu^{-1}\mathcal{J}(\mu, ss_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}})) \\
&= \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\nabla\gamma_{R,\varepsilon}(t)\|_2^2 + \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\gamma_{R,\varepsilon}(t)\|_2^2 - \frac{1}{2}\mu^{-\frac{N}{2}p}\mathcal{D}(ss_0\gamma_{R,\varepsilon}(t)) \\
&\leq \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\gamma_{R,\varepsilon}(t)\|_{H^1}^2 - \frac{1}{4}\mu^{-\frac{N}{2}p}(F(ss_0))^2 A.
\end{aligned}$$

Thus for μ small

$$\mathcal{J}(\mu, s_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}})) < 0 \quad \text{for } t \in \Sigma,$$

which implies that $st \mapsto s_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}})$ is a path belonging to $\Gamma_n(\mu)$. Moreover by (3.3.30)

$$\mu^{-1}a_n(\mu) \leq \max_{s\in[0,1],\, t\in\Sigma} \mu^{-1}\mathcal{J}(\mu, ss_0\gamma_{R,\varepsilon}(t)(\cdot/\mu^{-\frac{1}{2}}))$$

$$\begin{aligned}
&\leq \max_{s\in[0,1],\, t\in\Sigma} \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\gamma_{R,\varepsilon}(t)\|_{H^1}^2 - \frac{1}{4}\mu^{-\frac{N}{2}p}(F(ss_0))^2 A \\
&\leq \max_{s\in[0,1],\, t\in\Sigma} \frac{1}{2}\mu^{-\frac{N}{2}}(ss_0)^2\|\gamma_{R,\varepsilon}(t)\|_{H^1}^2 - \frac{1}{4}L_{s_0}(\mu^{-\frac{N}{2}}(ss_0)^2)^p A \\
&\leq C_{s_0},
\end{aligned}$$

where

$$C_{s_0} := \sup_{\tau\geq 0,\, t\in\Sigma}\left(\frac{1}{2}\tau\|\gamma_{R,\varepsilon}(t)\|_{H^1}^2 - \frac{1}{4}L_{s_0}A\tau^p\right) \in \mathbb{R}.$$

Thus we have

$$\limsup_{\mu\to 0^+} \mu^{-1}a_n(\mu) \leq C_{s_0}.$$

Since $C_{s_0} \to 0$ as $s_0 \to 0$, we have (ii) of Proposition 3.3.1. ∎

3.4 The Pohozaev mountain

In this Section we start studying the Lagrangian formulation, applying the previous asymptotic estimates to a Pohozaev geometry. We consider the functional $\mathcal{I}^m : \mathbb{R}\times H_r^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{I}^m(\lambda,u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\mathcal{D}(u) + \frac{e^\lambda}{2}(\|u\|_2^2 - m), \quad (\lambda,u)\in\mathbb{R}\times H_r^1(\mathbb{R}^N). \tag{3.4.31}$$

It is immediate that, for any $(\lambda,u)\in\mathbb{R}\times H_r^1(\mathbb{R}^N)$,

$$\mathcal{I}^m(\lambda,u) = \mathcal{J}(\lambda,u) - \frac{e^\lambda}{2}m.$$

If (F1)-(F2) hold, by [302, Theorems 2 and 3] we have that each solution u of (3.2.9) belongs to $W^{2,2}_{loc}(\mathbb{R}^N)$ and it satisfies the Pohozaev identity

$$\frac{N-2}{2}\|\nabla u\|_2^2 + \frac{N}{2}e^\lambda\|u\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(u) = 0 \tag{3.4.32}$$

or equivalently

$$\frac{1}{2^*_\alpha}\|\nabla u\|_2^2 + \frac{e^\lambda}{2^\#_\alpha}\|u\|_2^2 - \mathcal{D}(u) = 0$$

where $2^*_\alpha = \frac{N+\alpha}{N-2}$ and $2^\#_\alpha = \frac{N+\alpha}{N}$ are the upper and the lower critical exponents; again we see that essentially the identity means $\frac{d}{d\theta}\mathcal{J}(\lambda,u(\cdot/e^\theta))_{|\theta=0} = 0$. Inspired by this relation, we also introduce the Pohozaev functional $\mathcal{P}:\mathbb{R}\times H_r^1(\mathbb{R}^N)\to\mathbb{R}$ by setting

$$\mathcal{P}(\lambda,u) := \frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(u) + \frac{N}{2}e^\lambda\|u\|_2^2, \quad (\lambda,u)\in\mathbb{R}\times H_r^1(\mathbb{R}^N). \tag{3.4.33}$$

We consider the action of $\mathbb{G} := \mathbb{Z}_2$ on $\mathbb{R}^n$, $n\in\mathbb{N}^*$, and on $\mathbb{R}\times H_r^1(\mathbb{R}^N)$, given by

$$(\pm 1,\xi)\in\mathbb{G}\times\mathbb{R}^n \mapsto \pm\xi\in\mathbb{R}^n,$$

$$(\pm 1,\lambda,u)\in\mathbb{G}\times(\mathbb{R}\times H_r^1(\mathbb{R}^N)) \mapsto (\lambda,\pm u)\in\mathbb{R}\times H_r^1(\mathbb{R}^N).$$

We notice that, under the assumption (F5), $\mathcal{I}^m$, $\mathcal{J}$ and $\mathcal{P}$ are invariant under this action, i.e. they are even in u:

$$\mathcal{I}^m(\lambda,-u) = \mathcal{I}^m(\lambda,u), \quad \mathcal{J}(\lambda,-u) = \mathcal{J}(\lambda,u), \quad \mathcal{P}(\lambda,-u) = \mathcal{P}(\lambda,u).$$

In addition, we observe by the Principle of Symmetric Criticality of Palais [310] that every critical point of $\mathcal{I}^m$ restricted to $\mathbb{R}\times H_r^1(\mathbb{R}^N)$ is actually a critical point of $\mathcal{I}^m$ on the whole $\mathbb{R}\times H^1(\mathbb{R}^N)$. Finally, we denote by $P_2:\mathbb{R}\times H_r^1(\mathbb{R}^N)\to H_r^1(\mathbb{R}^N)$ the projection on the second component.

Moreover we consider the *Pohozaev set*

$$\Omega := \{(\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) > 0\} \cup \{(\lambda, 0) \mid \lambda \in \mathbb{R}\};$$

under the assumption (F5), Ω is symmetric with respect to the axis $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$, that is,

$$(\lambda, u) \in \Omega \implies (\lambda, -u) \in \Omega.$$

We start showing the following property, due to the fact that $\mathcal{D}(u) = o(\|u\|^2_{H^1})$ as $u \to 0$.

Lemma 3.4.1. *We have*

$$\{(\lambda, 0) \mid \lambda \in \mathbb{R}\} \subset int(\Omega). \tag{3.4.34}$$

Proof. By

$$|F(s)| \lesssim |s|^q + |s|^p$$

where $q = \frac{N+\alpha}{N}$ and $p = \frac{N+\alpha+2}{N} < 2^*$. Thus

$$\|F(u)\|_{\frac{2N}{N+\alpha}} \lesssim \||u|^q\|_{\frac{2N}{N+\alpha}} + \||u|^p\|_{\frac{2N}{N+\alpha}} = \|u\|_2^q + \|u\|^p_{\frac{2Np}{N+\alpha}}.$$

Therefore by Proposition 1.3.1 and Young's inequality we have

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * |F(u)|)|F(u)|\, dx &\lesssim \|F(u)\|^2_{\frac{2N}{N+\alpha}} \lesssim \left(\|u\|_2^q + \|u\|^p_{\frac{2Np}{N+\alpha}}\right)^2 \\ &\lesssim \|u\|_2^{2q} + \|u\|^{2p}_{\frac{2Np}{N+\alpha}} \leq \|u\|^{2q}_{H^1} + \|u\|^{2p}_{H^1}\end{aligned}$$

thus

$$\mathcal{P}(\lambda, u) \gtrsim \|u\|^2_{H^1} - \|u\|^{2q}_{H^1} - \|u\|^{2p}_{H^1} > 0$$

for $\|u\|_{H^1}$ small, $u \neq 0$. ∎

By (3.4.34) we detect the *Pohozaev mountain*

$$\partial\Omega = \{(\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N) \mid \mathcal{P}(\lambda, u) = 0,\ u \not\equiv 0\}.$$

We observe that $\partial\Omega \neq \emptyset$, for instance by [302, Theorems 1 and 3].

Proposition 3.4.2. *Assume* (F1)–(F4) *and* (F5)*. We have the following properties.*

(i) $\mathcal{J}(\lambda, u) \geq 0$ *for all* $(\lambda, u) \in \Omega$.

(ii) $\mathcal{J}(\lambda, u) \geq a_1(\lambda) > 0$ *for all* $(\lambda, u) \in \partial\Omega$.

(iii) Assume (CF3)*. For any* $m > 0$*, we set*

$$E^m := \inf_{(\lambda,u)\in\partial\Omega} \mathcal{I}^m(\lambda, u), \quad \text{and} \quad B^m := \inf_{\lambda\in\mathbb{R}} \left(a_1(\lambda) - \frac{e^\lambda}{2} m\right).$$

Then $E^m \geq B^m > -\infty$*. In particular* $B^m \in \mathbb{R}$ *and*

$$\mathcal{I}^m(\lambda, u) \geq B^m \quad \text{for every } (\lambda, u) \in \partial\Omega.$$

Proof. We notice that for all $(\lambda, u) \in \Omega$

$$\mathcal{J}(\lambda, u) \geq \mathcal{J}(\lambda, u) - \frac{\mathcal{P}(\lambda, u)}{N+\alpha} = \frac{\alpha+2}{2(N+\alpha)}\|\nabla u\|_2^2 + \frac{\alpha}{2(N+\alpha)} e^\lambda \|u\|_2^2 \geq 0$$

and thus (i) follows. Point (ii) follows from the fact that for each λ the mountain pass level $a_1(\lambda)$ coincides with the ground state energy level (see [301, Section 4.2] and Section 4.3 for details); see also Remark 3.4.3. Focus on (iii): the fact that $E^m \geq B^m$ is a direct consequence of (ii), while the fact that $B^m > -\infty$ comes from Proposition 3.3.1 (i). ∎

Remark 3.4.3. *In order to show that $a_1(\lambda) > 0$, without exploiting the existence result for the unconstrained problem, we argue as follows (see also [114]). Let $\gamma \in \Gamma_1(\lambda)$; by definition of $\Gamma_1(\lambda)$ and by Proposition 3.4.2 (i) there exists t^* such that $\gamma(t^*) \in \partial\Omega$ and $\gamma(t^*) \neq 0$, thus $\mathcal{P}(\lambda, \gamma(t^*)) = 0$. This means that*

$$\mathcal{J}(\lambda, \gamma(t^*)) = \frac{\alpha+2}{2(N+\alpha)} \|\nabla \gamma(t^*)\|_2^2 + \frac{\alpha\mu}{2(N+\alpha)} \|\gamma(t^*)\|_2^2 \simeq \|\gamma(t^*)\|_{H^1}^2$$

thus

$$a(\lambda) \gtrsim \inf_{u \in (\partial\Omega)_\lambda} \|u\|_{H^1}^2.$$

Since, by (3.4.34), $(\partial\Omega)_\lambda$ is far from the line $(\lambda, 0)$, we obtain that the right-hand side is strictly positive, which is the claim.

From now on we assume (CF3) to give sense to the quantity B^m. In view of Proposition 3.4.2 (iii), we set for $m > 0$ and $n \in \mathbb{N}^*$

$$\Gamma_n^m := \{\Theta \in C(D_n, \mathbb{R} \times H_r^1(\mathbb{R}^N)) \mid \Theta \text{ is } \mathbb{G}\text{-equivariant}, \ \mathcal{I}^m(\Theta(0)) \leq B^m - 1, \\ \Theta|_{\partial D_n} \notin \Omega, \ \mathcal{I}^m(\Theta|_{\partial D_n}) \leq B^m - 1\}$$

and

$$b_n^m := \inf_{\Theta \in \Gamma_n^m} \sup_{\xi \in D_n} \mathcal{I}(\Theta(\xi));$$

we point out that asking $\Theta = (\Theta_1, \Theta_2) \in \Gamma_n^m$ to be $\mathbb{G}$-equivariant means that Θ_1 is even and Θ_2 is odd, and in particular $\Theta_2(0) = 0$ which implies $\Theta(0) \in \Omega$.

Proposition 3.4.4. *Assume (F1)-(F2)-(CF3)-(F4)-(F5). We have the following properties.*

(i) For any $m > 0$ and $n \in \mathbb{N}^*$, we have $\Gamma_n^m \neq \emptyset$ and

$$b_n^m \leq a_n(\lambda) - e^\lambda \frac{m}{2}, \tag{3.4.35}$$

for each $\lambda \in \mathbb{R}$. Moreover, b_n^m increases with respect to n.

(ii) For any $k \in \mathbb{N}^*$ there exists $m_k \geq 0$, namely given by

$$m_k := 2 \inf_{\lambda \in \mathbb{R}} \frac{a_k(\lambda)}{e^\lambda}, \tag{3.4.36}$$

such that for $m > m_k$

$$b_n^m < 0 \quad \textit{for } n = 1, 2, \dots, k.$$

Moreover, m_k is increasing with respect to k.

(iii) If (CF4) holds, then $m_k = 0$ for each $k \in \mathbb{N}^*$. That is, for each $m > 0$ we have

$$b_n^m < 0 \quad \textit{for all } n \in \mathbb{N}^*.$$

Proof. For given $\lambda \in \mathbb{R}$ and $\zeta \in \Gamma_n(\lambda)$, we will find a $\psi \in \Gamma_n^m$ such that

$$\max_{\xi \in D_n} \mathcal{J}(\psi(\xi)) \leq \max_{\xi \in D_n} \mathcal{J}(\lambda, \zeta(\xi)), \tag{3.4.37}$$

so that we have

$$b_n^m \leq \max_{\xi \in D_n} \mathcal{I}^m(\psi(\xi)) \leq \max_{\xi \in D_n} \mathcal{J}(\lambda, \zeta(\xi)) - \frac{e^\lambda}{2} m$$

and, passing to the infimum over $\Gamma_n(\lambda)$, we gain (3.4.35).
To find $\psi \in \Gamma_n^m$ with (3.4.37), observe that, by definition of $\Gamma_n(\lambda)$ and compactness of $\zeta(\partial D_n)$,

there exists $C>0$ such that $\mathcal{D}(\zeta(\xi)) \geq C > 0$ for $\xi \in \partial D_n$. Thus, we have $\mathcal{I}^m(\lambda, \zeta(\xi)(\cdot/L)) \to -\infty$ and $\mathcal{P}(\lambda, \zeta(\xi)(\cdot/L)) \to -\infty$ as $L \to +\infty$, uniformly for $\xi \in \partial D_n$. Thus, for $L \gg 1$ we obtain, for every $\xi \in \partial D_n$,

$$\mathcal{I}^m(\lambda, \zeta(\xi)(\cdot/L)) \leq B^m - 1 \quad \text{and} \quad \mathcal{P}(\lambda, \zeta(\xi)(\cdot/L)) < 0. \tag{3.4.38}$$

We also note that $\mathcal{I}^m(\lambda+L, 0) = -\frac{e^{\lambda+L}}{2} m \to -\infty$ as $L \to +\infty$. Thus, for $L \gg 1$, we find that the path $\psi : D_n \to \mathbb{R} \times H^1_r(\mathbb{R}^N)$

$$\psi(\xi) := \begin{cases} (\lambda + L(1-2|\xi|), 0) & \text{if } |\xi| \in [0,1/2], \\ \left(\lambda, \zeta\left(\frac{\xi}{|\xi|}(2|\xi|-1)\right)(\cdot/L)\right) & \text{if } |\xi| \in (1/2, 1] \end{cases}$$

satisfies $\psi(0) = (\lambda+L, 0) \in \mathbb{R} \times \{0\}$, $\mathcal{I}^m(\psi(0)) \leq B^m - 1$ and $\mathcal{I}^m(\psi(\xi)) \leq B^m - 1$ for $\xi \in \partial D_n$. Thus, by (3.4.38), we obtain $\psi \in \Gamma^m_n$ and (3.4.37) holds.
The monotonicity of b^m_n with respect to n is a consequence of the definition. Point (ii) follows from (3.4.35) and (iii) follows from Proposition 3.3.1 (ii). ∎

As a corollary to Proposition 3.4.4, we have the following result.

Corollary 3.4.5. *For any $m>0$, we have*

$$B^m = E^m = b^m_1,$$

i.e. the first minimax value b^m_1 equals the Pohozaev minimum E^m on the product space.

Proof. Since any path in Γ^m_n passes through $\partial\Omega$, we have $b^m_n \geq E^m \geq B^m$ for each n. On the other hand, passing to the infimum (3.4.35) we obtain $b^m_1 \leq B^m$ and thus the claim. ∎

3.5 The Palais-Smale-Pohozaev condition

For every $b \in \mathbb{R}$ we set

$$K^m_b := \{(\lambda, u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N) \mid \mathcal{I}^m(\lambda, u) = b,\ \partial_\lambda \mathcal{I}^m(\lambda, u) = 0,\ \partial_u \mathcal{I}^m(\lambda, u) = 0\}.$$

As already observed, under (F1)-(F2) we have that $\mathcal{P}(\lambda, u) = 0$ for each $(\lambda, u) \in K^m_b$. We notice also that, assuming (F5), K^m_b is invariant under the $\mathbb{G}$-action, that is

$$(\lambda, u) \in K^m_b \implies (\lambda, -u) \in K^m_b.$$

Under our assumptions on F, it seems difficult to verify the standard Palais-Smale condition for the functional $\mathcal{I}^m$. Therefore we cannot recognize that K^m_b is compact.
Inspired by [125, 224, 231], we introduce the Palais-Smale-Pohozaev condition, a weaker compactness condition that takes into account the scaling properties of $\mathcal{I}^m$ through the Pohozaev functional $\mathcal{P}$. Through this tool we will show that K^m_b is compact when $b<0$.

Definition 3.5.1. *For $b \in \mathbb{R}$, we say that $(\lambda_n, u_n)_n \subset \mathbb{R} \times H^1_r(\mathbb{R}^N)$ is a* Palais-Smale-Pohozaev sequence *for $\mathcal{I}^m$ at level b (shortly a $(PSP)_b$ sequence) if*

$$\mathcal{I}^m(\lambda_n, u_n) \to b, \tag{3.5.39}$$

$$\partial_\lambda \mathcal{I}^m(\lambda_n, u_n) \to 0, \tag{3.5.40}$$

$$\|\partial_u \mathcal{I}^m(\lambda_n, u_n)\|_{(H^1_r(\mathbb{R}^N))^*} \to 0, \tag{3.5.41}$$

$$\mathcal{P}(\lambda_n, u_n) \to 0. \tag{3.5.42}$$

We say that $\mathcal{I}^m$ satisfies the Palais-Smale-Pohozaev condition *at level b (shortly the $(PSP)_b$ condition) if every $(PSP)_b$ sequence has a strongly convergent subsequence in $\mathbb{R} \times H^1_r(\mathbb{R}^N)$.*

We show now the following result.

Proposition 3.5.2. *Assume (F1)-(CF2)-(CF3) and let $b < 0$. Then $\mathcal{I}^m$ satisfies the $(PSP)_b$ condition.*

Proof. Let $b < 0$ and let $(\lambda_n, u_n) \subset \mathbb{R} \times H^1_r(\mathbb{R}^N)$ be a $(PSP)_b$ sequence, i.e. satisfying (3.5.39)–(3.5.42). First we note that by (3.5.40) we obtain

$$e^{\lambda_n}(\|u_n\|_2^2 - m) \to 0. \tag{3.5.43}$$

Step 1: *λ_n is bounded from below and $\|u_n\|_2^2 \to m$ as $n \to +\infty$.*
We have by (3.5.42), (3.5.39) and (3.5.43)

$$\begin{aligned}
o(1) &= \mathcal{P}(\lambda_n, u_n) \\
&= -\frac{\alpha+2}{2}\|\nabla u_n\|_2^2 + (N+\alpha)\Big(\mathcal{I}^m(\lambda_n, u_n) - \frac{e^{\lambda_n}}{2}(\|u_n\|_2^2 - m)\Big) + \frac{N}{2}e^{\lambda_n}\|u_n\|_2^2 \\
&= -\frac{\alpha+2}{2}\|\nabla u_n\|_2^2 + (N+\alpha)(b + o(1)) + \frac{N}{2}e^{\lambda_n}m + o(1).
\end{aligned}$$

Here we used (3.5.43). From the above identity, we derive boundedness of λ_n from below, since $b < 0$. This result joined to (3.5.43) finally gives $\|u_n\|_2^2 \to m$.
Step 2: *λ_n and $\|\nabla u_n\|_2^2$ are bounded.*
Since, by (3.5.41), $\varepsilon_n := \|\partial_u \mathcal{I}^m(\lambda_n, u_n)\|_{(H^1_r(\mathbb{R}^N))^*} \to 0$, we have

$$\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N}(I_\alpha * F(u_n))f(u_n)u_n dx + e^{\lambda_n}\|u_n\|_2^2 \le \varepsilon_n\|u_n\|_{H^1}. \tag{3.5.44}$$

We observe that by (CF3) for $\delta > 0$ fixed, there exists $C_\delta > 0$ such that

$$|F(s)| \le \delta|s|^p + C_\delta|s|^q$$

where we recall $p = \frac{N+\alpha+2}{N}$ and $q = \frac{N+\alpha}{N}$. Thus

$$\|F(u_n)\|_{\frac{2N}{N+\alpha}} \le \delta\||u_n|^p\|_{\frac{2N}{N+\alpha}} + C_\delta\||u_n|^q\|_{\frac{2N}{N+\alpha}} = \delta\|u_n\|^p_{\frac{2Np}{N+\alpha}} + C_\delta\|u_n\|_2^q.$$

Therefore by (CF2), Proposition 1.3.1 and Young's inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}(I_\alpha * |F(u_n)|)|f(u_n)u_n|\, dx \\
&\le C\|F(u_n)\|_{\frac{2N}{N+\alpha}}\|f(u_n)u_n\|_{\frac{2N}{N+\alpha}} \\
&\le C\left(\delta\|u_n\|^p_{\frac{2Np}{N+\alpha}} + C_\delta\|u_n\|_2^q\right)\cdot C'\left(\|u_n\|^p_{\frac{2Np}{N+\alpha}} + \|u_n\|_2^q\right) \\
&\le CC'\delta\|u_n\|^{2p}_{\frac{2Np}{N+\alpha}} + CC'(\delta + C_\delta)\left(\frac{\delta}{2}\|u_n\|^{2p}_{\frac{2Np}{N+\alpha}} + \frac{1}{2\delta}\|u_n\|_2^{2q}\right) + CC'C_\delta\|u_n\|_2^{2q} \\
&\le C''\delta\|u_n\|^{2p}_{\frac{2Np}{N+\alpha}} + C''_\delta\|u_n\|_2^{2q}
\end{aligned}$$

and thus, by the Gagliardo-Nirenberg inequality and (3.5.44),

$$\begin{aligned}
\|\nabla u_n\|_2^2 + e^{\lambda_n}\|u_n\|_2^2 &\le \int_{\mathbb{R}^N}(I_\alpha * |F(u_n)|)|f(u_n)u_n|dx + \varepsilon_n\|u_n\|_{H^1} \\
&\le C'''\delta\|\nabla u_n\|_2^2\|u_n\|_2^{2(p-1)} + C''_\delta\|u_n\|_2^{\frac{2(N+\alpha)}{N}} + \varepsilon_n\|u_n\|_{H^1}.
\end{aligned}$$

Since by Step 1 $\|u_n\|_2^2 = m + o(1)$, we obtain

$$(1 - C'''\delta(m + o(1))^{p-1})\|\nabla u_n\|_2^2 + e^{\lambda_n}(m + o(1))$$

$$\leq C''_\delta(m+o(1))^{\frac{N+\alpha}{N}} + \varepsilon_n(\|\nabla u_n\|_2^2 + m + o(1))^{1/2}.$$

For δ small enough, we have the boundedness of e^{λ_n} and $\|\nabla u_n\|_2$. Hence λ_n can not go to $+\infty$ and thus by Step 1 we infer that λ_n is bounded.

Step 3: *λ_n and u_n strongly converge.*
By Steps 1-2, the sequence $(\lambda_n, u_n)_n$ is bounded in $\mathbb{R}\times H^1_r(\mathbb{R}^N)$ and thus after extracting a subsequence, denoted in the same way, we may assume that $\lambda_n \to \lambda_0$ and $u_n \rightharpoonup u_0$ weakly in $H^1_r(\mathbb{R}^N)$ for some $(\lambda_0, u_0) \in \mathbb{R}\times H^1_r(\mathbb{R}^N)$. Taking into account the assumptions (F1)–(F3) and the compact embedding of $H^1_r(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ for $r\in(2,2^*)$, we have by Proposition 1.5.9

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n))f(u_n)u_0\,dx \to \int_{\mathbb{R}^N}(I_\alpha * F(u_0))f(u_0)u_0\,dx$$

and

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_n))f(u_n)u_n\,dx \to \int_{\mathbb{R}^N}(I_\alpha * F(u_0))f(u_0)u_0\,dx.$$

By (3.5.41) we derive that $\langle \partial_u \mathcal{I}^m(\lambda_n,u_n),u_n\rangle \to 0$ and $\langle \partial_u \mathcal{I}^m(\lambda_n,u_n),u_0\rangle \to 0$, and hence $(\nabla u_n, \nabla(u_n-u_0))_2 + (u_n, e^{\lambda_n}u_n - e^{\lambda_0}u_0)_2 \to 0$. Combining this with $u_n \rightharpoonup u_0$ and $\lambda_n\to\lambda_0$ we get

$$\|\nabla u_n\|_2^2 + e^{\lambda_n}\|u_n\|_2^2 \to \|\nabla u_0\|_2^2 + e^{\lambda_0}\|u_0\|_2^2$$

which implies $u_n \to u_0$ strongly in $H^1_r(\mathbb{R}^N)$. ∎

As a straightforward consequence we obtain the following result.

Corollary 3.5.3. *Assume (F1)-(CF2)-(CF3) and let $b<0$. Then $K_b^m \cap (\mathbb{R}\times\{0\}) = \emptyset$ and K_b^m is compact.*

Remark 3.5.4. *We emphasize that the $(PSP)_b$ condition does not hold at level $b=0$. Indeed we can consider a $(PSP)_0$ unbounded sequence $(\lambda_n, 0)$ with $\lambda_n \to -\infty$.*

3.6 Genus-shaped critical points

In this Section we essentially follow the lines of Sections 2.4.2–2.7. We give just an outline, avoiding details and proofs.

3.6.1 Augmented functional

We start by achieving a deformation lemma. In order to do this we define

$$M := \mathbb{R}\times\mathbb{R}\times H^1_r(\mathbb{R}^N)$$

and introduce the augmented functional $\mathcal{H}^m : M \to \mathbb{R}$

$$\begin{aligned}\mathcal{H}^m(\theta,\lambda,u) &:= \mathcal{I}^m(\lambda, u(e^{-\theta}\cdot)) \\ &= \frac{e^{(N-2)\theta}}{2}\|\nabla u\|_2^2 - \frac{e^{(N+\alpha)\theta}}{2}\mathcal{D}(u) + \frac{e^\lambda}{2}(e^{N\theta}\|u\|_2^2 - m)\end{aligned} \tag{3.6.45}$$

for all $(\theta,\lambda,u)\in M$, and thus $\partial_\theta \mathcal{H}^m(\theta,\lambda,u) = \mathcal{P}(\lambda, u(\cdot/e^\theta))$. We point out that, considered the action of $\mathbb{G}$ on M

$$\mathbb{G}\times M \to M;\ (\pm1,\theta,\lambda,u)\mapsto(\theta,\lambda,\pm u)$$

and assumed (F5), it results that $\mathcal{H}^m$ is $\mathbb{G}$-invariant. Introducing a metric on M by

$$\|(\alpha,\nu,h)\|^2_{(\theta,\lambda,u)} := \Big|\Big(\alpha,\nu,\|h(e^{-\theta}\cdot)\|_{H^1}\Big)\Big|^2$$

for any $(\alpha, \nu, h) \in T_{(\theta,\lambda,u)}M \equiv \mathbb{R} \times \mathbb{R} \times H^1_r(\mathbb{R}^N)$, we regard M as a Hilbert manifold. We also denote the dual norm on $T^*_{(\theta,\lambda,u)}M$ by $\|\cdot\|_{(\theta,\lambda,u),*}$, and observe that both $\|\cdot\|_{(\theta,\lambda,u)}$ and $\|\cdot\|_{(\theta,\lambda,u),*}$ actually depend only on θ.

Denote now $D := (\partial_\theta, \partial_\lambda, \partial_u)$ the gradient with respect to all the variables; a direct computation shows that for any $(\theta, \lambda, u) \in M$

$$\begin{aligned}&\|D\mathcal{H}^m(\theta,\lambda,u)\|^2_{(\theta,\lambda,u),*}\\ &\quad = |\mathcal{P}(\lambda, u(e^{-\theta}\cdot))|^2 + |\partial_\lambda \mathcal{I}^m(\lambda, u(e^{-\theta}\cdot))|^2 + \|\partial_u \mathcal{I}^m(\lambda, u(e^{-\theta}\cdot))\|^2_{(H^1_r(\mathbb{R}^N))^*}.\end{aligned}$$

We furthermore define the set of critical points of $\mathcal{H}^m$ at level b by

$$\begin{aligned}\tilde{K}^m_b :=& \{(\theta,\lambda,u) \in M \mid \mathcal{H}^m(\theta,\lambda,u) = b,\ D\mathcal{H}^m(\theta,\lambda,u) = 0\}\\ =& \{(\theta,\lambda,u(e^\theta\cdot)) \mid (\lambda,u) \in K^m_b,\ \theta \in \mathbb{R}\}.\end{aligned}$$

Finally we introduce the distance between two points as

$$\begin{aligned}&\mathrm{dist}_M((\theta_0,\lambda_0,h_0),(\theta_1,\lambda_1,h_1)) :=\\ &\quad \inf\left\{\int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}\,dt \mid \gamma \in C^1([0,1],M),\ \gamma(0) = (\theta_0,\lambda_0,h_0), \gamma(1) = (\theta_1,\lambda_1,h_1)\right\}.\end{aligned}$$

As a consequence of Proposition 3.5.2 we obtain the following.

Proposition 3.6.1. *Assume (F1)-(CF2)-(CF3) and let $b < 0$. Then $\mathcal{H}^m$ satisfies the following Palais-Smale-type condition $(\widetilde{PSP})_b$: for each sequence $(\theta_n, \lambda_n, u_n)_n \subset M$ such that*

$$\mathcal{H}^m(\theta_n,\lambda_n,u_n) \to b,$$

$$\|D\mathcal{H}^m(\theta_n,\lambda_n,u_n)\|_{(\theta_n,\lambda_n,u_n),*} \to 0$$

as $n \to +\infty$, we have, up to a subsequence,

$$\mathrm{dist}_M((\theta_n,\lambda_n,u_n), \tilde{K}^m_b) \to 0.$$

3.6.2 Deformation theory

We write, for $b \in \mathbb{R}$

$$[\mathcal{I}^m \le b] := \{(\lambda,u) \in \mathbb{R} \times H^1_r(\mathbb{R}^N) \mid \mathcal{I}^m(\lambda,u) \le b\},$$

$$[\mathcal{H}^m \le b]_M := \{(\theta,\lambda,u) \in M \mid \mathcal{H}^m(\theta,\lambda,u) \le b\}.$$

We state the following result.

Proposition 3.6.2. *Assume (F1)-(CF2)-(CF3). Let $b < 0$, and let $\mathcal{O}$ be a neighborhood of K^m_b with respect to the standard distance of $\mathbb{R} \times H^1_r(\mathbb{R}^N)$. Let $\bar\varepsilon > 0$, then there exist $\varepsilon \in (0, \bar\varepsilon)$ and $\eta : [0,1] \times (\mathbb{R} \times H^1_r(\mathbb{R}^N)) \to \mathbb{R} \times H^1_r(\mathbb{R}^N)$ continuous such that*

1. *$\eta(0,\cdot,\cdot) = id_{\mathbb{R}\times H^1_r(\mathbb{R}^N)}$;*
2. *η fixes $[\mathcal{I}^m \le b - \bar\varepsilon]$, that is, $\eta(t,\cdot,\cdot) = id_{[\mathcal{I}^m \le b-\bar\varepsilon]}$ for all $t \in [0,1]$;*
3. *$\mathcal{I}^m$ is non-increasing along η, and in particular $\mathcal{I}^m(\eta(t,\cdot,\cdot)) \le \mathcal{I}^m(\cdot,\cdot)$ for all $t \in [0,1]$;*
4. *if $K^m_b = \emptyset$, then $\eta(1, [\mathcal{I}^m \le b + \varepsilon]) \subseteq [\mathcal{I}^m \le b - \varepsilon]$;*
5. *if $K^m_b \neq \emptyset$, then*
$$\eta(1, [\mathcal{I}^m \le b + \varepsilon] \setminus \mathcal{O}) \subseteq [\mathcal{I}^m \le b - \varepsilon]$$
and
$$\eta(1, [\mathcal{I}^m \le b + \varepsilon]) \subseteq [\mathcal{I}^m \le b - \varepsilon] \cup \mathcal{O};$$

6. *if (F5) holds, then $\eta(t,\cdot,\cdot)$ is $\mathbb{G}$-equivariant, i.e. for $\eta = (\eta_1, \eta_2)$ we have η_1 even and η_2 odd in u.*

To prove this, we work first on the functional $\mathcal{H}$, for which we obtained a $(\widetilde{PSP})$ condition, which implies that for any $b < 0$ there exists ε, δ, $\nu > 0$ such that

$$\|D\mathcal{H}^m(\theta,\lambda,u)\|_{(\theta,\lambda,u),*} \geq \nu$$

for $(\theta,\lambda,u) \in M$ satisfying $\mathcal{H}^m(\theta,\lambda,u) \in [b-\varepsilon, b+\varepsilon]$ and $\mathrm{dist}_M((\theta,\lambda,u), \widetilde{K_b^m}) \geq \delta$.

Proposition 3.6.3. *Assume (F1)-(CF2)-(CF3). Let $b < 0$, and let $\tilde{\mathcal{O}}$ be a neighborhood of $\tilde{K}_b^m$ with respect to dist_M. Let $\bar{\varepsilon} > 0$, then there exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\tilde{\eta} : [0,1] \times M \to M$ continuous such that*

1. *$\tilde{\eta}(0,\cdot,\cdot,\cdot) = id_M$;*
2. *$\tilde{\eta}$ fixes $[\mathcal{H}^m \leq b - \bar{\varepsilon}]_M$, that is $\tilde{\eta}(t,\cdot,\cdot,\cdot) = id_{[\mathcal{H}^m \leq b-\bar{\varepsilon}]_M}$ for all $t \in [0,1]$;*
3. *$\mathcal{H}^m$ is non-increasing along $\tilde{\eta}$, and in particular $\mathcal{H}^m(\tilde{\eta}(t,\cdot,\cdot,\cdot)) \leq \mathcal{H}^m(\cdot,\cdot,\cdot)$ for all $t \in [0,1]$;*
4. *if $\tilde{K}_b^m = \emptyset$, then $\tilde{\eta}(1, [\mathcal{H}^m \leq b+\varepsilon]_M) \subseteq [\mathcal{H}^m \leq b-\varepsilon]_M$;*
5. *if $\tilde{K}_b^m \neq \emptyset$, then*
$$\tilde{\eta}(1, [\mathcal{H}^m \leq b+\varepsilon]_M \setminus \tilde{\mathcal{O}}) \subseteq [\mathcal{H}^m \leq b-\varepsilon]_M$$
and
$$\tilde{\eta}(1, [\mathcal{H}^m \leq b+\varepsilon]_M) \subseteq [\mathcal{H}^m \leq b-\varepsilon]_M \cup \tilde{\mathcal{O}};$$
6. *if (F5) holds, then $\tilde{\eta}(t,\cdot,\cdot)$ is $\mathbb{G}$-equivariant, i.e. for $\tilde{\eta} = (\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2)$ we have $\tilde{\eta}_0$, $\tilde{\eta}_1$ even and $\tilde{\eta}_2$ odd in u.*

To get Proposition 3.6.2 from Proposition 3.6.3 we introduce

$$\pi : (\theta,\lambda,u) \in M \mapsto (\lambda, u(e^{-\theta}\cdot)) \in \mathbb{R} \times H_r^1(\mathbb{R}^N),$$
$$\iota : (\lambda,u) \in \mathbb{R} \times H_r^1(\mathbb{R}^N) \mapsto (0,\lambda,u) \in M,$$

which are a kind of rescaling projection and immersion satisfying

$$\pi \circ \iota = id_{\mathbb{R}\times H_r^1(\mathbb{R}^N)}, \quad \pi(\tilde{K}_b^m) = K_b^m,$$
$$\mathcal{H}^m \circ \iota = \mathcal{I}^m, \quad \mathcal{I}^m \circ \pi = \mathcal{H}^m.$$

For a deformation $\tilde{\eta}$ obtained in Proposition 3.6.3 we thus define

$$\eta(t,\lambda,u) := \pi(\tilde{\eta}(t,\iota(\lambda,u))), \quad (t,\lambda,u) \in [0,1] \times (\mathbb{R} \times H_r^1(\mathbb{R}^N)). \tag{3.6.46}$$

3.6.3 Multiple critical points

For each $n \in \mathbb{N}^*$, define

$$\Lambda_n^m := \{A = \Theta(\overline{D_{n+l} \setminus Y}) \mid l \in \mathbb{N},\ \Theta \in \Gamma_{n+l}^m,\ Y \subseteq D_{n+l} \setminus \{0\} \text{ is closed, symmetric in } 0 \text{ and } \mathrm{genus}(Y) \leq l\}$$

and

$$c_n^m := \inf_{A \in \Lambda_n^m} \sup_A \mathcal{I}^m.$$

We notice that $\{\Theta(D_n)\}_{\Theta \in \Gamma_n^m} \subset \Lambda_n^m$. In the following lemma, we observe that Λ_n^m and c_n^m inherit the properties of Γ_n^m and b_n^m, also given by

$$A \cap \partial\Omega \neq \emptyset \quad \text{for all } A \in \Lambda_1^m,$$

together with the extra property (v).

Proposition 3.6.4. *Assume (F1)-(F2)-(CF3)-(F4). Let $n \in \mathbb{N}^*$ and $m > 0$. Then*

(i) $\Lambda_n^m \neq \emptyset$.

(ii) $\Lambda_{n+1}^m \subseteq \Lambda_n^m$*, and thus* $c_n^m \leq c_{n+1}^m$.

(iii) $c_n^m \leq b_n^m$.

(iv) $B^m = E^m \leq c_1^m$.

(v) Let $A \in \Lambda_n^m$ *and* $Z \subset \mathbb{R} \times H_r^1(\mathbb{R}^N)$ *be* $\mathbb{G}$*-invariant, closed, and such that* $0 \notin \overline{P_2(Z)}$ *and* $\mathrm{genus}(\overline{P_2(Z)}) \leq i < n$*. Then* $\overline{A \setminus Z} \in \Lambda_{n-i}^m$.

Fix $n \in \mathbb{N}^*$ and let Λ_n^m and c_n^m satisfying the properties of Proposition 3.6.4. We build now multiple solutions.

Proposition 3.6.5. *Assume (F1)-(CF2)-(CF3)-(F4)-(F5). Fix $k \in \mathbb{N}^*$ and assume $m > m_k$ (see (3.4.36)). Then*

$$c_1^m \leq c_2^m \leq \cdots \leq c_k^m < 0$$

are critical values of $\mathcal{I}^m$*. Moreover*

(i) if, for some $q \in \mathbb{N}^*$,

$$c_n^m < c_{n+1}^m < \cdots < c_{n+q}^m < 0$$

then we have $q+1$ different nonzero critical values, and thus $q+1$ different pairs of nontrivial solutions of (3.1.6);

(ii) if instead, for some $q \in \mathbb{N}^*$,

$$c_n^m = c_{n+1}^m = \cdots = c_{n+q}^m =: b < 0 \tag{3.6.47}$$

then

$$\mathrm{genus}(P_2(K_b^m)) \geq q+1 \tag{3.6.48}$$

and thus $\# P_2(K_b^m) = +\infty$*, which means that we have infinite different solutions of (3.1.6).*

Summing up, we have at least k different pairs of nontrivial solutions of (3.1.6).

3.7 The unconstrained problem

In this Section we show how to exploit some of the developed tools also to obtain infinitely many radial solutions for the *unconstrained problem* (3.1.1), and give a sketch of the proof of Theorem 3.1.5. Here we assume (F1)–(F5). We fix $\lambda \in \mathbb{R}$ and write $\mu = e^\lambda$; omitting λ, we denote $\mathcal{J}(\cdot) := \mathcal{J}(\lambda, \cdot) : H_r^1(\mathbb{R}^N) \to \mathbb{R}$, i.e.

$$\mathcal{J}(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\mathcal{D}(u) + \frac{\mu}{2}\|u\|_2^2, \quad u \in H_r^1(\mathbb{R}^N). \tag{3.7.49}$$

Similarly we write $\mathcal{P}(\cdot) := \mathcal{P}(\lambda, \cdot)$. For every $b \in \mathbb{R}$ we set

$$K_b := \{u \in H_r^1(\mathbb{R}^N) \mid \mathcal{J}(u) = b,\ \mathcal{J}'(u) = 0\}.$$

We have the following result.

Proposition 3.7.1. *Assume (F1)–(F3) and let $b \in \mathbb{R}$. Then $\mathcal{J}$ satisfies the Palais-Smale-Pohozaev condition at level b (shortly $(PSP)_b$), that is every sequence $(u_n)_n \subset H_r^1(\mathbb{R}^N)$ satisfying*

$$\mathcal{J}(u_n) \to b, \tag{3.7.50}$$

$$\|\mathcal{J}'(u_n)\|_{(H_r^1(\mathbb{R}^N))^*} \to 0, \tag{3.7.51}$$

$$\mathcal{P}(u_n) \to 0, \tag{3.7.52}$$

admits a strongly convergent subsequence in $H_r^1(\mathbb{R}^N)$. In particular, K_b is compact in $H_r^1(\mathbb{R}^N)$.

Proof. First observe that, by (3.7.50) and (3.7.52) we obtain

$$\frac{\alpha+2}{2}\|\nabla u_n\|_2^2 + \frac{\alpha}{2}\mu\|u_n\|_2^2 = (N+\alpha)b + o(1). \tag{3.7.53}$$

We observe that $b \geq 0$ and the boundedness of u_n in $H_r^1(\mathbb{R}^N)$. Thus by (F2)-(F3), $\mathcal{D}'(u_n)$ has a strongly convergent subsequence in $(H_r^1(\mathbb{R}^N))^*$ and by (3.7.51), u_n has a strongly convergent subsequence in $H_r^1(\mathbb{R}^N)$. Here we make use of Proposition 1.5.9. ∎

Set $[\mathcal{J} \leq b] := \{u \in H_r^1(\mathbb{R}^N) \mid \mathcal{J}_\lambda(u) \leq b\}$. Following the arguments of Section 3.6 and 3.6.2, we prove the following deformation result by means of an augmented functional.

Proposition 3.7.2. *Assume (F1)–(F3). Let $b \in \mathbb{R}$ and let $\mathcal{O}$ be a neighborhood of $K_b(\lambda)$. Let $\bar{\varepsilon} > 0$, then there exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta : [0,1] \times H_r^1(\mathbb{R}^N) \to H_r^1(\mathbb{R}^N)$ continuous such that*

1. *$\eta(0, \cdot) = id_{H_r^1(\mathbb{R}^N)}$;*
2. *η fixes $[\mathcal{J} \leq b - \bar{\varepsilon}]$, that is, $\eta(t,u) = u$ for all $t \in [0,1]$ and $\mathcal{J}(u) \leq b - \bar{\varepsilon}$;*
3. *$\mathcal{J}$ is non-increasing along η, and in particular $\mathcal{J}(\eta(t,\cdot)) \leq \mathcal{J}(\cdot)$ for all $t \in [0,1]$;*
4. *if $K_b = \emptyset$, then $\eta(1, [\mathcal{J} \leq b + \varepsilon]) \subseteq [\mathcal{J} \leq b - \varepsilon]$;*
5. *if $K_b \neq \emptyset$, then*
$$\eta(1, [\mathcal{J} \leq b + \varepsilon] \setminus \mathcal{O}) \subseteq [\mathcal{J} \leq b - \varepsilon]$$
and
$$\eta(1, [\mathcal{J} \leq b + \varepsilon]) \subseteq [\mathcal{J} \leq b - \varepsilon] \cup \mathcal{O};$$
6. *if (F5) holds, then $\eta(t,\cdot)$ is $\mathbb{G}$-equivariant, i.e. it is odd.*

As in Section 3.2, for any $n \in \mathbb{N}^*$ we define $\Gamma_n := \Gamma_n(\lambda)$. We note that $\Gamma_n \neq \emptyset$ is shown in Proposition 3.2.1. Now our Theorem 3.1.5 can be obtained through the arguments given in [325]. Here we just give the definition of another minimax classes Λ_n^m, which ensures the multiplicity of solutions. We set for $n \in \mathbb{N}^*$

$$\begin{aligned}\Lambda_n := \{A = \Theta(\overline{D_{n+l} \setminus Y}) \mid l \in \mathbb{N}^*,\ \Theta \in \Gamma_{n+l},\\ Y \subseteq D_{n+l} \setminus \{0\} \ \text{ is closed, symmetric in } 0\\ \text{and genus}(Y) \leq l\}\end{aligned}$$

and

$$c_n := \inf_{A \in \Lambda_n} \sup_A \mathcal{J}.$$

Then we have $\{\gamma(D_n) | \, \gamma \in \Gamma_n\} \subset \Lambda_n$ and we can also see that

$$0 < c_1 \leq c_2 \leq \cdots \leq c_n \leq \cdots.$$

Thus we have the following result.

Proposition 3.7.3. *Assume (F1)–(F5). Let $n \in \mathbb{N}^*$ and $m > 0$. Then*

(i) *$\Lambda_n \neq \emptyset$ and $c_n \leq c_{n+1}$.*

(ii) *Let $A \in \Lambda_n$ and $Z \subset H_r^1(\mathbb{R}^N)$ be $\mathbb{G}$-invariant, closed, and such that $0 \notin \overline{Z}$ and genus$(\overline{Z}) \leq i < n$. Then $\overline{A \setminus Z} \in \Lambda_{n-i}$.*

(iii) *c_n is a critical value of $\mathcal{J}$. Moreover*
$$c_n \to +\infty \quad \text{as } n \to +\infty.$$
In particular, $\mathcal{J}$ has an unbounded sequence of critical values.

Proof. Using Proposition 3.7.2, the proof can be given along the lines in [325]. See also [125]. ∎

Proof of Theorem 3.1.5. Theorem 3.1.5 follows from Proposition 3.7.3. ∎

Chapter 4

Doubly nonlocal equations: qualitative and quantitative results

This Chapter is dedicated to the study of the following *fractional Choquard equation*

$$(-\Delta)^s u + \mu u = \big(I_\alpha * F(u)\big)F'(u) \quad \text{in } \mathbb{R}^N$$

where $N \geq 2$, $s \in (0,1)$, $\alpha \in (0,N)$, $\mu > 0$ and $F \in C^1(\mathbb{R})$ is a general nonlinearity, in the spirit of Berestycki and Lions assumptions. After having achieved existence of positive solutions and ground states, we will focus on the study of some qualitative properties of these solutions: boundedness, regularity, L^1-summability, positivity, radial symmetry and asymptotic decay. We will stress how the interplay between a fractional framework and a nonlocal nonlinearity, generally nonhomogeneous, obstructs the application of classical techniques. Some results generalize the ones presented in [138] and extend [79, 302]; in particular, some new results are stated also for the limiting case $s = 1$. In addition, we will see that the interaction of the two nonlocalities arises a new critical threshold.

This Chapter is mainly based on the papers: [114] (see also [113]) for Section 4.2, [115] for Sections 4.2, 4.3, 4.4.1–4.4.3, 4.6.1, [112] for Sections 4.3, 4.4.4-4.4.5, 4.5, [197] for Sections 4.6.2–4.6.7, and [117] for Section 4.7.

4.1 An example of double nonlocality: collapse of boson stars

In Sections 2.1 and 3.1 we highlighted the importance in physics of the fractional Laplacian and of the Hartree-type terms. Combinations of the two arise as well in different frameworks: for example equations of the type

$$(-\Delta)^s u + \mu u = \big(I_\alpha * F(u)\big)f(u) \quad \text{in } \mathbb{R}^N \tag{4.1.1}$$

where $N \geq 2$, $s \in (0,1)$, $\alpha \in (0,N)$, $\mu > 0$ and $f = F' \in C(\mathbb{R})$, can be found in quantum chemistry [24,142,215] (see also [103] for some orbital stability results): here (4.1.1) appears in the study of the mean field limit of weakly interacting molecules and in the physics of multi-particle systems. In particular the equation applies to the study of graphene [276], where the nonlocal nonlinearity describes the short time interactions between particles. Doubly nonlocal equations appear also in the dynamics of populations [85], where small or large values of s better model specific environments.

One of the main applications anyway arises in the study of exotic stars: minimization properties related to (4.1.1) play indeed a fundamental role in the mathematical description of the dynamics of pseudorelativistic boson stars and their gravitational collapse [169, 192–195, 222, 256–258, 269], as well as the evolution of attractive fermionic systems, such as white dwarf stars [214]. In fact, the study of the ground states to (4.1.1) gives information on the size of the critical initial conditions for the solutions of the corresponding pseudorelativistic equation [256], where a critical value is given by the Chandrasekhar limiting mass. In particular, when $s = \frac{1}{2}$, $N = 3$, $\alpha = 2$ and $f(u) = u$, we obtain

$$\sqrt{-\Delta} u + \mu u = \left(\frac{1}{4\pi|x|} * u^2 \right) u \quad \text{in } \mathbb{R}^3 \tag{4.1.2}$$

related to the so called *massless boson stars equation* [189, 222, 258], where the pseudorelativistic operator $\sqrt{-\Delta + m}$ collapses to the square root of the Laplacian. Here $f(t) = |t|^{r-2}t$ with $r = 2$ is L^2-critical: in this Chapter, when dealing with the mass-constrained problem, we essentially address the subcritical case $r \in (\frac{5}{3}, 2)$, but we believe that this result, together with the developed minimax tools, can be a first step towards the study of the L^2-mass critical (and supercritical) case, since for these problems the minimization approach is generally not well posed. Moreover, the high generality assumed on the function f could be useful in the study of different physical problems.

Mathematically, concerning the fractional Schrödinger equation with Hartree nonlinearity, we mention the papers [138, 139] where D'Avenia, Siciliano and Squassina considered the case of pure power nonlinearities and obtained existence and qualitative properties of the solutions. We mention also [103, 202] for some orbital stability results, [104] for a Strichartz estimates approach, and [129] for the unidimensional case. Other results can be found in [41, 277, 342] for superlinear nonlinearities, in [219] for some local perturbation, in [218, 305] for critical equations and in [386] for concentration phenomena with strictly noncritical and monotone sources.

The existence of L^2-normalized solutions was investigated when $F(t) = |t|^p$ in [382] (see also [203, 212] for L^2-supercritical Cauchy problems by scattering), while in [102] it has been addressed the non-autonomous unconstrained case. In [141], symmetry and monotonicity of positive solutions are shown for the fractional Hartree equation for $\mu = 0$ and a critical power nonlinearity, by means of the direct method of moving planes. Regularity results for a class of doubly nonlocal equations on bounded domains are obtained in [207].

Some theoretical aspects related to the study of doubly nonlocal equations, both in the operator and in the source, remain open for general nonlinearities F, in particular when F is not a power function or F is odd.

In the present Chapter we are interested to derive some qualitative properties of the solutions to (4.1.1), also in these special cases. In particular, after having stated existence of free and normalized solutions, we will focus our attention on the study of regularity of solutions (boundedness, L^1-summability, Hölder continuity, differentiability), moving then to positivity and symmetry of ground states, to tackle at the end the asymptotic behaviour at infinity. The precise statements will be presented throughout the Chapter: these results generalize some of the ones in [138] from the case of power functions to general nonlinearities of Berestycki-Lions type; moreover, we extend some results of [302] to the fractional framework, and some results of [79] to the Choquard framework.

The achieving of these results requires some technical effort in order to deal with the two nonlocalities and their interaction, as well as the nonhomogeneity and the nonregularity of the function f. In particular, we highlight some of the difficulties that arise in this general framework.

In the proof of the positivity, for instance (as well as in the proof of the existence), the presence of the fractional power of the Laplacian does not allow to use the fact that every solution satisfies the Pohozaev identity to conclude that, if $|u|$ is a solution, then it satisfies the Pohozaev identity; moreover, the conservation of the norm of the gradient does not hold anymore, i.e. $\|\nabla|u|\|_2 = \|\nabla u\|_2$ is not generally true in the fractional framework, and an inequality is needed. In addition, when dealing with f even other information about u are lost through inequalities;

this is not the case when dealing with f odd [302]. Furthermore, the presence of the Choquard term, which scales differently from the L^2-norm term, does not allow to implement the classical minimization argument of [50, 131] (see (5.5.78)), which is useful to deal with the absolute value of u. Similarly, the nonhomogeneity of the nonlinearity f obstructs the minimization approach of [138, 300]. Thus, a new approach is needed, and it relies on a fiber map which sends solutions to the Pohozaev set (see Proposition 4.5.5). When dealing with f even, this technique allows to treat also the case $s = 1$, generalizing [302].

Regarding the L^1-summability, the possibility of including a critical behaviour in zero (that is, $F(t) \sim t^{2^\#_\alpha}$) is not relevant when dealing with pure power functions [138, 300], since no solution exists in this case: this growth is instead relevant for general f (for example, suitable sum of powers). Contrary to the case of noncritical nonlinearities, when f is critical it is not possible to implement a simple bootstrap argument to achieve that every solution is in L^1: a new method is thus needed, and it is based on a suitable combination of bootstrap argument and fixed point theorems (see Proposition 4.4.10). The study of this case is new even for $s = 1$, improving [300, 302].

When studying the asymptotic behaviour of solutions, especially when f has a sublinear growth, the interaction of the two nonlocalities is quite strong, and new phenomena arise: indeed, contrary to the local case $s = 1$ [300], here the effect of the fractional Laplacian and of the Choquard term give rise to a new threshold depicting the qualitative profile of ground states at infinity (see Theorem 4.6.11). From a technical point of view, new difficulties arise related to the explicit computation of the fractional Laplacian, and to the computation of concave powers, requiring a more delicate analysis and the implementation of new inequalities (see Sections 1.2.2 and 1.2.4). This result is new even for power functions, improving [138].

We refer to the following Sections for the detailed statements of the results.

The Chapter is organized as follows. In the remaining part of the Section we will briefly give a physical interpretation of equation (4.1.1) in the framework of gravitational collapses. In Section 4.2 we will deal with existence of solutions, both for the unconstrained problem and the constrained one, by highlighting some approach different from the ones developed in Chapters 2 and 3; some properties related to the energy minimum levels and existence of positive solutions will be then investigated in Section 4.3. Section 4.4 will be devoted to the study of regularity of positive solutions, including boundedness and Hölder regularity; moreover we will gain L^1-summability of solutions through a combination of bootstrap and fixed point maps arguments. Then in Section 4.5 we will exploit these results in order to gain positivity and radial symmetry of Pohozaev minima, by the implementation of maximum principles on some fiber maps. Afterwards, we will investigate in Section 4.6 the asymptotic decay of ground states, focusing especially on the case of f sublinear, which raises some new phenomenon. Finally in Section 4.7 we furnish a proof of the Pohozaev identity in the doubly nonlocal framework, by assuming the solutions merely C^1.

Physical derivation

Here we want to show how equation (4.1.1), in the particular case $N = 3$, $s = \frac{1}{2}$, $\alpha = 2$ and $F(u) = \frac{1}{2}u^2$, can be derived from a significant physical framework regarding boson stars. Aim of this Section is just to give an idea of the process, without any aim of accuracy or rigors. We refer to [93, 169, 192, 196, 213, 257, 258, 268, 269, 296, 307] (see also [189, 270, 308, 360] and [193, 214, 256]) for complete expositions on the topic.

The goal is to show how the equation (4.1.2) is strictly connected with the self-gravitational collapse of boson stars. Actually a similar derivation holds also for neutron stars and white dwarfs, with some little complications.

Let us consider thus a group of n *bosons* (i.e. particles with entire spin, described by symmetric functions, and which do not respond to the Pauli exclusion principle). We assume these bosons to form a *boson star*, i.e. $n \gg 0$ and we assume most of them (i.e. up to $o(N)$ particles) to be close one to each other and moving at a fast speed: these particles are at a same coherent state ψ (and for this reason called *condensate*) and create a trap for the remaining particles.

Due to the high speed of the particles, we cannot ignore the special relativistic effect; on the other hand, since the masses are not too big, we can ignore the effect of general relativity (this is not the case, instead, of neutron stars). Thus we consider the total relativistic energy

$$E^2 = (pc)^2 + (mc^2)^2$$

summation of the kinetic energy and the energy at rest; here p is the momentum, m the mass of the boson particle at rest, c the speed of light. Thus we obtain, in momentum representation,

$$E = \sqrt{|\xi|^2 c^2 + m^2 c^4};$$

passing through a quantization $p \mapsto -i\hbar\nabla$ to the coordinate representation (and setting $\hbar := 1$) we obtain the *pseudorelativistic* operator

$$E = \sqrt{-c^2\Delta + m^2 c^4}.$$

We observe that, letting $c \to +\infty$ (that is, the velocities are far from the one of the light) we obtain the operator $-\frac{1}{2m}\Delta$, that is the nonrelativistic operator (i.e. the classical Laplacian). We set instead, from now on, $c := 1$ for the sake of simplicity. Thus

$$E = \sqrt{-\Delta + m^2}$$

which is formally defined through the Fourier symbol $\mathcal{F}^{-1}\big((|\xi|^2 + m^2)^{1/2}\widehat{u}\big)$ (see also Remark 1.2.6).

We consider now the interaction between the particles: this interaction can be treated classically as a Newton two-bodies interaction, and thus given by the quantity

$$-k\frac{1}{|x_i - x_j|}$$

where k is a coupling constant (proportional to G, the gravitational constant).

Thus we come up with the Hamiltonian of the system

$$\mathcal{H}_n := \sum_{i=1}^{n} \sqrt{-\Delta_i + m^2} - \frac{k}{2}\sum_{i\neq j}^{n} \frac{1}{|x_i - x_j|}.$$

When dealing with dwarf stars, additional pieces given by the interaction (due to the Pauli exclusion principle) appear; anyway, for $n \gg 0$, one can ignore these pieces: this is called the *Hartree approximation of Hartree-Fock theory*.

Now we are interested in what happens when $n \gg 0$, that is, when the particles act like a single body, in what is called the *mean field limit*:

$$n \to +\infty, \quad k \to 0, \quad nk = \mathit{const};$$

formally this is given by assuming that the state of motion $\psi_n(t)$ can be factorized at each t – fact that is not generally true, even if one starts from a factorized state at $t = 0$. We highlight that the powers appearing in the relation $nk = \mathit{const}$ are typical of the boson star framework, and are indeed different in other frameworks (for instance, in the case of white dwarfs, we have $n^{3/2}k = \mathit{const}$).

By letting $n \to +\infty$ one can formally prove that, in some precise sense, the motion of the (single body) boson star converges to the motion of the following (time-dependent) PDE

$$iu_t = \sqrt{-\Delta + m^2}u - \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy\right) u \quad \text{in } (0, +\infty) \times \mathbb{R}^3 \tag{4.1.3}$$

that is (up to constant)

$$iu_t = \sqrt{-\Delta + m^2}u - (I_2 * u^2)u \quad \text{in } (0, +\infty) \times \mathbb{R}^3.$$

We focus now on this equation, and on the corresponding energy functional

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta + m)^{1/4}u|^2 - \frac{1}{2}\int_{\mathbb{R}^3} (I_2 * u^2)u^2.$$

By exploiting the Hardy-Littlewood-Sobolev and the fractional Gagliardo-Nirenberg inequalities we obtain

$$\int_{\mathbb{R}^3} (I_2 * u^2)u^2 \le \overline{C}\|(-\Delta)^{1/4}u\|_2^2\|u\|_2^2, \tag{4.1.4}$$

which combined with the trivial inequality $|\xi|^2 + m^2 \ge |\xi|^2$ gives

$$E(u) \ge \frac{1}{2}\|(-\Delta)^{1/4}u\|_2^2(1 - \overline{C}\|u\|_2^2)$$

for some $\overline{C} > 0$. Set

$$\int_{\mathbb{R}^3} u^2 =: M$$

the total mass of the boson star (interpreting $u^2(x)$ as the density in $x \in \mathbb{R}^3$) we obtain

$$E(u) \ge \frac{1}{2}\|(-\Delta)^{1/4}u\|_2^2(1 - \overline{C}M);$$

from this we see that $E(u)$ could be or be not bounded from below on the sphere $\{u \in H^{1/2}(\mathbb{R}^3) \mid \|u\|_2^2 = M\}$ depending on the size of M: this is actually a phenomenon related to the L^2-*critical growth* $2 = \frac{3+2+2\frac{1}{2}}{3} = 2^m_{2,\frac{1}{2}}$ in $N = 3$. More precisely, one can prove that there exists a constant M_*, related to the best constant of the inequality (4.1.4), such that

$$\inf_{\|u\|_2^2 = M} E(u) \begin{cases} \ge 0 & \text{if } M < M_*, \\ = -\infty & \text{if } M > M_*. \end{cases}$$

As a further consequence, one might study the dynamical properties of $u(x,t) = e^{it}u(x)$, solution of (4.1.3), showing that

$$u = u(x,t) \begin{cases} \text{exists for each } t > 0 & \text{if } M < M_*, \\ \text{explodes in finite time} & \text{if } M > M_*. \end{cases}$$

This is why M_*, called *Chandrasekhar mass*, is related to the *self-gravitational collapse* of boson stars (i.e., the collapse due to their own gravity). One could show that M_* is related to a number of particles of the size of $\sim 10^{38}$, that is, the number of particles that can be approximately found in a mountain.

As already highlighted, M_* is related to the best constant of the inequality (4.1.4). And one can show that the optimizers Q of this inequality satisfy the following equation

$$\sqrt{-\Delta}Q + \mu Q = (I_2 * Q^2)Q \quad \text{in } \mathbb{R}^3$$

for some $\mu > 0$ (actually M_* equals the L^2 norm squared of Q). And this is the equation we study.

4.2 Different approaches for the existence problem

In this Section we briefly sketch how to get existence of *free* and *constrained* solutions. The techniques are based on the ideas of Chapters 2 and 3. Anyway we present here a different approach to handle the boundary of $\mathbb{R}_+$, instead of considering the change of variable $\mu = e^{\lambda}$. With this aim, we will give a proof of some details, referring to Chapters 2-3 for all the other proofs.

This first Section is based on the paper [114] and [115] (see also [113]). For multiplicity results we refer to [117].

The first goal we address is to study the unconstrained problem of (4.1.1) when f satisfies the following set of assumptions of Berestycki-Lions type [50]:

(F1) $f \in C(\mathbb{R}, \mathbb{R})$;

(F2) we have

$$i) \ \limsup_{t \to 0} \frac{|tf(t)|}{|t|^{2^{\#}_{\alpha}}} < +\infty, \quad ii) \ \limsup_{|t| \to +\infty} \frac{|tf(t)|}{|t|^{2^{*}_{\alpha,s}}} < +\infty;$$

(F3) $F(t) = \int_0^t f(\tau) d\tau$ satisfies

$$i) \ \lim_{t \to 0} \frac{F(t)}{|t|^{2^{\#}_{\alpha}}} = 0, \quad ii) \ \lim_{|t| \to +\infty} \frac{F(t)}{|t|^{2^{*}_{\alpha,s}}} = 0;$$

(F4) there exists $t_0 \in \mathbb{R}$, $t_0 \neq 0$ such that $F(t_0) \neq 0$.

We observe again that (F3) implies that we are in a *noncritical* setting: indeed the exponents $2^{\#}_{\alpha} = \frac{N+\alpha}{N}$ and $2^{*}_{\alpha,s} = \frac{N+\alpha}{N-2s}$ have been addressed in [300] as critical for Choquard-type equations when $s = 1$, and then generalized to $s \in (0, 1)$ in [138]; we will assume the noncriticality in order to obtain the existence of a solution, while all the qualitative results in the following Sections will be given in a *possibly critical* setting.

This unconstrained case was studied by [138] for a power nonlinearity and by [53] in the case of combined local and nonlocal power-type nonlinearities; see also [199, 277, 342].

We obtain the following result.

Theorem 4.2.1. *Assume (F1)–(F4). Then there exists a radially symmetric weak solution u of (4.1.1), which satisfies the Pohozaev identity:*

$$\frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \frac{N}{2} \mu \int_{\mathbb{R}^N} u^2 - \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) = 0 \tag{4.2.5}$$

or equivalently

$$\frac{1}{2^{*}_{\alpha,s}} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \frac{\mu}{2^{\#}_{\alpha}} \int_{\mathbb{R}^N} u^2 - \int_{\mathbb{R}^N} (I_\alpha * F(u)) F(u) = 0.$$

This solution is of Mountain Pass type.

We point out some difficulties which arise in this framework. Indeed, the presence of the fractional Laplacian does not allow to use the fact that every solution satisfies the Pohozaev identity to conclude that a Mountain Pass solution is actually a (Pohozaev) ground state, as in [237] (see Remark 4.3.3). On the other hand, the presence of the Choquard term, which scales differently from the L^2-norm term, does not allow to implement the classical minimization argument by [50, 131]. Finally, the nonhomogeneity of the nonlinearity f obstructs the minimization approach of [138, 302]. Thus, we need a new approach to get existence of solutions, and this can be done in the spirit of Chapters 2-3. We omit the details, refering to [115].

The next goal is to study the constrained problem, i.e. we study the existence of solutions $(\mu, u) \in (0, +\infty) \times H^s_r(\mathbb{R}^N)$ to the nonlocal problem

$$\begin{cases} (-\Delta)^s u + \mu u = (I_\alpha * F(u))f(u) & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2\, dx = m, \end{cases} \tag{4.2.6}$$

where $\mu > 0$ is a Lagrange multiplier, part of the unknowns.

In particular we assume (F1), (F4) together with the stronger assumptions

(CF2)

$$i)\ \limsup_{t\to 0} \frac{|tf(t)|}{|t|^{2^\#_\alpha}} < +\infty, \quad ii)\ \limsup_{|t|\to+\infty} \frac{|tf(t)|}{|t|^{2^m_{\alpha,s}}} < +\infty;$$

(CF3)

$$i)\ \lim_{t\to 0} \frac{F(t)}{|t|^{2^\#_\alpha}} = 0, \quad ii)\ \lim_{|t|\to+\infty} \frac{F(t)}{|t|^{2^m_{\alpha,s}}} = 0;$$

we remark that the exponent $2^m_{\alpha,s} = \frac{N+\alpha+2s}{N}$ appears as an L^2-critical exponent for the fractional Choquard equations and the conditions (F1)-(CF2)-(CF3)-(F4) correspond to L^2-subcritical growth.

For this general class of nonlinearities of the Berestycki–Lions type [50, 302] we introduce a Lagrangian formulation: namely, set $\mathbb{R}_+ = (0, +\infty)$, a radially symmetric solution $(\mu, u) \in \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ of (4.2.6) corresponds to a critical point of the functional $\mathcal{I}^m : \mathbb{R}_+ \times H^s_r(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{I}^m(\mu, u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\, dx + \frac{\mu}{2}(\|u\|_2^2 - m).$$

Using a variant of the Palais–Smale condition [224, 231], which takes into account the Pohozaev identity, we will prove a deformation theorem which enables us to detect minimax structures in the product space $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ by means of a *Pohozaev mountain*. Our deformation arguments show that solutions without Pohozaev identity are suitably deformable, and thus they *do not influence the topology* of the sublevels of the functional. This information could be relevant in a fractional framework since it is not known if the Pohozaev identity holds for general continuous f and general values of $s \in (0, 1)$.

We state our main results.

Theorem 4.2.2. *Assume (F1)-(CF2)-(CF3)-(F4). Then there exists $m_0 \geq 0$ such that, for any $m > m_0$, the problem (4.2.6) has a radially symmetric solution, which satisfies the Pohozaev identity (4.2.5).*

Theorem 4.2.3. *Assume (F1)-(CF2)-(CF3), together with an L^2-subcritical growth at zero, i.e.,*

(CF4') $\lim_{t\to 0} \frac{F(t)}{|t|^{2^m_{\alpha,s}}} = +\infty.$

Then, for any $m > 0$, the problem (4.2.6) has a radially symmetric solution, which satisfies the Pohozaev identity (4.2.5).

We naively notice that (CF4') automatically implies (F4). We remark that, as in the local unconstrained case [237], the Mountain Pass solutions obtained in the above theorems are ground state solutions, that is, they have the least energy among all solutions; see Section 4.2.2 for details. This fact gives a strong indication on the stability properties of the found solution [103, 180].

Here we find solutions satisfying automatically the Pohozaev identity: in Section 4.7 we will prove that a general C^1 solution actually satisfies such relation.

4.2.1 Dealing with the boundary

In what follows, we will often denote

$$q = 2^{\#}_{\alpha} = \frac{N+\alpha}{N}, \quad p = 2^m_{\alpha,s} = \frac{N+\alpha+2s}{N}.$$

Consider the functional

$$\mathcal{J}_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\, dx - \frac{1}{2}\mathcal{D}(u) + \frac{\mu}{2}\|u\|_2^2$$

with $\mathcal{D}(u) = \int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)$. We notice that, by the Principle of Symmetric Criticality of Palais, the critical points of $\mathcal{J}_\mu$ are *weak solutions* of (4.1.1). Moreover, inspired by the Pohozaev identity (4.2.5), we define also the Pohozaev functional $\mathcal{P}_\mu : H^s_r(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{P}_\mu(u) := \frac{N-2s}{2}\|(-\Delta)^{s/2}u\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(u) + \frac{N}{2}\mu\|u\|_2^2.$$

Here we highlight how to deal with the boundary of $\mathbb{R}_+$ without implementing the change of variable $\mu = e^\lambda$. More details can be found in [114].

As a matter of fact, we notice that $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ with the standard metric induced by $\mathbb{R} \times H^s_r(\mathbb{R}^N)$ is not complete, and thus it is not suitable for a deformation argument. Since $(\mathbb{R}_+, \frac{1}{x^2}dx^2)$ is instead complete, it is natural to introduce a related metric on $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$. That is, we regard

$$R := \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$$

as a Riemannian manifold with the metric

$$((\nu_1, w_1), (\nu_2, w_2))_{T_{(\mu,u)}R} := \frac{1}{\mu^2}\nu_1\nu_2 + (w_1, w_2)_{H^s_r}$$

for (ν_1, w_1), $(\nu_2, w_2) \in T_{(\mu,u)}R$, $(\mu, u) \in R$; it is standard to see that $(R, (\cdot,\cdot)_{TR})$ is a complete Riemannian manifold. We regard thus $\mathcal{I}^m$ as a functional defined on R, and obtain

$$\|(\partial_\mu \mathcal{I}^m(\mu,u), \partial_u \mathcal{I}^m(\mu,u))\|^2_{(T_{(\mu,u)}R)^*} = \mu^2|\partial_\mu \mathcal{I}^m(\mu,u)|^2 + \|\partial_u \mathcal{I}^m(\mu,u)\|^2_{(H^s_r)^*}.$$

Definition 4.2.4. *For $b \in \mathbb{R}$, we say that $(\mu_j, u_j)_j \subset R = \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ is a* Palais-Smale-Pohozaev sequence *at level b (in short, the $(PSP)_b$ sequence) if, as $j \to +\infty$,*

$$\begin{aligned}
&\mathcal{I}^m(\mu_j, u_j) \to b,\\
&\|(\partial_\mu \mathcal{I}^m(\mu,u), \partial_u \mathcal{I}^m(\mu,u))\|_{(T_{(\mu,u)}M)^*} \to 0,\\
&\mathcal{P}(\mu_j, u_j) \to 0,
\end{aligned}$$

or equivalently

$$\mathcal{I}^m(\mu_j, u_j) \to b, \tag{4.2.7}$$
$$\mu_j \cdot \partial_\mu \mathcal{I}^m(\mu_j, u_j) \to 0, \tag{4.2.8}$$
$$\partial_u \mathcal{I}^m(\mu_j, u_j) \to 0 \quad \textit{strongly in } (H^s_r(\mathbb{R}^N))^*, \tag{4.2.9}$$
$$\mathcal{P}(\mu_j, u_j) \to 0. \tag{4.2.10}$$

We say that $\mathcal{I}^m$ satisfies the $(PSP)_b$ condition *if, for any $(PSP)_b$ sequence $(\mu_j, u_j)_j \subset \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$, it happens that $(\mu_j, u_j)_j$ has a strongly convergent subsequence in $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$.*

Remark 4.2.5. *Clearly, setting*

$$\widetilde{\mathcal{I}}^m(\lambda, u) := \mathcal{I}^m(e^\lambda, u), \quad \widetilde{\mathcal{I}} : \mathbb{R} \times H^s_r(\mathbb{R}^N) \to \mathbb{R},$$

we can observe that $\widetilde{\mathcal{I}}^m$ satisfies the $(PSP)_b$ in the sense of Definitions 2.4.1 and 3.5.1 if and only if $\mathcal{I}^m$ satisfies the $(PSP)_b$ condition in the sense of Definition 4.2.4 with $\mu_j := e^{\lambda_j}$.

For the sake of completeness, we give here some details on the proof of the $(PSP)_b$ condition at strictly negative levels. We emphasize again indeed that the $(PSP)_b$ condition does not hold at level $b = 0$: it is sufficient to consider an infinitesimal sequence $(\mu_j, 0)$ with $\mu_j \to 0$.

Theorem 4.2.6. *Assume (F1)-(CF2)-(CF3). Let $b < 0$. Then $\mathcal{I}^m$ satisfies the $(PSP)_b$ condition on $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$.*

Proof. Let $b < 0$ and $(\mu_j, u_j)_j \subset \mathbb{R} \times H^s_r(\mathbb{R}^N)$ be a sequence satisfying (4.2.7)–(4.2.10). First we note that, by (4.2.8), we have

$$\mu_j(\|u_j\|_2^2 - m) \to 0. \tag{4.2.11}$$

Step 1: $\liminf_{j\to\infty} \mu_j > 0$ *and* $\|u_j\|_2^2 \to m$.
By (4.2.10) and (4.2.7), we have

$$\begin{aligned} o(1) = \mathcal{P}(\mu_j, u_j) &= \frac{N-2s}{2}\|(-\Delta)^{s/2}u_j\|_2^2 + \\ &\quad + (N+\alpha)\Big(\mathcal{I}^m(\mu_j,u_j) - \frac{1}{2}\|(-\Delta)^{s/2}u_j\|_2^2 - \frac{\mu_j}{2}(\|u_j\|_2^2 - m)\Big) + \frac{N}{2}\mu_j\|u_j\|_2^2 \\ &= -\frac{\alpha+2s}{2}\|(-\Delta)^{s/2}u_j\|_2^2 + (N+\alpha)(b+o(1)) + \frac{N}{2}\mu_j m + o(1); \end{aligned}$$

here we have used (4.2.11). Since $b < 0$, we have $\liminf_{j\to\infty} \mu_j > 0$. Thus (4.2.11) implies $\|u_j\|_2^2 \to m$.
Step 2: $\|(-\Delta)^{s/2}u_j\|_2^2$ *and* μ_j *are bounded.*
Since $\varepsilon_j \equiv \|\partial_u \mathcal{I}^m(\mu_j, u_j)\|_{(H^s_r(\mathbb{R}^N))^*} \to 0$, we have

$$\|(-\Delta)^{s/2}u_j\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * F(u_j))f(u_j)u_j\,dx + \mu_j\|u_j\|_2^2 \le \varepsilon_j\|u_j\|_{H^s_r}. \tag{4.2.12}$$

Note that $\frac{2Np}{N+\alpha} \in (2, 2^*_s)$. Moreover, we observe that, by (CF3), for $\delta > 0$ fixed, there exists $C_\delta > 0$ such that

$$|F(t)| \le \delta|t|^p + C_\delta|t|^{\frac{N+\alpha}{N}}, \quad t \in \mathbb{R},$$

where $p = \frac{N+\alpha+2s}{N}$, and thus

$$\|F(u_j)\|_{\frac{2N}{N+\alpha}} \le \delta\||u_j|^p\|_{\frac{2N}{N+\alpha}} + C_\delta\||u_j|^{\frac{N+\alpha}{N}}\|_{\frac{2N}{N+\alpha}} = \delta\|u_j\|^p_{\frac{2Np}{N+\alpha}} + C_\delta\|u_j\|_2^{\frac{N+\alpha}{N}}.$$

Therefore, by (CF2) we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * |F(u_j)|)|f(u_j)u_j|\,dx \\ &\le C\|F(u_j)\|_{\frac{2N}{N+\alpha}}\|f(u_j)u_j\|_{\frac{2N}{N+\alpha}} \\ &\le C'\left(\delta\|u_j\|^p_{\frac{2Np}{N+\alpha}} + C_\delta\|u_j\|_2^{\frac{N+\alpha}{N}}\right)\cdot\left(\|u_j\|^p_{\frac{2Np}{N+\alpha}} + \|u_j\|_2^{\frac{N+\alpha}{N}}\right) \\ &= C'\delta\|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + C'(\delta+C_\delta)\|u_j\|^p_{\frac{2Np}{N+\alpha}}\|u_j\|_2^{\frac{N+\alpha}{N}} + C'C_\delta\|u_j\|_2^{\frac{2(N+\alpha)}{N}} \\ &= C'\delta\|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + C'(\delta+C_\delta)\left(\frac{\delta}{2}\|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + \frac{1}{2\delta}\|u_j\|_2^{\frac{2(N+\alpha)}{N}}\right) + C'C_\delta\|u_j\|_2^{\frac{2(N+\alpha)}{N}} \\ &\le C''\delta\|u_j\|^{2p}_{\frac{2Np}{N+\alpha}} + C''_\delta\|u_j\|_2^{\frac{2(N+\alpha)}{N}} \end{aligned}$$

and thus, by the fractional Gagliardo–Nirenberg inequality (1.2.8), with $r = \frac{2Np}{N+\alpha}$ and $\beta = \frac{1}{p}$, we derive

$$\|(-\Delta)^{s/2}u_j\|_2^2 + \mu_j\|u_j\|_2^2 \le \int_{\mathbb{R}^N}(I_\alpha * |F(u_j)|)|f(u_j)u_j|\,dx + \varepsilon_j\|u_j\|_{H^s_r}$$

$$\leq \quad C''\delta\|(-\Delta)^{s/2}u_j\|_2^2\|u_j\|_2^{2(p-1)} + C''_\delta\|u_j\|_2^{\frac{2(N+\alpha)}{N}} + \varepsilon_j\|u_j\|_{H^s_r}.$$

Since $\|u_j\|_2^2 = m + o(1)$, we get

$$\begin{aligned}&(1 - C''\delta(c+o(1))^{p-1})\|(-\Delta)^{s/2}u_j\|_2^2 + \mu_j(m+o(1))\\ &\quad\leq \quad C''_\delta(m+o(1))^{\frac{N+\alpha}{N}} + \varepsilon_j\big(\|(-\Delta)^{s/2}u_j\|_2^2 + m + o(1)\big)^{1/2}.\end{aligned}$$

For a small enough δ, we have the boundedness of $\|(-\Delta)^{s/2}u_j\|_2$ and μ_j.

Step 3: *Convergence in* $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$*.*

By Steps 1-2, the sequence $(\mu_j, u_j)_j$ is bounded in $\mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$ and thus, after extracting a subsequence denoted in the same way, we may assume that $\mu_j \to \mu_0 > 0$ and $u_j \rightharpoonup u_0$ weakly in $H^s_r(\mathbb{R}^N)$ for some $(\mu_0, u_0) \in \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$.

Step 4: *Conclusion.*

Taking into account the assumptions (F1)–(F4), we obtain by Proposition 1.5.9

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_j))f(u_j)u_0\,dx \to \int_{\mathbb{R}^N}(I_\alpha * F(u_0))f(u_0)u_0\,dx$$

and

$$\int_{\mathbb{R}^N}(I_\alpha * F(u_j))f(u_j)u_j\,dx \to \int_{\mathbb{R}^N}(I_\alpha * F(u_0))f(u_0)u_0\,dx.$$

Thus, we derive that $\langle \partial_u \mathcal{I}^m(\mu_j, u_j), u_j\rangle \to 0$ and $\langle \partial_u \mathcal{I}^m(\mu_j, u_j), u_0\rangle \to 0$, and hence

$$\|(-\Delta)^{s/2}u_j\|_2^2 + \mu_0\|u_j\|_2^2 \to \|(-\Delta)^{s/2}u_0\|_2^2 + \mu_0\|u_0\|_2^2$$

which implies $u_j \to u_0$ strongly in $H^s_r(\mathbb{R}^N)$. ∎

Now we define a metric on the Hilbert manifold

$$M := \mathbb{R} \times R = \mathbb{R} \times \mathbb{R}_+ \times H^s_r(\mathbb{R}^N)$$

by setting

$$\|(\alpha,\nu,h)\|^2_{(\theta,\mu,u)} := \alpha^2 + \frac{1}{\mu^2}\nu^2 + e^{N\theta}\|h\|_2^2 + e^{(N-2s)\theta}\|(-\Delta)^{s/2}h\|_2^2$$

for any $(\alpha,\nu,h) \in T_{(\theta,\mu,u)}M \equiv \mathbb{R}\times\mathbb{R}_+\times H^s_r(\mathbb{R}^N)$. We also denote the dual norm on $T^*_{(\theta,\mu,u)}M$ by $\|\cdot\|_{(\theta,\mu,u),*}$. We notice that $\|(\cdot,\cdot,\cdot)\|^2_{(\theta,\mu,u)}$ depends both on θ and μ (but not on u). Furthermore we define the standard distance between two points dist_M as the infimum of length of curves connecting the two points.

On M we consider the augmented functional

$$\mathcal{H}^m(\theta,\mu,u) := \mathcal{I}^m(\mu, u(e^{-\theta}\cdot));$$

denoted $D := (\partial_\theta, \partial_\mu, \partial_u)$, we obtain

$$\begin{aligned}&\|D\mathcal{H}^m(\theta,\mu,u)\|^2_{(\theta,\mu,u),*}\\ &\quad= \quad |\mathcal{P}(\mu,u(e^{-\theta}\cdot))|^2 + \mu^2|\partial_\mu\mathcal{I}^m(\mu,u(e^{-\theta}\cdot))|^2 + \|\partial_u\mathcal{I}^m(\mu,u(e^{-\theta}\cdot))\|^2_{(H^s_r)^*}.\end{aligned}$$

Finally, defined

$$\tilde{K}_b := \{(\theta,\lambda,u) \in M \mid \mathcal{H}^m(\theta,\lambda,u) = b,\ D\mathcal{H}^m(\theta,\lambda,u) = 0\}$$

the set of critical points at level b of $\mathcal{H}^m$, we deduce the following.

Proposition 4.2.7. *Let $b \in \mathbb{R}$, $b < 0$. Then the functional $\mathcal{H}^m$ satisfies the following Palais-Smale type condition $(\widetilde{PSP})_b$. That is, for each sequence $(\theta_j, \mu_j, u_j)_j$ such that*

$$\mathcal{H}^m(\theta_j, \mu_j, u_j) \to b,$$

$$\|D\mathcal{H}^m(\theta_j, \mu_j, u_j)\|_{(\theta_j,\mu_j,u_j),*} \to 0,$$

we have, up to a subsequence,

$$\text{dist}_M((\theta_j, \mu_j, u_j), \tilde{K}_b) \to 0.$$

Through the use of the augmented functional we can obtain again a deformation result. We write here the statement for the unconstrained case (similarly to Proposition 3.7.2), since it will be used afterwards. Set

$$K_b^{PSP} := \{u \in H^s_r(\mathbb{R}^N) \mid \mathcal{J}_\mu(u) = 0,\ \mathcal{J}_\mu'(u) = 0,\ \mathcal{P}_\mu(u) = 0\}.$$

Lemma 4.2.8. *For any $b \in \mathbb{R}$, $\bar{\varepsilon} > 0$ and any U open neighborhood of K_b^{PSP}, there exist an $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous map $\eta : [0,1] \times H^s_r(\mathbb{R}^N) \to H^s_r(\mathbb{R}^N)$ such that*

(1^o) $\eta(0, u) = u \quad \forall u \in H^s_r(\mathbb{R}^N)$;

(2^o) $\eta(t, u) = u \quad \forall (t, u) \in [0,1] \times [\mathcal{J}_\mu \le b - \bar{\varepsilon}]$;

(3^o) $\mathcal{J}_\mu(\eta(t, u)) \le \mathcal{J}_\mu(u) \qquad \forall (t, u) \in [0,1] \times H^s_r(\mathbb{R}^N)$;

(4^o) $\eta(1, [\mathcal{J}_\mu \le b + \varepsilon] \setminus U) \subset [\mathcal{J}_\mu \le b - \varepsilon]$;

(5^o) $\eta(1, [\mathcal{J}_\mu \le b + \varepsilon]) \subset [\mathcal{J}_\mu \le b - \varepsilon] \cup U$;

(6^o) *if* $K_b^{PSP} = \emptyset$, *then* $\eta(1, [\mathcal{J}_\mu \le b + \varepsilon]) \subset [\mathcal{J}_\mu \le b - \varepsilon]$.

The remaining part of the proof follows the lines of the previous Chapters, so that we obtain the existence of a (normalized) Mountain Pass solution: this proves Theorems 4.2.2 and 4.2.3.

4.2.2 Existence of L^2-ground states

In this Section we show (with an approach different from Section 2.8) how to obtain the existence of an L^2 ground state, by assuming that this energy level is negative and by exploiting Ekeland variational principle together with our Palais-Smale-Pohozaev condition; then we relate this solution to our Mountain Pass solution of Theorem 4.2.2.

More precisely, for any $m > 0$, we introduce the functional $\mathcal{L} : \mathcal{S}_m \to \mathbb{R}$ defined by

$$\mathcal{L}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx - \frac{1}{2} \mathcal{D}(u) \tag{4.2.13}$$

on the sphere

$$\mathcal{S}_m := \{u \in H^s_r(\mathbb{R}^N) \mid \|u\|_2^2 = m\}$$

and we consider the L^2 ground state level

$$\kappa_m := \inf_{u \in \mathcal{S}_m} \mathcal{L}(u).$$

We have the following result.

Proposition 4.2.9. *Under the assumption of Theorem 4.2.2, we have, for any $m > m_0$,*

(i) $-\infty < \kappa_m < 0$ *and* κ_m *is attained;*

(ii) $\kappa_m = b_m$, *where* b_m *is defined in* (2.6.36).

Moreover, in the assumptions of Theorem 4.2.3, $m_0 = 0$.

Proof. We split in some steps.
Step 1: $\kappa_m > -\infty$.
By arguing as in Step 2 of Theorem 2.4.2 we obtain

$$\mathcal{L}(u) \geq \left(\frac{1}{2} - \delta C m^{2(p-1)}\right) \|(-\Delta)^{s/2}u\|_2^2 - C_\delta m^{2\frac{N+\alpha}{N}}.$$

Choosing $\delta > 0$ small so that $\frac{1}{2} - \delta C m^{2(p-1)} > 0$, we have $\kappa_m \geq -C_\delta m^{2\frac{N+\alpha}{N}} > -\infty$.
Step 2: *For* $m > m_0$, $\kappa_m < 0$.
Since the solution $u_* \in \mathcal{S}_m$ obtained in Theorem 4.2.2 satisfies, for $m > m_0$,

$$0 > b_m = \mathcal{L}(u_*) \geq \kappa_m,$$

we have the claim.
Step 3: *For* $m > m_0$, κ_m *is attained.*
To show the existence of a minimizer of $\mathcal{L}$ on $\mathcal{S}_m$, we use a linear action $\Phi : \mathbb{R} \to L(H^s_r(\mathbb{R}^N))$ defined by

$$\Phi_\theta v := e^{\frac{N}{2}\theta} v(e^\theta \cdot).$$

We note that $\mathcal{S}_m$ is invariant under Φ_θ, that is, $\Phi_\theta(\mathcal{S}_m) = \mathcal{S}_m$. Let

$$N := \mathbb{R} \times \mathcal{S}_m$$

and on the tangent bundle $TN = \mathbb{R} \times T\mathcal{S}_m = \coprod_{(\theta,u)\in N}(\mathbb{R} \times T_u\mathcal{S}_m)$ we introduce a C^2-metric

$$\|(\kappa, v)\|_{(\theta,u)} := \left(\kappa^2 + \|\Phi_\theta v\|^2_{H^s(\mathbb{R}^N)}\right)^{1/2}$$

for all $(\theta, u) \in N$ and $(\kappa, v) \in TN$. We also introduce $\tilde{\mathcal{L}} : N \to \mathbb{R}$ by

$$\begin{aligned}\tilde{\mathcal{L}}(\theta, u) &:= \mathcal{L}(\Phi_\theta u)\\ &= \frac{1}{2}e^{2s\theta}\|(-\Delta)^{s/2}u\|_2^2 - \frac{1}{2}e^{-(N+\alpha)\theta}\mathcal{D}(e^{\frac{N}{2}\theta}u_0).\end{aligned}$$

We note that

$$\inf_{(\theta,u)\in N} \tilde{\mathcal{L}}(\theta, u) = \kappa_m.$$

Since $\kappa_m \in \mathbb{R}$ by Step 1, applying Ekeland's principle, there exists a sequence $(\theta_j, u_j)_{j=1}^\infty \subset N$ such that

$$\begin{aligned}\tilde{\mathcal{L}}(\theta_j, u_j) &\to \kappa_m,\\ \|D\tilde{\mathcal{L}}(\theta_j, u_j)\|_{T^*_{(\theta_j,u_j)}N} &\to 0.\end{aligned}$$

That is, noticing that $T_u\mathcal{S}_m \equiv \{v \in H^s_r(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} uv = 0\}$, we have

$$\partial_\theta \tilde{\mathcal{L}}(\theta_j, u_j) \to 0,$$

$$\|\partial_u \tilde{\mathcal{L}}(\theta_j, u_j)\|_{T^*_{u_j}\mathcal{S}_m} = \sup_{\substack{v\in T_{u_j}\mathcal{S}_m\\ \|\Phi_{\theta_j} v\|_{H^s(\mathbb{R}^N)}\leq 1}} |\partial_u \tilde{\mathcal{L}}(\theta_j, u_j)v| \to 0.$$

Setting $\tilde{u}_j := \Phi_{\theta_j} u_j$, we thus have

$$\|\tilde{u}_j\|_2^2 = m, \tag{4.2.14}$$
$$\mathcal{L}(\tilde{u}_j) = \tfrac{1}{2}\|(-\Delta)^{s/2}\tilde{u}_j\|_2^2 - \tfrac{1}{2}\mathcal{D}(\tilde{u}_j) \to \kappa_m, \tag{4.2.15}$$

$$s\|(-\Delta)^{s/2}\tilde{u}_j\|_2^2 + \tfrac{N+\alpha}{2}\mathcal{D}(\tilde{u}_j) - \tfrac{N}{2}\int_{\mathbb{R}^N}(I_\alpha * F(\tilde{u}_j))f(\tilde{u}_j)\tilde{u}_j \to 0 \tag{4.2.16}$$

and for a suitable $\mu_j \in \mathbb{R}$

$$\mathcal{L}'(\tilde{u}_j)\tilde{v} + \mu_j \int_{\mathbb{R}^N} \tilde{u}_j\tilde{v} = o(1)\|\tilde{v}\|_{H^s(\mathbb{R}^N)} \quad \text{for all } \tilde{v} \in H^s_r(\mathbb{R}^N). \tag{4.2.17}$$

By using (4.2.15) and arguing as in Step 1 we see that $\tilde{u}_j$ is bounded in $H^s_r(\mathbb{R}^N)$. Thus, choosing $\tilde{v} = \tilde{u}_j$ in (4.2.17), we have

$$\|(-\Delta)^{s/2}\tilde{u}_j\|_2^2 - \int_{\mathbb{R}^N}(I_\alpha * F(\tilde{u}_j))f(\tilde{u}_j)\tilde{u}_j + \mu_j m = o(1),$$

which, joined to (4.2.16), gives a Pohozaev identity in the limit

$$\frac{N-2s}{2}\|(-\Delta)^{s/2}\tilde{u}_j\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(\tilde{u}_j) + \frac{N}{2}\mu_j m = o(1). \tag{4.2.18}$$

From this relation and (4.2.15) we have

$$\mu_j = \frac{2}{Nm}\left(\frac{\alpha+2s}{2}\|(-\Delta)^{s/2}\tilde{u}_j\|_2^2 - (N+\alpha)\kappa_m\right) + o(1)$$

which implies, by Step 1, that $\mu_j > 0$ for j large.

Relations (4.2.14), (4.2.15), (4.2.17) and (4.2.18) imply that $(\tilde{\mu}_j, \tilde{u}_j)$ is a $(PSP)_{\kappa_m}$ sequence. Thanks to the Palais-Smale-Pohozaev condition given in Proposition 4.2.6, $(\tilde{\mu}_j, \tilde{u}_j)$ has a strongly convergent subsequence to some $(\tilde{\mu}_*, \tilde{u}_*) \in N$, which shows the existence of a minimizer $\tilde{u}_*$. Thus (i) is proved.

Step 4: *For $m > m_0$, $\kappa_m = b_m$.*
In Step 2 we showed $b_m \geq \kappa_m$. On the other hand by the argument in Step 3, for the minimizer $\tilde{u}_*$ of $\mathcal{L}$ on $\mathcal{S}_m$, there exists $\tilde{\mu}_* \in \mathbb{R}$ such that

$$\mathcal{I}^m(\tilde{\mu}_*, \tilde{u}_*) = \kappa_m, \quad \partial_u\mathcal{I}^m(\tilde{\mu}_*, \tilde{u}_*) = 0,$$
$$\partial_\mu\mathcal{I}^m(\tilde{\mu}_*, \tilde{u}_*) = 0, \quad \mathcal{P}(\tilde{\mu}_*, \tilde{u}_*) = 0.$$

Set $\xi_*(t) := \tilde{u}_*(\cdot/t)$ we have, by the Pohozaev identity, $\mathcal{I}^m(\tilde{\mu}_*, \xi_*(t)) \to -\infty$ as $t \to +\infty$; thus, up to a rescaling, we obtain $\xi_* \in \Gamma^m$ and

$$\max_{t\in[0,1]} \mathcal{I}^m(\xi_*(t)) = \mathcal{I}^m(\tilde{\mu}_*, \tilde{u}_*) = \kappa_m,$$

which implies $b_m \leq \kappa_m$ and the proof is completed. ∎

4.3 Preliminary properties of Pohozaev energy levels

As highlighted, the goal of this Chapter is to study qualitative properties of solutions and, in particular, of Pohozaev minima. In this Section, thus, we start by observing that the solution found in Theorem 4.2.1 is actually a Pohozaev minimum. Since, afterwards, we will be interested in studying symmetric properties of general ground states, in this Section we highlight the dependence of some sets and energy levels from the subspace of radially symmetric functions. Moreover, we show existence of positive solutions.

Energy levels in radially symmetric spaces

We introduce the set of paths

$$\Gamma_r(\mu) := \{\gamma \in C([0,1], H^s_r(\mathbb{R}^N)) \mid \gamma(0) = 0,\ \mathcal{J}_\mu(\gamma(1)) < 0\}$$

and the *Mountain Pass* (MP for short) value

$$a_r(\mu) := \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,1]} \mathcal{J}_\mu(\gamma(t)). \tag{4.3.19}$$

Then we introduce

$$p_r(\mu) := \inf \{\mathcal{J}_\mu(u) \mid u \in H^s_r(\mathbb{R}^N) \setminus \{0\},\ \mathcal{P}_\mu(u) = 0\}$$

the *least energy* of $\mathcal{J}_\mu$ on the Pohozaev set of radially symmetric functions.

Proposition 4.3.1. *The Mountain Pass level and the Pohozaev minimum level coincide, that is*

$$a_r(\mu) = p_r(\mu) > 0.$$

In particular, the solution found in Theorem 4.2.1 is a Pohozaev minimum.

Proof. Let $u \in H^s_r(\mathbb{R}^N) \setminus \{0\}$ such that $\mathcal{P}_\mu(u) = 0$; observe that $\mathcal{D}(u) > 0$. We define $\bar{\gamma}(t) := u(\cdot/t)$ for $t \neq 0$ and $\bar{\gamma}(0) := 0$ so that $t \in (0, +\infty) \mapsto \mathcal{J}_\mu(\bar{\gamma}(t))$ is negative for large values of t, and it attains the maximum in $t = 1$. After a suitable rescaling we have $\bar{\gamma} \in \Gamma_r(\mu)$ and thus

$$\mathcal{J}_\mu(u) = \max_{t \in [0,1]} \mathcal{J}_\mu(\bar{\gamma}(t)) \geq a_r(\mu). \tag{4.3.20}$$

Passing to the infimum in (4.3.20) we have $p_r(\mu) \geq a_r(\mu)$. Let now $\gamma \in \Gamma_r(\mu)$. By definition we have $\mathcal{J}_\mu(\gamma(1)) < 0$, thus by

$$\mathcal{P}_\mu(v) = N\mathcal{J}_\mu(v) - s\|(-\Delta)^{s/2} v\|_2^2 - \frac{\alpha}{2}\mathcal{D}(v), \quad v \in H^s_r(\mathbb{R}^N),$$

we obtain $\mathcal{P}_\mu(\gamma(1)) < 0$. In addition, since $\mathcal{D}(u) = o(\|u\|^2_{H^s})$ as $u \to 0$ and $\gamma(t) \to 0$ as $t \to 0$ in $H^s_r(\mathbb{R}^N)$, we have

$$\mathcal{P}_\mu(\gamma(t)) > 0 \quad \text{for small } t > 0.$$

Thus there exists a t^* such that $\mathcal{P}_\mu(\gamma(t^*)) = 0$, and hence

$$p_r(\mu) \leq \mathcal{J}_\mu(\gamma(t^*)) \leq \max_{t \in [0,1]} \mathcal{J}_\mu(\gamma(t));$$

passing to the infimum we come up with $p_r(\mu) \leq a_r(\mu)$, and hence the claim. ∎

We pass to investigate more in details Pohozaev minima, showing that it is a general fact that they are solutions of equation (4.1.1).

Proposition 4.3.2. *Every Pohozaev minimum is a solution of* (4.1.1), *i.e.*

$$\mathcal{J}_\mu(u) = p_r(\mu) \ \ \text{and} \ \ \mathcal{P}_\mu(u) = 0$$

imply

$$\mathcal{J}'_\mu(u) = 0.$$

As a consequence

$$p_r(\mu) = \inf \{\mathcal{J}_\mu(u) \mid u \in H^s_r(\mathbb{R}^N) \setminus \{0\},\ \mathcal{P}_\mu(u) = 0,\ \mathcal{J}'_\mu(u) = 0\}.$$

Proof. Let u be such that $\mathcal{J}_\mu(u) = p_r(\mu)$ and $\mathcal{P}_\mu(u) = 0$. In particular, considered $\gamma(t) := u(\cdot/t)$, we have that $\mathcal{J}_\mu(\gamma(t))$ is negative for large values of t and its maximum value is $p(\mu)$ attained only in $t = 1$.

Assume by contradiction that u is not critical. Let $I := [1-\delta, 1+\delta]$ be such that $\gamma(I) \cap K_{p(\mu)} = \emptyset$, and set $\bar{\varepsilon} := p(\mu) - \max_{t\notin I} \mathcal{J}_\mu(\gamma(t)) > 0$. Let now U be a neighborhood of $K_{p(\mu)}$ verifying $\gamma(I) \cap U = \emptyset$: by the Deformation Lemma 4.2.8 there exists an $\eta : [0,1] \times H^s_r(\mathbb{R}^N) \to H^s_r(\mathbb{R}^N)$ at level $p_r(\mu) \in \mathbb{R}$ with properties (1^o)-(6^o). Define then $\tilde{\gamma}(t) := \eta(1, \gamma(t))$ a deformed path.

For $t \notin I$ we have $\mathcal{J}_\mu(\gamma(t)) < p_r(\mu) - \bar{\varepsilon}$, and thus by ($2^o$) we gain

$$\mathcal{J}_\mu(\tilde{\gamma}(t)) = \mathcal{J}_\mu(\gamma(t)) < p_r(\mu) - \bar{\varepsilon}, \qquad \text{for } t \notin I. \tag{4.3.21}$$

Let now $t \in I$: we have $\gamma(t) \notin U$ and $\mathcal{J}_\mu(\gamma(t)) \leq p_r(\mu) \leq p_r(\mu) + \varepsilon$, thus by ($4^o$) we obtain

$$\mathcal{J}_\mu(\tilde{\gamma}(t)) \leq p_r(\mu) - \varepsilon. \tag{4.3.22}$$

Joining (4.3.21) and (4.3.22) we have

$$\max_{t\geq 0} \mathcal{J}_\mu(\tilde{\gamma}(t)) < p_r(\mu) = a_r(\mu)$$

which is an absurd, since after a suitable rescaling it results that $\tilde{\gamma} \in \Gamma_r(\mu)$, thanks to ($3^o$). ∎

Remark 4.3.3. *We point out that it is not known, even in the case of local nonlinearities [79], if*

$$p_r(\mu) \overset{?}{=} \inf\{\mathcal{J}_\mu(u) \mid u \in H^s_r(\mathbb{R}^N) \setminus \{0\},\ \mathcal{J}'_\mu(u) = 0\}.$$

On the other hand, by assuming that every solution of (4.1.1) *satisfies the Pohozaev identity (see e.g. [342, Proposition 2] and [138, Eq (6.1)] and Section 4.7), the claim holds true. We point out that the equality may hold even if it is not true that every solution satisfies the Pohozaev identity. The fact that Deformation Lemma 4.2.8 allows to deform the functional near critical points not satisfying the Pohozaev identity might be useful in the investigation of these facts.*

Energy levels in the whole space

We pass studying general Pohozaev minima on the whole space $H^s(\mathbb{R}^N)$. We start defining the least energy of $\mathcal{J}_\mu$ on the Pohozaev set, and call every minimizer a *Pohozaev minimum* (or *ground state*)

$$p(\mu) := \inf\{\mathcal{J}_\mu(u) \mid u \in H^s(\mathbb{R}^N) \setminus \{0\},\ \mathcal{P}_\mu(u) = 0\}. \tag{4.3.23}$$

We start by showing that Proposition 4.3.2 holds also in a nonradial setting, providing here the proof. To do this, we get advantage of the minimax paths and level of $\mathcal{J}_\mu$. Set

$$a(\mu) := \inf_{\gamma\in\Gamma_\mu} \sup_{t\in[0,1]} \mathcal{J}_\mu(\gamma(t))$$

where

$$\Gamma(\mu) := \{\gamma \in C([0,1], H^s(\mathbb{R}^N)) \mid \gamma(0) = 0, |\ \mathcal{J}_\mu(\gamma(1)) < 0\}.$$

Notice that, with the same proof of Proposition 4.3.1 we obtain

$$a(\mu) = p(\mu) > 0. \tag{4.3.24}$$

Proposition 4.3.4. *Assume* (F1)-(F2). *Then every Pohozaev minimum of $\mathcal{J}_\mu$ is a solution of* (4.1.1), *i.e.*

$$\mathcal{J}_\mu(u) = p(\mu) \ \text{ and } \ \mathcal{P}_\mu(u) = 0$$

imply

$$\mathcal{J}'_\mu(u) = 0.$$

As a consequence

$$p(\mu) = \inf\{\mathcal{J}_\mu(u) \mid u \in H^s(\mathbb{R}^N) \setminus \{0\},\ \mathcal{P}_\mu(u) = 0,\ \mathcal{J}'_\mu(u) = 0\}.$$

Proof. Assume by contradiction that $\mathcal{J}_\mu'(u) \neq 0$. Thus $\mathcal{J}_\mu'$ remains far from zero in a neighborhood of u, that is there exist $\delta > 0$ and $\lambda > 0$ such that

$$v \in B_{3\delta}(u) \implies \|\mathcal{J}_\mu'(v)\|_* \geq \lambda.$$

Consider the path $\gamma(t) := u(\cdot/t)$; it is straightforward to show that $t \in \mathbb{R}_+ \mapsto \mathcal{J}_\mu(\gamma(t))$ is negative for $t \gg 0$ and it has a unique strict maximum, equal to $\mathcal{J}_\mu(u) = p(\mu) > 0$, attained in $t = 1$. Let now $I := [1-\omega, 1+\omega]$, ω small, be such that

$$S := \gamma(I) \subset B_\delta(u);$$

we can also assume that

$$\max_{t \notin I} \mathcal{J}_\mu(\gamma(t)) \in (0, p(\mu)).$$

Introduce moreover

$$0 < \varepsilon < \min\left\{\frac{p(\mu) - \max_{t\notin I} \mathcal{J}_\mu(\gamma(t))}{2}, \frac{\lambda\delta}{8}\right\}.$$

By writing $S_{2\delta} := \{v \in H^s(\mathbb{R}^N) \mid d(v, S) \leq 2\delta\}$, we see that

$$v \in S_{2\delta} \implies \|\mathcal{J}_\mu'(v)\|_* \geq \frac{8\varepsilon}{\delta}$$

and in particular

$$v \in \mathcal{J}_\mu^{-1}([p(\mu) - 2\varepsilon, p(\mu) + 2\varepsilon]) \cap S_{2\delta} \implies \|\mathcal{J}_\mu'(v)\|_* \geq \frac{8\varepsilon}{\delta},$$

where we observe that $p(\mu) - 2\varepsilon > 0$. We are thus in the assumptions of [379, Lemma 2.3], and we have the existence of a *local* continuous deformation $\eta : [0,1] \times H^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$ such that (we write $\mathcal{J}_\mu^b := \mathcal{J}_\mu^{-1}((-\infty, b]))$

(a) $\eta(0, v) = v$,

(b) $\eta(t, v) = v$ if $v \notin \mathcal{J}_\mu^{-1}([p(\mu) - 2\varepsilon, p(\mu) + 2\varepsilon]) \cap S_{2\delta}$,

(c) $\mathcal{J}_\mu(\eta(\cdot, v))$ is non increasing for each $v \in H^s(\mathbb{R}^N)$,

(d) $\eta(1, \mathcal{J}_\mu^{p(\mu)+\varepsilon} \cap S) \subset \mathcal{J}_\mu^{p(\mu)-\varepsilon}$.

We thus define a deformed path

$$\tilde{\gamma}(t) := \eta(1, \gamma(t)).$$

Consider first $t \notin I$. By (c) and the definition of ε, we have

$$\mathcal{J}_\mu(\tilde{\gamma}(t)) \leq \mathcal{J}_\mu(\gamma(t)) < p(\mu) - 2\varepsilon < p(\mu).$$

Assume instead $t \in I$. Then $\gamma(t) \in \gamma(I) = S$ and $\mathcal{J}_\mu(\gamma(t)) \leq \mathcal{J}_\mu(\gamma(1)) = p(\mu) \leq p(\mu) + \varepsilon$, thus by (d) we have

$$\mathcal{J}_\mu(\tilde{\gamma}(t)) \leq p(\mu) - \varepsilon < p(\mu).$$

Joining together the two inequalities we obtain

$$\max_{t>0} \mathcal{J}_\mu(\tilde{\gamma}(t)) < p(\mu). \tag{4.3.25}$$

On the other hand, we have $\tilde{\gamma}(0) = \eta(1, \gamma(0)) = \eta(1, 0) = 0$ since $0 \notin \mathcal{J}_\mu^{-1}([p(\mu) - 2\varepsilon, p(\mu) + 2\varepsilon])$, and $\mathcal{J}_\mu(\tilde{\gamma}(t)) \leq \mathcal{J}_\mu(\eta(1, \gamma(t)) \leq \mathcal{J}_\mu(\gamma(t)) < 0$ for $t \gg 0$. Up to a rescaling, we can assume $\tilde{\gamma} \in \Gamma(\mu)$ and hence, by (4.3.24)

$$\max_{t\in[0,1]} \mathcal{J}_\mu(\tilde{\gamma}(t)) \geq a(\mu) = p(\mu),$$

which is in contradiction with (4.3.25). The proof is thus concluded. ∎

Remark 4.3.5. *As in Remark 4.3.3, we point out that it is not known, even in the case of local nonlinearities, if*

$$p(\mu) \stackrel{?}{=} \inf\{\mathcal{J}_\mu(u) \mid u \in H^s(\mathbb{R}^N)\setminus\{0\},\ \mathcal{J}'_\mu(u)=0\},$$

unless some additional assumptions on s or f are assumed.

In Corollary 4.5.8 we will state some relation between $p_r(\mu)$ and $p(\mu)$.

Most of the qualitative properties that we will investigate, will be stated in the case of positive solutions. Thus it is important to highlight the existence of a solution of constant sign.

Proposition 4.3.6. *Assume (F1)–(F4) and that $F \not\equiv 0$ on $(0,+\infty)$ (i.e., t_0 in assumption (F4) can be chosen positive). Then there exists a positive radially symmetric solution of (4.1.1), which is minimum over all the positive functions on the Pohozaev set.*

Proof. Let us define

$$\tilde{f} := \chi_{(0,+\infty)} f.$$

We have that $\tilde{f}$ still satisfies (F1)–(F4). Thus, by Theorem 4.2.1 there exists a solution u of

$$(-\Delta)^s u + \mu u = (I_\alpha * \tilde{F}(u))\tilde{f}(u) \quad \text{in } \mathbb{R}^N$$

where $\tilde{F}(t) := \int_0^t \tilde{f}(\tau)d\tau$, $\tilde{F} = \chi_{(0,+\infty)}F$. We show now that u is positive. Recall by Lemma 1.4.1 that $u_- = \frac{|u|-u}{2} \in H^s_r(\mathbb{R}^N)$. Thus, chosen u_- as test function, we obtain

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}u\,(-\Delta)^{s/2}u_-\,dx + \mu\int_{\mathbb{R}^N} u\,u_-\,dx = \int_{\mathbb{R}^N}(I_\alpha * \tilde{F}(u))\tilde{f}(u)u_-\,dx.$$

By definition of $\tilde{f}$ and (1.2.5) we have

$$C_{N,s}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+2s}}\,dx\,dy - \mu\int_{\mathbb{R}^N}u_-^2\,dx = 0. \tag{4.3.26}$$

Splitting the domain, we gain

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N}&\frac{(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+2s}}\,dx\,dy =\\ &-\int_{\{u(x)\geq 0\}\times\{u(y)<0\}}\frac{(u_+(x)+u_-(y))(u_-(y))}{|x-y|^{N+2s}}\,dx\,dy -\\ &-\int_{\{u(x)<0\}\times\{u(y)\geq 0\}}\frac{(u_-(x)+u_+(y))(u_-(x))}{|x-y|^{N+2s}}\,dx\,dy -\\ &-\int_{\{u(x)<0\}\times\{u(y)<0\}}\frac{(u_-(x)-u_-(y))^2}{|x-y|^{N+2s}}\,dx\,dy.\end{aligned}$$

Since the left-hand side of (4.3.26) is sum of nonpositive pieces, we have $u_- \equiv 0$, that is $u \geq 0$. Hence $\tilde{f}(u) = f(u)$ and $\tilde{F}(u) = F(u)$, which imply that u is a positive solution of (4.1.1). ∎

4.4 Regularity

In this Section we investigate regularity of solutions, focusing in particular on boundedness, Hölder regularity and L^1-summability.

The discussed results generalize some of the ones in [138] to the case of general, not homogeneous, nonlinearities; in particular, we do not even assume f to satisfy Ambrosetti-Rabinowitz type conditions nor monotonicity conditions. Moreover, we improve the results in [277, 342] since we do not assume f to be superlinear, and we have no restriction on the parameter α.

Some of these results extend the ones in [79, 302] to the fractional, Choquard framework.

4.4.1 Boundedness by splitting

Here we prove that solutions of (4.1.1) are bounded. In particular, when dealing with sign-changing solutions, we will consider also the following stronger assumption:

(F6) $\limsup_{t\to 0} \frac{|f(t)|}{|t|} < +\infty$,

which says that f is linear or superlinear in the origin. Observe that

$$\text{(F6)} \implies \text{(F2,i) and (F3,i)}.$$

Theorem 4.4.1. *Assume (F1)-(F2). Let $u \in H^s(\mathbb{R}^N)$ be a weak positive solution of (4.1.1). Then $u \in L^\infty(\mathbb{R}^N)$. The same conclusion holds for generally (possibly sign-changing) solutions by assuming also (F6).*

We start from the following lemma, that can be found in [302, Lemma 3.3].

Lemma 4.4.2 ([302])**.** *Let $N \geq 2$ and $\alpha \in (0, N)$. Let $\lambda \in [0, 2]$ and $q, r, h, k \in [1, +\infty)$ be such that*

$$1 + \frac{\alpha}{N} - \frac{1}{h} - \frac{1}{k} = \frac{\lambda}{q} + \frac{2-\lambda}{r}.$$

Let $\theta \in (0, 2)$ satisfying

$$\min\{q, r\}\left(\frac{\alpha}{N} - \frac{1}{h}\right) < \theta < \max\{q, r\}\left(1 - \frac{1}{h}\right),$$

$$\min\{q, r\}\left(\frac{\alpha}{N} - \frac{1}{k}\right) < 2 - \theta < \max\{q, r\}\left(1 - \frac{1}{k}\right).$$

Let $H \in L^h(\mathbb{R}^N)$, $K \in L^k(\mathbb{R}^N)$ and $u \in L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} \left(I_\alpha * (H|u|^\theta)\right) K|u|^{2-\theta}\, dx \leq C\|H\|_h \|K\|_k \|u\|_q^\lambda \|u\|_r^{2-\lambda}$$

for some $C > 0$ (depending on θ).

By a proper use of Lemma 4.4.2 we obtain now an estimate on the Choquard term depending on H^s-norm of the function.

Lemma 4.4.3. *Let $N \geq 2$, $s \in (0, 1)$ and $\alpha \in (0, N)$. Let moreover $\theta \in (\frac{\alpha}{N}, 2 - \frac{\alpha}{N})$ and $H, K \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$. Then for every $\varepsilon > 0$ there exists $C_{\varepsilon,\theta} > 0$ such that*

$$\int_{\mathbb{R}^N} \left(I_\alpha * (H|u|^\theta)\right) K|u|^{2-\theta}\, dx \leq \varepsilon^2 \|(-\Delta)^{s/2} u\|_2^2 + C_{\varepsilon,\theta}\|u\|_2^2$$

for every $u \in H^s(\mathbb{R}^N)$.

Proof. Observe that $2 - \theta \in (\frac{\alpha}{N}, 2 - \frac{\alpha}{N})$ as well. We write

$$H = H^* + H_* \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N),$$

$$K = K^* + K_* \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N).$$

We split $\int_{\mathbb{R}^N} \left(I_\alpha * (H|u|^\theta)\right) K|u|^{2-\theta}\, dx$ in four pieces and choose

$$q = r = 2, \quad h = k = \frac{2N}{\alpha}, \quad \lambda = 2,$$

$$q = 2,\ r = \frac{2N}{N-2s}, \quad h = \frac{2N}{\alpha},\ k = \frac{2N}{\alpha+2s}, \quad \lambda = 1,$$

$$q=2,\ r=\frac{2N}{N-2s},\quad h=\frac{2N}{\alpha+2s},\ k=\frac{2N}{\alpha},\quad \lambda=1,$$
$$q=r=\frac{2N}{N-2s},\quad h=k=\frac{2N}{\alpha+2s},\quad \lambda=0,$$

in Lemma 4.4.2, to obtain

$$\int_{\mathbb{R}^N}\left(I_\alpha*(H|u|^\theta)\right)K|u|^{2-\theta}\,dx\lesssim$$
$$\|H^*\|_{\frac{2N}{\alpha}}\|K^*\|_{\frac{2N}{\alpha}}\|u\|_2^2+\|H^*\|_{\frac{2N}{\alpha}}\|K_*\|_{\frac{2N}{\alpha+2s}}\|u\|_2\|u\|_{\frac{2N}{N-2s}}+$$
$$+\|H_*\|_{\frac{2N}{\alpha+2s}}\|K^*\|_{\frac{2N}{\alpha}}\|u\|_2\|u\|_{\frac{2N}{N-2s}}+\|H_*\|_{\frac{2N}{\alpha+2s}}\|K_*\|_{\frac{2N}{\alpha+2s}}\|u\|^2_{\frac{2N}{N-2s}}.$$

Recalled that $\frac{2N}{N-2s}=2^*_s$ and the Sobolev embedding (1.2.7), we obtain

$$\int_{\mathbb{R}^N}\left(I_\alpha*(H|u|^\theta)\right)K|u|^{2-\theta}\,dx\lesssim$$
$$\left(\|H^*\|_{\frac{2N}{\alpha}}\|K^*\|_{\frac{2N}{\alpha}}\right)\|u\|_2^2+\left(\|H_*\|_{\frac{2N}{\alpha+2s}}\|K_*\|_{\frac{2N}{\alpha+2s}}\right)\|(-\Delta)^{s/2}u\|_2^2+$$
$$+\left(\|H^*\|_{\frac{2N}{\alpha}}\|K_*\|_{\frac{2N}{\alpha+2s}}+\|H_*\|_{\frac{2N}{\alpha+2s}}\|K^*\|_{\frac{2N}{\alpha}}\right)\|u\|_2\|(-\Delta)^{s/2}u\|_2.\tag{4.4.27}$$

We want to show now that, since $\frac{2N}{\alpha}>\frac{2N}{\alpha+2s}$, we can choose the decomposition of H and K such that the $L^{\frac{2N}{\alpha+2s}}$-pieces are arbitrary small (see [71, Lemma 2.1]). Indeed, let

$$H=H_1+H_2\in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)+L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$$

be a first decomposition. Let $M>0$ to be fixed, and write

$$H=\left(H_1+H_2\chi_{\{|H_2|\le M\}}\right)+H_2\chi_{\{|H_2|>M\}}.$$

Since $H_2\chi_{\{|H_2|\le M\}}\in L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and $\frac{2N}{\alpha}\in(\frac{2N}{\alpha+2s},\infty)$, we have $H_2\chi_{\{|H_2|\le M\}}\in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$, and thus

$$H^*:=H_1+H_2\chi_{\{|H_2|\le M\}}\in L^{\frac{2N}{\alpha}}(\mathbb{R}^N),\quad H_*:=H_2\chi_{\{|H_2|>M\}}\in L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N).$$

On the other hand

$$\|H_*\|_{\frac{2N}{\alpha+2s}}=\left(\int_{|H_2|>M}|H_2|^{\frac{2N}{\alpha+2s}}\,dx\right)^{\frac{\alpha+2s}{2N}}$$

which can be made arbitrary small for $M\gg 0$. In particular we choose the decomposition so that

$$\left(\|H_*\|_{\frac{2N}{\alpha+2s}}\|K_*\|_{\frac{2N}{\alpha+2s}}\right)\lesssim\varepsilon^2$$

and thus

$$C'(\varepsilon):\approx\left(\|H^*\|_{\frac{2N}{\alpha}}\|K^*\|_{\frac{2N}{\alpha}}\right).$$

In the last term of (4.4.27) we use the generalized Young's inequality $ab\le\frac{\delta}{2}a^2+\frac{1}{2\delta}b^2$, with

$$\delta:=\varepsilon^2\left(\|H^*\|_{\frac{2N}{\alpha}}\|K_*\|_{\frac{2N}{\alpha+2s}}+\|H_*\|_{\frac{2N}{\alpha+2s}}\|K^*\|_{\frac{2N}{\alpha}}\right)^{-1}$$

so that

$$\left(\|H^*\|_{\frac{2N}{\alpha}}\|K_*\|_{\frac{2N}{\alpha+2s}}+\|H_*\|_{\frac{2N}{\alpha+2s}}\|K^*\|_{\frac{2N}{\alpha}}\right)\|u\|_2\|(-\Delta)^{s/2}u\|_2$$
$$\le\ \tfrac{1}{2}\varepsilon^2\|u\|_2^2+C''(\varepsilon)\|(-\Delta)^{s/2}u\|_2^2.$$

Merging the pieces, we have the claim. ∎

The following technical result can be found in [207, Lemma 3.5].

Lemma 4.4.4 ([207]). *Let $a, b \in \mathbb{R}$, $r \geq 2$ and $k \geq 0$. Set $T_k : \mathbb{R} \to [-k, k]$ the truncation in k, that is*

$$T_k(t) := \begin{cases} -k & \text{if } t \leq -k, \\ t & \text{if } t \in (-k, k), \\ k & \text{if } t \geq k, \end{cases}$$

and write $a_k := T_k(a)$, $b_k := T_k(b)$. Then

$$\frac{4(r-1)}{r^2}\left(|a_k|^{r/2} - |b_k|^{r/2}\right)^2 \leq (a-b)\left(a_k|a_k|^{r-2} - b_k|b_k|^{r-2}\right).$$

Notice that the (optimal) Sobolev embedding tells us that $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$. In what follows we show that u belongs to some $L^r(\mathbb{R}^N)$ with $r > 2_s^*$; we highlight that we make no use of the Caffarelli-Silvestre s-harmonic extension method, and work directly in the fractional framework.

Proposition 4.4.5. *Let $H, K \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$. Assume that $u \in H^s(\mathbb{R}^N)$ solves*

$$(-\Delta)^s u + \mu u = (I_\alpha * (Hu))K, \quad \text{in } \mathbb{R}^N$$

in the weak sense. Then

$$u \in L^r(\mathbb{R}^N) \quad \text{for all } r \in \Big[2, \tfrac{N}{\alpha}\tfrac{2N}{N-2s}\Big).$$

Moreover, for each of these r, we have

$$\|u\|_r \leq C_r \|u\|_2$$

with $C_r > 0$ not depending on u.

Proof. By Lemma 4.4.3 there exists $\lambda > \mu$ (that we can assume large) such that

$$\int_{\mathbb{R}^N} (I_\alpha * (H|u|))\, K|u|\, dx \leq \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{\lambda}{2}\|u\|_2^2. \tag{4.4.28}$$

Let us set

$$H_n := H\chi_{\{|H|\leq n\}}, \quad K_n := K\chi_{\{|K|\leq n\}}, \quad \text{for } n \in \mathbb{N}$$

and observe that

$$H_n,\ K_n \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N),$$

$$H_n \to H, \quad K_n \to K \quad \text{almost everywhere, as } n \to +\infty$$

and

$$|H_n| \leq |H|, \quad |K_n| \leq |K| \quad \text{for every } n \in \mathbb{N}. \tag{4.4.29}$$

We thus define the bilinear form

$$a_n(\varphi, \psi) := \int_{\mathbb{R}^N} (-\Delta)^{s/2}\varphi\, (-\Delta)^{s/2}\psi\, dx + \lambda \int_{\mathbb{R}^N} \varphi\psi\, dx - \int_{\mathbb{R}^N} (I_\alpha * (H_n\varphi))\, K_n\psi\, dx$$

for every $\varphi, \psi \in H^s(\mathbb{R}^N)$. Since, by (4.4.29) and (4.4.28), we have

$$a_n(\varphi, \varphi) \geq \frac{1}{2}\|(-\Delta)^{s/2}\varphi\|_2^2 + \frac{\lambda}{2}\|\varphi\|_2^2 \geq \frac{1}{2}\|\varphi\|_{H^s(\mathbb{R}^N)}^2 \tag{4.4.30}$$

for each $\varphi \in H^s(\mathbb{R}^N)$, we obtain that a_n is coercive. Set

$$f := (\lambda - \mu)u \in H^s(\mathbb{R}^N)$$

we obtain by Lax-Milgram theorem that, for each $n \in \mathbb{N}$, there exists a unique $u_n \in H^s(\mathbb{R}^N)$ solution of

$$a_n(u_n, \varphi) = (f, \varphi)_2, \quad \varphi \in H^s(\mathbb{R}^N),$$

that is

$$(-\Delta)^s u_n + \lambda u_n - \big(I_\alpha * (H_n u_n)\big) K_n = (\lambda - \mu) u, \quad \text{in } \mathbb{R}^N \tag{4.4.31}$$

in the weak sense; moreover the theorem tells us that

$$\|u_n\|_{H^s} \leq \frac{\|f\|_2}{1/2} = 2(\lambda - \mu)\|u\|_2$$

(since $1/2$ appears as coercivity coefficient in (4.4.30)), and thus u_n is bounded. Hence $u_n \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^N)$ up to a subsequence for some $\bar{u}$. This means in particular that $u_n \to \bar{u}$ almost everywhere pointwise.

Thus we can pass to the limit in

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n \, (-\Delta)^{s/2} \varphi + \lambda \int_{\mathbb{R}^N} u_n \varphi - \int_{\mathbb{R}^N} \left(I_\alpha * (H_n u_n)\right) K_n \varphi = (\lambda - \mu) \int_{\mathbb{R}^N} u \varphi;$$

we need to check only the Choquard term. We first see by the continuous embedding that $u_n \rightharpoonup \bar{u}$ in $L^q(\mathbb{R}^N)$, for $q \in [2, 2^*_s]$. Split again $H = H^* + H_*$, $K = K^* + K_*$ and work separately in the four combinations; we assume to work generally with $\tilde{H} \in \{H^*, H_*\}$, $\tilde{H} \in L^\beta(\mathbb{R}^N)$ and $\tilde{K} \in \{K^*, K_*\}$, $\tilde{K} \in L^\gamma(\mathbb{R}^N)$, where $\beta, \gamma \in \{\frac{2N}{\alpha}, \frac{2N}{\alpha+2s}\}$. Then one can easily prove that $\tilde{H}_n u_n \rightharpoonup \tilde{H}\bar{u}$ in $L^r(\mathbb{R}^N)$ with $\frac{1}{r} = \frac{1}{\beta} + \frac{1}{q}$. By the continuity and linearity of the Riesz potential we have $I_\alpha * (H_n u_n) \rightharpoonup I_\alpha * (H\bar{u})$ in $L^h(\mathbb{R}^N)$, where $\frac{1}{h} = \frac{1}{r} - \frac{\alpha}{n}$. As before, we obtain $\left(I_\alpha * (H_n u_n)\right) K_n \rightharpoonup \left(I_\alpha * (H\bar{u})\right) K$ in $L^k(\mathbb{R}^N)$, where $\frac{1}{k} = \frac{1}{\gamma} + \frac{1}{h}$. Simple computations show that if $\beta = \gamma = \frac{2N}{\alpha}$ and $q = 2$, then $k' = 2$; if $\beta = \frac{2N}{\alpha}$, $\gamma = \frac{2N}{\alpha+2s}$ (or viceversa) and $q = 2$, then $k' = 2^*_s$; if $\beta = \gamma = \frac{2N}{\alpha+2s}$ and $q = 2^*_s$, then $k' = 2^*_s$. Therefore $H^s(\mathbb{R}^N) \subset L^{k'}(\mathbb{R}^N)$ and we can pass to the limit in all the four pieces, obtaining

$$\int_{\mathbb{R}^N} \left(I_\alpha * (H_n u_n)\right) K_n \varphi \, dx \to \int_{\mathbb{R}^N} \left(I_\alpha * (H\bar{u})\right) K \varphi \, dx.$$

Therefore, $\bar{u}$ satisfies

$$(-\Delta)^s \bar{u} + \lambda \bar{u} - \big(I_\alpha * (H\bar{u})\big) K = (\lambda - \mu) u, \quad \text{in } \mathbb{R}^N$$

as well as u. But we can see this problem, similarly as before, with a Lax-Milgram formulation and obtain the uniqueness of the solution. Thus $\bar{u} = u$ and hence, as $n \to +\infty$,

$$u_n \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^N)$$

and almost everywhere pointwise. Let now $k \geq 0$ and write

$$u_{n,k} := T_k(u_n) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$$

where T_k is the truncation introduced in Lemma 4.4.4. Let $r \geq 2$. We have $|u_{n,k}|^{r/2} \in H^s(\mathbb{R}^N)$, by exploiting (1.2.5) and the fact that $h(t) := (T_k(t))^{r/2}$ is a Lipschitz function with $h(0) = 0$. By (1.2.5) and by Lemma 4.4.4 we have

$$\begin{aligned}\frac{4(r-1)}{r^2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(|u_{n,k}|^{r/2})|^2 &= C_{N,s} \int_{\mathbb{R}^{2N}} \frac{\frac{4(r-1)}{r^2}\left(|u_{n,k}(x)|^{r/2} - |u_{n,k}(y)|^{r/2}\right)^2}{|x-y|^{N+2s}} \\ &\leq C_{N,s} \int_{\mathbb{R}^{2N}} \frac{\left(u_n(x) - u_n(y)\right)\left(u_{n,k}(x)|u_{n,k}(x)|^{r-2} - u_{n,k}(y)|u_{n,k}(y)|^{r-2}\right)}{|x-y|^{N+2s}}\end{aligned}$$

Set

$$\varphi := u_{n,k}|u_{n,k}|^{r-2}$$

it results that $\varphi \in H^s(\mathbb{R}^N)$, since again $h(t) := T_k(t)|T_k(t)|^{r-2}$ is a Lipschitz function with $h(0) = 0$. Thus we can choose it as a test function in (4.4.31) and obtain, by (1.2.6),

$$\begin{aligned}\frac{4(r-1)}{r^2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}(|u_{n,k}|^{r/2})|^2 &\le C_{N,s}\int_{\mathbb{R}^{2N}}\frac{(u_n(x)-u_n(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\\ &= -\lambda\int_{\mathbb{R}^N}u_n\varphi+\int_{\mathbb{R}^N}(I_\alpha*(H_nu_n))\,K_n\varphi+(\lambda-\mu)\int_{\mathbb{R}^N}u\varphi\end{aligned}$$

and since $u_n\varphi \ge |u_{n,k}|^r$ we gain

$$\begin{aligned}&\frac{4(r-1)}{r^2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}(|u_{n,k}|^{r/2})|^2 \le\\ &\le -\lambda\int_{\mathbb{R}^N}|u_{n,k}|^r+\int_{\mathbb{R}^N}(I_\alpha*(H_nu_n))K_n\varphi+(\lambda-\mu)\int_{\mathbb{R}^N}u\varphi.\end{aligned} \tag{4.4.32}$$

Focus on the Choquard term on the right-hand side. We have, by using (4.4.29),

$$\begin{aligned}&\int_{\mathbb{R}^N}(I_\alpha*(H_nu_n))K_n\varphi \le\\ &\le \int_{\mathbb{R}^N}(I_\alpha*(|H_n||u_n|\chi_{\{|u_n|\le k\}}))|K_n||u_{n,k}|^{r-1}+\\ &\quad+\int_{\mathbb{R}^N}(I_\alpha*(|H_n||u_n|\chi_{\{|u_n|>k\}}))|K_n||u_{n,k}|^{r-1}\\ &\le \int_{\mathbb{R}^N}(I_\alpha*(|H_n||u_{n,k}|))|K_n||u_{n,k}|^{r-1}+\int_{\mathbb{R}^N}(I_\alpha*(|H_n||u_n|\chi_{\{|u_n|>k\}}))|K_n||u_n|^{r-1}\\ &\le \int_{\mathbb{R}^N}(I_\alpha*(|H||u_{n,k}|))|K||u_{n,k}|^{r-1}+\int_{\mathbb{R}^N}(I_\alpha*(|H_n||u_n|\chi_{\{|u_n|>k\}}))|K_n||u_n|^{r-1}\\ &=: (I)+(II).\end{aligned} \tag{4.4.33}$$

Focus on (I). Consider $r \in [2, \frac{2N}{\alpha})$, so that $\theta := \frac{2}{r} \in (\frac{\alpha}{N}, 2-\frac{\alpha}{N})$. Choose moreover $v := |u_{n,k}|^{r/2} \in H^s(\mathbb{R}^N)$ and $\varepsilon^2 := \frac{2(r-1)}{r^2} > 0$. Thus, observed that if a function belongs to a sum of Lebesgue spaces then its absolute value does the same (see Remark 1.5.3), by Lemma 4.4.3 we obtain

$$(I) \le \frac{2(r-1)}{r^2}\|(-\Delta)^{s/2}(|u_{n,k}|^{r/2})\|_2^2 + C(r)\||u_{n,k}|^{r/2}\|_2^2. \tag{4.4.34}$$

Focus on (II). Assuming $r < \min\{\frac{2N}{\alpha}, \frac{2N}{N-2s}\}$, we have $u_n \in L^r(\mathbb{R}^N)$ and $H_n \in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$, thus

$$|H_n||u_n| \in L^a(\mathbb{R}^N), \quad \text{with } \tfrac{1}{a} = \tfrac{\alpha}{2N} + \tfrac{1}{r}$$

for the Hölder inequality. Similarly

$$|K_n||u_n|^{r-1} \in L^b(\mathbb{R}^N), \quad \text{with } \tfrac{1}{b} = \tfrac{\alpha}{2N} + 1 - \tfrac{1}{r}.$$

Thus, since $\frac{1}{a}+\frac{1}{b} = \frac{N+\alpha}{N}$, we have by the Hardy-Littlewood-Sobolev inequality (see Proposition 1.3.1) that

$$\begin{aligned}&\int_{\mathbb{R}^N}(I_\alpha*(|H_n||u_n|\chi_{\{|u_n|>k\}}))|K_n||u_n|^{r-1}\,dx\\ &\le C\left(\int_{\{|u_n|>k\}}||H_n||u_n||^a\,dx\right)^{1/a}\left(\int_{\mathbb{R}^N}||K_n||u_n|^{r-1}|^b\,dx\right)^{1/b}.\end{aligned}$$

With respect to k, the second factor on the right-hand side is bounded, while the first factor goes to zero thanks to the dominated convergence theorem, thus

$$(II) = o_k(1), \quad \text{as } k \to +\infty. \tag{4.4.35}$$

Joining (4.4.32), (4.4.33), (4.4.34), (4.4.35) we obtain

$$\frac{2(r-1)}{r^2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(|u_{n,k}|^{r/2})|^2\,dx \leq$$
$$\leq -\lambda\int_{\mathbb{R}^N}|u_{n,k}|^r\,dx + C(r)\int_{\mathbb{R}^N}|u_{n,k}|^r\,dx + (\lambda-\mu)\int_{\mathbb{R}^N} u\varphi\,dx + o_k(1).$$

That is, by Sobolev inequality (1.2.7)

$$C'(r)\left(\int_{\mathbb{R}^N}|u_{n,k}|^{\frac{r}{2}2^*_s}\right)^{2/2^*_s} \leq (C(r)-\lambda)\int_{\mathbb{R}^N}|u_{n,k}|^r + (\lambda-\mu)\int_{\mathbb{R}^N}|u|\,|u_{n,k}|^{r-1} + o_k(1).$$

Letting $k\to+\infty$ by the monotone convergence theorem (since $u_{n,k}$ are monotone with respect to k and $u_{n,k}\to u_n$ pointwise) we have

$$C'(r)\left(\int_{\mathbb{R}^N}|u_n|^{\frac{r}{2}2^*_s}\right)^{2/2^*_s} \leq (C(r)-\lambda)\int_{\mathbb{R}^N}|u_n|^r + (\lambda-\mu)\int_{\mathbb{R}^N}|u|\,|u_n|^{r-1} \tag{4.4.36}$$

and thus $u_n\in L^{\frac{r}{2}2^*_s}(\mathbb{R}^N)$. Notice that $\frac{r}{2}\in\big[1,\min\{\frac{N}{\alpha},\frac{N}{N-2s}\}\big)$. If $N-2s<\alpha$ we are done. Otherwise, set $r_1 := r$, we can now repeat the argument with

$$r_2 \in \left(\frac{2N}{N-2s}, \min\left\{\frac{2N}{\alpha}, 2\left(\frac{N}{N-2s}\right)^2\right\}\right).$$

Again, if $\frac{2N}{\alpha} < 2\left(\frac{N}{N-2s}\right)^2$ we are done, otherwise we repeat the argument. Inductively, we have

$$\left(\frac{N}{N-2s}\right)^m \to +\infty, \quad \text{as } m\to+\infty$$

thus $\frac{2N}{\alpha} < 2\left(\frac{N}{N-2s}\right)^m$ after a finite number of steps. For such $r=r_m$, consider again (4.4.36): by the almost everywhere convergence of u_n to u and Fatou's lemma

$$\begin{aligned} C''(r)\left(\int_{\mathbb{R}^N}|u|^{\frac{r}{2}2^*_s}\right)^{2/2^*_s} dx &\leq \liminf_n C''(r)\left(\int_{\mathbb{R}^N}|u_n|^{\frac{r}{2}2^*_s}\,dx\right)^{2/2^*_s}\\ &\leq \liminf_n\left((C(r)-\lambda)\int_{\mathbb{R}^N}|u_n|^r\,dx + (\lambda-\mu)\int_{\mathbb{R}^N}|u|\,|u_n|^{r-1}\,dx\right)\\ &\leq (C(r)-\lambda)\limsup_n\int_{\mathbb{R}^N}|u_n|^r\,dx + (\lambda-\mu)\limsup_n\int_{\mathbb{R}^N}|u|\,|u_n|^{r-1}\,dx. \end{aligned}$$

Being u_n equibounded in $H^s(\mathbb{R}^N)$ and thus in $L^{2^*_s}(\mathbb{R}^N)$, by the iteration argument we have that it is equibounded also in $L^r(\mathbb{R}^N)$; in particular, the bound is given by $\|u\|_2$ times a constant $C(r)$. Thus the right-hand side is a finite quantity, and we gain $u\in L^{\frac{r}{2}2^*_s}(\mathbb{R}^N)$, which is the claim. ∎

The following lemma states that $I_\alpha * g\in L^\infty(\mathbb{R}^N)$ whenever g lies in $L^q(\mathbb{R}^N)$ with q in a neighborhood of $\frac{N}{\alpha}$; in particular, it extends Proposition 1.3.1 (see also Remark 1.5.8).

In addition, it shows the decay at infinity of the Riesz potential, which will be useful in Section 4.6.

Proposition 4.4.6. *Assume that (F1)-(F2) hold. Let $u\in H^s(\mathbb{R}^N)$ be a solution of (4.1.1). Then $u\in L^q(\mathbb{R}^N)$ for $q\in\big[2,\frac{N}{\alpha}\frac{2N}{N-2s}\big)$, and*

$$I_\alpha * F(u) \in C_0(\mathbb{R}^N),$$

that is, continuous and zero at infinity. In particular,

$$I_\alpha * F(u)\in L^\infty(\mathbb{R}^N)$$

and

$$(I_\alpha * F(u))(x)\to 0 \quad \text{as } |x|\to+\infty.$$

Proof. We first check to be in the assumptions of Proposition 4.4.5. Indeed, by (F1)-(F2) and the fact that $u \in H^s(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \cap L^{2_s^*}(\mathbb{R}^N)$ we obtain that

$$H := \frac{F(u)}{u}, \quad K := f(u)$$

lie in $L^{\frac{2N}{\alpha}}(\mathbb{R}^N) + L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N)$, since bounded by functions in this sum space (see Remark 1.5.3). Now by Proposition 4.4.5 we have $u \in L^q(\mathbb{R}^N)$ for $q \in [2, \frac{N}{\alpha}\frac{2N}{N-2s})$; the claim follows by Remark 1.5.8. ∎

Once obtained the boundedness of the Choquard term, we can finally gain the boundedness of the solution.

Proposition 4.4.7. *Assume that (F1)-(F2) hold. Let $u \in H^s(\mathbb{R}^N)$ be a positive solution of (4.1.1). Then $u \in L^\infty(\mathbb{R}^N)$.*

Proof. By Lemma 4.4.6 we obtain

$$a := I_\alpha * F(u) \in L^\infty(\mathbb{R}^N).$$

Thus u satisfies the following nonautonomous problem, with a local nonlinearity

$$(-\Delta)^{s/2} u + \mu u = a(x) f(u), \quad \text{ in } \mathbb{R}^N$$

with a bounded. In particular

$$(-\Delta)^{s/2} u = g(x,u) := -\mu u + a(x) f(u), \quad \text{ in } \mathbb{R}^N$$

where

$$|g(x,t)| \leq \mu|t| + C\|a\|_\infty \left(|t|^{\frac{\alpha}{N}} + |t|^{\frac{\alpha+2s}{N-2s}}\right).$$

Set $\gamma := \max\{1, \frac{\alpha+2s}{N-2s}\} \in [1, 2_s^*)$, we thus have

$$|g(x,t)| \leq C(1+|t|^\gamma).$$

Hence we are in the assumptions of [157, Proposition 5.1.1] and we can conclude. ∎

Proof of Theorem 4.4.1. The first part of the claim comes from Proposition 4.4.7. In the case of sign-changing solutions, we may apply Proposition 1.2.24 with

$$g(x,t) := (I_\alpha * F(u))(x) f(t) - \mu u,$$

whenever u is a fixed solution and (F6) holds (together with (F1)–(F2)), thanks to Proposition 4.4.6. ∎

4.4.2 Hölder regularity: strong solutions

Gained the boundedness, we obtain now that solutions are Hölder continuous and satisfy the equation in the strong sense. This extra regularity will be also implemented in some bootstrap argument for the L^1-summability, see Section 4.4.3.

Proposition 4.4.8. *Assume that (F1)-(F2) hold. Let $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a weak solution of (4.1.1). Then $u \in H^{2s}(\mathbb{R}^N) \cap C^{0,\gamma}(\mathbb{R}^N)$ for any $\gamma \in (0, \min\{1, 2s\})$, and u is a strong solution, i.e. u satisfies (4.1.1) almost everywhere.*

In addition, if $s \in (\frac{1}{2}, 1)$, then $u \in C^{1,\gamma}(\mathbb{R}^N)$ for any $\gamma \in (0, 2s-1)$.

Proof. By Proposition 4.4.7, Proposition 4.4.6 and (F2) we have that $u \in L^\infty(\mathbb{R}^N)$ satisfies

$$(-\Delta)^s u = g \in L^\infty(\mathbb{R}^N)$$

where $g(x) := (I_\alpha * F(u))(x) f(u(x)) - \mu u(x)$. We prove first that $u \in H^{2s}(\mathbb{R}^N)$. Indeed, we already know that $f(u)$, $F(u)$ and $I_\alpha * F(u)$ belong to $L^\infty(\mathbb{R}^N)$. By Remark 1.5.7, we obtain

$$f(u) \in L^{\frac{2N}{\alpha+2s}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad F(u) \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

$$I_\alpha * F(u) \in L^{\frac{2N}{N-2s}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad (I_\alpha * F(u)) f(u) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

In particular,

$$g = (I_\alpha * F(u)) f(u) - \mu u \in L^2(\mathbb{R}^N).$$

Since u is a weak solution, we have, fixed $\varphi \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u \, (-\Delta)^{s/2} \varphi \, dx = \int_{\mathbb{R}^N} g \, \varphi \, dx. \tag{4.4.37}$$

Since $g \in L^2(\mathbb{R}^N)$, we can apply Plancharel theorem and obtain

$$\int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u} \, \widehat{\varphi} \, d\xi = \int_{\mathbb{R}^N} \widehat{g} \, \widehat{\varphi} \, d\xi. \tag{4.4.38}$$

Since $H^s(\mathbb{R}^N) = \mathcal{F}(H^s(\mathbb{R}^N))$ and φ is arbitrary, we gain

$$|\xi|^{2s} \widehat{u} = \widehat{g} \in L^2(\mathbb{R}^N).$$

By definition, we obtain $u \in H^{2s}(\mathbb{R}^N)$, which concludes the proof. Observe moreover that $\mathcal{F}^{-1}\big((1+|\xi|^{2s})\widehat{u}\big) = u + g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, thus by definition $u \in H^{2s}(\mathbb{R}^N) \cap W^{2s,\infty}(\mathbb{R}^N)$. By the embedding (1.2.13) (see also Proposition 1.2.25) we obtain $u \in C^{0,\gamma}(\mathbb{R}^N)$ if $2s < 1$ and $\gamma \in (0, 2s)$, while $u \in C^{1,\gamma}(\mathbb{R}^N)$ if $2s > 1$ and $\gamma \in (0, 2s-1)$.

It remains to show that u is an almost everywhere pointwise solution. Thanks to the fact that $u \in H^{2s}(\mathbb{R}^N)$, we use again (4.4.38), where we can apply Plancharel theorem (that is, we are integrating by parts (4.4.37)) and thus

$$\int_{\mathbb{R}^N} (-\Delta)^s u \, \varphi \, dx = \int_{\mathbb{R}^N} g \, \varphi \, dx.$$

Since $\varphi \in H^s(\mathbb{R}^N)$ is arbitrary, we obtain

$$(-\Delta)^s u = g \quad \text{almost everywhere.}$$

This concludes the proof. ∎

We observe, by the proof, that if $s \in (\frac{1}{2}, 1)$, then u is a classical solution, with $(-\Delta)^s u \in C(\mathbb{R}^N)$ and equation (4.1.1) satisfied pointwise. We will further investigate these aspects in Section 4.4.4.

4.4.3 L^1-summability: fixed point maps

We deal now with the summability of u in Lebesgue spaces $L^r(\mathbb{R}^N)$ for $r < 2$. We observe that the information $u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is new even in the power-type setting: indeed in [138] the authors, in order to ensure existence of solutions, assume the nonlinearity to be not critical, while here we can include the possibility of criticality. Moreover, this result is new even for $s = 1$, improving [302]. The L^1-summability will be then used also to gain the asymptotic behaviour of the solutions in Section 4.6.

Remark 4.4.9. *We start noticing that, if a solution u belongs to some $L^q(\mathbb{R}^N)$ with $q<2$, then $u\in L^1(\mathbb{R}^N)$. Assume thus $q\in(1,2)$ and let $u\in L^q(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, then we have*

$$f(u)\in L^{\frac{qN}{\alpha}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\quad F(u)\in L^{\frac{qN}{N+\alpha}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),$$

$$I_\alpha * F(u)\in L^{\frac{qN}{N+\alpha(1-q)}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N),\quad (I_\alpha * F(u))f(u)\in L^{\frac{qN}{N+\alpha(2-q)}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N).$$

Thanks to Proposition 4.4.8, u satisfies (4.1.1) almost everywhere, thus we have

$$\mathcal{F}^{-1}\big((|\xi|^{2s}+\mu)\,\widehat{u}\big)=(-\Delta)^s u+\mu u=(I_\alpha * F(u))f(u)\in L^{\frac{qN}{N+\alpha(2-q)}}(\mathbb{R}^N)$$

hence by the properties of the Bessel operator (1.2.12) we obtain that u itself lies in the same Lebesgue space, that is

$$u\in L^{\frac{qN}{N+\alpha(2-q)}}(\mathbb{R}^N).$$

If $\frac{qN}{N+\alpha(2-q)}<1$, we mean that $(I_\alpha * F(u))f(u)\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, and thus $u\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. We convey this when we deal with exponents less than 1.

If $q<2$, then

$$\frac{qN}{N+\alpha(2-q)}<q$$

and we can implement a bootstrap argument to gain $u\in L^1(\mathbb{R}^N)$. More precisely

$$\begin{cases} q_0\in[1,2) \\ q_{n+1}=\dfrac{q_n N}{N+\alpha(2-q_n)} \end{cases}$$

where $q_n\to 0$ (but we stop at 1).

We show now that $u\in L^1(\mathbb{R}^N)$. It is easy to see that, if the problem is (strictly) not lower-critical, i.e., (F2) holds together with

$$\lim_{t\to 0}\frac{F(t)}{|t|^\beta}=0$$

for some $\beta\in(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2s})$, then $u\in L^1(\mathbb{R}^N)$. Indeed $u\in H^s(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\subset L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and

$$(I_\alpha * F(u))f(u)\in L^q(\mathbb{R}^N),$$

where $\frac{1}{q}=\frac{\beta}{2}-\frac{\alpha}{2N}$; noticed that $q<2$, we can implement the bootstrap argument of Remark 4.4.9.

We will show that the same conclusion can be reached by assuming only (F2).

Proposition 4.4.10. *Assume that (F1)-(F2) hold. Let $u\in H^s(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ be a weak solution of (4.1.1). Then $u\in L^1(\mathbb{R}^N)$.*

Proof. For a given solution $u\in H^s(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ we set again

$$H:=\frac{F(u)}{u},\quad K:=f(u).$$

Since $u\in L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, by (F2) we have $H,\,K\in L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$. For $n\in\mathbb{N}$, we set

$$H_n:=H\chi_{\{|x|\geq n\}}.$$

Then we have

$$\|H_n\|_{\frac{2N}{\alpha}}\to 0\quad\text{as } n\to\infty. \tag{4.4.39}$$

Since $\mathrm{supp}(H - H_n) \subset \{|x| \le n\}$ is a bounded set, we have for any $\beta \in [1, \frac{2N}{\alpha}]$

$$H - H_n \in L^\beta(\mathbb{R}^N) \quad \text{for all } n \in \mathbb{N}. \tag{4.4.40}$$

We write our equation (4.1.1) as

$$(-\Delta)^s u + \mu u = (I_\alpha * H_n u)K + R_n \quad \text{in } \mathbb{R}^N,$$

where we introduced the function R_n by

$$R_n := (I_\alpha * (H - H_n)u)K.$$

Now we consider the following linear equation:

$$(-\Delta)^s v + \mu v = (I_\alpha * H_n v)K + R_n \quad \text{in } \mathbb{R}^N. \tag{4.4.41}$$

We have the following facts:

(i) The given solution u solves (4.4.41).

(ii) By the property (4.4.40) with $\beta \in (\frac{2N}{N+\alpha}, \frac{2N}{\alpha})$, there exists $q_1 \in (1, 2)$, namely $\frac{1}{q_1} = \frac{1}{\beta} + \frac{1}{2} - \frac{\alpha}{2N}$, such that $R_n \in L^{q_1}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$.

(iii) By the property (4.4.39), for any $r \in (\frac{2N}{2N-\alpha}, 2] \subset (1, 2]$

$$v \in L^r(\mathbb{R}^N) \mapsto A_n(v) := (I_\alpha * H_n v)K \in L^r(\mathbb{R}^N)$$

is well defined and verifies

$$\|A_n(v)\|_r \le C_{r,n} \|v\|_r. \tag{4.4.42}$$

Here $C_{r,n}$ satisfies $C_{r,n} \to 0$ as $n \to \infty$.

We show only (iii). Since $v \in L^r(\mathbb{R}^N)$, by Hardy-Littlewood-Sobolev inequality and Hölder inequality we obtain

$$\|A_n(v)\|_r \le C_r \|H_n\|_{\frac{2N}{\alpha}} \|K\|_{\frac{2N}{\alpha}} \|v\|_r,$$

where $C_r > 0$ is independent of n, v. Thus by (4.4.39) we have $C_{r,n} := C_r \|H_n\|_{\frac{2N}{\alpha}} \|K\|_{\frac{2N}{\alpha}} \to 0$ as $n \to \infty$.

Now we show $u \in L^{q_1}(\mathbb{R}^N)$, where $q_1 \in (1, 2)$ is given in (ii). Since $((-\Delta)^s + \mu)^{-1} : L^r(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$ is a bounded linear operator for $r \in (1, 2]$ (see (1.2.11)), (4.4.41) can be rewritten as

$$v = T_n(v),$$

where

$$T_n(v) := ((-\Delta)^s + \mu)^{-1}(A_n(v) + R_n).$$

By choosing $\beta \in (2, \frac{2N}{\alpha})$ we have $q_1 \in (\frac{2N}{2N-\alpha}, 2) \subset (1, 2)$, thus we observe that for n large, T_n is a contraction in $L^2(\mathbb{R}^N)$ and in $L^{q_1}(\mathbb{R}^N)$. We fix such an n.

Since T_n is a contraction in $L^2(\mathbb{R}^N)$, we can see that $u \in H^s(\mathbb{R}^N)$ is a unique fixed point of T_n. In particular, we have

$$u = \lim_{k \to \infty} T_n^k(0) \quad \text{in } L^2(\mathbb{R}^N).$$

On the other hand, since T_n is a contraction in $L^{q_1}(\mathbb{R}^N)$, $(T_n^k(0))_{k=1}^\infty$ also converges in $L^{q_1}(\mathbb{R}^N)$. Thus the limit u belongs to $L^{q_1}(\mathbb{R}^N)$.

Since $q_1 < 2$ we can use the bootstrap argument of Remark 4.4.9 to get $u \in L^1(\mathbb{R}^N)$, and reach the claim. ∎

With similar arguments we obtain also the following result for $s = 1$.

Proposition 4.4.11. *Let $s = 1$ and assume $N \ge 3$ and (F1)-(F2). Let $u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a weak solution of (3.1.1) Then $u \in L^1(\mathbb{R}^N)$.*

4.4.4 C^γ-regularity: classical solutions

We continue the analysis of the regularity started in Proposition 4.4.8 and we infer the following result. This extra regularity will be exploited in the discussion of the positivity of solutions, see Section 4.5.1; some more results about the regularity of the solutions will be stated in Section 4.4.5.

Consider the condition

(F7) $f \in C^{0,\sigma}_{loc}(\mathbb{R})$ for some $\sigma \in (0,1]$.

Proposition 4.4.12. *Assume (F1)-(F2). Let $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a weak solution of (4.1.1). If $s \in (\frac{1}{2}, 1)$, then $u \in C^{1,\gamma}(\mathbb{R}^N)$ for any $\gamma \in (0, 2s-1)$ and u is a classical solution.*

Assume now instead $s \in (0,1)$ and in addition (F7). Then u is a classical solution, that is a pointwise solution lying in

- $C^{0,\gamma}(\mathbb{R}^N) \cap H^{2s}(\mathbb{R}^N)$ *for some $\gamma > 2s$, if $2s < 1$,*
- $C^{1,\gamma-1}(\mathbb{R}^N) \cap H^{2s}(\mathbb{R}^N)$ *for some $\gamma > 2s$, if $2s \geq 1$.*

More specifically, set $\omega := \min\{\sigma, 2s\sigma, \alpha\}$, we have

- *if $\omega + 2s \in (0,1]$, then $u \in C^{0,\gamma}(\mathbb{R}^N)$ for each $\gamma \in (0, \omega+2s] \cap (0,1)$,*
- *if $\omega + 2s \in (1,2]$, then $u \in C^{1,\gamma-1}(\mathbb{R}^N)$ for each $\gamma \in (0, \omega+2s] \cap (0,2)$,*
- *if $\omega + 2s \in (2,3)$, then $u \in C^{2,\omega+2s-2}(\mathbb{R}^N)$.*

Proof. Start noticing that by Proposition 4.4.6 we have $I_\alpha * F(u) \in C_0(\mathbb{R}^N)$; in particular $I_\alpha * F(u)$ is pointwise finite. Moreover, by (F2) we have $F(u) \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If we choose

$$\begin{cases} q \in [\frac{2N}{N+\alpha}, \infty) & \text{if } \alpha \in (0,1], \\ q \in [\frac{2N}{N+\alpha}, \frac{N}{\alpha-1}) & \text{if } \alpha \in (1,N) \end{cases}$$

we obtain

$$F(u) \in L^q(\mathbb{R}^N), \quad \tfrac{N}{q} < \alpha < 1 + \tfrac{N}{q}$$

and thus we can apply Proposition 1.3.6 to conclude that

$$I_\alpha * F(u) \in C^{0,\alpha - \frac{N}{q}}(\mathbb{R}^N).$$

In particular, by suitable choices of q, we gain

$$I_\alpha * F(u) \in C^{0,\omega}(\mathbb{R}^N) \quad \text{for every } \omega \in (0, \min\{1,\alpha\}).$$

Notice that up to now we did not use the regularity on f. Assume (F7) now. By Proposition 4.4.8 we have that u is bounded and $u \in C^{0,\gamma}(\mathbb{R}^N)$ for every $\gamma \in (0, \min\{1, 2s\})$. By composition, we obtain

$$f(u) \in C^{0,\theta}(\mathbb{R}^N), \quad \text{for } \theta \in (0, \min\{\sigma, 2s\sigma\}).$$

Chosen

$$\omega \equiv \theta \in (0, \min\{\sigma, 2s\sigma, \alpha\})$$

then, since both $f(u)$ and $I_\alpha * F(u)$ are bounded and Hölder continuous, we have

$$(I_\alpha * F(u)) f(u) \in C^{0,\omega}(\mathbb{R}^N).$$

At this point we can use Proposition 1.2.25 to gain

- if $\omega + 2s \in (0,1]$, then $u \in C^{0,\gamma}(\mathbb{R}^N)$ for each $\gamma \leq \omega + 2s$, $\gamma < 1$,

- if $\omega + 2s \in (1, 2]$, then $u \in C^{1,\gamma-1}(\mathbb{R}^N)$ for each $\gamma \le \omega + 2s$, $\gamma < 2$,
- if $\omega + 2s \in (2, 3)$, then $u \in C^{2,\omega+2s-2}(\mathbb{R}^N)$,

and thus the regularity claim. Finally, again by Proposition 4.4.8 u satisfies (4.1.1) almost everywhere; moreover, by the achieved regularity and Proposition 1.2.1, we have that all the appearing functions in (4.1.1) are continuous; thus the equation must be satisfied everywhere pointwise. This concludes the proof. ∎

4.4.5 C^1 and C^2 regularity

We prove now that, under some more restrictive conditions on s, α and σ, where $f \in C^{0,\sigma}_{loc}(\mathbb{R}^N)$, we can prove that $u \in C^1(\mathbb{R}^N)$. We notice that partial results for $s \in [\frac{1}{4}, 1)$ are already contained in Proposition 4.4.8 and Proposition 4.4.12. This C^1-regularity will be implemented then in the study of the Pohozaev identity in Section 4.7.

Proposition 4.4.13. *Assume (F1)-(F2). Let $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a weak solution of (4.1.1). Then*

i) if $s \in (\frac{1}{2}, 1)$, then $u \in C^{1,\gamma}(\mathbb{R}^N)$ for any $\gamma \in (0, 2s-1)$.

Assume now (F7) in addition. Then

ii) if $s \in [\frac{1}{2}, 1)$ and $\omega := \min\{\sigma, \alpha\} \le 2 - 2s$, then $u \in C^{1,\gamma}(\mathbb{R}^N)$ for any $\gamma \in (0, \omega + 2s - 1)$; if instead $\omega > 2 - 2s$, then $u \in C^{2,\omega+2s-2}(\mathbb{R}^N)$;

iii) if $s \in [\frac{1}{4}, \frac{1}{2})$, $\alpha > 1 - 2s$ and $\sigma > \frac{1-2s}{2s}$, then $u \in C^{1,\gamma}(\mathbb{R}^N)$ for every $\gamma \in (0, \omega + 2s - 1)$, where $\omega := \min\{2s\sigma, \alpha\}$;

iv) if $\alpha < 2$ and $\sigma > 1 - 2s$, then $u \in C^{1,\gamma}(\mathbb{R}^N)$ for every $\gamma \in (0, 1)$.

Proof. We need to check only the fourth case. We aim to prove

$$(I_\alpha * F(u))f(u) \in C^{0,\omega}_{loc}(\mathbb{R}^N), \quad \text{for some } \omega + 2s > 1$$

in order to apply Proposition 1.2.25. We want to show thus that $I_\alpha * F(u)$ is Hölder continuous; more precisely, we will show that it belongs to $C^{0,\omega}(\mathbb{R}^N)$ for some ω that increases according to γ, where $u \in C^{0,\gamma}(\mathbb{R}^N)$, so that we can employ a bootstrap argument.

Thanks to Proposition 4.4.12, set

$$\begin{aligned} \theta_0 :=& \min\{\sigma + 2s, 2s\sigma + 2s, \alpha + 2s, 1\} \\ =& \min\{2s\sigma + 2s, \alpha + 2s, 1\} \end{aligned}$$

we have

$$u \in C^{0,\gamma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \qquad \text{for } \gamma \in (0, \theta_0).$$

By composition we obtain

$$f(u) \in C^{0,\gamma}(\mathbb{R}^N), \quad \text{for } \gamma \in (0, \sigma\theta_0),$$

and

$$F(u) \in C^{0,\gamma}(\mathbb{R}^N), \quad \text{for } \gamma \in (0, \theta_0),$$

which implies, by Proposition 1.3.6 (possible since $\alpha < 2$) and Remark 1.1.1 (recall that $I_\alpha * F(u) \in L^\infty(\mathbb{R}^N)$), that

$$I_\alpha * F(u) \in C^{0,\alpha+\gamma}(\mathbb{R}^N), \quad \text{for } \gamma \in (0, \min\{\theta_0, 1 - \alpha\})$$

that is

$$I_\alpha * F(u) \in C^{0,\gamma}(\mathbb{R}^N), \quad \text{for } \gamma \in (0, \min\{\theta_0 + \alpha, 1\}).$$

Since $\omega_0 := \sigma\theta_0 < \min\{\theta_0 + \alpha, 1\}$, we have

$$(I_\alpha * F(u))f(u) \in C^{0,\gamma}(\mathbb{R}^N) \quad \text{for } \gamma \in (0, \omega_0).$$

We implement now the bootstrap argument. By Proposition 1.2.25 we gain

- if $\theta_1 := \omega_0 + 2s > 1$, then $u \in C^{1,\gamma}(\mathbb{R}^N)$ for $\gamma \in (0, \theta_1 - 1)$;
- if $\theta_1 = \omega_0 + 2s \le 1$, then $u \in C^{0,\gamma}(\mathbb{R}^N)$ for $\gamma \in (0, \theta_1)$.

In the first case, we stop. Otherwise,

$$(I_\alpha * F(u))f(u) \in C^{0,\omega_1}(\mathbb{R}^N), \quad \omega_1 := \sigma\theta_1,$$

and

- if $\theta_2 := \omega_1 + 2s > 1$, then $u \in C^{1,\gamma}(\mathbb{R}^N)$ for $\gamma \in (0, \theta_2 - 1)$;
- if $\theta_2 = \omega_1 + 2s \le 1$, then $u \in C^{0,\gamma}(\mathbb{R}^N)$ for $\gamma \in (0, \theta_2)$.

We proceed inductively by setting

$$\begin{cases} \omega_i := \sigma\theta_i, \\ \theta_i := \omega_{i-1} + 2s, \end{cases}$$

that is

$$\theta_i = \sigma\theta_{i-1} + 2s.$$

We need to show that $\theta_i > 1$ at some point. We observe that

$$\theta_i > \theta_{i-1} \iff \theta_{i-1} < \frac{2s}{1-\sigma}.$$

If for some i we have

$$\theta_i \ge \frac{2s}{1-\sigma} > 1$$

then we stop. Otherwise, θ_i is increasing, and thus its limit $\theta_i \to l$ satisfies

$$l = \sigma l + 2s$$

which means that $l = +\infty$ or $l = \frac{2s}{1-\sigma} > 1$. This concludes the proof. ∎

Remark 4.4.14. *We notice that, by assuming $F \ge 0$, we can use Proposition 1.3.6 to implement a bootstrap argument (namely $\omega_0 := \min\{\sigma, \frac{N}{N+\alpha}\}\theta_0$ with the notations of the above proof) to get additional regularity for a generic $\alpha \in (0, N)$. We leave the details to the interested reader. Similar arguments can be developed by assuming*

$$|F(t) - F(s)| \lesssim |t-s|^\theta |f(t) - f(s)|, \quad \textit{for } t, s \in \mathbb{R}$$

for some $\theta \in (0, 1]$.

Remark 4.4.15. *Let us consider $u > 0$ radially symmetric decreasing and assume $f \in C^1((0,+\infty))$ with*

$$|f'(t)| \lesssim |t|^{-\frac{N-\alpha}{N}} + |t|^{\frac{2\alpha}{N-\alpha}}, \quad \text{for } t > 0$$

and

$$|f(t) - f(s)| \lesssim |t-s|^{\theta} |f'(t) - f'(s)|, \quad \text{for } t, s > 0$$

for some $\theta \in (0,1]$, then we can refine the regularity argument of Remark 4.4.14 by exploiting some asymptotic estimates. Indeed, better regularity on $(I_\alpha * F(u))f(u)$ can be deduced as follows: for $x, y \in \mathbb{R}^N$ we have

$$\begin{aligned}&\left|(I_\alpha * F(u))(x) f(u(x)) - (I_\alpha * F(u))(y) f(u(y))\right| \\ &\quad \leq \left|(I_\alpha * F(u))(x) - (I_\alpha * F(u))(y)\right| \|f(u)\|_\infty + |u(x)-u(y)|^{\theta} \left|(I_\alpha * F(u))(y)\right| |f'(u(x)) - f'(u(y))|\end{aligned}$$

where, by Corollary 4.6.20 and Remark 4.6.17

$$\left|(I_\alpha * F(u))(y)\right| |u(x)|^{-\frac{N-\alpha}{N}} \leq \frac{1+|x|^{N-\alpha}}{1+|y|^{N-\alpha}} \leq C$$

whenever $|x-y| \leq 1$. Thus the Hölder regularity exponent of $(I_\alpha * F(u))f(u)$ directly depends on the ones of $I_\alpha * F(u)$, u and on θ. We leave the details to the interested reader.

Finally, we exploit the L^1-summability in order to further investigate the C^2-regularity of the solution u. We notice that some results are already contained in Proposition 4.4.12, whenever $s \in (\frac{1}{2}, 1)$, with some restriction on the regularity of f and on α: for instance, if $f \in C^{0,1}_{loc}(\mathbb{R})$ we need $\alpha + 2s > 2$ (e.g., $\alpha \geq 1$). We prove now that, if $f \in C^1(\mathbb{R})$, then no restriction on α is needed. Notice that $f \in C^1(\mathbb{R})$ implies f non sublinear in zero, that is (F6).

Proposition 4.4.16. *Assume (F1)-(F2) and $s \in (\frac{1}{2}, 1)$. Let $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a weak solution of (4.1.1). Then we have*

- *if (F7) holds with $\omega := \min\{\sigma, \alpha\} > 2 - 2s$, then $u \in C^{2,\omega+2s-2}(\mathbb{R}^N)$,*
- *if $f \in C^1(\mathbb{R})$, then $u \in C^{2,\gamma-2}(\mathbb{R}^N)$ for every $\gamma < 2s+1$.*

Proof. We need to prove only the second point. First we show that $I_\alpha * F(u)$ is in $C^1(\mathbb{R}^N)$. Indeed, considered $\eta \in C^\infty_c(\mathbb{R}^N)$ a smooth mollification of χ_{B_1}, we have

$$(I_\alpha \eta) * F(u) \in C^1(\mathbb{R}^N)$$

since $I_\alpha \eta \in L^1(\mathbb{R}^N)$ has compact support and $u \in C^1(\mathbb{R}^N)$ (by Proposition 4.4.8), while

$$(I_\alpha(1-\eta)) * F(u) \in C^1(\mathbb{R}^N)$$

since $I_\alpha(1-\eta)$ has support far from the origin and thus belongs to $C^1_b(\mathbb{R}^N)$, while $F(u) \in L^1(\mathbb{R}^N)$ by Proposition 4.4.10 (since $u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \supset L^{\frac{N+\alpha}{N}}(\mathbb{R}^N)$, see also Remark 4.4.9).

In particular

$$(-\Delta)^s u = -\mu u + (I_\alpha * F(u)) f(u) \in C^1(\mathbb{R}^N).$$

Since $2s > 1$, we gain $u \in H^{2s}(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N)$, and in particular $\partial_j u \in L^2(\mathbb{R}^N) \cap C^{0,\gamma}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ for each $j = 1 \dots N$. Moreover we have

$$\partial_j (I_\alpha * F(u)) = (I_\alpha \eta) * (\partial_j F(u)) + (\partial_j (I_\alpha (1-\eta))) * F(u).$$

We want to show that the derivative can be moved to $F(u)$ in the second term. Indeed, set $h := I_\alpha(1-\eta)$ for brevity, and let ϕ_n be a cut-off function with $\phi_n \equiv 1$ in B_n and support in B_{n+1}; thus

$$\int_{\mathbb{R}^N} \partial_j h(x-y) F(u(y)) \phi_n(y) = \int_{\mathbb{R}^N} h(x-y) \partial_j F(u(y)) \phi_n(y) +$$

$$+ \int_{\mathbb{R}^N} h(x-y)F(u(y))\partial_j \phi_n(y);$$

being $\phi_n \to 1$, $\partial_j \phi_n \to 0$ as $n \to +\infty$, and $|h|, |\partial_j h| \leq C$ together with $F(u) \in L^1(\mathbb{R}^N)$ and $\partial_j F(u) = f(u)\partial_j u \in L^1(\mathbb{R}^N)$ (notice that $f(u) \in L^2(\mathbb{R}^N)$ since $u \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$), by dominated convergence theorem we reach the claim. Thus we obtain

$$\partial_j \big((I_\alpha * F(u)) f(u) \big) = \big(I_\alpha * (f(u)\partial_j u)\big) f(u) + (I_\alpha * F(u)) f'(u) \partial_j u.$$

Since $u \in L^\infty(\mathbb{R}^N)$ and f' is continuous, we have $f'(u)$ is bounded. Thus the right hand side belongs to $L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

If we prove that

$$\partial_j((-\Delta)^s u) = (-\Delta)^s (\partial_j u) \tag{4.4.43}$$

then we have

$$(-\Delta)^s (\partial_j u) = -\mu \partial_j u + \partial_j \big((I_\alpha * F(u)) f(u) \big) \in L^\infty(\mathbb{R}^N);$$

by Proposition 1.2.25 and again $2s > 1$, we obtain that $\partial_j u \in C^{1,\gamma}(\mathbb{R}^N)$ for any $\gamma \in (0, 2s-1)$, which is the claim.

We deal thus with (4.4.43). Since $\partial_j((-\Delta)^s u) \in L^2(\mathbb{R}^N)$, we can evaluate the Fourier transform $\mathcal{F}(\partial_j((-\Delta)^s u))$, and since $(-\Delta)^s u \in C^1(\mathbb{R}^N)$ we have

$$\mathcal{F}(\partial_j((-\Delta)^s u)) = i\xi_j \mathcal{F}((-\Delta)^s u) = i\xi_j \big(|\xi|^{2s} \mathcal{F}(u)\big).$$

Since $u \in L^2(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ we obtain

$$\mathcal{F}(\partial_j((-\Delta)^s u)) = |\xi|^{2s} (i\xi_j \mathcal{F}(u)) = |\xi|^{2s} \mathcal{F}(\partial_j u);$$

taking back the Fourier transform, we obtain (4.4.43). This concludes the proof. ∎

4.5 Shape of ground states

In this Section we exploit the regularity of the solutions gained in Proposition 4.4.12 to deduce the following theorem concerning the sign and the symmetry of the ground state solutions.

Theorem 4.5.1. *Assume $N \geq 2$ and (F7) in addition to (F1)-(F2). Assume moreover*

(F8) *(i) f is odd or even,*

(ii) f has constant sign on $(0, +\infty)$.

Then every Pohozaev minimum of (4.1.1) has strict constant sign (strictly positive or negative), is radially symmetric and decreasing.

This last result is obtained also for *constrained* problem with fixed mass, see Remark 4.5.9 for details.

Remark 4.5.2. *We observe that the qualitative results in Theorem 4.5.1 holds also for least energy solutions, when the Pohozaev identity holds for every solution, see Section 4.7 (see also [138, Eq (6.1)] and [342]).*

This theorem extends the result in Theorem 3.1.1 to the fractional case; in particular, [302] deals with the case F even. Here we address also the study of the case F odd: as already highlighted in Chapter 3, this case is generally less studied in literature, even if (in the nonlocal framework) this assumption makes the functional symmetric as well as the odd case. Mathematically, F odd reveals to be more challenging, since the interactions in the nonlocal term among positive and negative contributions is stronger and more difficult to manage.

Specifically, we highlight that this result is new even in the limiting local case $s = 1$, $N \geq 3$, when F is odd, extending some results in [302]. Notice that in this framework the regularity results hold for f merely continuous, and moreover every Pohozaev minimum is a least energy solution, since every solution satisfies the Pohozaev identity (3.4.32).

Theorem 4.5.3. *Let $s = 1$ and assume $N \geq 3$ and (F1)-(F2). Assume moreover (F8). Then every least energy solution of (3.1.1) has strict constant sign (strictly positive or negative), is radially symmetric and decreasing.*

4.5.1 Positivity through fibers

We want to show now that every Pohozaev ground state has constant sign. This result requires some additional symmetric condition on f.

We start by providing some trivial but useful inequalities, consequence of Lemma 1.4.1.

Lemma 4.5.4. *Let $u \in H^s(\mathbb{R}^N)$. Then*

$$\|(-\Delta)^{s/2}|u|\|_2 \leq \|(-\Delta)^{s/2}u\|_2.$$

Assume moreover that

- *f is odd, or*
- *f is even, and F has constant sign on $(0, +\infty)$,*

then

$$\mathcal{D}(|u|) \geq \mathcal{D}(u);$$

if f is odd, equality holds. As a consequence

$$\mathcal{J}_\mu(|u|) \leq \mathcal{J}_\mu(u), \quad \mathcal{P}_\mu(|u|) \leq \mathcal{P}_\mu(u).$$

To prove the positivity of Pohozaev ground states, we need to get information about the absolute value of the function. This analysis is simplified when dealing with local operators $s = 1$ (since $\|\nabla|u|\|_2 = \|\nabla u\|_2$ and the Pohozaev identity holds for every solution, see [302]), or when dealing with local nonlinearities (since the source scales in the argument in the same way as $|u|^2$ and an equivalent minimization approach can be exploited, see [50]), or when dealing with homogeneous nonlinearities (since another minimization approach holds, see [138, 300]). In order to implement a different approach, we start observing the following fact.

For every $u \in H^s(\mathbb{R}^N)$, $u \not\equiv 0$, we define the fiber $g_u : (0, +\infty) \to \mathbb{R}$ as follows

$$g_u(t) := \mathcal{J}_\mu(u(\cdot/t)) = \frac{t^{N-2s}}{2}\|(-\Delta)^{s/2}u\|_2^2 + \mu\frac{t^N}{2}\|u\|_2^2 - \frac{t^{N+\alpha}}{2}\mathcal{D}(u), \quad t \in (0, +\infty).$$

By a straightforward computation we notice that

$$g_u'(1) = \mathcal{P}_\mu(u).$$

Since $N + \alpha > N > N - 2s$ it is immediate showing that there exists a single critical point for g_u, that we call $\lambda(u)$, which is a global maximum. That is

$$g_u'(\lambda(u)) = 0, \quad g_u(\lambda(u)) \geq g_u(t) \text{ for each } t \in (0, +\infty).$$

Noticed that $\lambda(u) > 0$, we set

$$v := u(\cdot/\lambda(u));$$

by the fact that $g_u(\lambda(u)t) = g_v(t)$, we obtain $g_v'(1) = 0$, that is

$$\mathcal{P}_\mu(v) = 0.$$

In other words, the scaling through $\lambda(u)$ brings u to the Pohozaev manifold; moreover, the energy is maximized in $\lambda(u)$ all over the scaling.

Proposition 4.5.5. *Assume*

- *f is odd, or*
- *f is even, and F has constant sign on $(0, +\infty)$,*

in addition to (F1)-(F2). Assume moreover (F7). Let u be a Pohozaev minimum of (4.1.1). Then u has strict constant sign (strictly positive or negative).

Proof. Since u satisfies $\mathcal{P}_\mu(u) = 0$, we obtain $\lambda(u) = 1$ and thus

$$g_u(t) \leq g_u(1) \quad \text{for each } t \in (0, +\infty). \tag{4.5.44}$$

Consider $|u|$ and $\lambda(|u|)$. Define

$$v := |u|(\cdot/\lambda(|u|))$$

which satisfies $\mathcal{P}_\mu(v) = 0$. Since u is a Pohozaev minimum we obtain

$$\mathcal{J}_\mu(u) \leq \mathcal{J}_\mu(v).$$

We then use Lemma 4.5.4 to gain

$$\begin{aligned}
\mathcal{J}_\mu(u) &\leq \mathcal{J}_\mu(v) \\
&= \frac{(\lambda(|u|))^{N-2s}}{2} \|(-\Delta)^{s/2}|u|\|_2^2 + \mu \frac{(\lambda(|u|))^N}{2} \||u|\|_2^2 - \frac{(\lambda(|u|))^{N+\alpha}}{2} \mathcal{D}(|u|) \\
&\leq \frac{(\lambda(|u|))^{N-2s}}{2} \|(-\Delta)^{s/2}u\|_2^2 + \mu \frac{(\lambda(|u|))^N}{2} \|u\|_2^2 - \frac{(\lambda(|u|))^{N+\alpha}}{2} \mathcal{D}(u) \\
&= g_u(\lambda(|u|)).
\end{aligned}$$

We finally use (4.5.44) with $t = \lambda(|u|)$ and obtain

$$\mathcal{J}_\mu(u) \leq \mathcal{J}_\mu(v) \leq g_u(\lambda(|u|)) \leq g_u(1) = \mathcal{J}_\mu(u).$$

Thus

$$\mathcal{J}_\mu(v) = \mathcal{J}_\mu(u) = p(\mu)$$

which, together with $\mathcal{P}_\mu(v) = 0$, implies that v is also a Pohozaev minimum of (4.1.1). By Proposition 4.3.4 we obtain that v is a weak solution of (4.1.1), positive by definition. Thus by Proposition 4.4.7 we have $v \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$; this implies, by Proposition 4.4.12, that v is a classical solution, and in particular well defined pointwise. Thus, if by contradiction there exists an $x_0 \in \mathbb{R}^N$ such that $v(x_0) = 0$, then computing

$$(-\Delta)^s v(x_0) + \mu v(x_0) = (I_\alpha * F(v))(x_0) f(v(x_0))$$

we obtain, by definition of fractional Laplacian and $f(0) = 0$,

$$-\int_{\mathbb{R}^N} \frac{v(y)}{|x_0 - y|^{N+2s}} \, dy = 0$$

and hence $v \equiv 0$, which is absurd. Thus $|u| \neq 0$. Being $v \in L^\infty(\mathbb{R}^N)$, we obtain $u \in L^\infty(\mathbb{R}^N)$, and hence u continuous by Proposition 4.4.8. As a consequence, u does not change sign. This concludes the proof. ∎

Remark 4.5.6. *We point out that, without assuming (F7), we can achieve*

$$p(\mu) = \inf \{\mathcal{J}_\mu(u) \mid u \in H^s(\mathbb{R}^N) \setminus \{0\}, \; \mathcal{P}_\mu(u) = 0, \; u \text{ positive}\}$$

and the same for $p_r(\mu)$. Indeed, let $(u_n)_n \subset H^s(\mathbb{R}^N) \setminus \{0\}$, $\mathcal{P}_\mu(u_n) = 0$, $\mathcal{J}_\mu(u_n) \to p(\mu)$ be a minimizing sequence. Set $v_n := |u_n|(\cdot/\lambda(|u_n|))$ we have $\mathcal{P}_\mu(u_n) = 0$ and, arguing as in the first part of Proposition 4.5.5, we obtain

$$\lim_{n \to +\infty} \mathcal{J}_\mu(u_n) = p(\mu) \leq \mathcal{J}_\mu(v_n) \leq g_{u_n}(\lambda(|u_n|)) \leq \mathcal{J}_\mu(u_n);$$

thus $\mathcal{J}_\mu(v_n) \to p(\mu)$, which means that v_n is a positive minimizing sequence.

4.5.2 Radial symmetry

The solution found in Theorem 4.2.1 is radially symmetric by construction. We show now that, under some condition on f, every Pohozaev ground state is actually radially symmetric. To this aim, we will exploit the polarization introduced in Section 1.4. We remark that other techniques could be investigated (with different assumptions on f, see e.g. [275, 371]), but this goes beyond the scope of this thesis.

Proposition 4.5.7. *Assume that f has constant sign on $(0,+\infty)$ in addition to (F1)-(F2). Let u be a positive Pohozaev minimum of (4.1.1). Then u is radially symmetric and decreasing with respect to some point.*

Proof. Let u^H be the polarization of u with respect to a closed half-space $H \subset \mathbb{R}^N$. By Proposition 1.4.5 we have

$$\|(-\Delta)^{s/2}u^H\|_2 \leq \|(-\Delta)^{s/2}u\|_2.$$

Assume moreover that $f \geq 0$ on $(0,+\infty)$ (if we substitute f with $-f$ the Hartree-type terms are conserved). Observed that F is nondecreasing on $(0,+\infty)$, we have by (1.4.40)

$$F(v^H) = (F(v))^H \quad \text{whenever } v \geq 0.$$

Thanks to these facts, we can argue as in [302, Section 5.3] to reach that $\mathcal{J}_\mu(u^H) = \mathcal{J}_\mu(u)$, which implies $\mathcal{D}(u^H) \geq \mathcal{D}(u)$; on the other hand, the inverse inequality is always true, and hence $\mathcal{D}(u^H) = \mathcal{D}(u)$; again by the argument in [302] we have the claim. ∎

Corollary 4.5.8. *In the assumptions of Theorem 4.5.1, every Pohozaev minimum of (4.1.1) has constant sign, is radially symmetric and decreasing. Moreover, assuming also (F3)-(F4), we have*

$$p_r(\mu) = p(\mu) = \inf\{\mathcal{J}_\mu(u) \mid u \in H^s_r(\mathbb{R}^N) \setminus \{0\},\ \mathcal{P}_\mu(u) = 0,\ u \text{ positive}\}.$$

Proof Theorems 4.5.1 and 4.5.3. The claim of Theorem 4.5.1 is contained in Corollary 4.5.8. The proof of Theorem 4.5.3 can be obtained arguing in the same way, obtaining regularity of solutions by standard results (see e.g. [302]). ∎

Remark 4.5.9. *In Section 4.2.2 we found a Mountain Pass solution $(\bar{\mu}, \bar{u})$ for the L^2-mass prescribed problem by assuming L^2-subcriticality of the nonlinearity. This solution is a ground state of*

$$\mathcal{L}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)\,dx,$$

restricted to the set

$$\mathcal{S}_m = \{u \in H^s_r(\mathbb{R}^N) \mid \|u\|_2^2 = m\};$$

moreover, this solution $(\bar{\mu}, \bar{u})$ is a minimum over the Pohozaev set in the product space, that is

$$\mathcal{L}(\mu, u) = \inf_{\mathcal{P}_\nu(v)=0} \mathcal{L}(\nu, v).$$

This property easily implies that $\bar{u}$ is a ground state (in the unconstrained case) of $\mathcal{J}_{\bar{\mu}}$ over the Pohozaev set, that is

$$\mathcal{J}_{\bar{\mu}}(\bar{u}) = p_r(\bar{\mu}).$$

Thus, the positivity result in Proposition 4.5.5 applies to $\bar{u}$.

Actually, all the positivity and symmetry results gained in this Section hold also for this constrained mass problem, up to simple adaptations. Indeed, in this case the proof of the positivity is even easier, since

$$u \in \{\|v\|_2^2 = m\} \implies |u| \in \{\|v\|_2^2 = m\},$$

which means that if u is a ground state, then $|u|$ is a ground state as well. We highlight again that this simplified approach can be not implemented in the unconstrained case. In addition, under these symmetric and regularity assumptions (F7)-(F8), also this L^2-minimum is actually an L^2-minimum all over the whole $H^s(\mathbb{R}^N)$ (and not only restricting the functional on $H^s_r(\mathbb{R}^N)$).

4.6 Asymptotic decay

In this Section we exploit the L^1-summability of the solutions to study the asymptotic behaviour of solutions for $|x| \to +\infty$. Recall that $2^{\#}_{\alpha} = \frac{N+\alpha}{N}$ and $2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}$.

When $s = 1$ and $f(u) = |u|^{r-2}u$, that is

$$-\Delta u + \mu u = (I_\alpha * |u|^r)|u|^{r-2}u \quad \text{in } \mathbb{R}^N \tag{4.6.45}$$

Cingolani, Clapp and Secchi in [109, Proposition A.2] obtained an exponential decay of positive solutions whenever $r \geq 2$, which means that the effect of the classical Laplacian prevails. Afterwards, Moroz and Van Schaftingen in [300] (see also [301, 304] and [101, 128]) extended the previous analysis in the case of ground state solutions to all the possible values of r in the range $[2^{\#}_{\alpha}, 2^*_{\alpha,1}]$, in particular by finding a polynomial decay when f is sublinear (i.e., the Choquard term effect prevails). They prove the following result [300, Theorem 4].

Theorem 4.6.1 ([300]). *Let $s = 1$ and let $u \in H^1(\mathbb{R}^N)$ be a nonnegative ground state of (4.6.45), and $r \in [2^{\#}_{\alpha}, 2^*_{\alpha,1}]$. Assume $\mu = 1$. Then*

- *if $r > 2$, then*
$$\lim_{|x|\to+\infty} u(x)|x|^{\frac{N-1}{2}} e^{|x|} \in (0, +\infty);$$
- *if $r = 2$, then*
$$\lim_{|x|\to+\infty} u(x)|x|^{\frac{N-1}{2}} e^{\int_\nu^{|x|} \sqrt{1-\frac{\nu^{N-\alpha}}{t^{N-\alpha}}}\,dt} \in (0, +\infty)$$
for some explicit $\nu = \nu(u)$;
- *if $r < 2$, then*
$$\lim_{|x|\to+\infty} u(x)|x|^{\frac{N-\alpha}{2-r}} = C(N, \alpha, r, u) \in (0, +\infty)$$
where
$$C(N, \alpha, r, u) := \left(C_{N,\alpha}\|u\|_r^r\right)^{\frac{1}{2-r}} \tag{4.6.46}$$
with $C_{N,\alpha} := \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{N/2}\Gamma(\frac{\alpha}{2})}$.

Notice that, when $\mu \neq 1$, μ influences both the limiting constants and the speed of the exponential decays. We refer also to [135, Section 8.2] for some results on convolution equations with non-variational structure.

The case of the fractional Choquard equation $s \in (0, 1)$ with homogeneous f, that is

$$(-\Delta)^s u + \mu u = (I_\alpha * |u|^r)|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \tag{4.6.47}$$

has been studied by D'Avenia, Siciliano and Squassina in [138] (see also [139] and [280, 395] for other related results). In this paper the authors gain existence of ground states, multiplicity and qualitative properties of solutions. In particular they obtain asymptotic decay of solutions whenever the source is linear or superlinear, that is when $r \geq 2$ (see also [41] for the p-fractional Laplacian counterpart): in this case the rate is polynomial, as one can expect dealing with the fractional Laplacian; more specifically, it does not depend on α, and they prove the following theorem.

Theorem 4.6.2 ([138]). *Let $u \in H^s(\mathbb{R}^N)$ be a solution of (4.6.47), and assume $r \in [2, 2^*_{\alpha,s}]$. Then*

$$0 < \liminf_{|x|\to+\infty} |u(x)||x|^{N+2s} \leq \limsup_{|x|\to+\infty} |u(x)||x|^{N+2s} < +\infty.$$

In this Section we study the asymptotic profile of solutions of equation (4.1.1), starting by the case f linear or superlinear. In the remeaning part we will develop the more tricky case of f sublinear: the found decay is of polynomial type, with a rate possibly slower than $\sim \frac{1}{|x|^{N+2s}}$; the result is new even for homogeneous functions $f(u)=|u|^{r-2}u$, $r\in[\frac{N+\alpha}{N},2)$, and, differently from the local case $s=1$ in [300], new phenomena arise connected to a new s-sublinear threshold that we detect on r.

This Section is mainly based on papers [115] and [198].

We show first some conditions which imply the decay at infinity of the solutions.

Lemma 4.6.3. *Assume that (F1)-(F2) hold. Let $u\in H^s(\mathbb{R}^N)$ be a weak solution of (4.1.1). Assume*

$$(I_\alpha * F(u))f(u)\in L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N).$$

Then we have

$$u(x)\to 0 \quad \text{as } |x|\to+\infty.$$

Proof. Being u solution of

$$(-\Delta)^s u+\mu u=\big(I_\alpha * F(u)\big)f(u)=:\chi \quad \text{in } \mathbb{R}^N,$$

where $\chi\in L^2(\mathbb{R}^N)\cap L^{\frac{N}{2s}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, we have the representation formula (being $\chi\in L^2(\mathbb{R}^N)$)

$$u=\mathcal{K}*\chi$$

where $\mathcal{K}:=\mathcal{K}_{2s,\mu}$ is the Bessel kernel; we recall that $\mathcal{K}$ is positive, it satisfies $\mathcal{K}(x)\le\frac{C}{|x|^{N+2s}}$ for $|x|\ge 1$ and $\mathcal{K}\in L^q(\mathbb{R}^N)$ for $q\in[1,1+\frac{2s}{N-2s})$ (see Lemma 1.2.29). Let us fix $\eta>0$; we have, for $x\in\mathbb{R}^N$,

$$\begin{aligned} u(x)&=\int_{\mathbb{R}^N}\mathcal{K}(x-y)\chi(y)dy\\ &=\int_{|x-y|\ge 1/\eta}\mathcal{K}(x-y)\chi(y)dy+\int_{|x-y|<1/\eta}\mathcal{K}(x-y)\chi(y)dy. \end{aligned}$$

As regards the first piece

$$\int_{|x-y|\ge 1/\eta}\mathcal{K}(x-y)\chi(y)dy\le\|\chi\|_\infty\int_{|x-y|\ge 1/\eta}\frac{C}{|x-y|^{N+2s}}dy\le C\eta^{2s}$$

while for the second piece, fixed a whatever $q\in(1,\min\{2,\frac{N}{N-2s}\})$ and its conjugate exponent $q'\in(\max\{2,\frac{N}{2s}\},+\infty)$ we have by Hölder inequality

$$\int_{|x-y|<1/\eta}\mathcal{K}(x-y)\chi(y)dy\le\|\mathcal{K}\|_q\|\chi\|_{L^{q'}(B_{1/\eta}(x))}$$

where the second factor can be made small for $|x|\gg 0$. Joining the pieces, we conclude the proof. ∎

Notice that $u\in L^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ implies the assumptions of Lemma 4.6.3.

4.6.1 The (super)linear case

By assuming the condition in zero (F6) for the function f, we obtain the following polynomial decay, as stated in paper [115].

Theorem 4.6.4. *Assume (F1)-(F2) and (F6). Let $u \in H^s(\mathbb{R}^N)$ be a positive weak solution of (4.1.1). Then there exists $C', C'' > 0$ such that*

$$\frac{C'}{1+|x|^{N+2s}} \leq u(x) \leq \frac{C''}{1+|x|^{N+2s}}, \qquad \text{for } x \in \mathbb{R}^N.$$

We are now ready to prove the polynomial decay of the solutions.

Proof of Theorem 4.6.4. Observe that, by (F6) and Theorem 4.4.1

$$\frac{f(u)}{u} \in L^\infty(\mathbb{R}^N). \tag{4.6.48}$$

Moreover, by Proposition 4.4.6 we obtain

$$(I_\alpha * F(u))(x)\frac{f(u(x))}{u(x)} \to 0 \quad \text{ as } |x| \to +\infty. \tag{4.6.49}$$

As a consequence, by (4.6.49) and the positivity of u, we have for some $R' \gg 0$

$$(-\Delta)^s u + \tfrac{1}{2}\mu u = (I_\alpha * F(u))f(u) - \tfrac{1}{2}\mu u = \Big((I_\alpha * F(u))\tfrac{f(u)}{u} - \tfrac{1}{2}\mu\Big) u \leq 0 \quad \text{in } \mathbb{R}^N \setminus B_{R'}.$$

Similarly

$$(-\Delta)^s u + \tfrac{3}{2}\mu u = (I_\alpha * F(u))f(u) + \tfrac{1}{2}\mu u = \Big((I_\alpha * F(u))\tfrac{f(u)}{u} + \tfrac{1}{2}\mu\Big) u \geq 0 \quad \text{in } \mathbb{R}^N \setminus B_{R'}.$$

Notice that we always intend differential inequalities in the weak sense.

In addition, by Lemma 1.2.30 we have that there exist two positive functions $\underline{W}'$, $\overline{W}'$ and three positive constants R'', C' and C'' depending only on μ, such that

$$\begin{cases} (-\Delta)^s \underline{W}' + \dfrac{3}{2}\mu \underline{W}' = 0 & \text{in } \mathbb{R}^N \setminus B_{R''}, \\ \dfrac{C'}{|x|^{N+2s}} < \underline{W}'(x), & \text{for } |x| > 2R''. \end{cases}$$

and

$$\begin{cases} (-\Delta)^s \overline{W}' + \dfrac{1}{2}\mu \overline{W}' = 0 & \text{in } \mathbb{R}^N \setminus B_{R''}, \\ \overline{W}'(x) < \dfrac{C''}{|x|^{N+2s}}, & \text{for } |x| > 2R''. \end{cases}$$

Set $R := \max\{R', 2R''\}$. Let $\underline{C}_1$ and $\overline{C}_1$ be some lower and upper bounds for u on B_R, $\underline{C}_2 := \min_{B_R} \overline{W}'$ and $\overline{C}_2 := \max_{B_R} \underline{W}'$, all strictly positive. Define

$$\underline{W} := \underline{C}_1 \overline{C}_2^{-1} \underline{W}', \quad \overline{W} := \overline{C}_1 \underline{C}_2^{-1} \overline{W}'$$

so that

$$\underline{W}(x) \leq u(x) \leq \overline{W}(x), \qquad \text{for } |x| \leq R.$$

Thanks to the comparison principle in Lemma 1.2.34, and redefining C' and C'', we obtain

$$\frac{C'}{|x|^{N+2s}} < \underline{W}(x) \leq u(x) \leq \overline{W}(x) < \frac{C''}{|x|^{N+2s}}, \qquad \text{for } |x| > R.$$

By the boundedness of u, we obtain the claim. ∎

We see that, for non sublinear f (that is, (F6)), the decay is essentially given by the fractional operator. It is important to remark that, contrary to the limiting local case $s = 1$ (Theorem 4.6.1), the Choquard term in case of linear f seems not to affect the decay of the solution.

Remark 4.6.5. *We observe that the conclusion of the proof of Theorem 4.6.4 can be substituted by exploiting the results in [190] through a Kato's inequality (see also [19, Theorem 3.2]). Indeed write* $V := -(I_\alpha * F(u))\frac{f(u)}{u}$*, which is bounded and zero at infinity as observed in* (4.6.48)–(4.6.49)*, and gain*

$$(-\Delta)^s u + V(x)u = -\mu u \quad \text{in } \mathbb{R}^N.$$

Up to dividing for $\|u\|_2$*, we may assume* $\|u\|_2 = 1$*. Thus we are in the assumptions of [190, Lemma C.2] and obtain, for constant sign or sign-changing solutions of* (4.1.1)*,*

$$|u(x)| \leq \frac{C_1}{(1+|x|^2)^{\frac{N+2s}{2}}}$$

together with

$$|u(x)| = \frac{C_2}{|x|^{N+2s}} + o\left(\frac{1}{|x|^{N+2s}}\right) \qquad \text{as } |x| \to +\infty$$

for some $C_1, C_2 > 0$*.*

4.6.2 The sublinear case: fractional Laplacian versus Riesz potential

We focus now on the case f sublinear: we aim to study the fractional Choquard case $s \in (0,1)$, $\alpha \in (0,N)$, in presence of general, sublinear nonlinearities. We point out that the arguments in [300] cannot be directly adapted to the fractional framework: for instance, we see that the explicit computation of the fractional Laplacian of some comparison function is not possible, and the choice of the comparison functions itself is hindered by some growth condition typical of the nonlocal framework; moreover, it is not obvious that all the weak solutions are pointwise solutions, and neither one can deduce that the concave power of a pointwise solution is indeed a solution (of a different equation) itself.

We start by presenting the case of homogeneous powers f, which has an interest on its own. Since in the superlinear case the rate of convergence is of the type $\sim \frac{1}{|x|^{N+2s}}$, in the sublinear case we generally expect a slower decay. Actually this is what we find, as the following theorem states.

Theorem 4.6.6. *Let* $u \in H^s(\mathbb{R}^N)$*, strictly positive, radially symmetric and decreasing, be a weak solution of* (4.6.47)*. Let* $r \in [2^\#_\alpha, 2)$ *and set*

$$\beta := \min\left\{\frac{N-\alpha}{2-r}, N+2s\right\} \geq N.$$

Then

$$0 < \liminf_{|x|\to+\infty} u(x)|x|^\beta \leq \limsup_{|x|\to+\infty} u(x)|x|^\beta < +\infty.$$

We refer to Remark 4.6.12 and Corollary 4.6.32 for some comments and generalizations on the assumptions. This result in particular applies to ground states solutions.

Corollary 4.6.7. *Let* u *be a positive ground state of* (4.6.47)*. Then the conclusions of Theorem 4.6.6 hold.*

We highlight that the found decay of the ground states might give information, when $r < 2$, also on the twice Gateaux differentiability of the corresponding functional and on the nondegeneracy of the ground state solution itself, see [300] (see also [304, Section 3.3.5]). Moreover this information on the decay may be exploited to study fractional Choquard equations with potentials $V = V(x)$ approaching, as $|x| \to +\infty$, some $V_\infty > 0$ from above or oscillating, in the spirit of [282, 283]. It might be further used, for example, in the semiclassical analysis of concentration phenomena, see e.g. Chapter 5.

Joining the results in Theorem 4.6.2 and Theorem 4.6.6 we obtain the following picture of the asymptotic decay of fractional Choquard equations.

Corollary 4.6.8. *Let u be a positive ground state of (4.6.47), with $r \in [2^{\#}_{\alpha}, 2^*_{\alpha,s}]$ and $\mu > r-1$.*

- *If $r \in [2^{\#}_{\alpha}, \frac{N+\alpha+4s}{N+2s}]$, then*

$$0 < \liminf_{|x|\to+\infty} u(x)|x|^{\frac{N-\alpha}{2-r}} \leq \limsup_{|x|\to+\infty} u(x)|x|^{\frac{N-\alpha}{2-r}} < +\infty;$$

in particular, $\frac{N-\alpha}{2-r} = N$ in the lower critical case $r = 2^{\#}_{\alpha}$.

- *If $r \in [\frac{N+\alpha+4s}{N+2s}, 2^*_{\alpha,s}]$, then*

$$0 < \liminf_{|x|\to+\infty} u(x)|x|^{N+2s} \leq \limsup_{|x|\to+\infty} u(x)|x|^{N+2s} < +\infty.$$

By the previous Corollary we see that the exponent

$$r^*_{\alpha,s} := \frac{N+\alpha+4s}{N+2s},$$

$r^*_{\alpha,s} \in (2^{\#}_{\alpha}, 2)$, separates the cases where the fractional Laplacian influences more the rate of convergence (which does not depend on α), from the cases where the asymptotic behaviour is dictated by the Choquard term (which does not depend on s). This phenomenon seems to highlight a difference between the fractional and the local case, where the separating exponent is $r = 2$ (see Theorem 4.6.1): indeed, when $r \in (r^*_{\alpha,1}, 2)$, the arbitrary big (as $r \to 2$) polynomial behaviour $\sim \frac{1}{|x|^{\frac{N-\alpha}{2-r}}}$ keeps being slower than the exponential decay induced by the classical Laplacian; this is not the case when compared with the polynomial decay induced by the fractional Laplacian, and this is why this new phenomenon appears in this range. Thus $r^*_{\alpha,s}$ can be seen as a kind of *s-subquadratic* threshold for the growth of F; set instead

$$p^*_{\alpha,s} := r^*_{\alpha,s} - 1 = \frac{\alpha+2s}{N+2s},$$

it can be seen as a *s-sublinear* threshold for the growth of f. Notice that

$$r^*_{\alpha,s} \overset{s\to 0}{\to} 2^{\#}_{\alpha}, \quad r^*_{\alpha,s} \overset{\alpha\to N}{\to} 2,$$

while

$$r^*_{\alpha,s} \overset{s\to 1}{\to} \frac{N+\alpha+4}{N+2} \in (2^{\#}_{\alpha}, 2), \quad r^*_{\alpha,s} \overset{\alpha\to 0}{\to} \frac{N+4s}{N+2s} \in (1,2).$$

We refer also to the recent paper [209, Theorem 1.4] where asymptotic decay results are studied in a different framework (still involving the fractional Laplacian and the Riesz potential); here a threshold different from the classical case $s = 1$ is detected as well.

When $r \in [2^{\#}_{\alpha}, r^*_{\alpha,s})$ we are also able to find a *sharp decay* for u.

Corollary 4.6.9. *Let $u \in H^s(\mathbb{R}^N)$, strictly positive, radially symmetric and decreasing, be a weak solution of (4.6.47); in particular, u may be a ground state. If $r \in [2^{\#}_{\alpha}, r^*_{\alpha,s})$, we have*

$$\lim_{|x|\to+\infty} u(x)|x|^{\frac{N-\alpha}{2-r}} = \left(\frac{C_{N,\alpha}\|u\|_r^r}{\mu}\right)^{\frac{1}{2-r}};$$

notice that, if $\mu = 1$, the constant is coherent with (4.6.46).

We finally highlight that, for $s \in (0,1]$, the rate of convergence of the solutions for $r \leq r^*_{\alpha,s}$ is $\sim \frac{1}{|x|^{\frac{N-\alpha}{2-r}}}$: for bigger values of r, the rate stabilizes to $\sim \frac{1}{|x|^{N+2s}}$ when $s < 1$, while it keeps getting faster when $s = 1$ (up to the threshold $r = 2$, where it gets constantly exponential). It might be interesting to investigate other possible phenomena on fractional Choquard equations when r is above and below this exponent $r^*_{\alpha,s}$, or also possible phenomena in $(r^*_{\alpha,1}, 2)$ for the local Choquard equation.

Remark 4.6.10. *We notice that, fixed a positive solution u, by setting*

$$\rho := I_\alpha * u^r$$

equation (4.6.47) can be rewritten as

$$(-\Delta)^s u + \mu u = \rho(x) u^{r-1}.$$

When $\mu = 0$ and $\rho(x) \leq \frac{1}{|x|^\gamma}$ with $\gamma > N$, this fractional sublinear equation ($r \in (0,2)$) has been studied in [321] (see also [211, Theorem 4.4] where they extend the result to $\gamma > 2s$): here the authors find an estimate from above of the asymptotic decay of the solutions, which is strictly slower than $\sim \frac{1}{|x|^N}$. Notice that, in our case, $\rho = I_\alpha * u^r$ decays at most as $\sim \frac{1}{|x|^{N-\alpha}}$ (see (1.3.34)), and we discuss the strict positive mass case $\mu > 0$. See also [138, 253] for more results on the zero mass case.

We pass now to more general nonlinearities, and study (4.1.1). We will assume (F1)-(F2), which in particular imply

$$i) \ \limsup_{t \to 0} \frac{|F(t)|}{|t|^{2^\#_\alpha}} < +\infty, \quad ii) \ \limsup_{|t| \to +\infty} \frac{|F(t)|}{|t|^{2^*_{\alpha,s}}} < +\infty, \tag{4.6.50}$$

or equivalently that there exists $C > 0$ such that for every $t \in \mathbb{R}$,

$$|F(t)| \leq C\big(|t|^{2^\#_\alpha} + |t|^{2^*_{\alpha,s}}\big).$$

In addition we consider f *sublinear in the origin*, given by the following assumptions:

(F9) there exists $r \in [2^\#_\alpha, 2)$ such that

$$\limsup_{t \to 0^+} \frac{|f(t)|}{t^{r-1}} \in [0, +\infty),$$

i.e., for some $\bar{C} > 0$ and $\delta \in (0,1)$ we have

$$|f(t)| \leq \bar{C} t^{r-1} \quad \text{for } t \in (0, \delta); \tag{4.6.51}$$

(F10) there exists $r \in [2^\#_\alpha, 2)$ such that

$$\liminf_{t \to 0^+} \frac{f(t)}{t^{r-1}} \in (0, +\infty),$$

i.e., for some $\underline{C} > 0$ and $\delta \in (0,1)$ we have

$$f(t) \geq \underline{C} t^{r-1} \quad \text{for } t \in (0, \delta). \tag{4.6.52}$$

A sufficient condition for (F9) is clearly given by

$$\limsup_{t \to 0^+} \frac{f(t)}{t^{r-1}} = 0 \quad \text{for some } r \in [2^\#_\alpha, 2), \tag{4.6.53}$$

which means that $\bar{C}$ can be taken arbitrary small in (4.6.51); in particular it includes logarithmic nonlinearities $f(t) = t \log(t^2)$, where r can be chosen arbitrary close to 2. A sufficient condition for (F10) is instead given (for example) by a local Ambrosetti-Rabinowitz condition of the type

$$f(t)t \geq rF(t) > 0 \quad \text{for } t \in (0, \delta).$$

The restriction in (F9) and (F10) to right neighborhoods of zero is due to the fact we deal with positive solutions.

We eventually come up with the following generalization of Theorem 4.6.6.

Theorem 4.6.11. *Assume (F1)-(F2), and let $u \in H^s(\mathbb{R}^N)$, strictly positive, radially symmetric and decreasing, be a weak solution of (4.1.1). Let $r \in [2^\#_\alpha, 2)$ and set*

$$\beta := \min\left\{\frac{N-\alpha}{2-r}, N+2s\right\} \geq N.$$

(i) *Assume (F9). Then*

$$\limsup_{|x|\to+\infty} u(x)|x|^\beta \in (0,+\infty).$$

(ii) *Assume (F10), f locally Hölder continuous and $\int_{\mathbb{R}^N} F(u) > 0$ (e.g. $F \geq 0$ on $(0,+\infty)$). Then*

$$\liminf_{|x|\to+\infty} u(x)|x|^\beta \in (0,+\infty).$$

If both conditions in (i) and (ii) hold, together with $\overline{C} = \underline{C}$ (i.e., f is a power near the origin) and $r \in [\frac{N+\alpha}{N}, \frac{N+\alpha+4s}{N+2s})$, then we have the sharp decay

$$\lim_{|x|\to+\infty} u(x)|x|^{\frac{N-\alpha}{2-r}} = \left(\frac{C_{N,\alpha}\big(\lim_{t\to 0^+}\frac{f(t)}{t^{r-1}}\big)\int_{\mathbb{R}^N} F(u)}{\mu}\right)^{\frac{1}{2-r}} \tag{4.6.54}$$

where $C_{N,\alpha} > 0$ is given in (4.6.46).

Remark 4.6.12. *We highlight that the conclusions of Theorem 4.6.11 (as well as of Theorem 4.6.6) hold in more general cases. Indeed:*

- *The case*

$$\lim_{t\to 0^+}\frac{f(t)}{t} = +\infty$$

in a non-strict sense (i.e. $\lim_{t\to 0}\frac{|f(t)|}{|t|^{r-1}} = 0$ for each $r \in [1+\frac{\alpha}{N}, 2)$, for example $f(t) \sim -t\log(t^2)$) is included, and as we expect the decay is of order $\sim \frac{1}{|x|^{N+2s}}$. See Corollary 4.6.31.

- *The conclusions hold also without assuming radial symmetry and monotonicity of u, but by assuming a priori that*

$$\limsup_{|x|\to+\infty} |u(x)||x|^\omega < +\infty$$

for some $\omega > \frac{N^2}{N+\alpha}$: see Remark 4.6.19. When $u \in L^q(\mathbb{R}^N)$, $q < \frac{N+\alpha}{N}$, is radially symmetric and decreasing, this is the case with $\omega = \frac{N}{q}$ (see Remark 4.6.17); in particular, if $q = 1$, we have $\omega = N$. Notice that u is automatically radially symmetric and decreasing when $u \in C^{1,1}_{loc}(\mathbb{R}^N)$, $f(u) = |u|^{r-2}u$ and $\omega > \frac{\alpha}{r-1}$ thanks to [254, Theorem 1] (see also [371, Theorem 1.3]).

- *In light of the previous remark, we highlight that the estimate from above actually holds true also for nonnegative solutions $u \geq 0$; see Proposition 4.6.23; moreover, it can be further extended to $|u|$ in the case of changing sign solutions, by applying a Kato's inequality [19, Theorem 3.2].*

- *The conclusions hold also for solutions $u \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ in the viscosity sense, without assuming f Hölder continuous (which is needed in (ii) only to pass from weak to viscosity solutions): see Section 4.6.6.*

- *When (F10) holds, we actually have $F(t) \geq \underline{C}\frac{t^r}{r}$ for $t \in (0,\delta)$; thus, being also $u \in L^\infty(\mathbb{R}^N)$, the condition $\int_{\mathbb{R}^N} F(u) > 0$ means that F is not* too negative *in $[\delta, \|u\|_\infty]$. We highlight that the energy term $\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)$ is always positive (see e.g. Proposition 1.3.2).*

- *We find some estimates on the asymptotic constants, which are coherent, when $r \in [2^\#_\alpha, r^*_{\alpha,s})$, with the one found in Theorem 4.6.1 and Theorem 4.6.11: see Propositions 4.6.23 and 4.6.26, and Corollary 4.6.9. We notice that (4.6.47) is obtained by (4.1.1) formally choosing $f(t) = \sqrt{r}|t|^{r-2}t$. In our proofs – up to well posedness and regularity – we do not use that F is the primitive of f: in particular, we do not apply (F9) and (F10) to F. Thus we can arbitrary move constants from f to F in our arguments to adjust – for example – the value of $\underline{C}$, and this allows to gain the result for every $\mu > 0$ (see also Corollary 4.6.30).*

Our results apply in particular to Pohozaev minima of the equation, whenever some symmetric assumption is assumed on f, that is (F7)-(F8). We notice that, since every Pohozaev minimum has constant sign, it is not restrictive to assume a priori the sign of u.

Corollary 4.6.13. *Assume (F1)-(F2) and (F7)-(F8). Let u be a (positive) Pohozaev minimum of (4.1.1). Then the conclusions of Theorem 4.6.11 hold.*

We finally want to highlight that our results may be adapted to the local case $s = 1$, extending Theorem 4.6.1 to general nonlinearities, studied in [302]. We leave the details to the reader, observing that in this case the rate of decaying is simply given by $\beta = \frac{N-\alpha}{2-r}$, since, as already observed, the solutions of the homogeneous linear (associated) equation decay exponentially.

Theorem 4.6.14. *Let $s = 1$ and $N \geq 3$, and assume (F1)-(F2). Let $u \in H^1(\mathbb{R}^N)$, strictly positive, radially symmetric and decreasing, be a solution of (3.1.1); in particular, u may be a ground state. Let $r \in [2^\#_\alpha, 2)$.*

(i) *Assume (F9) and $\mu > (r-1)\bar{C}^{\frac{1}{r-1}}$. Then*

$$\limsup_{|x|\to+\infty} u(x)|x|^{\frac{N-\alpha}{2-r}} \in (0, +\infty).$$

(ii) *Assume (F10) and $\int_{\mathbb{R}^N} F(u) > 0$ (e.g. $F \geq 0$ on $(0, +\infty)$). Then*

$$\liminf_{|x|\to+\infty} u(x)|x|^{\frac{N-\alpha}{2-r}} \in (0, +\infty).$$

If both conditions (i) and (ii) hold, together with $\overline{C} = \underline{C}$, then (4.6.54) holds.

In both the estimates from above and below in Theorem 4.6.11 we rely on some comparison principle and the use of some auxiliary function whose fractional Laplacian is related to the Gauss hypergeometric function. For the estimate from above we succeed in working with the weak formulation of the problem; on the other hand, in order to deal with the estimate from below, we find the necessity of working with u^{2-r}, where $2-r \in (0,1)$: this concave power of the solution may fail to lie in $H^s(\mathbb{R}^N)$, and thus we cannot treat the problem with its weak formulation. The pointwise formulation seems to arise some problems as well, since the fractional Laplacian of u^{2-r} needs some restrictive assumption on α, s, N and r in order to be well defined. This is why we work with a viscosity formulation of the problem: in this case, to pass from weak to viscosity solutions, we ask only a bit of Hölder regularity on f. We remark that the estimate from above may be treated with the viscosity formulation as well.

The remaining part of the Chapter is organized as follows. In Section 4.6.3 we recall the suitable auxiliary function introduced in Section 1.2.2, and establish some asymptotic behaviour on suitable comparison functions; other preliminary estimates are studied in Section 4.6.4. Then in Section 4.6.5 we deal with the estimate from above, by working with the weak formulation, while in Section 4.6.6 we study the asymptotic behaviour from below, by exploiting a viscosity formulation. Finally in Section 4.6.7 we conclude the proofs of Theorem 4.6.11 and its corollaries.

4.6.3 Fractional auxiliary functions

In order to implement some comparison argument, in Section 1.2.2 we introduced the function

$$h_\beta(x) := \frac{1}{(1+|x|^2)^{\frac{\beta}{2}}},$$

which behaves, at infinity, like $\sim \frac{1}{|x|^\beta}$, $\beta > 0$, but lies in $H^s(\mathbb{R}^N)$, avoiding the pole in the origin when $\beta \geq N$. This function verifies

$$(-\Delta)^s h_\beta(x) = C_{\beta,N,s}\, {}_2F_1\left(\frac{N}{2}+s, \frac{\beta}{2}+s, \frac{N}{2}; -|x|^2\right)$$

where $C_{\beta,N,s} := 2^{2s}\frac{\Gamma\left(\frac{N}{2}+s\right)\Gamma\left(\frac{\beta}{2}+s\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{\beta}{2}\right)} > 0$ and ${}_2F_1$ denotes the Gauss hypergeometric function. Notice that we will be interested in $\beta \in (0, N+2s]$. In Section 1.2.2 we collected some results on Gauss hypergeometric functions and their asymptotic behaviour at infinity. We use now this auxiliary function to study some comparison at infinity.

Lemma 4.6.15 (Comparison for weak equation)**.** *Let $u \in C(\mathbb{R}^N)$ be a weak solution of*

$$(-\Delta)^s u + \lambda u = \gamma h_\beta \quad \text{in } \mathbb{R}^N \setminus B_\rho(0)$$

for some $\lambda, \gamma > 0$, $\rho > 0$ and

$$\beta \in \left(\frac{N}{2}, N+2s\right].$$

Then

$$\limsup_{|x|\to+\infty} u(x)|x|^\beta < \infty.$$

Moreover, if $\beta \in (\frac{N}{2}, N+2s)$, we have

$$\lim_{|x|\to+\infty} u(x)|x|^\beta = \frac{\gamma}{\lambda}.$$

Proof. We start noticing that, since $\beta > \frac{N}{2}$, then the equation is well posed from a weak point of view. By Lemma 1.2.30 there exists a continuous function $w \in H^{2s}(\mathbb{R}^N)$, such that

$$(-\Delta)^s w + \lambda w = 0 \quad \text{in } \mathbb{R}^N \setminus B_\rho(0)$$

in the weak sense and pointwise, and moreover, for some $C_1'', C_2'' > 0$,

$$\frac{C_1''}{|x|^{N+2s}} < w(x) \leq \frac{C_2''}{|x|^{N+2s}}, \quad \text{for every } |x| > \rho.$$

Let thus define, for some $\tau, \sigma \in \mathbb{R}$ and $\theta \in [\beta, N+2s]$ to be chosen,

$$v_{\tau,\sigma}(x) := \frac{\gamma}{\lambda} h_\beta(x) + \sigma h_\theta(x) + \tau w(x)$$

for every $x \in \mathbb{R}^N$. We have, for $|x| > \rho$,

$$\begin{aligned}(-\Delta)^s v_{\tau,\sigma}(x) + \lambda v_{\tau,\sigma}(x) &= \gamma h_\beta(x) + \left(\frac{\gamma}{\lambda}(-\Delta)^s h_\beta(x) + \sigma(-\Delta)^s h_\theta(x) + \lambda\sigma h_\theta(x)\right) \\ &=: \gamma h_\beta(x) + g_{\sigma,\theta}(x).\end{aligned}$$

By Lemma 1.2.13 we obtain

- if $\beta \in (\frac{N}{2}, N) \setminus \{N-2s\}$,

$$g_{\sigma,\theta}(x) \sim \frac{\gamma}{\lambda} C'_{\beta,N,s} h_{\beta+2s}(x) + \sigma C'_{\theta,N,s} h_{\theta+2s}(x) + \lambda\sigma h_\theta(x) \quad \text{as } |x| \to +\infty;$$

  in this case we assume $\theta \in (\beta, \min\{N, \beta+2s\}) \setminus \{N-2s\}$;

- if $\beta = N$,

$$g_{\sigma,\theta}(x) \sim \frac{\gamma}{\lambda} C'_{N,N,s} \log(x) h_{N+2s}(x) + \sigma C'_{\theta,N,s} h_{N+2s}(x) + \lambda\sigma h_\theta(x) \quad \text{as } |x| \to +\infty;$$

  in this case we assume $\theta \in (N, N+2s)$;

- otherwise

$$g_{\sigma,\theta}(x) \sim \frac{\gamma}{\lambda} C'_{\beta,N,s} h_{N+2s}(x) + \sigma C'_{\theta,N,s} h_{N+2s}(x) + \lambda\sigma h_\theta(x) \quad \text{as } |x| \to +\infty,$$

  and in this case

  - if $\beta = N-2s$ (possible only if $N > 4s$), we choose $\theta \in (N, N+2s)$;
  - if $\beta \in (N, N+2s)$, we choose $\theta \in (\beta, N+2s)$;
  - if $\beta = N+2s$, we simply assume $\theta = N+2s$.

Assume first $\beta < N+2s$. By the abovementioned choices of $\theta > \beta$ we obtain

$$g_{\sigma,\theta}(x) \sim \lambda\sigma h_\theta(x) \quad \text{as } |x| \to +\infty.$$

In particular, fixed $\varepsilon > 0$, for some $R = R_\varepsilon(\gamma, \lambda, \beta, \theta, \sigma) \gg 0$ (we may assume $R > \rho$) we obtain

$$(1-\varepsilon)\lambda\sigma h_\theta(x) \leq g_{\sigma,\theta}(x) \leq (1+\varepsilon)\lambda\sigma h_\theta(x) \quad \text{for } |x| \geq R$$

if $\sigma > 0$, and

$$(1+\varepsilon)\lambda\sigma h_\theta(x) \leq g_{\sigma,\theta}(x) \leq (1-\varepsilon)\lambda\sigma h_\theta(x) \quad \text{for } |x| \geq R$$

if $\sigma < 0$. Notice that R does not depend on τ. Thus

$$(-\Delta)^s v_{\tau,\overline{\sigma}}(x) + \lambda v_{\tau,\overline{\sigma}}(x) \geq \gamma h_\beta(x) + (1-\varepsilon)\lambda\overline{\sigma} h_\theta(x) \geq \gamma h_\beta(x) \quad \text{in } \mathbb{R}^N \setminus B_R(0)$$

by choosing a whatever $\overline{\sigma} > 0$, and

$$(-\Delta)^s v_{\tau,\underline{\sigma}}(x) + \lambda v_{\tau,\underline{\sigma}}(x) \leq \gamma h_\beta(x) + (1-\varepsilon)\lambda\underline{\sigma} h_\theta(x) \leq \gamma h_\beta(x) \quad \text{in } \mathbb{R}^N \setminus B_R(0)$$

by choosing a whatever $\underline{\sigma} < 0$. Summing up

$$\begin{cases} (-\Delta)^s v_{\tau,\overline{\sigma}}(x) + \lambda v_{\tau,\overline{\sigma}}(x) \geq \gamma h_\beta(x) & \text{in } \mathbb{R}^N \setminus B_R(0), \\ (-\Delta)^s v_{\tau,\underline{\sigma}}(x) + \lambda v_{\tau,\underline{\sigma}}(x) \leq \gamma h_\beta(x) & \text{in } \mathbb{R}^N \setminus B_R(0). \end{cases} \tag{4.6.55}$$

We choose now $\overline{\tau} > 0$ such that

$$v_{\overline{\tau},\overline{\sigma}} - u \geq 0 \quad \text{on } B_R(0).$$

Indeed, we impose

$$\frac{\gamma}{\lambda} h_\beta(x) + \overline{\sigma} h_\theta(x) + \tau w(x) \geq u(x) \quad \text{on } B_R(0)$$

that is

$$\tau w(x) \geq u(x) - \frac{\gamma}{\lambda} h_\beta(x) - \overline{\sigma} h_\theta(x) \quad \text{on } B_R(0)$$

which is satisfied if we impose (recall that $\overline{\sigma} > 0$)

$$\tau \min_{B_R} w \geq \max_{B_R} u - \frac{\gamma}{\lambda} h_\beta(R) \geq u(x) - \frac{\gamma}{\lambda} h_\beta(x) - \overline{\sigma} h_\theta(x) \quad \text{on } B_R(0)$$

that is

$$\overline{\tau} \geq \frac{\max_{B_R} u - \frac{\gamma}{\lambda} h_\beta(R)}{\min_{B_R} w}.$$

Similarly, we choose $\underline{\tau} \in \mathbb{R}$ such that

$$v_{\underline{\tau},\underline{\sigma}} - u \leq 0 \quad \text{on } B_R(0),$$

given by

$$\underline{\tau} \leq \frac{\min_{B_R} u - \frac{\gamma}{\lambda} h_\beta(R)}{\max_{B_R} w}.$$

We notice that both the minimum and the maximum of w in the ball are finite and strictly positive, since $w > 0$ is continuous. Thus, summing up

$$\begin{cases} v_{\overline{\tau},\overline{\sigma}} - u \geq 0 & \text{on } B_R(0), \\ v_{\underline{\tau},\underline{\sigma}} - u \leq 0 & \text{on } B_R(0). \end{cases} \tag{4.6.56}$$

By joining (4.6.55) with the assumption on u, we obtain

$$\begin{cases} (-\Delta)^s (v_{\overline{\tau},\overline{\sigma}} - u)(x) + \lambda (v_{\overline{\tau},\overline{\sigma}} - u)(x) \geq 0 & \text{in } \mathbb{R}^N \setminus B_R(0), \\ (-\Delta)^s (v_{\underline{\tau},\underline{\sigma}} - u)(x) + \lambda (v_{\underline{\tau},\underline{\sigma}} - u)(x) \leq 0 & \text{in } \mathbb{R}^N \setminus B_R(0). \end{cases} \tag{4.6.57}$$

By the weak version of the Comparison Principle (Lemma 1.2.34) we obtain

$$\begin{cases} v_{\overline{\tau},\overline{\sigma}} - u \geq 0 & \text{on } \mathbb{R}^N, \\ v_{\underline{\tau},\underline{\sigma}} - u \leq 0 & \text{on } \mathbb{R}^N. \end{cases}$$

that is

$$\frac{\gamma}{\lambda} h_\beta(x) + \underline{\sigma} h_\theta(x) + \underline{\tau} w(x) \leq u(x) \leq \frac{\gamma}{\lambda} h_\beta(x) + \overline{\sigma} h_\theta(x) + \overline{\tau} w(x)$$

and hence, by the assumption on w,

$$\frac{\gamma}{\lambda} h_\beta(x) + \underline{\sigma} h_\theta(x) + \underline{\tau} \frac{C_1''}{|x|^{N+2s}} \leq u(x) \leq \frac{\gamma}{\lambda} h_\beta(x) + \overline{\sigma} h_\theta(x) + \overline{\tau} \frac{C_2''}{|x|^{N+2s}}$$

for each $x \in \mathbb{R}^N$, $x \neq 0$. Thus

$$\begin{aligned} \frac{\gamma}{\lambda} \frac{|x|^\beta}{(1+|x|^2)^{\frac{\beta}{2}}} + \underline{\sigma} \frac{|x|^\beta}{(1+|x|^2)^{\frac{\theta}{2}}} + \underline{\tau} \frac{C_1''}{|x|^{N+2s-\beta}} \leq \\ \leq u(x)|x|^\beta \leq \frac{\gamma}{\lambda} \frac{|x|^\beta}{(1+|x|^2)^{\frac{\beta}{2}}} + \overline{\sigma} \frac{|x|^\beta}{(1+|x|^2)^{\frac{\theta}{2}}} + \overline{\tau} \frac{C_2''}{|x|^{N+2s-\beta}}, \end{aligned}$$

which gives the claim passing to the limit $|x| \to +\infty$, since $\theta > \beta$ and $\beta < N + 2s$.

Assume now $\beta = N + 2s$, and choose $\theta = \beta = N + 2s$. Now we have

$$g_{\sigma,\theta}(x) \sim \overline{C}_\sigma h_{N+2s}(x) \quad \text{as } |x| \to +\infty$$

where

$$\overline{C}_\sigma := \frac{\gamma}{\lambda} C'_{N+2s,N,s} + \sigma C'_{N+2s,N,s} + \lambda \sigma;$$

recall that $C'_{N+2s,N,s}, C'_{\theta,N,s} < 0$. We can choose proper $\overline{\sigma} \in \mathbb{R}$ such that $\overline{C_{\overline{\sigma}}} < 0$ and thus the first equation in (4.6.55) still hold. Since the sign of $\overline{\sigma}$ may be now different, we choose

$$\overline{\tau} \geq \frac{\max_{B_R} u - \frac{\gamma}{\lambda} h_\beta(R) - \min\{\overline{\sigma}, 0\}}{\min_{B_R} w}.$$

We come up then with the same proof, obtaining

$$\limsup_{|x|\to+\infty} u(x)|x|^\beta \leq \frac{\gamma}{\lambda} + \overline{\sigma} + \overline{\tau} C''_2.$$

Notice that the appearing constants depend on $u, \gamma, \lambda, \rho, \beta, N, s$. ∎

Lemma 4.6.16 (Comparison for pointwise equation)**.** *Let $u \in C(\mathbb{R}^N)$ be a pointwise solution of*

$$(-\Delta)^s u + \lambda u = \gamma h_\beta \quad \text{in } \mathbb{R}^N \setminus B_\rho(0)$$

for some $\lambda, \gamma > 0$, $\rho > 0$ and

$$\beta \in (0, N + 2s].$$

Then

$$0 < \liminf_{|x|\to+\infty} u(x)|x|^\beta \leq \limsup_{|x|\to+\infty} u(x)|x|^\beta < \infty.$$

More precisely, if $\beta \in (0, N + 2s)$, we have

$$\lim_{|x|\to+\infty} u(x)|x|^\beta = \frac{\gamma}{\lambda}.$$

Proof. The proof goes as the previous Lemma, with the difference that at the end we apply the pointwise version of the Comparison Principle (Lemma 1.2.36). ∎

4.6.4 A preliminary range

We start with some observations.

Remark 4.6.17. *Let $u \in L^q(\mathbb{R}^N)$, for some $q \in [1, +\infty)$, be continuous and such that $|u|$ is radially symmetric and decreasing. Then, for every $x \in \mathbb{R}^N$,*

$$\begin{aligned} |u(|x|)|^q|x|^N &= N|u(|x|)| \int_0^{|x|} t^{N-1} dt = N \int_0^{|x|} |u(|x|)|^q t^{N-1} dt \\ &\leq N \int_0^{|x|} |u(t)|^q t^{N-1} dt = \frac{N}{\omega_{N-1}} \int_{B_{|x|}(0)} |u(y)|^q dy \leq \frac{N}{\omega_{N-1}} \|u\|^q_{L^q(\mathbb{R}^N)} \end{aligned}$$

where ω_{N-1} denotes the area of the $N-1$ dimensional sphere. Thus

$$|u(x)| \leq \frac{C_u^2}{|x|^{\frac{N}{q}}}, \quad x \neq 0$$

where $C_u^2 := C_N \|u\|_q^q > 0$. In particular, if $u \in L^1(\mathbb{R}^N)$, we have

$$|u(x)| \leq \frac{C_u^2}{|x|^N}, \quad x \neq 0.$$

We keep with some preliminary results.

Lemma 4.6.18. *Let $u \in L^1(\mathbb{R}^N)$ continuous be such that $|u|$ is radially symmetric and decreasing. Let f satisfy (F1)-(F2,i), and let $\theta \in (N, N+\alpha]$. Then there exists $C = C(N,\alpha) > 0$ such that*

$$\left|(I_\alpha * F(u))(x) - I_\alpha(x)\int_{\mathbb{R}^N} F(u)\right| \leq C\|F(u)\|_{\infty,\theta} I_\alpha(x)\left(\frac{1}{1+|x|} + \frac{1}{1+|x|^{\theta-N}}\right)$$

for each $x \in \mathbb{R}^N$, $x \neq 0$.

Proof. First notice that $u \in L^\infty(\mathbb{R}^N)$, $F(u) \in L^\infty(\mathbb{R}^N)$, and that $I_\alpha * F(u)$ and $\int_{\mathbb{R}^N} F(u)$ are finite and well defined. By Remark 4.6.17 we have

$$|u(x)| \leq \frac{C_u^2}{|x|^N} \to 0.$$

Thus $|F(u(x))||x|^\theta$ is bounded on a ball B_R (since $F(u)$ is bounded), and it is bounded on the complement of this ball since

$$|F(u(x))||x|^\theta = \frac{|F(u(x))|}{|u(x)|^{\frac{N+\alpha}{N}}}|u(x)|^{\frac{N+\alpha}{N}}|x|^\theta \leq \frac{|F(u(x))|}{|u(x)|^{\frac{N+\alpha}{N}}}\frac{C}{|x|^{N+\alpha-\theta}}$$

by considering the growth condition (F2,i) of F in zero (when $R \gg 0$, not depending on θ) and the restriction on θ. Thus

$$\sup_{x\in\mathbb{R}^N} |F(u(x))||x|^\theta < +\infty$$

and Lemma 1.3.3 applies with $g(x) := F(u(x))$, which concludes the proof. We further notice that

$$\|F(u)\|_{\infty,\theta} \leq \|F(u)\|_\infty (1+R^\theta) + \left(\limsup_{t\to 0}\frac{|F(t)|}{|t|^{\frac{N+\alpha}{N}}}\right)\frac{1+R^\theta}{R^{N+\alpha}}$$

for any $\theta \in (N, N+\alpha]$ and any $R \gg 0$ (not depending on θ, but depending on u). ∎

Remark 4.6.19. *In what follows, for the sake of exposition we will restrict our analysis to the space of radially symmetric and decreasing functions in $L^1(\mathbb{R}^N)$, but we highlight that this assumption is needed only to get the a priori asymptotic decay of Remark 4.6.17. By the above proof, actually we see that we may ask only*

$$|u(x)| \leq \frac{C}{|x|^\omega}$$

for some ω such that

$$\omega > \frac{N^2}{N+\alpha}.$$

In particular $\omega = N$, obtained in Remark 4.6.17, fits this condition. Alternatively, one may assume this a priori asymptotic decay on u (and adapt the restrictions on θ by $\theta \in (N, \frac{N+\alpha}{N}\omega]$).

Corollary 4.6.20. *Let $u \in L^1(\mathbb{R}^N)$ continuous be such that $|u|$ is radially symmetric and decreasing. Let f satisfy (F1)-(F2,i), and let $\theta \in (N, N+\alpha]$. Then for any $\varepsilon > 0$, there exists $R_\varepsilon = R_\varepsilon(N,\alpha,\theta) \gg 0$ such that*

$$\left|(I_\alpha * F(u))(x)\right| \leq I_\alpha(x)\left(\left|\int_{\mathbb{R}^N} F(u)\right| + \varepsilon\|F(u)\|_{\infty,\theta}\right)$$

and

$$(I_\alpha * F(u))(x) \geq I_\alpha(x)\left(\int_{\mathbb{R}^N} F(u) - \varepsilon\|F(u)\|_{\infty,\theta}\right)$$

for each $|x| \geq R_\varepsilon$.

Remark 4.6.21. *In Section 4.6.1 it was showed that the solutions decay as fast as* $\sim \frac{1}{|x|^{N+2s}}$ *when the nonlinearity is linear or superlinear. In the sublinear case, we expect a slower decay. Indeed, assume (F1)-(F2) and (F10), and let* u *be a strictly positive solution of (1.3.38). By Lemma 4.6.3 we obtain* $u(x) \to 0$ *as* $|x| \to +\infty$[1]*. Thus there exists* $R \gg 0$ *such that* $0 \le u(x) \le \delta < 1$ *for* $|x| \ge R$ *and thus*

$$f(u(x)) \ge \underline{C} u^{r-1}(x) \quad \textit{for } |x| \ge R$$

together with

$$u^{r-1}(x) \ge u(x) \quad \textit{for } |x| \ge R.$$

If we assume $(I_\alpha * F(u))(x) \ge 0$ *for* $|x| \ge R$*, we gain*

$$(-\Delta)^s u + \mu u \ge (I_\alpha * F(u)) u \quad \textit{on } \mathbb{R}^N \setminus B_R(0)$$

which implies

$$(-\Delta)^s u + \tfrac{3}{2}\mu u \ge \Big(I_\alpha * F(u) + \tfrac{1}{2}\mu\Big) u \quad \textit{on } \mathbb{R}^N \setminus B_R(0).$$

By Proposition 4.4.6 we have that $(I_\alpha * F(u))(x) \to 0$ *as* $|x| \to +\infty$,[2]*, thus for some* $R' \ge R \gg 0$ *we get*

$$(-\Delta)^s u + \tfrac{3}{2}\mu u \ge 0 \quad \textit{on } \mathbb{R}^N \setminus B_{R'}(0).$$

At this point (being u *strictly positive) we conclude as in the proof of Theorem 4.6.4 and obtain*

$$u(x) \ge \frac{C_u^1}{|x|^{N+2s}} \quad \textit{for } |x| \ge R$$

for some constant $C_u^1 = C_{N,\alpha,R,\mu} \min_{B_R} u > 0$ *and some sufficiently large* $R \gg 0$.

By Remarks 4.6.21 and 4.6.17, we obtain that every strictly positive, continuous, radially symmetric and decreasing solution of (1.3.38) in $L^1(\mathbb{R}^N)$ satisfies

$$\frac{C_u^1}{|x|^{N+2s}} \le u(x) \le \frac{C_u^2}{|x|^N} \quad \text{for } |x| \ge R \gg 0, \tag{4.6.58}$$

whenever f satisfies (F1)-(F2) and (F10), together with $\int_{\mathbb{R}^N} F(u) > 0$: indeed in this case, by Lemma 4.6.18, we have $(I_\alpha * F(u))(x) \sim I_\alpha(x) \int_{\mathbb{R}^N} F(u) > 0$ for $|x|$ large. Thus the goal is to improve the asymptotic decay (4.6.58) in the case of sublinear nonlinearities.

Remark 4.6.22. *By Lemma 4.6.15, Corollary 4.6.20, and a bootstrap argument we can give some first qualitative proofs of the main result. Indeed, by*

$$(-\Delta)^s u + \mu u = (I_\alpha * F(u)) f(u) \lesssim I_\alpha(x) u^{r-1} \lesssim \frac{1}{|x|^{N-\alpha}} \frac{1}{|x|^{\gamma_0(r-1)}} = \frac{1}{|x|^{\gamma_1}}$$

where $\gamma_0 := N$ *and* $\gamma_1 := \gamma_0(r-1) + N - \alpha$*, and a comparison argument, we obtain* $u(x) \lesssim \frac{1}{|x|^{\gamma_1}}$*. By induction, set*

$$\gamma_{i+1} := \gamma_i(r-1) + N - \alpha$$

we obtain $u(x) \lesssim \frac{1}{|x|^{\gamma_i}}$ *and* $\gamma_i \nearrow \frac{N-\alpha}{2-r}$ *(but the argument works only for* $\gamma_i \le N + 2s$*). Similarly,*

$$(-\Delta)^s u + \mu u = (I_\alpha * F(u)) f(u) \gtrsim I_\alpha(x) u^{r-1} \gtrsim \frac{1}{|x|^{N-\alpha}} \frac{1}{|x|^{\gamma_0(r-1)}} = \frac{1}{|x|^{\gamma_1}}$$

where now $\gamma_0 := N + 2s$*, which implies* $u(x) \gtrsim \frac{1}{|x|^{\gamma_i}}$ *and* $\gamma_i \searrow \frac{N-\alpha}{2-r}$ *if* $r \le r^*_{s,\alpha}$ *(while the case* $r \ge r^*_{s,\alpha}$ *implying* $\gamma_i \nearrow \frac{N-\alpha}{2-r}$ *cannot be set in motion).*

In order to pass to the limit we have to take care of the bounding constants (or, equivalently, of the radii related to the complements of the balls where the inequalities hold), which is not an easy task; see anyway Corollary 4.6.30. This suggests the implementation of more direct approaches, as done in following Sections.

[1] If u is assumed radially symmetric and $s \in (\frac{1}{2}, 1)$, this is actually a consequence of the Strauss radial lemma. If u is radially symmetric and decreasing, it is a consequence of the monotonicity and (a whatever) summability.

[2] Notice that the claim is obtained by assuming that u is a weak solution or, alternatively, that u is in the right Lebesgue spaces; in particular it is true if $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ a priori.

4.6.5 Estimate from above

First, we deal with the estimate from above. In this case we succeed in arguing in the weak sense with no additional assumption on f.

Proposition 4.6.23. *Assume (F1) and (F9). Let $u \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, continuous, nonnegative, radially symmetric and decreasing, be a weak solution of (1.3.38). Assume moreover*

$$\mu > (r-1)\bar{C}^{\frac{1}{r-1}}.$$

Then, set $\beta := \min\left\{\frac{N-\alpha}{2-r}, N+2s\right\}$, we have, for some $C_u \geq 0$,

$$\limsup_{|x|\to+\infty} u(x)|x|^{\beta} \leq C_u;$$

if $\beta < N+2s$, the constant C_u depends on u in the following way:

$$C_u = \frac{(2-r)\left(C_{N,\alpha}\left|\int_{\mathbb{R}^N} F(u)\right|\right)^{\frac{1}{2-r}}}{\mu - (r-1)\bar{C}^{\frac{1}{r-1}}}$$

where $C_{N,\alpha} > 0$ is given in (1.3.32).

Remark 4.6.24. *We observe that*

$$r = 1 + \frac{\alpha}{N} \implies \beta = N,$$
$$r \in \left(1+\frac{\alpha}{N}, 1+\frac{\alpha+2s}{N+2s}\right) \implies \beta = \frac{N-\alpha}{2-r} \in (N, N+2s),$$
$$r \in \left[1+\frac{\alpha+2s}{N+2s}, 2\right) \implies \beta = N+2s;$$

actually, as already observed, the asymptotic decay with $\beta = N+2s$ applies for general $r \in [1+\frac{\alpha+2s}{N+2s}, 1+\frac{\alpha+2s}{N-2s}]$, including linear and superlinear cases, thanks to the results in Section 4.6.1. We notice that, when $r > 1+\frac{\alpha}{N}$, we are actually improving (4.6.58).

Proof. We start noticing that, by the Young product inequality, we obtain

$$(I_\alpha * F(u))f(u) \leq \frac{1}{a}|I_\alpha * F(u)|^a + \frac{1}{b}|f(u)|^b$$

when $a, b > 0$, $\frac{1}{a} + \frac{1}{b} = 1$. In particular we choose $b = \frac{1}{r-1}$ and thus $a = \frac{1}{2-r} > 0$ (possible thanks to the sublinearity restriction on r); with this choice, by (4.6.51) and the fact that $u(x) \to 0$ as $|x| \to +\infty$, we obtain

$$(I_\alpha * F(u))f(u) \leq (2-r)|I_\alpha * F(u)|^{\frac{1}{2-r}} + (r-1)\bar{C}^{\frac{1}{r-1}} u$$

for $|x| \geq R$, where $R = R(u) \gg 0$ is sufficiently large. By Corollary 4.6.20, for a whatever fixed $\theta \in (N, N+\alpha]$ and any $\varepsilon > 0$ we obtain

$$\begin{aligned}(I_\alpha * F(u))f(u) &\leq (2-r)\left(I_\alpha(x)\left(\left|\int_{\mathbb{R}^N} F(u)\right| + \varepsilon\|F(u)\|_{\infty,\theta}\right)\right)^{\frac{1}{2-r}} + (r-1)\bar{C}^{\frac{1}{r-1}} u \\ &= (2-r)\left(\left|\int_{\mathbb{R}^N} F(u)\right| + \varepsilon\|F(u)\|_{\infty,\theta}\right)^{\frac{1}{2-r}} \frac{C_{N,\alpha}^{\frac{1}{2-r}}}{|x|^{\frac{N-\alpha}{2-r}}} + (r-1)\bar{C}^{\frac{1}{r-1}} u\end{aligned}$$

for every $|x| \geq R_\varepsilon = R_\varepsilon(u, N, \alpha, \theta)$, thus

$$(-\Delta)^s u + \mu u \leq (2-r)C_{N,\alpha}^{\frac{1}{2-r}}\left(\left|\int_{\mathbb{R}^N} F(u)\right| + \varepsilon\|F(u)\|_{\infty,\theta}\right)^{\frac{1}{2-r}} \frac{1}{|x|^{\frac{N-\alpha}{2-r}}} + (r-1)\bar{C}^{\frac{1}{r-1}} u.$$

Notice that $F(u) \not\equiv 0$ (otherwise, by the equation, $u \equiv 0$ and the claim is trivial), thus we set

$$\gamma_{u,\varepsilon} := (2-r)C_{N,\alpha}^{\frac{1}{2-r}} \left(\left| \int_{\mathbb{R}^N} F(u) \right| + \varepsilon \|F(u)\|_{\infty,\theta} \right)^{\frac{1}{2-r}} > 0$$

and

$$\lambda := \mu - (r-1)\bar{C}^{\frac{1}{r-1}} > 0$$

we obtain

$$(-\Delta)^s u + \lambda u \leq \frac{\gamma_{u,\varepsilon}}{|x|^\beta} \quad \text{ in } \mathbb{R}^N \setminus B_{R_\varepsilon}(0);$$

notice that we use the fact that $\frac{1}{|x|^{\frac{N-\alpha}{2-r}}} \leq \frac{1}{|x|^\beta}$ for $|x|$ large. For each $\delta > 1$ we have $\frac{1}{|x|^\beta} \leq \delta h_\beta(x)$ when $|x| > R_\delta := (\delta^{\frac{2}{\beta}} - 1)^{-\frac{1}{2}}$; we may choose $R_{\delta,\varepsilon} > \max\{R_\delta, R_\varepsilon\}$. Thus

$$(-\Delta)^s u + \lambda u \leq \delta \gamma_{u,\varepsilon} h_\beta(x) \quad \text{ in } \mathbb{R}^N \setminus B_{R_{\delta,\varepsilon}}(0). \tag{4.6.59}$$

We have $h_\beta \in L^2(\mathbb{R}^N)$, since $2\frac{N-\alpha}{2-r} > N$. By Lemma 1.2.31, being $u \in H^s(\mathbb{R}^N)$, there exists $v \in H^s(\mathbb{R}^N)$ such that

$$\begin{cases} (-\Delta)^s v + \lambda v = \delta \gamma_{u,\varepsilon} h_\beta(x) & \text{ in } \mathbb{R}^N \setminus B_{R_{\delta,\varepsilon}}(0), \\ v = u & \text{ on } B_{R_{\delta,\varepsilon}}(0). \end{cases}$$

Joining the first equation with (4.6.59) we obtain

$$(-\Delta)^s (u-v) + \lambda(u-v) \leq 0 \quad \text{ in } \mathbb{R}^N \setminus B_{R_{\delta,\varepsilon}}(0)$$

and thus, by the weak version of the Comparison Principle (Lemma 1.2.34) we gain

$$u \leq v \quad \text{ on } \mathbb{R}^N. \tag{4.6.60}$$

By Lemma 4.6.15, if $\beta < N + 2s$, we can estimate v by

$$\limsup_{|x| \to +\infty} v(x)|x|^\beta \leq \frac{\delta \gamma_{u,\varepsilon}}{\lambda}.$$

This relation, combined with (4.6.60), gives

$$\limsup_{|x| \to +\infty} u(x)|x|^\beta \leq \frac{\delta \gamma_{u,\varepsilon}}{\lambda}$$

for each $\delta > 1$. In particular, as $\delta \to 1^+$ and $\varepsilon \to 0^+$, we obtain the claim. If $\beta = N + 2s$, we argue similarly (without moving δ and ε). ∎

Notice that, if we assume (4.6.53), then one can choose every $\bar{C} > 0$, and thus in particular every $\mu > 0$ is allowed.

Corollary 4.6.25. *Assume (F1) and the condition (4.6.53). Let $u \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, continuous, nonnegative, radially symmetric and decreasing, be a solution of (1.3.38). Then, set $\beta := \min\left\{ \frac{N-\alpha}{2-r}, N+2s \right\}$, we have*

$$\limsup_{|x| \to +\infty} u(x)|x|^\beta \leq C_u;$$

if $\beta < N + 2s$ the constant $C_u > 0$ depends on u in the following way:

$$C_u := \frac{(2-r)\left(C_{N,\alpha} \left| \int_{\mathbb{R}^N} F(u) \right|\right)^{\frac{1}{2-r}}}{\mu}.$$

We observe that the previous estimate from above is still valid by considering viscosity solutions $u \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, see Section 4.6.6. We leave the details to the reader.

4.6.6 Estimate from below

Next, we deal with the estimate from below. Here we need to deal with the fractional Laplacian of the concave power of a function: since it might happen that $u^\theta \notin H^s(\mathbb{R}^N)$ when $u \in H^s(\mathbb{R}^N)$ and $\theta \in (0,1)$, the weak formulation seems not to be appropriate. Similarly, $(-\Delta)^s u^\theta$ might be not well defined pointwise, even if u is regular enough. Notice that knowing a priori that u is continuous, radially symmetric and decreasing seems of no use. The idea is thus to treat the problem via viscosity formulation, and we do it by exploiting the concave chain rule obtained in Section 1.2.4.

Proposition 4.6.26. *Assume (F1)-(F2,i) and the sublinear condition (F10). Let $u \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, strictly positive, radially symmetric and decreasing, be a viscosity solution of (1.3.38). Assume $\int_{\mathbb{R}^N} F(u) > 0$. Then,*

$$\liminf_{|x|\to+\infty} u(x)|x|^{\frac{N-\alpha}{2-r}} \geq C'_u$$

where

$$C'_u := \left(\frac{\underline{C} C_{N,\alpha} \int_{\mathbb{R}^N} F(u)dx}{\mu} \right)^{\frac{1}{2-r}}$$

and $C_{N,\alpha} > 0$ is given in (1.3.32). Moreover, set $\beta := \min\left\{\frac{N-\alpha}{2-r}, N+2s\right\}$, we have, for some $C''_u > 0$,

$$\liminf_{|x|\to+\infty} u(x)|x|^{\beta} \geq C''_u;$$

if $\frac{N-\alpha}{2-r} \leq N+2s$ (i.e. $\beta = \frac{N-\alpha}{2-r}$), we have $C''_u := C'_u$, otherwise we have $C''_u := C^1_u$ (see Remark 4.6.21).

Proof. First notice that, by the assumptions, $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and thus, by Remark 1.5.8, $I_\alpha * F(u)$ is pointwise well defined.

By Corollary 1.2.23, since $2-r \in (0, 1-\frac{\alpha}{N}] \subset (0,1)$ we have

$$(-\Delta)^s u^{2-r} \geq \frac{2-r}{u^{r-1}} \Big(-\mu u + (I_\alpha * F(u)) f(u) \Big)$$

on $\mathbb{R}^N$, in the viscosity sense. Thus

$$(-\Delta)^s u^{2-r} + \mu(2-r)u^{2-r} \geq (2-r)\frac{(I_\alpha * F(u))f(u)}{u^{r-1}}.$$

For a fixed $\theta \in (N, N+\alpha]$ and any $\varepsilon > 0$ small, by Corollary 4.6.20 and (4.6.52) (since $u(x) \to 0$ as $|x| \to +\infty$, being u decreasing and in $L^1(\mathbb{R}^N)$) we obtain – we use here that $\int_{\mathbb{R}^N} F(u) > 0$ –

$$(I_\alpha * F(u))f(u) \geq \underline{C} \left(\int_{\mathbb{R}^N} F(u) - \varepsilon \|F(u)\|_{\infty,\theta} \right) I_\alpha u^{r-1} \quad \text{in } \mathbb{R}^N \setminus B_{R_\varepsilon}(0)$$

for some $R_\varepsilon \gg 0$, thus

$$(-\Delta)^s u^{2-r} + \mu(2-r)u^{2-r} \geq (2-r)\underline{C} \left(\int_{\mathbb{R}^N} F(u) - \varepsilon \|F(u)\|_{\infty,\theta} \right) I_\alpha \quad \text{in } \mathbb{R}^N \setminus B_{R_\varepsilon}(0);$$

that is, exploiting $\frac{1}{|x|^{N-\alpha}} \geq \frac{1}{(1+|x|^2)^{\frac{N-\alpha}{2}}}$, we get

$$(-\Delta)^s u^{2-r} + \lambda' u^{2-r} \geq \gamma'_{u,\varepsilon} h_{N-\alpha} \quad \text{in } \mathbb{R}^N \setminus B_{R_\varepsilon}(0)$$

in the viscosity sense, where

$$\gamma'_{u,\varepsilon} := (2-r)\underline{C} C_{N,\alpha} \left(\int_{\mathbb{R}^N} F(u) - \varepsilon \|F(u)\|_{\infty,\theta} \right) > 0$$

and

$$\lambda' := \mu(2-r).$$

We observe that $u^{2-r} \in L^\infty(B_R(0)) \cap C(B_{R_\varepsilon}(0))$, while $h_{N-\alpha} \in L^\infty(\mathbb{R}^N) \cap C^\sigma_{loc}(\mathbb{R}^N)$ (for any σ), thus by Lemma 1.2.33, there exists $v \in C^\omega_{loc}(\mathbb{R}^N)$, for some $\omega > 2s$ such that

$$\begin{cases} (-\Delta)^s v + \lambda' v = \gamma'_{u,\varepsilon} h_{N-\alpha} & \text{in } \mathbb{R}^N \setminus B_{R_\varepsilon}(0), \\ v = u^{2-r} & \text{on } B_{R_\varepsilon}(0), \end{cases}$$

pointwise. Thus

$$(-\Delta)^s(u^{2-r} - v) + \lambda'(u^{2-r} - v) \geq 0 \quad \text{in } \mathbb{R}^N \setminus B_{R_\varepsilon}(0)$$

in the viscosity sense, with

$$u^{2-r} - v \geq 0 \quad \text{on } B_{R_\varepsilon}(0).$$

Observe that, by Lemma 4.6.16, we have $v(x) \to 0$ as $|x| \to +\infty$. Since $(u^{r-2} - v)(x) \to 0$ as $|x| \to +\infty$, by the viscosity version of the Comparison Principle (Lemma 1.2.36) we obtain

$$u^{2-r} \geq v \quad \text{on } \mathbb{R}^N.$$

By Lemma 4.6.16 we gain

$$\liminf_{|x| \to +\infty} v(x)|x|^{N-\alpha} \geq \frac{\gamma'_{u,\varepsilon}}{\lambda'}.$$

Combining the previous inequalities and sending $\varepsilon \to 0^+$, we have the first claim. We conclude by adapting Remark 4.6.21 to the viscosity case (notice that $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$). ∎

The above estimate applies, in particular, to pointwise solutions.

Corollary 4.6.27. *Assume (F1)-(F2,i) and the sublinear condition (F10). Let $u \in L^1(\mathbb{R}^N) \cap C^\gamma_{loc}(\mathbb{R}^N)$ for some $\gamma > 2s$, strictly positive, radially symmetric and decreasing, be a pointwise solution of (1.3.38), such that $\int_{\mathbb{R}^N} F(u) > 0$. Then the conclusions of Proposition 4.6.26 hold.*

By the results of the previous Sections (see Proposition 4.4.12), we gain sufficient conditions in order to state that a weak solution is a pointwise solution.

Corollary 4.6.28. *Assume (F1)-(F2,i) together with (F7), and the sublinear condition (F10). Let $u \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, strictly positive, radially symmetric and decreasing, be a weak solution of (1.3.38), such that $\int_{\mathbb{R}^N} F(u) > 0$. Then u is a classical solution and the conclusions of Proposition 4.6.26 hold.*

Notice that, by the sublinearity in zero, the Hölder exponent σ can lie only in $(0, r-1]$. We conjecture anyway that the conclusion of Corollary 4.6.28 holds in more general cases, by assuming merely f continuous.

4.6.7 An s-sublinear threshold

We can sum up some of the results of the previous Sections in what follows.

Corollary 4.6.29. *Assume (F1)-(F2,i) and the sublinear conditions (F9)-(F10), in particular*

$$0 < \liminf_{t \to 0} \frac{f(t)}{|t|^{r-1}} \leq \limsup_{t \to 0} \frac{f(t)}{|t|^{r-1}} \leq \bar{C} < +\infty.$$

Let $u \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, strictly positive, radially symmetric and decreasing, be a weak solution of (1.3.38). Finally assume (F7), i.e.

$$f \in C^{0,\sigma}(\mathbb{R}) \ \ \textit{for some } \sigma \in (0, r-1],$$

and $\int_{\mathbb{R}^N} F(u) > 0$. *Then, if*

$$\mu > (r-1)\bar{C}^{\frac{1}{r-1}}$$

we have

$$0 < \liminf_{|x|\to+\infty} u(x)|x|^{\beta} \leq \limsup_{|x|\to+\infty} u(x)|x|^{\beta} < +\infty$$

where $\beta := \min\left\{\frac{N-\alpha}{2-r}, N+2s\right\}$.

We notice that, by assuming

$$\limsup_{t\to 0} \frac{f(t)}{|t|^{r-1}} \in (0, +\infty)$$

we obtain that

$$\limsup_{t\to 0} \frac{f(t)}{|t|^{r-\varepsilon-1}} = 0$$

for any $\varepsilon > 0$. Thus we may directly extend the estimate from above to a whatever $\mu > 0$ by paying the cost of a slower decay at infinity; this was essentially contained already in Remark 4.6.22. Notice that we still need $r - \varepsilon \geq 2^{\#}_{\alpha}$.

Corollary 4.6.30. *Assume (F1)-(F2,i) and the sublinear conditions (F9)-(F10), in particular*

$$0 < \liminf_{t\to 0} \frac{f(t)}{|t|^{r-1}} \leq \limsup_{t\to 0} \frac{f(t)}{|t|^{r-1}} < +\infty$$

with $r \in (2^{\#}_{\alpha}, 2)$. *Let* $u \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, *strictly positive, radially symmetric and decreasing, be a weak solution of (1.3.38). Finally assume (F7), i.e.*

$$f \in C^{0,\sigma}(\mathbb{R}) \ \ \textit{for some } \sigma \in (0, r-1],$$

and $\int_{\mathbb{R}^N} F(u) > 0$. *Then, if* $\mu > 0$ *is arbitrary and* $\varepsilon > 0$ *is small, we have*

$$0 < \liminf_{|x|\to+\infty} u(x)|x|^{\beta_0} \leq \limsup_{|x|\to+\infty} u(x)|x|^{\beta_\varepsilon} < +\infty$$

where

$$\beta_\varepsilon := \min\left\{\frac{N-\alpha}{2-r+\varepsilon}, N+2s\right\}.$$

We can now conclude the proof of the main theorem.

Proof of Theorem 4.6.11. First, we show how to remove the restriction on μ in Proposition 4.6.23. Indeed, for any $\kappa > 0$ we can write $\big(I_\alpha * F(u)\big)f(u) \equiv \big(I_\alpha * F_\kappa(u)\big)f_\eta(u)$, where $f_\kappa := \frac{1}{\kappa}f$ and $F_\kappa := \kappa F$. We can thus rewrite (f3) as

$$|f_\kappa(t)| \leq \frac{1}{\kappa}\overline{C}t^{r-1} \quad \text{for } t \in (0,\delta).$$

Since in Proposition 4.6.23 we did not use that F is the primitive of f (in particular, we did not apply (f3) to F), fixed a whatever $\mu > 0$ we can choose κ such that

$$\mu > (r-1)\left(\frac{\overline{C}}{\kappa}\right)^{\frac{1}{r-1}} > 0,$$

that is a large κ given by $\kappa > \left(\frac{r-1}{\mu}\right)^{r-1}\overline{C}$, and obtain

$$\limsup_{|x|\to+\infty} u(x)|x|^{\beta} \leq C_{u,\kappa}$$

where, if $\beta < N + 2s$,

$$C_{u,\kappa} := \frac{(2-r)\left(C_{N,\alpha}\kappa\left|\int_{\mathbb{R}^N} F(u)\right|\right)^{\frac{1}{2-r}}}{\mu - (r-1)\left(\frac{\bar{C}}{\kappa}\right)^{\frac{1}{r-1}}}.$$

We notice, as we expect, that as $\mu \to 0$ then $\kappa \to +\infty$ and $C_{u,\kappa} \to +\infty$, while C'_u defined in Proposition 4.6.26 is invariant under κ-transformations.

We show now the sharp decay. Indeed, we search for a κ such that $C_{u,\kappa} = C'_u$. By a straightforward analysis of $g(\kappa) := C_{u,\kappa} - C'_u$, $\kappa > \left(\frac{r-1}{\mu}\right)^{r-1}\overline{C}$, we find a (unique, explicit) zero κ^* (which actually is a point of minimum) if only if $\overline{C} = \underline{C}$, i.e. if f is exactly a power near the origin.

By the results of the previous Sections (Theorem 4.4.1, Proposition 4.4.8, Proposition 4.4.10), we have that every positive solution is bounded, and every bounded solution is in $H^{2s}(\mathbb{R}^N) \cap C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. By the previous results we conclude the proof. ∎

The conditions on f in the previous results imply that f is sublinear, but in a strict sense. We see that the results actually generalize to sublinear functions in a non strict sense.

Corollary 4.6.31. *Assume (F1)-(F2,i). Assume moreover that f is sublinear in a non-strict sense, i.e.*

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty$$

but

$$\lim_{t \to 0} \frac{f(t)}{|t|^{r-1}} = 0 \quad \textit{for each } r \in (2^\#_\alpha, 2).$$

Let $u \in H^s(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, strictly positive, radially symmetric and decreasing, be a weak solution of (1.3.38). Finally assume

$$f \in C^{0,\sigma}(\mathbb{R}) \ \ \textit{for some } \sigma \in (0, r-1].$$

Then, if $\mu > 0$, we have

$$0 < \liminf_{|x| \to +\infty} u(x)|x|^{N+2s} \leq \limsup_{|x| \to +\infty} u(x)|x|^{N+2s} < +\infty.$$

Proof. The estimate from below comes from the argument in Remark 4.6.21 (since $f(t) \geq \underline{C}t$ for t small and positive). The estimate from above comes from Proposition 4.6.23, after having chosing a whatever $r \in [r^*_{\alpha,s}, 2)$. ∎

Proof of Corollary 4.6.13. By the results in the previous Sections (Theorem 4.5.1), we have that every Pohozev minimum has constant sign – e.g., it is strictly positive – (if f is odd or even, and Hölder continuous), and it is radially symmetric and decreasing (if in addition f has constant sign on $(0, +\infty)$). Thus we conclude by the previous results. ∎

All the previous theorems particularly apply to homogeneous nonlinearities $f(u) = |u|^{r-2}u$; notice that in this case we have $f \in C^{r-1}_{loc}(\mathbb{R}^N)$.

Corollary 4.6.32. *Let $u \in H^s(\mathbb{R}^N)$, strictly positive, radially symmetric and decreasing, be a solution of*

$$(-\Delta)^s u + \mu u = (I_\alpha * |u|^r)|u|^{r-2}u \quad \textit{in } \mathbb{R}^N$$

with $r \in [2^\#_\alpha, 2)$. Set, for every $\varepsilon \geq 0$,

$$\beta_\varepsilon := \min\left\{\frac{N-\alpha}{2-r+\varepsilon}, N+2s\right\}.$$

We have

- *if* $\mu > r-1$ *then*

$$0 < \liminf_{|x|\to+\infty} u(x)|x|^{\beta_0} \leq \limsup_{|x|\to+\infty} u(x)|x|^{\beta_0} < +\infty;$$

- *if* $r \in (2^{\#}_{\alpha}, 2)$ *and* $\mu \in (0, r-1]$ *then, for any* $\varepsilon > 0$ *small,*

$$0 < \liminf_{|x|\to+\infty} u(x)|x|^{\beta_0} \leq \limsup_{|x|\to+\infty} u(x)|x|^{\beta_\varepsilon} < +\infty.$$

Proof of Theorem 4.6.6, Corollary 4.6.7 and Corollary 4.6.9. Theorem 4.6.6 is a direct consequence of the above result. By [138, Theorems 3.2 and 4.2] we have that every ground state satisfies all the assumptions of the previous results; thus we have the claims of Corollary 4.6.7 and Corollary 4.6.9. ∎

4.7 The Pohozaev identity

In [138, equation (6.1)], in presence of power nonlinearities, it is proved that every weak solution u is C^2 and thus satisfies the Pohozaev identity (4.2.5), and this relation is extended to general superlinear nonlinearities $f \in C^1(\mathbb{R})$ in [342, Proposition 2]. Here we want to further extend the identity to more general nonlinearities and to more general solutions $u \in C^1$, without employing the Caffarelli-Silvestre s-harmonic extension.

This Section is mainly based on the paper [117].

First, we collect the results of the previous Sections to highlight the conditions that ensure the right regularity of the solutions.

Corollary 4.7.1. *Assume that (F1)-(F2) hold. Let* $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *be a weak solution of* (4.1.1). *Assume in addition one of the following*

- $s \in (\frac{1}{2}, 1)$,
- $s = \frac{1}{2}$ *and* (F7),
- $s \in [\frac{1}{4}, \frac{1}{2})$, $\alpha \in (1-2s, N)$ *and* (F7) *with* $\sigma \in (\frac{1-2s}{2s}, 1]$,
- $s \in (0, \frac{1}{2})$, $\alpha \in (0, 2)$ *and* (F7) *with* $\sigma \in (1-2s, 1]$.

Then $u \in C^{1,\gamma}(\mathbb{R}^N)$ *for some* $\gamma \in (0,1)$. *If* $s \in (\frac{1}{2}, 1)$ *and* (F7) *holds too, then we can choose* $\gamma \in (2s-1, 1)$.

Thus we want to prove the following result.

Theorem 4.7.2. *Let* $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *be a weak solution of* (4.1.1), *and assume* (F1)-(F2). *Assume moreover* (F7) *and one of the following:*

- $s \in [\frac{1}{2}, 1)$,
- $s \in [\frac{1}{4}, \frac{1}{2})$, $\alpha \in (1-2s, N)$ *and* $\sigma \in (\frac{1-2s}{2s}, 1]$,
- $s \in (0, \frac{1}{2})$, $\alpha \in (0, 2)$ *and* $\sigma \in (1-2s, 1]$.

Then $u \in C^{1,\gamma}(\mathbb{R}^N)$ *for some* $\gamma \in (\max\{0, 2s-1\}, 1)$, *and* u *satisfies the Pohozaev identity* (4.2.5), *or equivalently*

$$\frac{1}{2^*_{\alpha,s}} \|(-\Delta)^{s/2} u\|_2^2 + \frac{\mu}{2^{\#}_{\alpha}} \|u\|_2^2 - \mathcal{D}(u) = 0.$$

The result in particular applies to positive weak solutions $u \in H^s(\mathbb{R}^N)$ *of* (4.1.1).

We start by the following integration by parts rule, inspired by [155, Lemma 4.2], obtained under a pointwise well posedness of the fractional Laplacian and the existence of a weak gradient.

Proposition 4.7.3. *Let $s \in (0,1)$. Let $u \in \dot{H}^s(\mathbb{R}^N) \cap C^{\gamma}_{loc}(\mathbb{R}^N) \cap Lip_{loc}(\mathbb{R}^N)$ for some $\gamma > 2s$, and assume (1.2.1). Let moreover $X \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$ be a vector field, and define, for $x, y \in \mathbb{R}^N$, $x \neq y$,*

$$\mathcal{K}^s_X(x,y) := \frac{(\mathrm{div}(X))(x) + (\mathrm{div}(X))(y)}{2} - \frac{N+2s}{2}\frac{(X(x)-X(y))\cdot(x-y)}{|x-y|^2}$$

the fractional divergence kernel *related to X. Then it holds*

$$\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathcal{K}^s_X(x,y)dxdy = -\int_{\mathbb{R}^N}(-\Delta)^s u\,(\nabla u\cdot X)\,dx;$$

noticed that the left-hand side is the weighted Gagliardo seminorm with weight $\mathcal{K}^s_X$, set

$$\mathcal{G}^s_u(x,y) := \frac{C_{N,s}}{2}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}$$

we can write

$$(\mathcal{G}^s_u, \mathcal{K}^s_X)_{L^2(\mathbb{R}^{2N})} = -\big((-\Delta)^s u\,\nabla u, X\big)_{L^2(\mathbb{R}^N)}.$$

Proof. For the proof, we follow the lines of [154]. We start noticing that, being $u \in \dot{H}^s(\mathbb{R}^N)$, by the assumptions we have

$$\mathcal{G}^s_u \in L^1(\mathbb{R}^{2N}), \quad \mathcal{K}^s_X \in L^\infty(\mathbb{R}^{2N})$$

so that the product is summable. By dominated convergence theorem, the symmetry of the kernel, and the Fubini theorem, we obtain

$$\begin{aligned}
&\frac{2}{C_{N,s}}(\mathcal{G}^s_u, \mathcal{K}^s_X)_{L^2(\mathbb{R}^{2N})}\\
&= \lim_{\varepsilon\to 0}\iint_{|x-y|>\varepsilon}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathcal{K}^s_X(x,y)dxdy\\
&= \lim_{\varepsilon\to 0}\iint_{|x-y|>\varepsilon}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\left(\mathrm{div}(X)(x) - (N+2s)\frac{(x-y)\cdot X(x)}{|x-y|^2}\right)dxdy\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,(\mathrm{div}(X)(x)-\right.\\
&\qquad \left.\left.-(N+2s)\frac{(x-y)\cdot X(x)}{|x-y|^2}\right)dx\right)dy.
\end{aligned}$$

Exploiting that, for $x \neq y$, $\nabla_x \frac{1}{|x-y|^{N+2s}} = -(N+2s)\frac{x-y}{|x-y|^{N+2s+2}}$, and the divergence theorem (possible because $\frac{X}{|\cdot-y|^{N+2s}} \in C^1_c(\overline{\mathbb{R}^N\setminus B_\varepsilon(y)})$ and $u \in Lip_{loc}(\mathbb{R}^N) \subset W^{1,\infty}(\mathrm{supp}(X))$, see [170, Theorem 4.6])

$$\begin{aligned}
&\tfrac{2}{C_{N,s}}(\mathcal{G}^s_u, \mathcal{K}^s_X)_{L^2(\mathbb{R}^{2N})}\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}|u(x)-u(y)|^2\left(\frac{\mathrm{div}(X)(x)}{|x-y|^{N+2s}}-\right.\right.\\
&\qquad \left.\left.-(N+2s)\frac{(x-y)\cdot X(x)}{|x-y|^{N+2s+2}}\right)dx\right)dy\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}|u(x)-u(y)|^2\mathrm{div}_x\left(\frac{X}{|x-y|^{N+2s}}\right)(x)dx\right)dy
\end{aligned}$$

$$
\begin{aligned}
&= -2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}(u(x)-u(y))\nabla u(x)\cdot\frac{X(x)}{|x-y|^{N+2s}}dx\right)dy+\\
&\quad+\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\int_{\partial B_\varepsilon(y)}|u(x)-u(y)|^2\frac{X(x)}{|x-y|^{N+2s}}\cdot\frac{x-y}{|x-y|}d\sigma(x)\right)dy\\
&= -2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\nabla u(x)\cdot X(x)dx\right)dy+\\
&\quad+\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N+2s+1}}\int_{\mathbb{R}^N}\left(\int_{\partial B_\varepsilon(y)}|u(x)-u(y)|^2X(x)\cdot(x-y)d\sigma(x)\right)dy.\\
&=: -2\lim_{\varepsilon\to 0}I_\varepsilon+\lim_{\varepsilon\to 0}E_\varepsilon;
\end{aligned}
$$

here we split the limits since we will prove the existence of both.

For the first integral, we notice that $x\mapsto\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|}{|x-y|^{N+2s}}|\nabla u(x)\cdot X(x)|dy\leq C_\varepsilon\|u\|_\infty|\nabla u(x)\cdot X(x)|\in L^1(\mathbb{R}^N)$ so that we can apply Fubini theorem, then we perform a symmetrization substitution and apply again Fubini theorem, and finally dominated convergence theorem (since $y\mapsto\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{N+2s}}\in L^1(\mathbb{R}^N)$ by Proposition 1.2.2), obtaining

$$
\begin{aligned}
&C_{N,s}\lim_{\varepsilon\to 0}I_\varepsilon\\
&= C_{N,s}\lim_{\varepsilon\to 0}\iint_{|x-y|>\varepsilon}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\nabla u(x)\cdot X(x)dxdy\\
&= \frac{C_{N,s}}{2}\lim_{\varepsilon\to 0}\iint_{|y|>\varepsilon}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{N+2s}}\nabla u(x)\cdot X(x)dxdy\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\nabla u(x)\cdot X(x)\left(\frac{C_{N,s}}{2}\int_{\mathbb{R}^N\setminus B_\varepsilon(0)}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{N+2s}}dy\right)dx\\
&= \int_{\mathbb{R}^N}\nabla u\cdot X(-\Delta)^s u.
\end{aligned}
$$

For the second integral, notice that the set $\{(x,y)\in\mathbb{R}^{2N}\mid x\in\operatorname{supp}(X),\,|x-y|=\varepsilon\}$ is bounded. Thus the integrand (being bounded) is summable, which allows us to implement the Fubini theorem and obtain, by exploiting also a symmetrization argument,

$$
\begin{aligned}
(N+2s)E_\varepsilon &= \frac{1}{\varepsilon^{N+2s+1}}\iint_{|x-y|=\varepsilon}|u(x)-u(y)|^2X(x)\cdot(x-y)d\sigma(x)\times dy\\
&= \frac{1}{2\varepsilon^{N+2s+1}}\iint_{|x-y|=\varepsilon}|u(x)-u(y)|^2(X(x)-X(y))\cdot(x-y)d\sigma(x)\times dy.
\end{aligned}
$$

If $\operatorname{supp}(X)\subset B_R(0)$, then out of the set

$$A_{R,\varepsilon}:=\{(x,y)\in B_R(0)\times B_R(0)\mid |x-y|=\varepsilon\}$$

the integrand is null. Thus, being $u\in Lip_{loc}(\mathbb{R}^N)$ (actually it is sufficient $u\in C^{0,\theta}(\mathbb{R}^N)$ for some $\theta>s$) and $X\in Lip(\mathbb{R}^N,\mathbb{R}^N)$, we get

$$
\begin{aligned}
E_\varepsilon &\lesssim \frac{1}{\varepsilon^{N+2s+1}}\iint_{A_{R,\varepsilon}}|x-y|^4d\sigma(x)\times dy\\
&= \varepsilon^{-N-2s+3}m_{2N-1}(A_{R,\varepsilon}).
\end{aligned}
$$

Observed that $m_{2N-1}(A_{R,\varepsilon})\lesssim m_N(B_R)m_{N-1}(\partial B_\varepsilon)\sim\varepsilon^{N-1}$, we obtain $E_\varepsilon\lesssim\varepsilon^{-2s+2}\to 0$. Joining the pieces, we reach the claim. ∎

Corollary 4.7.4. *In the assumptions of Proposition 4.7.3, let* $G \in C^1(\mathbb{R}^N)$ *with* $G(u) \in L^1(\mathbb{R}^N)$ *and*

$$(-\Delta)^s u = g(u) \quad \text{in } \mathbb{R}^N$$

in the pointwise sense, where $G' = g$. *Then*

$$\begin{aligned}\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathcal{K}_X^s(x,y)dxdy &= -\int_{\mathbb{R}^N}\nabla G(u)\cdot X\,dx\\ &= \int_{\mathbb{R}^N} G(u)\,\mathrm{div}(X)dx,\end{aligned}$$

i.e.

$$(\mathcal{G}_u^s, \mathcal{K}_X^s)_{L^2(\mathbb{R}^{2N})} = (G(u), \mathrm{div}(X))_{L^2(\mathbb{R}^N)}.$$

We deal now with the Riesz kernel right-hand side of the equation.

Proposition 4.7.5. *Let* $\alpha \in (0,N)$ *and* $H \in Lip_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *be such that*

$$(I_\alpha * |H|)|H| \in L^1(\mathbb{R}^N), \quad (I_\alpha * |H|)|\nabla H| \in L^1_{loc}(\mathbb{R}^N).$$

Let moreover $X \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$ *be a vector field and set, for* $x, y \in \mathbb{R}^N$, $x \neq y$,

$$\mathcal{K}_X^{-\frac{\alpha}{2}}(x,y) := \frac{\big(\mathrm{div}(X)\big)(x) + \big(\mathrm{div}(X)\big)(y)}{2} - \frac{N-\alpha}{2}\frac{\big(X(x)-X(y)\big)\cdot(x-y)}{|x-y|^2}.$$

Then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)H(x)H(y)\mathcal{K}_X^{-\frac{\alpha}{2}}(x,y)dxdy = -\int_{\mathbb{R}^N}(I_\alpha * H)\nabla H\cdot X\,dx,$$

i.e. set

$$\mathcal{R}_H^\alpha(x,y) := I_\alpha(x-y)H(x)H(y)$$

we have

$$(\mathcal{R}_H^\alpha, \mathcal{K}_X^{-\frac{\alpha}{2}})_{L^2(\mathbb{R}^{2N})} = -\big((I_\alpha * H)\nabla H, X\big)_{L^2(\mathbb{R}^N)}.$$

Proof. We proceed as in the proof of Proposition 4.7.3. We start noticing that

$$\mathcal{R}_H^\alpha \in L^1(\mathbb{R}^{2N}), \quad \mathcal{K}_X^{-\frac{\alpha}{2}} \in L^\infty(\mathbb{R}^{2N})$$

by the assumptions, so that the product is summable. Thus

$$\begin{aligned}&(\mathcal{R}_H^\alpha, \mathcal{K}_X^{-\frac{\alpha}{2}})_{L^2(\mathbb{R}^{2N})}\\ &\quad= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\bigg(\int_{\mathbb{R}^N\setminus B_\varepsilon(y)} I_\alpha(x-y)H(x)H(y)\,(\mathrm{div}(X)(x) -\\ &\qquad -(N-\alpha)\frac{(x-y)\cdot X(x)}{|x-y|^2}\bigg)dx\bigg)dy.\end{aligned}$$

Since $H \in Lip_{loc}(\mathbb{R}^N) \subset W^{1,\infty}(\mathrm{supp}(X))$, we have

$$\begin{aligned}&\frac{1}{C_{N,\alpha}}(\mathcal{R}_H^\alpha, \mathcal{K}_X^{-\frac{\alpha}{2}})_{L^2(\mathbb{R}^{2N})}\\ &\quad= -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\bigg(\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}\frac{1}{|x-y|^{N-\alpha}}H(y)\nabla H(x)\cdot X(x)dx\bigg)dy +\\ &\qquad +\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N-\alpha+1}}\int_{\mathbb{R}^N}\bigg(\int_{\partial B_\varepsilon(y)} H(x)H(y)X(x)\cdot(x-y)d\sigma(x)\bigg)dy.\end{aligned}$$

$$=: \quad -\lim_{\varepsilon\to 0} I_\varepsilon + \lim_{\varepsilon\to 0} E_\varepsilon.$$

For I_ε we notice that $(I_\alpha*|H|)|\nabla H||X| \in L^1(\mathbb{R}^N)$, thus $(x,y)\mapsto I_\alpha(x-y)H(y)\nabla H(x)\cdot X(x) \in L^1(\mathbb{R}^{2N})$ and we can apply (twice) Fubini theorem; moreover $I_\alpha(x-\cdot)H \in L^1(\mathbb{R}^N)$, and we can apply dominated convergence theorem. Hence we obtain

$$\begin{aligned} C_{N,\alpha}\lim_{\varepsilon\to 0} I_\varepsilon &= \lim_{\varepsilon\to 0}\iint_{|x-y|>\varepsilon} I_\alpha(x-y)H(y)\nabla H(x)\cdot X(x)dxdy \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\nabla H(x)\cdot X(x)\left(\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} I_\alpha(x-y)H(y)dy\right)dx \\ &= \int_{\mathbb{R}^N}\nabla H(x)\cdot X(x)\left(\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} I_\alpha(x-y)H(y)dy\right)dx \\ &= \int_{\mathbb{R}^N}\nabla H(x)\cdot X\,(I_\alpha * H). \end{aligned}$$

We can write E_ε instead as

$$E_\varepsilon = \frac{1}{2\varepsilon^{N-\alpha+1}}\iint_{|x-y|=\varepsilon} H(x)H(y)(X(x)-X(y))\cdot(x-y)d\sigma(x)\times dy.$$

If $\operatorname{supp}(X)\subset B_R(0)$, set $A_{R,\varepsilon} := \{(x,y)\in B_R(0)\times B_R(0) \mid |x-y|=\varepsilon\}$ and observed that $H\in L^\infty(\mathbb{R}^N)$, we obtain

$$E_\varepsilon \lesssim \frac{1}{\varepsilon^{N-\alpha+1}}\iint_{A_{R,\varepsilon}} |x-y|^2 d\sigma(x)\times dy = \varepsilon^{-N+\alpha+1} m_{2N-1}(A_{R,\varepsilon}) \lesssim \varepsilon^\alpha \to 0,$$

being $\alpha>0$. This concludes the proof. ∎

Theorem 4.7.6. *Let $s\in(0,1)$ and $\alpha\in(0,N)$ and assume that (F1)-(F2) hold. Let $u\in H^s(\mathbb{R}^N)\cap C^\gamma_{loc}(\mathbb{R}^N)\cap Lip_{loc}(\mathbb{R}^N)$ for some $\gamma>2s$, be a pointwise solution of (4.1.1). Then u satisfies the Pohozaev identity (4.2.5).*

Proof. We apply Proposition 4.7.3 and Proposition 4.7.5 with $H=F(u)$; notice that the assumptions on u and F imply the needed conditions on H (in particular we highlight that $f(u)\in L^{\frac{2N}{N+\alpha}}_{loc}(\mathbb{R}^N)$). Thus, for a generic $X\in C^1_c(\mathbb{R}^N,\mathbb{R}^N)$ we obtain

$$\begin{aligned} (\mathcal{G}^s_u,\mathcal{K}^s_X)_{L^2(\mathbb{R}^{2N})} &= -\big((-\Delta)^s u\,\nabla u, X\big)_{L^2(\mathbb{R}^N)} \\ &= \mu\big(u\,\nabla u, X\big)_{L^2(\mathbb{R}^N)} - \big((I_\alpha * F(u))f(u)\,\nabla u, X\big)_{L^2(\mathbb{R}^N)} \\ &= \frac{\mu}{2}\big(\nabla(u^2), X\big)_{L^2(\mathbb{R}^N)} - \big((I_\alpha * F(u))\nabla F(u), X\big)_{L^2(\mathbb{R}^N)} \\ &= -\frac{\mu}{2}\big(u^2, \operatorname{div}(X)\big)_{L^2(\mathbb{R}^N)} + (\mathcal{R}^\alpha_{F(u)}, \mathcal{K}^{-\frac{\alpha}{2}}_X)_{L^2(\mathbb{R}^{2N})}. \end{aligned}$$

In particular, we apply the result to

$$X_n(x) := \varphi_n(x)\,x,$$

where φ_n is a cut-off function with $\varphi_n\equiv 1$ in $B_n(0)$, $\operatorname{supp}(\varphi_n)\subset B_{n+1}(0)$, $\|\varphi_n\|_\infty = 1$ and $|x||\nabla\varphi_n(x)| \leq C$ for each $x\in\mathbb{R}^N$ and $n\in\mathbb{N}$; for instance, defined such φ_1, we can set $\varphi_n := \varphi_1(\cdot/n)$ and obtain

$$|x||\nabla\varphi_n(x)| = |x/n||\nabla\varphi_1(x/n)| \leq \||x||\nabla\varphi_1(x)|\|_\infty.$$

In particular, $x\mapsto x\varphi_n(x)$ is equi-Lipschitz. Noticed that $\operatorname{div}(X_n) = N\varphi_n + \nabla\varphi_n\cdot x$ we gain

$$(\mathcal{G}^s_u,\mathcal{K}^s_{X_n})_{L^2(\mathbb{R}^{2N})}$$

$$\begin{aligned}
&= \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\left(\frac{N(\varphi_n(x)+\varphi_n(y))}{2}\right)dxdy - \\
&\quad -\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\left(\frac{N+2s}{2}\frac{(\varphi_n(x)\,x-\varphi_n(y)y)\cdot(x-y)}{|x-y|^2}\right)dxdy + \\
&\quad +\frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\left(\frac{\nabla\varphi_n(x)\cdot x+\nabla\varphi_n(y)\cdot y}{2}\right)dxdy \\
&\to \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\left(N-\frac{N+2s}{2}\right) = \frac{N-2s}{2}[u]^2_{\mathbb{R}^N}
\end{aligned}$$

where we used $\varphi_n \to 1$, $\nabla\varphi_n \to 0$ and dominated convergence theorem. Similarly

$$\begin{aligned}
(\mathcal{R}^\alpha_{F(u)}, \mathcal{K}^\alpha_{X_n})_{L^2(\mathbb{R}^{2N})} &\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} I_\alpha(x-y)F(u(x))F(u(y))\left(N-\frac{N-\alpha}{2}\right)dxdy \\
&= \frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u).
\end{aligned}$$

and

$$\frac{\mu}{2}(u^2, \mathrm{div}(X_n))_{L^2(\mathbb{R}^N)} \to \mu\frac{N}{2}\|u\|_2^2.$$

Joining the pieces, we have the claim. ∎

Proof of Theorem 4.7.2. The theorem is a consequence of Corollary 4.7.1 and Theorem 4.7.6. ∎

Remark 4.7.7. *We comment the name of $\mathcal{K}^s_X$. Indeed, up to a multiplicative constant, we have, for any $\beta \in (0,1)$ and $X \in Lip_c(\mathbb{R}^N, \mathbb{R}^N)$, by [132, equations (2.9c) and (2.11)] (see also [345])*

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mathcal{K}^s_X(x,y)}{|x-y|^{N+\beta-1}} &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{\mathrm{div}(X(y))}{|x-y|^{N+\beta-1}}dy\right)dx - \\
&\quad -\frac{N+2s}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{(X(x)-X(y))\cdot(x-y)}{|x-y|^{N+\beta+1}}dy\right)dx \\
&=(N+\beta-1)\int_{\mathbb{R}^N}\mathrm{div}^\beta(X)(x) - \frac{N+2s}{2}\int_{\mathbb{R}^N}\mathrm{div}^\beta(X)(x) \\
&=(N+2\beta-2-2s)\int_{\mathbb{R}^N}\mathrm{div}^\beta(X);
\end{aligned}$$

in particular

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mathcal{K}^s_X(x,y)}{|x-y|^{N+s-1}} = (N-2)\int_{\mathbb{R}^N}\mathrm{div}^s(X).$$

We refer also to [154, Chapter 3] where $\mathcal{K}^s_X$ is seen as the derivative of a suitable family of deformations.

Chapter 5

Concentration phenomena: the effect of the fractional operator

In this Chapter we investigate how the nonlocalities interact with concentration phenomena. We consider the fractional, semiclassical nonlinear Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s v + V(x)v = f(v), \quad x \in \mathbb{R}^N$$

where $s \in (0,1)$, $N \geq 2$, $V \in C(\mathbb{R}^N, \mathbb{R})$ is a positive potential and f is a nonlinearity satisfying Berestycki-Lions type conditions. For $\varepsilon > 0$ small, we prove the existence of at least $\mathrm{cupl}(K) + 1$ positive solutions, where K is a set of local minima in a bounded potential well and $\mathrm{cupl}(K)$ denotes the cup-length of K. Due to the generality of f, we cannot implement a Lyapunov-Schimdt reduction, nor we can bound our functional on a Nehari manifold: thus, by means of variational methods, our approach is to analyze the topological difference between two levels of an indefinite functional in a neighborhood of expected solutions. Since the nonlocality comes in the decomposition of the space directly, we introduce also a new *fractional center of mass*, via a suitable seminorm. Some other delicate aspects arise strictly related to the presence of the nonlocal operator: in particular, L^∞-boundedness, regularity and polynomial decay have to be specifically investigated. We show then that the found solutions decay polynomially and concentrate around some point of K as $\varepsilon \to 0$.

The main discussion (Section 5.1–5.4) will be focused on the case f Sobolev-subcritical. This is based mainly on paper [111]. Afterwards, in Section 5.5, we will see how to treat the case f critical; the argument will be based mainly on paper [197].

5.1 From classical to quantum: semiclassical states

In Section 2.1 we highlighted the physical relevance of the fractional Laplacian operator. In particular we mentioned the study of standing waves of the fractional nonlinear Schrödinger (fNLS for short) equation

$$i\hbar\partial_t\psi = \hbar^{2s}(-\Delta)^s\psi + V(x)\psi - f(\psi), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N \tag{5.1.1}$$

i.e. factorized functions $\psi(t,x) = e^{\frac{i\mu t}{\hbar}} v(x)$, $\mu \in \mathbb{R}$. Instead of considering the fixed case $\hbar = 1$, we focus now on the study of small $\hbar > 0$: in this case standing waves are usually called *semiclassical* states and the transition from quantum physics to classical physics is somehow described letting $\hbar \to 0$.

Roughly speaking, when $s = 1$ the centers of mass $q_\varepsilon = q_\varepsilon(t)$ of the soliton solutions in (5.1.1), under suitable assumptions and initial conditions, converge as $\hbar \to 0$ to the solution of the Newton's equation of motion

$$\ddot{q}(t) = -\nabla V(q(t)), \quad t \in (0, +\infty); \tag{5.1.2}$$

for $s \in (0,1)$ a suitable power-type modification of equation (5.1.2) is needed. Here, considering small $\hbar$ roughly means that the size of the support of the soliton in (5.1.1) is considerably smaller than the size of the potential V; for details we refer to [46, 73, 191, 241], and to [337] for the fractional case (see also [57] for the Choquard case).

Similar problems arise also in the study of superconductivity in Ginzburg-Landau vortices, see [52] and references therein; here the point of concentration is indeed a point where a vortex is formed.

Without loss of generality, shifting μ to 0 and denoting $\hbar \equiv \varepsilon$, the search for semiclassical states leads to the investigation of the following nonlocal equation

$$\varepsilon^{2s}(-\Delta)^s v + V(x)v = f(v), \quad x \in \mathbb{R}^N \tag{5.1.3}$$

where V is positive and $\varepsilon > 0$ is small. Setting $u := v(\varepsilon \cdot)$, we observe that (5.1.3) can be rewritten as

$$(-\Delta)^s u + V(\varepsilon x)u = f(u), \quad x \in \mathbb{R}^N, \tag{5.1.4}$$

thus the equation

$$(-\Delta)^s U + aU = f(U), \quad x \in \mathbb{R}^N \tag{5.1.5}$$

becomes a formal limiting equation, as $\varepsilon \to 0$, of (5.1.4), for some $a > 0$. Indeed, if $x_0 \in \mathbb{R}^N$ and $r > 0$,

$$\sup_{x \in B(x_0, \varepsilon r)} |V(\varepsilon x) - V(x_0)| \to 0 \quad \text{as } \varepsilon \to 0.$$

Solutions of (5.1.3) usually exhibit concentration behaviour as $\varepsilon \to 0$: by *concentrating solutions* we mean a family v_ε of solutions of (5.1.3) which converges, up to rescaling, to a ground state of (5.1.5) and whose maximum points converge to some point $x_0 \in \mathbb{R}^N$ given by the topology of V (see Theorem 5.5.1 for a precise statement). This point x_0 reveals, generally, to be a critical point of V – i.e. an equilibrium of (5.1.2) – as shown in [174, 370].

In the limiting case $s = 1$ the semiclassical analysis of NLS equations has been largely investigated, starting from the seminal paper [184]: by means of a finite Lyapunov-Schmidt dimensional reduction argument, Floer and Weinstein proved the existence of positive spike solutions to the homogeneous 3D cubic NLS equation, concentrating at each nondegenerate critical point of the potential V (see also [309]); here the nondegeneracy of the ground states of the limiting problem (5.1.5) is crucial. Successively, refined variational techniques were implemented to study singularly perturbed elliptic problems in entire space: several existence results of positive spike solutions to the NLS equation in a semiclassical regime are derived under different assumptions on the potential and the nonlinear terms. We confine to mention [13, 66, 78, 81, 82, 140, 148, 149, 326, 370] and references therein.

Starting from the work [144], in [17, 96, 119–121, 143, 239] topological invariants were used to derive multiplicity results in singularly perturbed frameworks, in the spirit of well known results of Bahri, Coron [28] and Benci, Cerami [44] for semilinear elliptic problems with Dirichlet boundary condition. Precisely, in [120] it has been proved that the number of positive solutions of the stationary NLS equation is influenced by the topological richness of the set of global minima of V. Some years later, using a perturbative approach, Ambrosetti, Malchiodi and Secchi [17] obtained a multiplicity result for the NLS equation with power nonlinearity, assuming that the set of critical points of V is nondegenerate in the sense of Bott. More recently, in [119] Cingolani, Jeanjean and Tanaka improved the result in [120], relating the number of semiclassical standing waves solutions to the *cup-length* of K, where K is a set of local (possibly degenerate) minima of the potential, under almost optimal assumptions on the nonlinearity (see also the recent paper [123] in the context of nonlinear Choquard equations).

When $s \in (0,1)$, the search of semiclassical standing waves for the fNLS equation has been firstly considered by Dávila, Del Pino and Wei in [146] under the assumptions $f(t) = |t|^{p-2}t$, with $2 < p < 2^*_s$, where $2^*_s := \frac{2N}{N-2s}$ is the Sobolev critical exponent, and $V \in C^{1,\alpha}(\mathbb{R}^N)$ is bounded. Using a Lyapunov-Schmidt reduction inspired by [184, 309], they showed the existence of a positive spike solution whose maximum point concentrates at some nondegenerate critical point of V: this approach relies on the nondegeneracy property of the linearization at the positive ground state shown by Frank, Lenzmann and Silvestre [190]. Successively, inspired by [78, 148], variational techniques were employed to derive existence of spike solutions concentrating at local minima of V, see [12, 20, 338] and references therein (see also [336] where global assumptions on V are considered).

A first multiplicity result for the (fNLS) equation is obtained in [183], inspired by [120]. Precisely, letting K be the set of global minima of V, Figueiredo and Siciliano proved that the number of positive solutions of (5.1.3), when f satisfies monotonicity and Ambrosetti-Rabinowitz condition, is at least given by the Ljusternik-Schnirelmann category of K: here the search of solutions of (5.1.3) can be reduced to the study of the (global) level sets of the Nehari manifold, where the energy functional is restricted, and to deformation arguments valid on Hilbert manifolds without boundary. See also [10] where the Ambrosetti-Rabinowitz condition is dropped. In [96], moreover, Chen implemented a Lyapunov-Schmidt reduction for nondegenerate critical points of V and power-type functions f in order to get multiplicity results related to the cup-length, extending the results of [17].

In this first part of the Chapter we are interested to prove multiplicity of positive solutions for the fNLS equation (5.1.3) when ε is small, without monotonicity and Ambrosetti-Rabinowitz conditions on f, nor nondegeneracy and global conditions on V, concentrating at a local minimum of V.

On the potential V we assume

(V1) $V \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\underline{V} := \inf_{\mathbb{R}^N} V > 0$ (see also Remark 5.1.3);

(V2) there exists a bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$m_0 := \inf_{\Omega} V < \inf_{\partial\Omega} V;$$

by the strict inequality and the continuity of V, we can assume that $\partial\Omega$ is regular. We define K as the set of local minima

$$K := \{x \in \Omega \mid V(x) = m_0\}. \tag{5.1.6}$$

On f we assume the following *subritical* assumptions

(f1) Berestycki-Lions type assumptions with respect to m_0, that is

- (f1.1) $f \in C(\mathbb{R}, \mathbb{R})$;
- (f1.2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
- (f1.3) $\lim_{t\to +\infty} \frac{f(t)}{|t|^p} = 0$ for some $p \in (1, 2^*_s - 1)$, where we recall $2^*_s = \frac{2N}{N-2s}$;
- (f1.4) $F(t_0) > \frac{1}{2} m_0 t_0^2$ for some $t_0 > 0$, where $F(t) := \int_0^t f(s)ds$;

(f2) $f(t) = 0$ for $t \leq 0$.

On f we further assume

(f3) $f \in C^{0,\gamma}_{loc}(\mathbb{R})$ for some $\gamma \in (1-2s, 1)$ if $s \in (0, 1/2]$.

Remark 5.1.1. *We remark that (f3) is needed only to get a Pohozaev identity (see Proposition 2.2.2). See also Remark 5.4.3 below.*

It is standard that weak solutions to (5.1.4) correspond to critical points of the C^1-energy functional

$$I_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx - \int_{\mathbb{R}^N} F(u)dx, \quad u \in H^s(\mathbb{R}^N).$$

We remark that, because of the general assumptions on f, we can not take advantage of the boundedness of the functional from above and below, nor of Nehari type constraint. Therefore in the present paper we combine reduction methods and *penalization arguments* in a nonlocal setting: in particular, as in [119, 123], the analysis of the topological changes between two level sets of the indefinite energy functional I_ε in a small *neighborhood* $\mathcal{X}_{\varepsilon,\delta}$ *of expected solutions* is essential in our approach. With the aid of ε-independent pseudo-differential estimates, we detect such a neighborhood, which will be positively invariant under a pseudo-gradient flow, and we develop our deformation argument in the context of nonlocal operators. To this aim we introduce two maps Φ_ε and Ψ_ε between topological pairs: we emphasize that to define such maps, a center of mass Υ and a functional P_a which is inspired by the Pohozaev identity are crucial.

With respect to the local case, several difficulties arise linked to special features of the nonlocal nature of the problem: among them we have the polynomial decay of the least energy solutions of the limiting problems, the weak regularizing effect of the fractional Laplacian, the lack of general comparison arguments, the differences between the supports of a function and of its Fourier transform, and the lack of the standard Leibniz formula (see e.g. [54, 336]). Moreover we highlight that, for fractional equations, the nonlocal part strongly influences the decomposition of the space and this makes quite delicate to use truncating test functions and perform the localization of the centers of mass.

In the present Chapter we need to implement new ideas to overcome the above obstructions; in particular we introduce a new *fractional local center of mass* by means of a suitable seminorm, stronger than the usual Gagliardo seminorm in a bounded set.

Our main result is the following theorem.

Theorem 5.1.2. *Suppose $N \geq 2$ and that (V1)-(V2), (f1)–(f3) hold. Let K be defined by (5.1.6). Then, for sufficiently small $\varepsilon > 0$, equation (5.1.3) has at least* $\mathrm{cupl}(K) + 1$ *positive solutions, which belong to $C^{0,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $\sigma \in (0,1)$.*

Here $\mathrm{cupl}(K)$ denotes the *cup-length* of K defined by the Alexander-Spanier cohomology with coefficients in some field $\mathbb{F}$ (see Appendix A). Notice that the cup-length of a set K is strictly related to the *category* of K, see Lemma A.10 and Remark A.11.

Remark 5.1.3. *Observe that, arguing as in [78] and [81], we could omit the assumption that V is bounded from above in Theorem 5.1.2. For the sake of simplicity, we assume here the boundedness of V.*

The regularity statement in Theorem 5.1.2 relies on some recent regularity results based on fractional De Giorgi classes and tail functions (see Section 1.2.5); notice that the fact that the noncriticality is *strict* in (f1.3) (that is $p < 2_s^* - 1$) is here needed. Through these results we are able to prove also the concentration of the solutions.

Theorem 5.1.4. *In the assumptions of Theorem 5.1.2, let v_ε be one of the* $\mathrm{cupl}(K) + 1$ *family of solutions of equation (5.1.3). Then, $(v_\varepsilon)_{\varepsilon>0}$* concentrates *in K as $\varepsilon \to 0$, i.e. there exist a maximum points $x_\varepsilon \in \mathbb{R}^N$ of v_ε such that*

$$\lim_{\varepsilon \to 0} d(x_\varepsilon, K) = 0;$$

moreover, for some positive C', C'', we have the uniform polynomial decay

$$\frac{C'}{1+|\frac{x-x_\varepsilon}{\varepsilon}|^{N+2s}} \leq v_\varepsilon(x) \leq \frac{C''}{1+|\frac{x-x_\varepsilon}{\varepsilon}|^{N+2s}}, \quad \text{for } x \in \mathbb{R}^N.$$

In addition, let $(\varepsilon_n)_n$ with $\varepsilon_n \to 0^+$ as $n \to +\infty$. Then, up to a subsequence, there exists a point $x_0 \in K$ sucht that $x_{\varepsilon_n} \to x_0$ as $n \to +\infty$, and $v_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n})$ converges in $H^s(\mathbb{R}^N)$ and uniformly on compact sets to a least energy solution of

$$(-\Delta)^s U + m_0 U = f(U), \quad U > 0, \quad U \in H^s(\mathbb{R}^N). \tag{5.1.7}$$

This first part of the Chapter is organized as follows. In Section 5.1.1 we recall the mixed Gagliardo seminorm introduced in Section 1.2.1, while in Section 5.2 we show the uniform polynomial decay of the solutions of (5.1.5), and we introduce a new fractional center of mass Υ, by means of a suitable seminorm. Section 5.3 is the main core of the Chapter, where we introduce a penalized functional and prove a deformation lemma on a neighborhood of expected solutions; moreover, we build suitable maps Φ_ε, Ψ_ε essential in the proof of the multiplicity of solutions. Then in Section 5.4 we prove Theorem 5.1.2 by the use of the deformation lemma and the built maps applied to the theory of relative category and relative cup-length. Finally we prove Theorem 5.1.4 by using regularity results based on fractional De Giorgi classes.

Afterwards, in Section 5.5 we move to the study of the critical case.

5.1.1 A tail-controlling mixed norm

In this Chapter we will make use of the following norm

$$\|u\|^2_{H^s_\varepsilon(\mathbb{R}^N)} := \|(-\Delta)^{s/2}u\|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx$$

which is an equivalent norm on $H^s(\mathbb{R}^N)$ (once ε is fixed), thanks to the positivity and the boundedness of V; the space $H^s_\varepsilon(\mathbb{R}^N)$ is defined straightforwardly. Moreover we will make use of the *mixed* Gagliardo seminorm (introduced in Section 1.2.1)

$$[u]^2_{A_1,A_2} = \int_{A_1}\int_{A_2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dx\, dy, \quad [u]_A := [u]_{A,A}$$

for any $A_1, A_2, A \subset \mathbb{R}^N$ and $u \in H^s(\mathbb{R}^N)$. For any $u \in H^s(\mathbb{R}^N)$ and $A \subset \mathbb{R}^N$ it will be useful to work also with the following norms:

$$\|u\|_A^2 := \|u\|^2_{L^2(A)} + [u]^2_{A,\mathbb{R}^N} \tag{5.1.8}$$

and

$$|||u|||_A := \|u\|_A + \|u\|_{L^{p+1}(A)}, \tag{5.1.9}$$

where p is introduced in assumption (f1.3). We highlight that $\|u\|_{\mathbb{R}^N} = \|u\|_{H^s(\mathbb{R}^N)}$, but generally $\|u\|_A \geq \|u\|_{H^s(A)}$ for $A \neq \mathbb{R}^N$. By $H^s(A) \hookrightarrow L^{p+1}(A)$ the norms $\|\cdot\|_A$ and $|||\cdot|||_A$ are equivalent: on the other hand, the constant such that $|||u|||_A \leq C_A\|u\|_A$ depends on A, thus not useful for expanding sets $A = A(\varepsilon)$. This is why we will make direct use of $|||\cdot|||_A$.

Before ending this Section, we highlight that, by the assumptions on f, for each $q \geq p$ and $\beta > 0$ there exists a $C_\beta > 0$ such that

$$|f(t)| \leq \beta|t| + C_\beta|t|^q \quad \text{and} \quad |F(t)| \leq C\left(\beta|t|^2 + C_\beta|t|^{q+1}\right). \tag{5.1.10}$$

5.2 Limiting equation

In this Section we further investigate the autonomous equation studied in Section 2.2.

5.2.1 A single equation

Consider

$$(-\Delta)^s U + aU = f(U), \quad x \in \mathbb{R}^N \tag{5.2.11}$$

with $a > 0$. Weak solutions of (5.2.11) are known to be characterized as critical points of the C^1-functional $L_a : H^s(\mathbb{R}^N) \to \mathbb{R}$

$$L_a(U) := \frac{1}{2}\|(-\Delta)^{s/2}U\|_2^2 + \frac{a}{2}\|U\|_2^2 - \int_{\mathbb{R}^N} F(U)dx, \quad U \in H^s(\mathbb{R}^N).$$

Set moreover the Pohozaev functional $P_a : H^s(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}_+$

$$P_a(U) := \left(2_s^* \frac{\int_{\mathbb{R}^N} F(U)dx - \frac{a}{2}\|U\|_2^2}{\|(-\Delta)^{s/2}U\|_2^2}\right)_+^{\frac{1}{2s}}, \quad U \in H^s(\mathbb{R}^N),\ U \neq 0.$$

We further set

$$C_{po,a} := \inf\left\{L_a(U) \mid U \in H^s(\mathbb{R}^N) \setminus \{0\},\ P_a(U) = 1\right\}$$

the Pohozaev minimum energy, and

$$E_a := \inf\left\{L_a(U) \mid U \in H^s(\mathbb{R}^N) \setminus \{0\},\ L_a'(U) = 0\right\}$$

the least energy for L_a.

We recall the following result by Section 2.2, where we further highlight the regularity, the positivity and the decay at infinity (see [79, Theorems 1.1–1.3] and [177, Theorem 1.5]).

Theorem 5.2.1. *Assume (f1) with respect to $a > 0$ and (f2).*

- *There exists a positive minimizer for $C_{po,a} > 0$, which is a weak solution of (5.2.11).*
- *Every weak solution $U \in H^s(\mathbb{R}^N)$ of (5.2.11) is actually a strong solution, i.e. U satisfies (5.2.11) almost everywhere. Moreover $U \in H^{2s}(\mathbb{R}^N) \cap C^\sigma(\mathbb{R}^N)$ for every $\sigma \in (0, 2s)$.*
- *Every weak solution $U \in H^s(\mathbb{R}^N)$ of (5.2.11) is strictly positive and decays polynomially at infinity, that is there exist positive constants C_a', C_a'' such that*

$$\frac{C_a'}{1+|x|^{N+2s}} \le U(x) \le \frac{C_a''}{1+|x|^{N+2s}}, \quad \text{for } x \in \mathbb{R}^N. \tag{5.2.12}$$

  Observe that the bounding functions in (5.2.12) belong to $L^q(\mathbb{R}^N)$ for any $q \in [1, +\infty]$.

- *If (f3) holds, then the Pohozaev identity*

$$P_a(U) = 1$$

  holds for each nontrivial solution U of (5.2.11). As a consequence

$$E_a = C_{po,a}. \tag{5.2.13}$$

We observe that to reach the Pohozaev identity we need the solutions to be regular enough, fact that is given by (f3). The functional P_a will be of key importance for estimating L_a from below, see Lemma 5.2.3 and Lemma 5.2.6.

We highlight the polynomial decay of solutions of (5.2.11). This decay is much less slower than the one, exponential, of the local case $s = 1$. An alternative proof, which underlines some uniformity in a, can be found in Proposition 5.4.2.

Remark 5.2.2. *Since the equation is satisfied almost everywhere, we have also, by* (5.1.10)

$$|(-\Delta)^s U(x)| \leq |f(U(x))| + a|U(x)| \leq \beta|U(x)| + C_\beta |U(x)|^p + a|U(x)|$$
$$\leq C\left(\frac{1}{1+|x|^{(N+2s)p}} + \frac{1}{1+|x|^{N+2s}}\right) \leq C\frac{1}{1+|x|^{N+2s}}$$

for almost every $x \in \mathbb{R}^N$.

We end this Section by a technical lemma, which allows to link the level $L_a(u)$ of a whatever function u, having a functional $P_a(u) \approx 1$, with the ground state E_a; in particular, it provides a useful lower bound for L_a.

Lemma 5.2.3. *Let $u \in H^s(\mathbb{R}^N)$ and define*

$$g(t) := \frac{1}{2s}\left(Nt^{N-2s} - (N-2s)t^N\right), \quad t \in \mathbb{R}.$$

(a) If $P_a(u) \in \left(0, \left(\frac{N}{N-2s}\right)^{\frac{1}{2s}}\right)$, which we highlight is a neighborhood of 1, then

$$L_a(u) \geq g(P_a(u))E_a.$$

(b) If $u = U_a\left(\frac{\cdot - q}{t}\right)$ for some $q \in \mathbb{R}^N$ and $t \in \mathbb{R}$, with U_a being a ground state of (5.2.11)*, then the above inequality is indeed an equality, that is*

$$L_a\left(U_a\left(\frac{\cdot - q}{t}\right)\right) = g(t)E_a.$$

We highlight that the function g verifies

$$g(t) \leq 1 \quad \text{and} \quad g(t) = 1 \iff t = 1.$$

Proof. Let $\sigma := P_a(u)$ and set $v := u(\sigma\cdot)$. Then $P_a(v) = 1$. A straightforward computation shows, by using $P_a(v) = 1$ and $g(\sigma) > 0$, that

$$L_a(u) = g(\sigma)L_a(v) \geq g(\sigma)C_{po,a} = g(\sigma)E_a.$$

We see that, if $u = U_a\left(\frac{\cdot - q}{t}\right)$, then by $P_a(U_a) = 1$ we have $\sigma = P_a\left(U_a\left(\frac{\cdot - q}{t}\right)\right) = t$ and thus $v = U_a(\cdot - q)$, which by translation invariance is again a ground state of (5.2.11); thus $L_a(v) = C_{po,a}$. This concludes the proof. ∎

5.2.2 A family of equations: minimal radius map

In this Section we study equation (5.2.11) for variable values of $a > 0$. Introduce the notation

$$\Omega[a, b] := V^{-1}([m_0 + a, m_0 + b]) \cap \Omega$$

and similarly $\Omega(a, b)$ and mixed-brackets combinations. We choose now a small $\nu_0 > 0$ such that the minimum m_0 is not heavily perturbed, namely

- Berestycki-Lions type assumptions (f1) hold with respect to $a \in [m_0, m_0 + \nu_0]$ – i.e., in particular, $F(t_0) > \frac{1}{2}(m_0 + \nu_0)t_0^2$;
- assumption (V2) holds with respect to $m_0 + \nu_0$, i.e. $m_0 + \nu_0 < \inf_{\partial\Omega} V$;
- $\overline{\Omega[0, \nu_0]} \subset K_d \subset \Omega$ for a sufficiently small $d > 0$ subsequently fixed, see Lemma A.5;

- other conditions subsequently stated, see e.g. (5.2.14) and Lemma 5.2.6.

We observe that, by construction, for $a \in [m_0, m_0+\nu_0]$ the considerations of Section 5.2.1 apply. Moreover, by scaling arguments on $C_{po,a}$, we notice that

$$a \in [m_0, m_0+\nu_0] \mapsto E_a \in (0, +\infty)$$

is strictly increasing and that, up to choosing a smaller ν_0, we have[1] $E_{m_0+\nu_0} < 2E_{m_0}$ and thus we can find an $l_0 = l_0(\nu_0) \in \mathbb{R}$ such that

$$E_{m_0+\nu_0} < l_0 < 2E_{m_0}. \tag{5.2.14}$$

As a final step in the proof of the main Theorem, we will make ν_0 and l_0 moving such that $\nu_0^n \to 0$ and $l_0^n \to E_{m_0}$. We now define the set of *almost* ground states of (5.2.11)

$$S_a := \left\{ U \in H^s(\mathbb{R}^N) \setminus \{0\} \mid L_a'(U) = 0,\ L_a(U) \leq l_0,\ U(0) = \max_{\mathbb{R}^N} U \right\} \neq \emptyset.$$

We observe that we set the last condition in order to fix solutions in a point and prevent them to escape to infinity; the idea is to gain thickness and compactness (see [78, 119, 122]): notice indeed that, in the case of a proper ground state $U \in S_a$, then U is radially symmetric (see also (5.5.81)). We further define

$$\widehat{S} := \bigcup_{a \in [m_0, m_0+\nu_0]} S_a.$$

The following properties of the set $\widehat{S}$ will be of key importance in the whole paper.

Lemma 5.2.4. *The following properties hold.*

(a) There exist positive constants C', C'' such that, for each $U \in \widehat{S}$ we have

$$\frac{C'}{1+|x|^{N+2s}} \leq U(x) \leq \frac{C''}{1+|x|^{N+2s}}, \quad \text{for } x \in \mathbb{R}^N. \tag{5.2.15}$$

(b) $\widehat{S}$ is compact. Since it does not contain the zero function, we have

$$r^* := \min_{U \in \widehat{S}} \|U\|_{H^s(\mathbb{R}^N)} > 0;$$

the maximum is attained as well.

(c) We have

$$\lim_{R \to +\infty} \|U\|_{\mathbb{R}^N \setminus B_R} = 0, \quad \text{uniformly for } U \in \widehat{S},$$

where the norm $\|\cdot\|_{\mathbb{R}^N \setminus B_R}$ is defined in (5.1.8). Moreover, if $(U_n)_n \subset \widehat{S}$, and $(\theta_n)_n \subset \mathbb{R}^N$ is bounded, then

$$\lim_{n \to +\infty} \|U_n(\cdot + \theta_n)\|_{\mathbb{R}^N \setminus B_n} = 0.$$

Proof. We divide the proof in some steps.
Step 1. We see that $\widehat{S}$ is bounded. Indeed, by the Pohozaev identity, we have

$$\|(-\Delta)^{s/2} U\|_2^2 = \frac{N}{s} L_a(U) \leq \frac{N}{s} l_0.$$

[1] Let ν_0 to be fixed, and v be a m_0-Pohozaev minimum (i.e. $L_{m_0}(v) = C_{po,m_0}$). Let rescale v in such a way it belongs to the $(m_0+\nu_0)$-Pohozaev set, i.e. $u := v(\cdot/\theta)$ for some explicit θ: computation shows $\theta > 1$ and $\theta \to 1$ as $\nu_0 \to 0$. Thus $C_{po,m_0+\nu_0} \leq L_{m_0+\nu_0}(u) = \theta^{N-2s} L_{m_0}(v) = \theta^{N-2s} C_{po,m_0}$. By choosing ν_0 small we have $\theta^{N-2s} < 2$.

By (1.2.7) we have that also $\|U\|_{2_s^*}$ is uniformly bounded. Since $L_a'(U)U=0$, we have by (5.1.10)

$$\|(-\Delta)^{s/2}U\|_2^2 + a\|U\|_2^2 = \int_{\mathbb{R}^N} f(U)U\,dx \leq \beta\|U\|_2^2 + C_\beta\|U\|_{2_s^*}^{2_s^*}$$

which implies, by choosing $\beta < a$, that also $\|U\|_2^2$ is bounded.
Step 2. There exist uniform $C>0$ and $\sigma \in (0,1)$ such that

$$\|U\|_\infty \leq C, \quad [U]_{C^{0,\sigma}_{loc}(\mathbb{R}^N)} \leq C \tag{5.2.16}$$

for any $U \in \widehat{S}$. We postpone the proof of (5.2.16), as well as the proof of the uniform pointwise estimate (5.2.15) (where we use (f3)), since they will carry some arguments used in the proof of Theorem 5.1.4 in Section 5.4.1; see Proposition 5.4.2.

We show now (b), which is a refinement of the fact that S_a itself is compact.
Step 3. We observe first that $\widehat{S}$ is closed. Indeed, if $U_k \in S_{a_k} \subset \widehat{S}$ converges strongly to U, then up to a subsequence we have $a_k \to a \in [m_0, m_0+\nu_0]$ and, by the strong convergence, we have that the condition

$$E_{m_0} \leq L_a(U) \leq l_0$$

holds, which in particular implies that $U \not\equiv 0$. Moreover, exploiting the weak convergence $U_k \rightharpoonup U$, and the almost everywhere convergence (together with the estimate on f, the uniform estimate (5.2.15) and the dominated convergence theorem), we obtain that for each $v \in H^s(\mathbb{R}^N)$

$$\begin{aligned} 0 = L'_{a_k}(U_k)v &= \int_{\mathbb{R}^N} (-\Delta)^{s/2}U_k(-\Delta)^{s/2}v\,dx + a_k\int_{\mathbb{R}^N} U_k v\,dx - \int_{\mathbb{R}^N} f(U_k)v\,dx \\ &\to \int_{\mathbb{R}^N} (-\Delta)^{s/2}U(-\Delta)^{s/2}v\,dx + a\int_{\mathbb{R}^N} Uv\,dx - \int_{\mathbb{R}^N} f(U)v\,dx = L'_a(U)v, \end{aligned}$$

that is, $L'_a(U)=0$. As regards the maximum in zero, we need a pointwise convergence. In order to get it, we exploit the fact that, by (5.2.16), U_k are uniformly bounded in $L^\infty(\mathbb{R}^N)$ and in $C^{0,\sigma}_{loc}(\mathbb{R}^N)$ and we apply Ascoli-Arzelà theorem to get local uniform convergence. This shows that $U \in S_a \subset \widehat{S}$.
Step 4. Let now $U_k \in S_{a_k} \subset \widehat{S}$. By the boundedness, up to a subsequence we have $U_k \rightharpoonup U \in H^s(\mathbb{R}^N)$ and $a_k \to a \in [m_0, m_0+\nu_0]$. We need to show that $\|U_k\|_{H^s(\mathbb{R}^N)} \to \|U\|_{H^s(\mathbb{R}^N)}$; the closedness of $\widehat{S}$ will conclude the proof.

As observed in Step 3, we have by the weak convergence that $L'_{a_k}(U_k)U_k = 0 = L'_a(U)U$; hence, if $R>0$ is some radius to be fixed, we gain

$$\begin{aligned} &\Big|\Big(\|(-\Delta)^{s/2}U_k\|_2^2 + a_k\|U_k\|_2^2\Big) - \Big(\|(-\Delta)^{s/2}U\|_2^2 + a\|U\|_2^2\Big)\Big| \\ &\leq \left|\int_{|x|\leq R} f(U_k)U_k dx - \int_{|x|\leq R} f(U)U dx\right| + \\ &\quad + \int_{|x|>R} |f(U_k)U_k| dx + \int_{|x|>R} |f(U)U)| dx =: (I) + (II). \end{aligned}$$

Fix now a small $\eta>0$. As regards (II), we have by (5.1.10)

$$\int_{|x|>R} |f(U_k)U_k| dx \leq \beta \int_{|x|>R} |U_k|^2 dx + C_\beta \int_{|x|>R} |U_k|^{p+1} dx < \eta$$

for sufficiently (uniformly in k) large $R>0$ thanks to (5.2.15); up to taking a larger R, it holds also for U.

Fixed this $R>0$, focusing on (I), by Proposition 1.5.5 we have

$$\left|\int_{|x|\leq R} f(U_k)U_k dx - \int_{|x|\leq R} f(U)U dx\right| < \eta$$

for sufficiently large $k = k(R)$. Merging together, we obtain

$$\|(-\Delta)^{s/2}U_k\|_2^2 + a_k\|U_k\|_2^2 \to \|(-\Delta)^{s/2}U\|_2^2 + a\|U\|_2^2$$

which with elementary passages leads to the claim.

Step 5. Finally, we prove (c). By contradiction, there exists an $\eta > 0$ such that, for each $n \in \mathbb{N}$ there exists a $U_n \in \widehat{S}$ which satisfies

$$\|U_n\|_{\mathbb{R}^N \setminus B_n} > \eta.$$

By the compactness, we have, up to a subsequence, $U_n \to U \in \widehat{S}$ as $n \to +\infty$. Thus (notice that $\int_{\mathbb{R}^N} \frac{|U(x)-U(\cdot)|^2}{|x-\cdot|^{N+2s}} dx \in L^1(\mathbb{R}^N)$ and absolute integrability of the integral applies)

$$\eta < \|U_n\|_{\mathbb{R}^N \setminus B_n} \leq \|U_n - U\|_{H^s(\mathbb{R}^N)} + \|U\|_{\mathbb{R}^N \setminus B_n} \to 0$$

which is an absurd.

For the second part, we argue similarly. Indeed, up to a subsequence, $U_n \to U$ in $H^s(\mathbb{R}^N)$ and $\theta_n \to \theta$ in $\mathbb{R}^N$, thus

$$\begin{aligned} &\|U_n(\cdot - \theta_n)\|_{\mathbb{R}^N \setminus B_n} \\ &\quad \leq \|U_n - U\|_{H^s(\mathbb{R}^N)} + \|\tau_{\theta_n} U - \tau_\theta U\|_{H^s(\mathbb{R}^N)} + \|U(\cdot - \theta)\|_{\mathbb{R}^N \setminus B_n} \to 0, \end{aligned}$$

where τ_θ is the translation. This concludes the proof. ∎

Remark 5.2.5. *The compactness of the set $\widehat{S}$ of (almost) ground states is somehow expected by thinking at the power case $f(u) = |u|^{p-2}u$: in this case, indeed, the ground state is unique (and nongenerate) [190, 247]. On the other hand, in the general case (for examples for suitable sums of powers), uniqueness seems not to be the case [145, 376].*

Gained compactness, we turn back considering the set of all the solutions (with no restrictions in zero), that is

$$\widehat{S}' := \bigcup_{p \in \mathbb{R}^N} \tau_p(\widehat{S});$$

we observe that $\widehat{S}'$ is bounded. Moreover we define an open r-neighborhood of $\widehat{S}'$, reminiscent of the perturbation approach in [13, 146, 184],

$$S(r) := \{u \in H^s(\mathbb{R}^N) \mid d(u, \widehat{S}') < r\}$$

that is

$$S(r) = \Big\{u = U(\cdot - p) + \varphi \in H^s(\mathbb{R}^N) \mid U \in \widehat{S},\ p \in \mathbb{R}^N,\ \varphi \in H^s(\mathbb{R}^N),\ \|\varphi\|_{H^s(\mathbb{R}^N)} < r\Big\}.$$

In order to re-gain some compactness, we aim to detect and somehow bound the point of translation and the size of the error. To this last goal, we define a *minimal radius* map $\widehat{\rho} : H^s(\mathbb{R}^N) \to \mathbb{R}_+$ by

$$\widehat{\rho}(u) := \inf\Big\{\|u - U(\cdot - y)\|_{H^s(\mathbb{R}^N)} \mid U \in \widehat{S},\ y \in \mathbb{R}^N\Big\}.$$

We observe

$$u \in S(r) \implies \widehat{\rho}(u) < r, \tag{5.2.17}$$

and in addition

$$\widehat{\rho}(u) = \inf\{t \in \mathbb{R}_+ \mid u \in S(t)\},$$

where the infimum on the right-hand side is not attained. Finally, $\widehat{\rho} \in Lip(H^s(\mathbb{R}^N), \mathbb{R})$ with Lipschitz constant equal to 1, that is, for every $u, v \in H^s(\mathbb{R}^N)$,

$$|\widehat{\rho}(u) - \widehat{\rho}(v)| \le \|u - v\|_{H^s(\mathbb{R}^N)}. \tag{5.2.18}$$

The detection of the point of translation will be instead more tricky, and will be investigated in Section 5.2.3.

We end this Section with two technical lemmas. The first one is a direct consequence of Lemma 5.2.3, and allows to link the level $L_{m_0}(u)$ of a whatever function $u \in S(r)$ with the ground state E_{m_0}, once r is sufficiently small; this further gives a lower bound for the functional L_{m_0}.

Lemma 5.2.6. *Up to taking a smaller $\nu_0 = \nu_0(l_0) > 0$, there exists a sufficiently small $r' = r'(\nu_0, r^*) > 0$ such that, for every $u \in S(r')$, we have*

$$L_{m_0}(u) \ge g(P_{m_0}(u))E_{m_0}.$$

Proof. By Lemma 5.2.3 (a), we know that the inequality holds if $P_{m_0}(u)$ is in a neighborhood of the value 1. Observe that $P_a(U) = 1$ if $U \in S_a$: by continuity and compactness, $P_{m_0}(U) \approx 1$ if $U \in S_a$ and $a \approx m_0$. In particular, by choosing a small value of ν_0, $P_{m_0}(U) \approx 1$ for $U \in \widehat{S}$. Indeed

$$P_{m_0}(U) = 1 + \frac{1}{2s}\frac{N}{N-2s}\,(a - m_0)\frac{\|U\|_2^2}{\|(-\Delta)^{s/2}U\|_2^2} + o(1);$$

the addendum on the right-hand side can be bounded by the maximum and the minimum over $\widehat{S}$ (notice that $(-\Delta)^{s/2}U$ cannot be zero) and thus we can find a uniform small ν_0. Again by continuity we have that $P_{m_0}(u) \approx 1$ for $u \in S(r')$, r' sufficiently small. Indeed

$$P_{m_0}(u) = 1 - \frac{1}{2s}\frac{N}{N-2s}\frac{1}{\|(-\Delta)^{s/2}U\|_2^2}\big(L_{m_0}(U(\cdot - p) + \varphi) - L_{m_0}(U(\cdot - p))\big) + o(1).$$

This concludes the proof. ∎

We notice that the condition $E_{m_0+\nu_0} < l_0$ keeps holding by decreasing ν_0, so no ambiguity in the l_0-depending choice of ν_0 in Lemma 5.2.6 arises. We focus now on the second lemma.

Lemma 5.2.7. *There exist $\nu_1 \in (0, \nu_0)$ and $\delta_0 = \delta_0(\nu_1) > 0$ such that*

$$L_{m_0+\nu_1}(U) \ge E_{m_0} + \delta_0, \quad \textit{uniformly for } U \in \widehat{S}.$$

Proof. Observe first that, since $\widehat{S}$ is compact also in $L^2(\mathbb{R}^N)$, we have also finite and strictly positive minimum $\underline{M}$ and maximum $\overline{M}$ with respect to $\|\cdot\|_2$. Consider $\nu_1 \in (0, \nu_0)$ such that

$$E_{m_0+\nu_1} - E_{m_0} > \frac{1}{2}(\nu_0 - \nu_1)\overline{M};$$

we notice that such ν_1 exists since, as $\nu_1 \to \nu_0^+$, the left hand side positively increases while the right hand side goes to zero. Let now $a \in [m_0, m_0+\nu_0]$; we consider two cases. If $a \in [m_0, m_0+\nu_1]$ we argue as follow: for $U \in S_a$ we have

$$\begin{aligned} L_{m_0+\nu_1}(U) &= L_a(U) + \tfrac{1}{2}(m_0 + \nu_1 - a)\|U\|_2^2 \\ &\ge E_a + \tfrac{1}{2}(m_0 + \nu_1 - a)\underline{M} =: (I) + (II); \end{aligned}$$

now, the quantity (I) is minimum when $a = m_0$, while (II) is minimum when $a = m_0 + \nu_1$; if both could apply at the same time, we would have as a minimum the quantity E_{m_0}. Since it is not possible, we obtain

$$\inf_{U \in \bigcup_{a\in[m_0, m_0+\nu_1]} S_a} L_{m_0+\nu_1}(U) > E_{m_0}.$$

If $a \in (m_0+\nu_1, m_0+\nu_0]$ instead, we have

$$L_{m_0+\nu_1}(U) \geq E_a - \tfrac{1}{2}(a-(m_0+\nu_1))\overline{M} \geq E_{m_0+\nu_1} - \tfrac{1}{2}(\nu_0-\nu_1)\overline{M}$$

and thus, by the property on ν_1,

$$\inf_{U \in \bigcup_{a\in[m_0+\nu_1, m_0+\nu_0]} S_a} L_{m_0+\nu_1}(U) \geq E_{m_0+\nu_1} - \tfrac{1}{2}(\nu_0-\nu_1)\overline{M} > E_{m_0}.$$

This concludes the proof, by taking as $\delta_0 > 0$ the smallest of the two differences. ∎

5.2.3 Fractional center of mass

As in [119], inspired by [43, 81, 149], we want to define a *barycentric map* Υ which, given a function $u = U(\cdot - p) + \varphi \in S(r)$, gives an estimate on the maximum point p of $U(\cdot - p)$; since φ is small and U decays (polynomially) at infinity, p is, in some ways, the *center of mass* of u. The idea will be to bound $\Upsilon(u)$ in order to re-gain compactness.

Since the nonlocality comes into the very definition of the ambient space, we need the use of the norm (5.1.8), which we notice being stronger than the one induced by the Gagliardo seminorm.

Lemma 5.2.8. *Let r^* be as in Lemma 5.2.4. Then there exist a sufficiently large $R_0 > 0$, a sufficiently small radius $r_0 \in (0, r^*)$ and a continuous map*

$$\Upsilon : S(r_0) \to \mathbb{R}^N$$

such that, for each $u = U(\cdot - p) + \varphi \in S(r_0)$ we have

$$|\Upsilon(u) - p| \leq 2R_0.$$

Moreover, Υ is continuous and $-\Upsilon$ is shift-equivariant, *that is, $\Upsilon(u(\cdot + \xi)) = \Upsilon(u) - \xi$ for every $u \in S(r_0)$ and $\xi \in \mathbb{R}^N$.*

Proof. Recalled that $r^* = \min_{U\in\widehat{S}} \|U\|_{H^s(\mathbb{R}^N)} > 0$, we have by Lemma 5.2.4

$$\|U\|_{\mathbb{R}^N\setminus B_{R_0}} < \tfrac{1}{8}r^*, \qquad \text{uniformly for } U \in \widehat{S} \tag{5.2.19}$$

for $R_0 \gg 0$. Thus

$$r^* \leq \|U\|_{H^s(\mathbb{R}^N)} \leq \|U\|_{B_{R_0}} + \|U\|_{\mathbb{R}^N\setminus B_{R_0}} < \|U\|_{B_{R_0}} + \tfrac{1}{8}r^*$$

which implies

$$\|U\|_{B_{R_0}} > \tfrac{7}{8}r^* \quad \text{and} \quad \|U\|_{\mathbb{R}^N\setminus B_{R_0}} < \tfrac{1}{8}r^*$$

for each $U \in \widehat{S}$. Consider now a cutoff function $\psi \in C_c^\infty(\mathbb{R}_+)$ such that

$$[0, \tfrac{1}{4}r^*] \prec \psi \prec [\tfrac{1}{2}r^*, +\infty);$$

let $r_0 \in (0, \frac{1}{8}r^*)$ and define, for each $u \in S(r_0)$ and $q \in \mathbb{R}^N$, a *density* function

$$d(q,u) := \psi\left(\inf_{\tilde{U}\in\widehat{S}} \|u - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)}\right).$$

Notice that $d(\cdot, u)$ is an integrable function: indeed $[u(\cdot + \xi)]_{B_{R_0}(q)} = [u]_{B_{R_0}(q+\xi)}$ for $\xi \in \mathbb{R}^N$, and $q \mapsto \|u - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)} = \|\tau_q u - \tilde{U}\|_{B_{R_0}}$ is continuous by

$$|\|\tau_q u - \tilde{U}\|_{B_{R_0}} - \|\tau_p u - \tilde{U}\|_{B_{R_0}}| \leq \|\tau_q u - \tau_p u\|_{B_{R_0}} \leq \|\tau_q u - \tau_p u\|_{H^s(\mathbb{R}^N)} \to 0$$

as $p \to q$, and the infimum over continuous functions is upper semicontinuous.

If we show that $d(\cdot,u) \geq 0$ is not identically zero and it has compact support, then it will be well defined the quantity

$$\Upsilon(u) := \frac{\int_{\mathbb{R}^N} q\, d(q,u)dq}{\int_{\mathbb{R}^N} d(q,u)dq}.$$

We show first that $d(\cdot,u)$ has compact support. Indeed if $u = U(\cdot - p) + \varphi$ and $\tilde{U} \in \widehat{S}$ is arbitrary, then

$$\begin{aligned} &\|u - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)} \\ &\geq \ \|\tilde{U}(\cdot - q)\|_{B_{R_0}(q)} - \|U(\cdot - p)\|_{B_{R_0}(q)} - \|\varphi\|_{B_{R_0}(q)} \\ &\geq \ \|\tilde{U}\|_{B_{R_0}} - \|U\|_{B_{R_0}(q-p)} - \|\varphi\|_{H^s(\mathbb{R}^N)} \geq \tfrac{6}{8}r^* - \|U\|_{B_{R_0}(q-p)}; \end{aligned}$$

take now $q \notin B_{2R_0}(p)$: if $x \in B_{R_0}(q-p)$, by the fact that $|x-(q-p)| < R_0$ and $|q-p| \geq 2R_0$, we obtain that $|x| \geq R_0$, that is, $x \in \mathbb{R}^N \setminus B_{R_0}$. Therefore by (5.2.19)

$$\|u - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)} \geq \tfrac{6}{8}r^* - \|U\|_{\mathbb{R}^N \setminus B_{R_0}} \geq \tfrac{5}{8}r^* > \tfrac{1}{2}r^* \tag{5.2.20}$$

thus $\inf_{U \in \widehat{S}} \|u - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)} \geq \frac{1}{2}r^*$ and hence $d(q,u) = 0$ for $q \notin B_{2R_0}(p)$; this means that $\mathrm{supp}(d(\cdot,u)) \subset B_{2R_0}(p)$.

We show next that $d(\cdot,u)$ is equal to 1 on a ball. Indeed if $u = U(\cdot - p) + \varphi$

$$\begin{aligned} &\inf_{\tilde{U} \in \widehat{S}} \|u - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)} \leq \|u - U(\cdot - q)\|_{B_{R_0}(q)} \\ &\leq \ \|U(\cdot - p) - U(\cdot - q)\|_{B_{R_0}(q)} + \|\varphi\|_{B_{R_0}(q)} \leq \|\tau_{p-q}U - U\|_{B_{R_0}} + \tfrac{1}{8}r^*. \end{aligned}$$

We can make the first term as small as we want by taking $|p-q|$ small, that is

$$\inf_{U \in \widehat{S}} \|u - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)} \leq \tfrac{1}{4}r^*$$

for $q \in B_r(p)$, r small, which implies $d(q,u) = 1$.

By the fact that $B_r(p) \subset \mathrm{supp}(d(\cdot,u)) \subset B_{2R_0}(p)$ we have the well posedness of $\Upsilon(u)$ and

$$\Upsilon(u) = \frac{\int_{B_{2R_0}(p)} q\, d(q,u)dq}{\int_{B_{2R_0}(p)} d(q,u)dq}.$$

The main property comes straightforward, as well as the shift equivariance. We show now the continuity. Indeed, assume $\|u-v\|_{H^s(\mathbb{R}^N)} \leq \frac{1}{8}r^*$. Then, by (5.2.20),

$$\|v - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)} \geq \|u - \tilde{U}(\cdot - q)\|_{B_{R_0}(q)} - \|v-u\|_{B_{R_0}(q)} \geq \tfrac{1}{2}r^*$$

and again we can conclude that $\mathrm{supp}(d(\cdot,v)) \subset B_{2R_0}(p)$ for each $\|u-v\|_{H^s(\mathbb{R}^N)} \leq \frac{1}{8}r^*$, where p depends only on u. Moreover, observe that $\int_{B_{2R_0}(p)} d(q,u)dq \geq \int_{B_r(p)} 1\, dq \geq |B_r| =: C_1$ not depending on u and p (and similarly $C_2 := |B_{2R_0}|$), and that $d(q,\cdot)$ is Lipschitz (since ψ and the norm are so, and the infimum over a family of Lipschitz functions is still Lipschitz). Thus we have

$$\begin{aligned} |\Upsilon(u) - \Upsilon(v)| \leq \ & \frac{\int_{B_{2R_0}(p)} |q|\, |d(q,u) - d(q,v)| dq}{\int_{B_{2R_0}(p)} d(q,u)dq} + \\ & + \int_{B_{2R_0}(p)} |q|\, d(q,v)dq \frac{\int_{B_{2R_0}(p)} |d(q,v) - d(q,u)| dq}{\int_{B_{2R_0}(p)} d(q,u)dq \int_{B_{2R_0}(p)} d(q,v)dq} \\ \leq \ & \int_{B_{2R_0}(p)} |q| dq \frac{1}{C_1}\left(1 + \frac{C_2}{C_1}\right) \|u-v\|_{H^s(\mathbb{R}^N)} =: C_p \|u-v\|_{H^s(\mathbb{R}^N)}. \end{aligned}$$

Since C_p can be bounded above by a constant of the type $C(1+\Upsilon(u))$, we have

$$\|u-v\|_{H^s(\mathbb{R}^N)} \leq r_0 \implies |\Upsilon(u) - \Upsilon(v)| \leq C(1+\Upsilon(u))\|u-v\|_{H^s(\mathbb{R}^N)};$$

in particular this implies the continuity. ∎

5.3 Singularly perturbed equation

We come back now to our equation

$$(-\Delta)^s u + V(\varepsilon x)u = f(u), \quad x \in \mathbb{R}^N. \tag{5.3.21}$$

It is known that the solutions of (5.3.21) can be characterized as critical points of the functional $I_\varepsilon : H^s(\mathbb{R}^N) \to \mathbb{R}$

$$I_\varepsilon(u) := \frac{1}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx - \int_{\mathbb{R}^N} F(u)dx, \quad u \in H^s(\mathbb{R}^N)$$

where $I_\varepsilon \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$, since $\|\cdot\|_{H^s_\varepsilon(\mathbb{R}^N)}$ is a norm.

We start with a technical result. Let ν_1 be as in Lemma 5.2.7; we want to show that the claim of the lemma continues holding, for ε small, if we replace $L_{m_0+\nu_1}$ with I_ε, and $\widehat{S}$ with $S(r_0') \cap \{\varepsilon\Upsilon(u) \in \Omega[\nu_1, \nu_0]\}$, r_0' small.

Lemma 5.3.1. *Let ν_1 and δ_0 be as in Lemma 5.2.7. Then there exist $\delta_1 \in (0, \delta_0)$ and $r_0' = r_0'(\delta_1) \in (0, r_0)$ sufficiently small, such that for every ε small we have*

$$I_\varepsilon(u) \geq E_{m_0} + \delta_1$$

for each $u \in \{u \in S(r_0') \mid \varepsilon\Upsilon(u) \in \Omega[\nu_1, +\infty)\} \supset \{u \in S(r_0') \mid \varepsilon\Upsilon(u) \in \Omega[\nu_1, \nu_0]\}$.

Proof. First we improve Lemma 5.2.7 for L_a in the direction of the nonautonomous equation. Indeed, by the assumption, we have $V(\varepsilon\Upsilon(u)) \geq m_0 + \nu_1$, that is

$$L_{V(\varepsilon\Upsilon(u))}(U) \geq L_{m_0+\nu_1}(U) \geq E_{m_0} + \delta_0$$

for any $U \in \widehat{S}$. Moreover, if $u = \tilde{U}(\cdot - p) + \tilde{\varphi} \in S(r_0)$ then, by Lemma 5.2.8, $\varepsilon p \in \Omega_{2\varepsilon R_0} \subset \Omega_{2R_0}$ which is compact. By uniform continuity of V and boundedness from above of $\widehat{S}$, we have

$$L_{V(\varepsilon p)}(U) \geq E_{m_0} + \delta_0/2 \tag{5.3.22}$$

for all $U \in \widehat{S}$ and ε small enough.

Let now r_0' to be fixed and $u = U(\cdot - p) + \varphi \in S(r_1)$. Then we have

$$I_\varepsilon(u) = I_\varepsilon(U(\cdot - p) + \varphi) = I_\varepsilon(U(\cdot - p)) + I_\varepsilon'(v)\varphi$$

for some $v \in H^s(\mathbb{R}^N)$ in the segment $[U(\cdot - p), u]$. Notice that v lies in a ball of radius $\max \widehat{S} + r_0'$ and I_ε' sends bounded sets in bounded sets (uniformly on ε); thus there exists a constant C, not depending on U, p and φ, such that

$$I_\varepsilon(u) \geq I_\varepsilon(U(\cdot - p)) - C\|\varphi\|_{H^s(\mathbb{R}^N)} \geq I_\varepsilon(U(\cdot - p)) - \delta_1/2 \tag{5.3.23}$$

for $\|\varphi\|_{H^s(\mathbb{R}^N)} < r_0'$ sufficiently small. Recalled that $\varepsilon p \in \Omega_{2R_0}$ we have, by the uniform continuity of V and the uniform estimate (5.2.15), for sufficiently small ε,

$$I_\varepsilon(U(\cdot - p)) \geq L_{V(\varepsilon p)}(U) - \delta_1/2$$

and the claim comes from (5.3.23) and (5.3.22), since, for $\delta_1 < \delta_0/4$,

$$I_\varepsilon(u) \geq E_{m_0} + \delta_0/2 - \delta_1 \geq E_{m_0} + \delta_1.$$

∎

Before introducing the penalized functional, we state another technical lemma, which gives a (trivial, but useful) lower bound for $I_\varepsilon'(v)v$ for small values of $v \in H^s(\mathbb{R}^N)$.

Lemma 5.3.2. *There exists $r_1 > 0$ sufficiently small and a constant $C > 0$ such that*

$$I'_\varepsilon(v)v \geq C\|v\|^2_{H^s(\mathbb{R}^N)} \tag{5.3.24}$$

for every $\varepsilon > 0$ and $v \in H^s(\mathbb{R}^N)$ with $\|v\|_{H^s(\mathbb{R}^N)} \leq r_1$.

Proof. We have, by (5.1.10) with $\beta < \frac{1}{2}\underline{V}$,

$$\begin{aligned} I'_\varepsilon(v)v &\geq \|(-\Delta)^{s/2}v\|_2^2 + \int_{\mathbb{R}^N} \underline{V} v^2 dx - \beta\|v\|_2^2 - C_\beta\|v\|_{p+1}^{p+1} \\ &\geq \|(-\Delta)^{s/2}v\|_2^2 + \frac{1}{2}\underline{V}\|v\|_2^2 - C_\beta\|v\|_{p+1}^{p+1} \\ &\geq C\|v\|^2_{H^s(\mathbb{R}^N)} - C_\beta\|v\|^{p+1}_{H^s(\mathbb{R}^N)} \geq C'\|v\|^2_{H^s(\mathbb{R}^N)} \end{aligned} \tag{5.3.25}$$

where the last inequality holds for $\|v\|_{H^s(\mathbb{R}^N)}$ small, since $p + 1 > 2$. ∎

Remark 5.3.3. *For a later use, we observe that one can improve (5.3.25) by*

$$\|(-\Delta)^{s/2}v\|_2^2 + \frac{1}{2}\underline{V}\|v\|_2^2 - 2^p C_\beta\|v\|_{p+1}^{p+1} \geq C\|v\|^2_{H^s(\mathbb{R}^N)} \tag{5.3.26}$$

up to choosing a smaller r_1.

5.3.1 A mass-concentrating penalization

We want to study now a penalized functional (see [78, 83, 119]), that is I_ε plus a term which forces solutions to stay in Ω.

Since $V > m_0$ on $\partial\Omega$, we can find an annulus around $\partial\Omega$ where this relation keeps holding, that is

$$V(x) > m_0, \quad \text{for } x \in \overline{\Omega_{2h_0} \setminus \Omega}$$

for h_0 sufficiently small. We then define the *mass-concentrating* penalization functional $Q_\varepsilon : H^s(\mathbb{R}^N) \to \mathbb{R}$

$$Q_\varepsilon(u) := \left(\frac{1}{\varepsilon^\alpha}\|u\|^2_{L^2(\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon))} - 1\right)_+^{\frac{p+1}{2}}, \quad u \in H^s(\mathbb{R}^N)$$

where $\alpha \in (0, \min\{1/2, s\})$.

We observe that, for every $u, v \in H^s(\mathbb{R}^N)$,

$$Q'_\varepsilon(u)v = \frac{(p+1)}{\varepsilon^\alpha}\left(\frac{1}{\varepsilon^\alpha}\|u\|^2_{L^2(\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon))} - 1\right)_+^{\frac{p-1}{2}} \int_{\mathbb{R}^N \setminus (\Omega_{2h_0}/\varepsilon)} uv\, dx$$

and it is straightforward to prove the following estimate

$$Q'_\varepsilon(u)u \geq (p+1)Q_\varepsilon(u). \tag{5.3.27}$$

We thus set

$$J_\varepsilon := I_\varepsilon + Q_\varepsilon$$

the *penalized* functional. It results that Q_ε and J_ε are in $C^1(H^s(\mathbb{R}^N), \mathbb{R})$.

We want to find critical points of J_ε and show, afterwards, that these critical points, under suitable assumptions, are critical points of I_ε too, since Q_ε will be identically zero. Let $\varepsilon = 1$: observed that $Q_1(u)$ vanishes if u have much mass inside Ω, we see that $J_1(u) = I_1(u)$ holds when the mass of u *concentrates* in Ω; this is why we say that Q_ε *forces* u to stay in Ω. Similarly, as $\varepsilon \to 0$, much less mass must be found outside Ω/ε.

We start by two technical lemmas. The first one gives a sufficient condition to pass from weak to strong convergent sequences in a Hilbert space, similarly to the convergence of the norms.

Lemma 5.3.4. *Fix $\varepsilon > 0$ and let $(u_j)_j \subset H^s(\mathbb{R}^N)$ be such that*

$$\|J'_\varepsilon(u_j)\|_{(H^s(\mathbb{R}^N))^*} \to 0 \quad \text{as } j \to +\infty. \tag{5.3.28}$$

Assume moreover that $u_j \rightharpoonup u_0$ in $H^s(\mathbb{R}^N)$ as $j \to +\infty$, and that

$$\lim_{R,j\to+\infty} \|u_j\|_{L^q(\mathbb{R}^N\setminus B_R)} = 0 \tag{5.3.29}$$

for $q = 2$ and $q = p+1$. Then $u_j \to u_0$ in $H^s(\mathbb{R}^N)$ as $j \to +\infty$.

Proof. We have by the weak lower semicontinuity of the norm

$$\liminf_{j\to+\infty} \|u_j\|_{H^s_\varepsilon(\mathbb{R}^N)} \geq \|u_0\|_{H^s_\varepsilon(\mathbb{R}^N)}. \tag{5.3.30}$$

Moreover

$$\begin{aligned} \|u_j\|^2_{H^s_\varepsilon(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} f(u_j)u_j dx + I'_\varepsilon(u_j)u_j \\ &= \left(\int_{\mathbb{R}^N} f(u_j)u_j dx - \int_{\mathbb{R}^N} f(u_0)u_0 dx \right) + \Big(I'_\varepsilon(u_j)u_j - I'_\varepsilon(u_0)u_0 \Big) + \\ &\quad + I'_\varepsilon(u_0)u_0 + \int_{\mathbb{R}^N} f(u_0)u_0 dx =: (I) + (II) + \|u_0\|^2_{H^s_\varepsilon(\mathbb{R}^N)}; \end{aligned}$$

if we prove that

$$\limsup_{j\to+\infty} \big((I) + (II) \big) \leq 0$$

we are done, because together with (5.3.30) we obtain

$$\|u_j\|_{H^s_\varepsilon(\mathbb{R}^N)} \to \|u_0\|_{H^s_\varepsilon(\mathbb{R}^N)} \quad \text{as } j \to +\infty,$$

which implies the claim, since $H^s_\varepsilon(\mathbb{R}^N)$ is a Hilbert space.

Focus on (I); we have

$$\begin{aligned} \int_{\mathbb{R}^N} (f(u_j)u_j - f(u_0)u_0) dx &= \int_{B_R} (f(u_j)u_j - f(u_0)u_0) dx + \\ &+ \int_{\mathbb{R}^N\setminus B_R} (f(u_j)u_j - f(u_0)u_0) dx =: (I_1) + (I_2). \end{aligned}$$

The piece (I_2) can be made small for j and R sufficiently large, by exploiting the estimates on f, assumption (5.3.29) and the absolute continuity of the Lebesgue integral for u_0. For such large R and j, up to taking a larger j, we can make the piece (I_1) small by Proposition 1.5.5.

Focus now on (II); we first observe that by exploiting Hölder inequalities and again classical arguments we have $I'_\varepsilon(u_j)u_0 \to I'_\varepsilon(u_0)u_0$. Thus we have, by (5.3.28),

$$\begin{aligned} \limsup_{j\to+\infty} \big(I'_\varepsilon(u_j)u_j - I'_\varepsilon(u_0)u_0 \big) &= -\liminf_{j\to+\infty} \big(Q'_\varepsilon(u_j)u_j - Q'_\varepsilon(u_j)u_0 \big) \\ &= -\liminf_{j\to+\infty} \left(\left(\frac{1}{\varepsilon^\alpha} \|u_j\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))} - 1 \right)_+^{\frac{p-1}{2}} \cdot \right. \\ &\quad \left. \cdot \left(\int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)} u_j^2 dx - \int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)} u_j u_0 dx \right) \right) \leq 0 \end{aligned}$$

where the last inequality is due to the following fact: observe first that $u_j \rightharpoonup u_0$ in $H^s_\varepsilon(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ thus (by restriction) $u_j \rightharpoonup u_0$ in $L^2(\mathbb{R}^n \setminus (\Omega_{2h_0}/\varepsilon))$; by definition of weak convergence and by the lower semicontinuity of the norm, we have

$$\liminf_{j\to+\infty} \int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)} u_j^2 dx \geq \int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)} u_0^2 dx = \lim_{j\to+\infty} \int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)} u_j u_0 dx,$$

that is

$$\liminf_{j\to+\infty}\left(\int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)} u_j^2dx - \int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)} u_ju_0dx\right)\geq 0.$$

Noticed that $a_n\geq 0$ and $\liminf_n b_n\geq 0$ imply $\liminf_n(a_nb_n)\geq 0$, we conclude. ∎

The second Lemma is a lower bound for J_ε with respect to the functional L_{m_0}. We highlight that in what follows we understand that the case $m_0=\underline{V}$, i.e. m_0 *global* minimum, gives rise to a not-perturbed result.

Lemma 5.3.5. *Set $C_{min}:=\frac{1}{2}(m_0-\underline{V})\geq 0$ we have, for ε small and $u\in H^s(\mathbb{R}^N)$,*

$$J_\varepsilon(u)\geq L_{m_0}(u)-C_{min}\varepsilon^\alpha.$$

Proof. We have, recalling that m_0 is the infimum of V over Ω_{2h_0} and $\underline{V}$ is the infimum over $\mathbb{R}^N$,

$$\begin{aligned}
J_\varepsilon(u)&=L_{m_0}(u)+\frac{1}{2}\int_{\mathbb{R}^N}(V(\varepsilon x)-m_0)u^2dx+Q_\varepsilon(u)\\
&\geq L_{m_0}(u)+\frac{1}{2}\int_{\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon)}(V(\varepsilon x)-m_0)u^2dx+Q_\varepsilon(u)\\
&\geq L_{m_0}(u)-C_{min}\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))}+Q_\varepsilon(u).
\end{aligned}$$

If $\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))}\leq 2\varepsilon^\alpha$ we have the claim by the positivity of $Q_\varepsilon(u)$. If instead $\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))}\geq 2\varepsilon^\alpha$, then

$$Q_\varepsilon(u)\geq\left(\frac{1}{2\varepsilon^\alpha}\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))}\right)^{\frac{p+1}{2}}\geq\frac{1}{2\varepsilon^\alpha}\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))}.$$

Thus

$$J_\varepsilon(u)\geq L_{m_0}+\left(\frac{1}{2\varepsilon^\alpha}-C_{min}\right)\|u\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{2h_0}/\varepsilon))}\geq L_{m_0}$$

for ε small. This concludes the proof. ∎

5.3.2 Critical points and truncated Palais-Smale condition

In order to get critical points of J_ε we want to implement a deformation argument. As usual, we need a uniform estimate from below of $\|J'_\varepsilon(u)\|_{(H^s(\mathbb{R}^N))^*}$, and this is the next goal.

First, by the strict monotonicity of E_a, let us fix $l'_0=l'_0(\nu_1)>0$ such that

$$E_{m_0}<l'_0<E_{m_0+\nu_1};$$

as well as ν_0 and l_0, even l'_0 will be let vary as $(l'_0)^n\to E_{m_0}$ in the proof of the existence.

Lemma 5.3.6. *Let r_0 and r_1 be as in Lemma 5.2.8 and Lemma 5.3.2. There exists $r'_2\in(0,\min\{r_0,r_1\})$ sufficiently small with the following property: let $0<\rho_1<\rho_0\leq r'_2$ and $(u_\varepsilon)_\varepsilon\subset S(r'_2)$ be such that*

$$\|J'_\varepsilon(u_\varepsilon)\|_{(H^s(\mathbb{R}^N))^*}\to 0 \qquad \text{as } \varepsilon\to 0, \tag{5.3.31}$$

$$J_\varepsilon(u_\varepsilon)\leq l'_0<E_{m_0+\nu_1}, \qquad \text{for any } \varepsilon>0, \tag{5.3.32}$$

with the additional assumption

$$(\widehat{\rho}(u_\varepsilon))_\varepsilon\subset[0,\rho_0], \quad (\varepsilon\Upsilon(u_\varepsilon))_\varepsilon\subset\Omega[0,\nu_0].$$

Then, for ε small

$$\widehat{\rho}(u_\varepsilon)\in[0,\rho_1], \quad \varepsilon\Upsilon(u_\varepsilon)\in\Omega[0,\nu_1].$$

We notice, by (5.3.31) and (5.3.32), that $(u_\varepsilon)_\varepsilon$ resembles a particular (truncated) Palais-Smale sequence. As an immediate consequence of the Lemma, set the sublevel

$$J_\varepsilon^c := \{u \in H^s(\mathbb{R}^N) \mid J_\varepsilon(u) \le c\}$$

we have the following theorem.

Theorem 5.3.7. *There exists $r_2' \in (0, \min\{r_0, r_1\})$ sufficiently small with the following property: if $0 < \rho_1 < \rho_0 \le r_2'$, then there exists a $\delta_2 = \delta_2(\rho_0, \rho_1) > 0$ such that, for ε small*

$$\|J_\varepsilon'(u)\|_{(H^s(\mathbb{R}^N))^*} \ge \delta_2$$

for any

$$u \in \left\{ u \in S(r_2') \cap J_\varepsilon^{l_0'} \mid (\widehat{\rho}(u), \varepsilon\Upsilon(u)) \in \big((0,\rho_0] \times \Omega[0,\nu_0]\big) \setminus \big([0,\rho_1] \times \Omega[0,\nu_1]\big) \right\}$$
$$\supset \left\{ u \in S(r_2') \cap J_\varepsilon^{l_0'} \mid \rho_1 < \widehat{\rho}(u) \le \rho_0,\ \varepsilon\Upsilon(u) \in \Omega(\nu_1, \nu_0] \right\}.$$

Remark 5.3.8. *Arguing as in the last part of the proof of Lemma 5.2.4, noticed that $\widehat{S}$ is compact not only in $H^s(\mathbb{R}^N)$ but also in $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*]$, if $(U_n)_n \subset \widehat{S}$ and $(\theta_n)_n \subset \mathbb{R}^N$ is included in a compact, we have*

$$\lim_{n\to+\infty} |\!|\!| U_n(\cdot + \theta_n) |\!|\!|_{\mathbb{R}^N \setminus B_n} = 0,$$

where the norm $|\!|\!|\cdot|\!|\!|_{\mathbb{R}^N\setminus B_n}$ is defined in (5.1.9).

Proof of Lemma 5.3.6. We use the notation, for $h > 0$,

$$\Omega_h^\varepsilon := (\Omega_{\varepsilon h})/\varepsilon = (\Omega/\varepsilon)_h$$

and notice that if $h < h'$ then $\Omega/\varepsilon \subset \Omega_h^\varepsilon \subset \Omega_{h'}^\varepsilon$. Let $r_2' < \min\{r_0, r_1\}$ to be fixed.

Step 1. *An estimate for u_ε.*
We have, for $u_\varepsilon = U_\varepsilon(\cdot - p_\varepsilon) + \varphi_\varepsilon \in S(r_2')$,

$$\begin{aligned} |\!|\!| u_\varepsilon |\!|\!|_{\mathbb{R}^N\setminus(\Omega/\varepsilon)} &\le |\!|\!| U_\varepsilon(\cdot - p_\varepsilon) |\!|\!|_{\mathbb{R}^N\setminus(\Omega/\varepsilon)} + |\!|\!| \varphi_\varepsilon |\!|\!|_{\mathbb{R}^N\setminus(\Omega/\varepsilon)} \\ &\le |\!|\!| U_\varepsilon(\cdot - p_\varepsilon + \Upsilon(u_\varepsilon)) |\!|\!|_{\mathbb{R}^N\setminus(\Omega/\varepsilon - \Upsilon(u_\varepsilon))} + C\|\varphi_\varepsilon\|_{H^s(\mathbb{R}^N)} \\ &\le |\!|\!| U_\varepsilon(\cdot - p_\varepsilon + \Upsilon(u_\varepsilon)) |\!|\!|_{\mathbb{R}^N\setminus(\Omega/\varepsilon - \Upsilon(u_\varepsilon))} + C r_2'. \end{aligned}$$

By the fact that $\varepsilon\Upsilon(u_\varepsilon) \in \Omega[\nu_1, \nu_0] \subset \Omega$, we have that $0 \in \Omega/\varepsilon - \Upsilon(u_\varepsilon)$ and thus $\Omega/\varepsilon - \Upsilon(u_\varepsilon)$ expands in $\mathbb{R}^N$ as $\varepsilon \to 0$. Moreover by Lemma 5.2.8 we have $\theta_\varepsilon := \Upsilon(u_\varepsilon) - p_\varepsilon \in B_{2R_0}$ compact. By Remark 5.3.8, for ε small we have

$$|\!|\!| u_\varepsilon |\!|\!|_{\mathbb{R}^N\setminus(\Omega/\varepsilon)} \le (1 + C) r_2' = C' r_2'. \tag{5.3.33}$$

Let

$$n_\varepsilon := \left[\frac{\sqrt{1 + 4h_0/\varepsilon} + 1}{2} \right] \in \mathbb{N}$$

which by definition satisfies $\varepsilon n_\varepsilon(n_\varepsilon + 1) \le h_0$ and $n_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We have

$$\sum_{i=1}^{n_\varepsilon} \|u_\varepsilon\|^2_{L^2(\Omega^\varepsilon_{n_\varepsilon(i+1)} \setminus \Omega^\varepsilon_{n_\varepsilon i})} \le \|u_\varepsilon\|^2_{L^2(\Omega^\varepsilon_{n_\varepsilon(n_\varepsilon+1)} \setminus \Omega^\varepsilon_{n_\varepsilon})} \le (C' r_2')^2$$

and similarly

$$\sum_{i=1}^{n_\varepsilon} [u_\varepsilon]^2_{\Omega^\varepsilon_{n_\varepsilon(i+1)} \setminus \Omega^\varepsilon_{n_\varepsilon i}, \mathbb{R}^N} \le (C' r_2')^2, \qquad \sum_{i=1}^{n_\varepsilon} \|u_\varepsilon\|^{p+1}_{L^{p+1}(\Omega^\varepsilon_{n_\varepsilon(i+1)} \setminus \Omega^\varepsilon_{n_\varepsilon i})} \le (C' r_2')^{p+1}$$

thus, for some $C = C(r_2')$,

$$\sum_{i=1}^{n_\varepsilon}\left(\|u_\varepsilon\|^2_{\Omega^\varepsilon_{n_\varepsilon(i+1)}\setminus\Omega^\varepsilon_{n_\varepsilon i}} + \|u_\varepsilon\|^{p+1}_{L^{p+1}(\Omega^\varepsilon_{n_\varepsilon(i+1)}\setminus\Omega^\varepsilon_{n_\varepsilon i})}\right) \leq C.$$

This implies that there exists $i_\varepsilon \in \{1,\dots,n_\varepsilon\}$ such that

$$\|u_\varepsilon\|^2_{A^\varepsilon} + \|u_\varepsilon\|^{p+1}_{L^{p+1}(A^\varepsilon)} \leq \frac{C}{n_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0, \tag{5.3.34}$$

where

$$A^\varepsilon := \Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)} \setminus \Omega^\varepsilon_{n_\varepsilon i_\varepsilon}$$

and C depends on r_2' (we will omit this dependence).

Step 2. *Split the sequence.*

Consider cutoff functions $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^N)$

$$\Omega^\varepsilon_{n_\varepsilon i_\varepsilon} \prec \varphi_\varepsilon \prec \Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)}$$

such that $\|\nabla\varphi_\varepsilon\|_\infty \leq \frac{C}{n_\varepsilon} = o(1)$ as $\varepsilon \to 0$ (which is possible because the distance between $\Omega^\varepsilon_{n_\varepsilon i_\varepsilon}$ and $\Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)}$ is $n_\varepsilon \to +\infty$).

Define

$$u_\varepsilon^{(1)} := \varphi_\varepsilon u_\varepsilon, \quad u_\varepsilon^{(2)} := (1-\varphi_\varepsilon)u_\varepsilon \quad \text{and} \quad u_\varepsilon = u_\varepsilon^{(1)} + u_\varepsilon^{(2)};$$

notice that both $\operatorname{supp}\left(u_\varepsilon^{(1)}u_\varepsilon^{(2)}\right)$ and $\operatorname{supp}(F(u_\varepsilon) - F(u_\varepsilon^{(1)}) - F(u_\varepsilon^{(2)}))$ are contained in A^ε, that is where we gained the estimate of the norm. Moreover, since

$$\operatorname{supp}(u_\varepsilon^{(1)}) \subset \Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)} \subset \Omega_{\varepsilon n_\varepsilon(n_\varepsilon+1)}/\varepsilon \subset \Omega_{2h_0}/\varepsilon$$

we have, by definition of Q_ε, that $Q_\varepsilon(u_\varepsilon^{(1)}) = 0$, $Q_\varepsilon(u_\varepsilon) = Q_\varepsilon(u_\varepsilon^{(2)})$ and

$$Q_\varepsilon'(u_\varepsilon^{(1)}) = 0, \quad Q_\varepsilon'(u_\varepsilon) = Q_\varepsilon'(u_\varepsilon^{(2)}). \tag{5.3.35}$$

Step 3. *Relations of the functionals.*

We show that

$$|I_\varepsilon(u_\varepsilon) - I_\varepsilon(u_\varepsilon^{(1)}) - I_\varepsilon(u_\varepsilon^{(2)})| \to 0 \quad \text{as } \varepsilon \to 0$$

from which

$$J_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon^{(1)}) + I_\varepsilon(u_\varepsilon^{(2)}) + Q_\varepsilon(u_\varepsilon^{(2)}) + o(1). \tag{5.3.36}$$

Indeed

$$\begin{aligned}|I_\varepsilon(u_\varepsilon) - I_\varepsilon(u_\varepsilon^{(1)}) - I_\varepsilon(u_\varepsilon^{(2)})| &\leq \left|\int_{\mathbb{R}^N} (-\Delta)^{s/2}u_\varepsilon^{(1)}(-\Delta)^{s/2}u_\varepsilon^{(2)}\right| + \int_{A^\varepsilon} |V(\varepsilon x)u_\varepsilon^{(1)}u_\varepsilon^{(2)}| \\ &\quad + \int_{A^\varepsilon} |F(u_\varepsilon) - F(u_\varepsilon^{(1)}) - F(u_\varepsilon^{(2)})| \\ &=: (I) + (II) + (III).\end{aligned}$$

The second piece can be easily estimated by the boundedness of φ_ε and V, and the information on the L^2-norm given by (5.3.34), i.e $(II) \leq \frac{C}{n_\varepsilon}$. Similarly, as regards (III), we estimate each single piece separately, in the same way: use (5.1.10) and the information on the L^2-norm and the L^{p+1}-norm given by (5.3.34), obtaining $(III) \leq \frac{C}{n_\varepsilon}$.

Focus instead on (I). Recall that $(u_\varepsilon)_\varepsilon \subset S(r_2')$, and thus $\|u_\varepsilon\|_2$ is bounded. We have

$$(I) \leq C\int_{\mathbb{R}^{2N}} \frac{|u_\varepsilon^{(1)}(x) - u_\varepsilon^{(1)}(y)||u_\varepsilon^{(2)}(x) - u_\varepsilon^{(2)}(y)|}{|x-y|^{N+2s}}dx\,dy$$

$$\begin{aligned}
&\leq 2C\int_{\Omega^\varepsilon_{n_\varepsilon i_\varepsilon}\times \complement(\Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)})}\frac{|u^{(1)}_\varepsilon(x)-u^{(1)}_\varepsilon(y)||u^{(2)}_\varepsilon(x)-u^{(2)}_\varepsilon(y)|}{|x-y|^{N+2s}}dx\,dy+\\
&\quad+2C\int_{A^\varepsilon\times\mathbb{R}^N}\frac{|u^{(1)}_\varepsilon(x)-u^{(1)}_\varepsilon(y)||u^{(2)}_\varepsilon(x)-u^{(2)}_\varepsilon(y)|}{|x-y|^{N+2s}}dx\,dy\\
&=: 2C\big((I_1)+(I_2)\big)
\end{aligned}$$

since on $\Omega^\varepsilon_{n_\varepsilon i_\varepsilon}\times\Omega^\varepsilon_{n_\varepsilon i_\varepsilon}$ and $\complement(\Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)})\times\complement(\Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)})$ the integrand is null. Focusing on (I_1)

$$\begin{aligned}
(I_1)&=\int_{(\Omega^\varepsilon_{n_\varepsilon i_\varepsilon}\times \complement(\Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)}))\cap\{|x-y|>n_\varepsilon\}}\frac{|u_\varepsilon(x)u_\varepsilon(y)|}{|x-y|^{N+2s}}dx\,dy\\
&\leq\frac12\int_{(\Omega^\varepsilon_{n_\varepsilon i_\varepsilon}\times \complement(\Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)}))\cap\{|x-y|>n_\varepsilon\}}\frac{u^2_\varepsilon(x)+u^2_\varepsilon(y)}{|x-y|^{N+2s}}dx\,dy\\
&=\frac12\int_{\Omega^\varepsilon_{n_\varepsilon i_\varepsilon}}u^2_\varepsilon(x)\int_{\complement(\Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)})\cap\{|x-y|>n_\varepsilon\}}\frac{1}{|x-y|^{N+2s}}dy\,dx+\\
&\quad+\frac12\int_{\complement(\Omega^\varepsilon_{n_\varepsilon(i_\varepsilon+1)})}u^2_\varepsilon(y)\int_{\Omega^\varepsilon_{n_\varepsilon i_\varepsilon}\cap\{|x-y|>n_\varepsilon\}}\frac{1}{|x-y|^{N+2s}}dx\,dy\\
&\leq C\|u_\varepsilon\|_2^2\int_{|x-y|>n_\varepsilon}\frac{1}{|x-y|^{N+2s}}dx\,dy\leq\frac{C}{{n_\varepsilon}^{2s}}\to 0\qquad\text{as }\varepsilon\to 0.
\end{aligned}$$

Focusing on (I_2) we have

$$\begin{aligned}
&(I_2)\\
&\leq\int_{A^\varepsilon\times\mathbb{R}^N}\frac{1}{|x-y|^{N+2s}}\Big(|(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))u_\varepsilon(x)|+|\varphi_\varepsilon(y)(u_\varepsilon(x)-u_\varepsilon(y))|\Big)\cdot\\
&\quad\cdot\Big(|(\varphi_\varepsilon(y)-\varphi_\varepsilon(x))u_\varepsilon(x)|+|(1-\varphi_\varepsilon(y))(u_\varepsilon(x)-u_\varepsilon(y))|\Big)dx\,dy\\
&\leq\int_{A^\varepsilon\times\mathbb{R}^N}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2|u_\varepsilon(x)|^2}{|x-y|^{N+2s}}dx\,dy+\int_{A^\varepsilon\times\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dx\,dy+\\
&\quad+2\int_{A^\varepsilon\times\mathbb{R}^N}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)||u_\varepsilon(x)||u_\varepsilon(x)-u_\varepsilon(y)|}{|x-y|^{N+2s}}dx\,dy\\
&\leq\int_{A^\varepsilon\times\mathbb{R}^N}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2|u_\varepsilon(x)|^2}{|x-y|^{N+2s}}+\int_{A^\varepsilon\times\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}+\\
&\quad+2\Big(\int_{A^\varepsilon\times\mathbb{R}^N}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2|u_\varepsilon(x)|^2}{|x-y|^{N+2s}}\Big)^{\frac12}\Big(\int_{A^\varepsilon\times\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\Big)^{\frac12}\\
&=:\ A^2+B^2+2AB
\end{aligned}$$

and we see that both A and B go to zero: $B=[u_\varepsilon]_{A^\varepsilon,\mathbb{R}^N}\leq\frac{C}{n_\varepsilon^{1/2}}$ by (5.3.34), while for A we exploit that $\|\nabla\varphi_\varepsilon\|_\infty\to 0$. Indeed, let $\alpha_\varepsilon:=\frac{1}{\|\nabla\varphi_\varepsilon\|_\infty}$; we have

$$\begin{aligned}
A^2&=\int_{A^\varepsilon}|u_\varepsilon(x)|^2\int_{|x-y|\leq\alpha_\varepsilon}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}}dy\,dx+\\
&\quad+\int_{A^\varepsilon}|u_\varepsilon(x)|^2\int_{|x-y|>\alpha_\varepsilon}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2}{|x-y|^{N+2s}}dy\,dx\\
&\leq C\|u_\varepsilon\|^2_{L^2(A^\varepsilon)}\bigg(\|\nabla\varphi_\varepsilon\|^2_\infty\int_{|z|\leq\alpha_\varepsilon}\frac{1}{|z|^{N+2s-2}}dz+4\|\varphi_\varepsilon\|^2_\infty\int_{|z|>\alpha_\varepsilon}\frac{1}{|z|^{N+2s}}dz\bigg)
\end{aligned}$$

$$\leq \frac{C}{n_\varepsilon}\alpha_\varepsilon^{-2s}\left(\|\nabla\varphi_\varepsilon\|_\infty^2\alpha_\varepsilon^2+1\right)=\frac{C}{n_\varepsilon}\|\nabla\varphi_\varepsilon\|_\infty^{2s}\leq\frac{C}{n_\varepsilon^{2s+1}}\to 0 \quad \text{as } \varepsilon\to 0.$$

Thus $(I_2)\leq \frac{C}{n_\varepsilon^{2s+1}}+\frac{C}{n_\varepsilon^{s+1}}+\frac{C}{n_\varepsilon}\leq\frac{C'}{n_\varepsilon}\to 0$, which reaches the claim.

Step 4. *Relations of the derivatives.*
We have

$$\|I_\varepsilon'(u_\varepsilon)-I_\varepsilon'(u_\varepsilon^{(1)})-I_\varepsilon'(u_\varepsilon^{(2)})\|_{(H^s(\mathbb{R}^N))^*}\to 0 \quad \text{as } \varepsilon\to 0, \tag{5.3.37}$$

from which, joined to (5.3.35),

$$J_\varepsilon'(u_\varepsilon)=I_\varepsilon'(u_\varepsilon^{(1)})+I_\varepsilon'(u_\varepsilon^{(2)})+Q_\varepsilon'(u_\varepsilon^{(2)})+o(1). \tag{5.3.38}$$

Indeed by Hölder inequality, for any $v\in H^s(\mathbb{R}^N)$,

$$|I_\varepsilon'(u_\varepsilon)v-I_\varepsilon'(u_\varepsilon^{(1)})v-I_\varepsilon'(u_\varepsilon^{(2)})v|\leq\int_{A^\varepsilon}|f(u_\varepsilon)-f(u_\varepsilon^{(1)})-f(u_\varepsilon^{(2)})||v|dx$$

and again we argue in the same way as in the third piece of Step 3, observing that, by (5.1.10), $|f(u_\varepsilon)||v|\leq\beta|u_\varepsilon||v|+C_\beta|u_\varepsilon|^p|v|$ thus

$$\begin{aligned}\int_{A^\varepsilon}|f(u_\varepsilon)||v|dx&\leq\beta\|u_\varepsilon\|_{L^2(A^\varepsilon)}\|v\|_2+C_\beta\|u_\varepsilon\|^p_{L^{p+1}(A^\varepsilon)}\|v\|_{p+1}\\&\leq C\left(\beta\|u_\varepsilon\|_{L^2(A^\varepsilon)}+C_\beta\|u_\varepsilon\|^p_{L^{p+1}(A^\varepsilon)}\right)\|v\|_{H^s}\end{aligned}$$

and hence the claim. In particular, $|(I_\varepsilon'(u_\varepsilon)-I_\varepsilon'(u_\varepsilon^{(1)})-I_\varepsilon'(u_\varepsilon^{(2)}))u_\varepsilon^{(2)}|\leq\frac{C}{n_\varepsilon}$. We see also that

$$I_\varepsilon'(u_\varepsilon^{(1)})u_\varepsilon^{(2)}=o(1). \tag{5.3.39}$$

Indeed

$$\begin{aligned}|I_\varepsilon'(u_\varepsilon^{(1)})u_\varepsilon^{(2)}|&\leq\left|\int_{\mathbb{R}^N}(-\Delta)^{s/2}u_\varepsilon^{(1)}(-\Delta)^{s/2}u_\varepsilon^{(2)}dx\right|+\\&+\int_{A^\varepsilon}|V(\varepsilon x)u_\varepsilon^{(1)}u_\varepsilon^{(2)}dx|+\int_{A^\varepsilon}|f(u_\varepsilon^{(1)})u_\varepsilon^{(2)}dx|=:(I)+(II)+(III)\end{aligned}$$

where for (I) and (II) we argue as in Step 3 obtaining $(I)+(II)\leq\frac{C}{n_\varepsilon^{2s}}+\frac{C}{n_\varepsilon}$, while for (III) we argue as in (5.3.38) obtaining $(III)\leq\frac{C}{n_\varepsilon}$.

Step 5. *Convergence of* $u_\varepsilon^{(2)}$.
Observing that the support of $u_\varepsilon^{(2)}$ is outside Ω/ε, we have with arguments similar to Step 3 that, by (5.3.33),

$$\|u_\varepsilon^{(2)}\|_{H^s(\mathbb{R}^N)}\leq r_1. \tag{5.3.40}$$

Indeed, focusing only on the nonlocal part, we have (recall that supp $(u_\varepsilon^{(2)})\subset\complement(\Omega/\varepsilon)$)

$$\begin{aligned}\int_{\mathbb{R}^{2N}}\frac{|u_\varepsilon^{(2)}(x)-u_\varepsilon^{(2)}(y)|^2}{|x-y|^{N+2s}}&\leq 2\int_{\complement(\Omega/\varepsilon)\times\mathbb{R}^N}\frac{|u_\varepsilon^{(2)}(x)-u_\varepsilon^{(2)}(y)|^2}{|x-y|^{N+2s}}dx\,dy\\&\leq 4\int_{\complement(\Omega/\varepsilon)\times\mathbb{R}^N}\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^2|u_\varepsilon(x)|^2}{|x-y|^{N+2s}}dx\,dy+\\&\quad+4\int_{\complement(\Omega/\varepsilon)\times\mathbb{R}^N}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}dx\,dy\end{aligned}$$

and we use again the final argument in Step 3 and (5.3.33) to gain, for ε small,

$$\|u_\varepsilon^{(2)}\|^2_{H^s(\mathbb{R}^N)}\leq(C+o(1))\|u_\varepsilon\|^2_{L^2(\complement(\Omega/\varepsilon))}+C[u_\varepsilon]^2_{\complement(\Omega/\varepsilon),\mathbb{R}^N}\leq(Cr_2')^2$$

where C does not depend on r_2'. We choose thus r_2' such that (5.3.40) holds.

This allows us to use Lemma 5.3.2. By joining (5.3.31), (5.3.38), (5.3.39), (5.3.24), (5.3.27) we obtain

$$\begin{aligned} o(1) = J_\varepsilon'(u_\varepsilon)u_\varepsilon^{(2)} &= I_\varepsilon'(u_\varepsilon^{(1)})u_\varepsilon^{(2)} + I_\varepsilon'(u_\varepsilon^{(2)})u_\varepsilon^{(2)} + Q_\varepsilon'(u_\varepsilon^{(2)})u_\varepsilon^{(2)} + o(1) \\ &\geq C\|u_\varepsilon^{(2)}\|^2_{H^s(\mathbb{R}^N)} + (p+1)Q_\varepsilon(u_\varepsilon^{(2)}) + o(1) \end{aligned}$$

or more precisely (we highlight this for a later use), for some $C = C(r_2')$,

$$o(1) = J_\varepsilon'(u_\varepsilon)u_\varepsilon^{(2)} \geq C\|u_\varepsilon^{(2)}\|^2_{H^s(\mathbb{R}^N)} + (p+1)Q_\varepsilon(u_\varepsilon^{(2)}) - \left(\frac{C}{n_\varepsilon} + \frac{C}{n_\varepsilon^{2s}}\right), \tag{5.3.41}$$

which implies (since Q_ε is positive) that

$$\|u_\varepsilon^{(2)}\|_{H^s(\mathbb{R}^N)} \to 0 \quad \text{ as } \varepsilon \to 0 \tag{5.3.42}$$

and

$$Q_\varepsilon(u_\varepsilon^{(2)}) \to 0 \quad \text{ as } \varepsilon \to 0. \tag{5.3.43}$$

As a further consequence, (5.3.42) and the boundedness of V imply

$$I_\varepsilon(u_\varepsilon^{(2)}) \to 0 \quad \text{and} \quad I_\varepsilon'(u_\varepsilon^{(2)}) \to 0 \quad \text{ as } \varepsilon \to 0. \tag{5.3.44}$$

Step 6. *Convergence of* $I_\varepsilon'(u_\varepsilon^{(1)})$.
In particular we obtain from (5.3.44), together with (5.3.36) and (5.3.43), that

$$J_\varepsilon(u_\varepsilon) = I_\varepsilon(u_\varepsilon^{(1)}) + o(1). \tag{5.3.45}$$

We want now to show that

$$I_\varepsilon'(u_\varepsilon^{(1)}) \to 0 \quad \text{in } (H^s(\mathbb{R}^N))^* \quad \text{ as } \varepsilon \to 0. \tag{5.3.46}$$

Start observing that (5.3.44) together with (5.3.37) give

$$I_\varepsilon'(u_\varepsilon) = I_\varepsilon'(u_\varepsilon^{(1)}) + o(1); \tag{5.3.47}$$

let now $v \in H^s(\mathbb{R}^N)$ and evaluate $I_\varepsilon'(u_\varepsilon^{(1)})v$. We want to exploit (5.3.47) together again with the assumption (5.3.31). In order to do this we need to pass from $u_\varepsilon^{(2)}$ to u_ε, but getting rid of $Q_\varepsilon'(u_\varepsilon)$ on which we have no information. Thus we introduce a cutoff function $\tilde{\varphi} \in C_c^\infty(\mathbb{R}^N)$ such that

$$\Omega_{\frac{3}{2}h_0} \prec \tilde{\varphi} \prec \Omega_{2h_0}$$

and hence

$$\operatorname{supp}(u_\varepsilon^{(1)}) \subset \Omega_{h_0}/\varepsilon \subset \Omega_{\frac{3}{2}h_0}/\varepsilon \subset \{\tilde{\varphi}(\varepsilon\cdot) \equiv 1\} \subset \operatorname{supp}(\tilde{\varphi}(\varepsilon\cdot)) \subset \Omega_{2h_0}/\varepsilon.$$

Thus we have

$$\begin{aligned} I_\varepsilon'(u_\varepsilon^{(1)})v &\overset{(*)}{=} I_\varepsilon'(u_\varepsilon^{(1)})(\tilde{\varphi}(\varepsilon\cdot)v) + (1 + \|v\|_2)o(1) \\ &= I_\varepsilon'(u_\varepsilon)(\tilde{\varphi}(\varepsilon\cdot)v) - \Big(I_\varepsilon'(u_\varepsilon)(\tilde{\varphi}(\varepsilon\cdot)v) - I_\varepsilon'(u_\varepsilon^{(1)})(\tilde{\varphi}(\varepsilon\cdot)v)\Big) + (1 + \|v\|_2)o(1) \\ &= J_\varepsilon'(u_\varepsilon)(\tilde{\varphi}(\varepsilon\cdot)v) - \Big(I_\varepsilon'(u_\varepsilon) - I_\varepsilon'(u_\varepsilon^{(1)})\Big)(\tilde{\varphi}(\varepsilon\cdot)v) + (1 + \|v\|_2)o(1). \end{aligned}$$

Indeed, we justify $(*)$ as done in Step 3: notice that $u_\varepsilon^{(1)} \equiv 0$ outside Ω_{h_0}/ε and $1 - \tilde{\varphi}(\varepsilon\cdot) \equiv 0$ in $\Omega_{\frac{3}{2}h_0}/\varepsilon$, so in the annulus $(\Omega_{\frac{3}{2}h_0}/\varepsilon) \setminus (\Omega_{h_0}/\varepsilon)$ both $u_\varepsilon^{(1)}$ and $1 - \tilde{\varphi}(\varepsilon\cdot)$ are zero; notice also that $\complement(\Omega_{\frac{3}{2}h_0}/\varepsilon)$ and Ω_{h_0}/ε get far one from the other as $\varepsilon \to 0$. Thus we have

$$|I_\varepsilon'(u_\varepsilon^{(1)})((1 - \tilde{\varphi}(\varepsilon\cdot))v)| = 2\int_{(\Omega_{h_0}/\varepsilon) \times \complement(\Omega_{\frac{3}{2}h_0}/\varepsilon)} \frac{|u_\varepsilon^{(1)}(x)||(1 - \tilde{\varphi}(\varepsilon y))v(y)|}{|x-y|^{N+2s}} dx\, dy$$

$$\leq 2\int_{(\Omega_{h_0}/\varepsilon)\times \complement(\Omega_{\frac{3}{2}h_0}/\varepsilon)} \frac{|u_\varepsilon(x)||v(y)|}{|x-y|^{N+2s}} dx\,dy \leq (1+\|v\|_2)o(1)$$

where in the last passage we argue as for (I_1) in Step 3. Notice that $o(1)$ does not depend on v. Thus we obtain

$$\begin{aligned} &|I'_\varepsilon(u_\varepsilon^{(1)})v| \\ &\leq \left(\|J'_\varepsilon(u_\varepsilon)\|_{(H^s(\mathbb{R}^N))^*} + \|I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)})\|_{(H^s(\mathbb{R}^N))^*}\right) \|\tilde{\varphi}(\varepsilon\cdot)v\|_{H^s(\mathbb{R}^N)} \\ &\quad +(1+\|v\|_{H^s(\mathbb{R}^N)})o(1) \\ &\leq \left(\|J'_\varepsilon(u_\varepsilon)\|_{(H^s(\mathbb{R}^N))^*} + \|I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)})\|_{(H^s(\mathbb{R}^N))^*}\right) \|v\|_{H^s(\mathbb{R}^N)}(C+o(1)) + \\ &\quad +(1+\|v\|_{H^s(\mathbb{R}^N)})o(1) \end{aligned}$$

where in the last inequality we argue as in Step 5 (again $o(1)$ does not depend on v). Concluding, we have, by choosing $\|v\|_{H^s(\mathbb{R}^N)} = 1$, that

$$\begin{aligned} &\|I'_\varepsilon(u_\varepsilon^{(1)})\|_{(H^s(\mathbb{R}^N))^*} \\ &\quad \leq C\left(\|J'_\varepsilon(u_\varepsilon)\|_{(H^s(\mathbb{R}^N))^*} + \|I'_\varepsilon(u_\varepsilon) - I'_\varepsilon(u_\varepsilon^{(1)})\|_{(H^s(\mathbb{R}^N))^*}\right)(1+o(1)) + o(1) \to 0 \end{aligned}$$

by using (5.3.31) and (5.3.47).

Step 7. *Weak convergence of* $u_\varepsilon^{(1)}$.
Set $q_\varepsilon := \Upsilon(u_\varepsilon)$ to avoid cumbersome notation. Since $(\varepsilon q_\varepsilon)_\varepsilon \subset \Omega[0,\nu_0] \subset \Omega$ bounded in $\mathbb{R}^N$, we have that up to a subsequence

$$\varepsilon q_\varepsilon \to p_0 \in \overline{\Omega[0,\nu_0]} \subset K_d \subset \Omega.$$

Moreover, by estimating the norm of $u_\varepsilon^{(1)}$ with the norm of u_ε (as done before, in Step 5, for $u_\varepsilon^{(2)}$), where u_ε belongs to $S(r'_2)$ bounded in $H^s(\mathbb{R}^N)$, we have that also $u_\varepsilon^{(1)}$ is a bounded sequence, and thus is so $u_\varepsilon^{(1)}(\cdot+q_\varepsilon)$, which implies, up to a subsequence

$$u_\varepsilon^{(1)}(\cdot+q_\varepsilon) \rightharpoonup \tilde{U} \text{ in } H^s(\mathbb{R}^N) \quad \text{as } \varepsilon\to 0.$$

For each $v \in H^s(\mathbb{R}^N)$ we apply this weak convergence to the following equalities, derived from (5.3.46),

$$\begin{aligned} o(1) = I'_\varepsilon(u_\varepsilon^{(1)})v(\cdot - q_\varepsilon) &= \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_\varepsilon^{(1)}(y+q_\varepsilon)(-\Delta)^{s/2}v(y)dy + \\ &+ \int_{\mathbb{R}^N} V(\varepsilon y + \varepsilon q_\varepsilon)u_\varepsilon^{(1)}(y+q_\varepsilon)v(y)dy - \int_{\mathbb{R}^N} f(u_\varepsilon^{(1)}(y+q_\varepsilon))v(y)dy \\ &=: (I) + (II) + (III). \end{aligned}$$

For (I) and (III) we obtain by the weak convergence and by Proposition 1.5.5

$$(I) \to \int_{\mathbb{R}^N} (-\Delta)^{s/2}\tilde{U}\,(-\Delta)^{s/2}v\,dy \quad \text{and} \quad (III) \to -\int_{\mathbb{R}^N} f(\tilde{U})v\,dy \quad \text{as } \varepsilon \to 0.$$

For (II) instead we have

$$\begin{aligned} \left|(II) - \int_{\mathbb{R}^N} V(p_0)\tilde{U}v\right| &\leq \left(\int_{\mathbb{R}^N} (V(\varepsilon y+\varepsilon q_\varepsilon) - V(p_0))^2 v^2(y)\right)^{1/2} \|u_\varepsilon^{(1)}(\cdot+q_\varepsilon)\|_2 + \\ &+ V(p_0)\left|\int_{\mathbb{R}^N} u_\varepsilon^{(1)}(y+q_\varepsilon)v(y) - \int_{\mathbb{R}^N}\tilde{U}v\right| \end{aligned}$$

where the first term goes to zero (thanks to the boundedness of $u_\varepsilon^{(1)}$) by the dominated convergence theorem, while the second thanks to the weak convergence. Thus we finally obtain

$$L'_{V(p_0)}(\tilde{U})v = \int_{\mathbb{R}^N} (-\Delta)^{s/2}\tilde{U}\,(-\Delta)^{s/2}v\,dy + \int_{\mathbb{R}^N} V(p_0)\tilde{U}v\,dy - \int_{\mathbb{R}^N} f(\tilde{U})v\,dy = 0$$

for each $v \in H^s(\mathbb{R}^N)$, that is

$$L'_{V(p_0)}(\tilde{U}) = 0. \tag{5.3.48}$$

Step 8. *Strong convergence of* $u_\varepsilon^{(1)}$.
We want to show the strong convergence of $u_\varepsilon^{(1)}(\cdot + q_\varepsilon)$, that is

$$u_\varepsilon^{(1)}(\cdot + q_\varepsilon) \to \tilde{U} \quad \text{in } H^s(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0. \tag{5.3.49}$$

Set $\tilde{w}_\varepsilon := u_\varepsilon^{(1)}(\cdot + q_\varepsilon) - \tilde{U} \rightharpoonup 0$, again by (5.3.46) we have

$$\begin{aligned}
o(1) &= I'_\varepsilon(u_\varepsilon^{(1)})\tilde{w}_\varepsilon(\cdot - q_\varepsilon)\\
&= L'_{V(p_0)}(\tilde{U})\tilde{w}_\varepsilon + \left(\|(-\Delta)^{s/2}\tilde{w}_\varepsilon\|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon y + \varepsilon q_\varepsilon)\tilde{w}_\varepsilon^2 dy\right) +\\
&\quad + \int_{\mathbb{R}^N} (V(\varepsilon y + \varepsilon q_\varepsilon) - V(p_0))\tilde{U}\tilde{w}_\varepsilon dy + \int_{\mathbb{R}^N} (f(\tilde{U}) - f(\tilde{U} + \tilde{w}_\varepsilon))\tilde{w}_\varepsilon dy\\
&=: \left(\|(-\Delta)^{s/2}\tilde{w}_\varepsilon\|_2^2 + \int_{\mathbb{R}^N} V(\varepsilon y + \varepsilon q_\varepsilon)\tilde{w}_\varepsilon^2 dy\right) + (I)\\
&\geq \|(-\Delta)^{s/2}\tilde{w}_\varepsilon\|_2^2 + \underline{V}\|\tilde{w}_\varepsilon\|_2^2 + (I)
\end{aligned}$$

where we have used (5.3.48). We obtain by the boundedness of V and (5.1.10)

$$\begin{aligned}
(I) \geq &- 2\|V\|_\infty \int_{\mathbb{R}^N} |\tilde{U}||\tilde{w}_\varepsilon|dy - \int_{\mathbb{R}^N} \left(2\beta|\tilde{U}| + C_\beta(2^p+1)|\tilde{U}|^p\right)|\tilde{w}_\varepsilon|dy -\\
&- \int_{\mathbb{R}^N} \left(\beta|\tilde{w}_\varepsilon|^2 + 2^p C_\beta|\tilde{w}_\varepsilon|^{p+1}\right) dy = o(1) - \beta\|\tilde{w}_\varepsilon\|_2^2 - 2^p C_\beta\|\tilde{w}_\varepsilon\|_{p+1}^{p+1};
\end{aligned}$$

in the last passage we have used that $\tilde{w}_\varepsilon \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, thus by Remark 1.4.2 $|\tilde{w}_\varepsilon| \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ and hence in $L^2(\mathbb{R}^N)$ and in $L^{p+1}(\mathbb{R}^N)$ (observing that $\tilde{U}^p \in L^{1+\frac{1}{p}}(\mathbb{R}^N)$). Merging together all the things we have, by (5.3.26) and choosing $\beta < \frac{1}{2}\underline{V}$,

$$o(1) \geq \|(-\Delta)^{s/2}\tilde{w}_\varepsilon\|_2^2 + (\underline{V} - \beta)\|\tilde{w}_\varepsilon\|_2^2 - 2^p C_\beta\|\tilde{w}_\varepsilon\|_{p+1}^{p+1} \geq C\|\tilde{w}_\varepsilon\|_{H^s(\mathbb{R}^N)}^2$$

and thus $\tilde{w}_\varepsilon \to 0$ strongly in $H^s(\mathbb{R}^N)$, that is the claim.

Step 9. *Localization.*
Observe first that $\tilde{U} \not\equiv 0$. Indeed, if not, by (5.3.42), (5.3.49) and translation invariance of the norm we would have

$$\begin{aligned}
r^* &\leq \liminf_{\varepsilon\to 0} \|U_\varepsilon\|_{H^s(\mathbb{R}^N)}\\
&\leq \liminf_{\varepsilon\to 0} \|U_\varepsilon(\cdot - p_\varepsilon) + \varphi_\varepsilon\|_{H^s(\mathbb{R}^N)} + \liminf_{\varepsilon\to 0} \|\varphi_\varepsilon\|_{H^s(\mathbb{R}^N)}\\
&\leq \lim_{\varepsilon\to 0} \left(\|u_\varepsilon^{(1)}\|_{H^s(\mathbb{R}^N)} + \|u_\varepsilon^{(2)}\|_{H^s(\mathbb{R}^N)}\right) + r'_2 \leq r'_2 < r^*,
\end{aligned}$$

impossible. By (5.3.49) we obtain also

$$I_\varepsilon(u_\varepsilon^{(1)}) \to L_{V(p_0)}(\tilde{U}) \quad \text{as } \varepsilon \to 0.$$

Thus we find, by using also (5.3.45) and (5.3.32),

$$L_{V(p_0)}(\tilde{U}) = I_\varepsilon(u_\varepsilon^{(1)}) + o(1) = J_\varepsilon(u_\varepsilon) + o(1) \leq l'_0 + o(1)$$

and hence, letting $\varepsilon \to 0$,

$$L_{V(p_0)}(\tilde{U}) \le l_0' < E_{m_0+\nu_1}. \tag{5.3.50}$$

Moreover by (5.3.48) and $\tilde{U} \not\equiv 0$, we have

$$E_{V(p_0)} \le L_{V(p_0)}(\tilde{U});$$

joining together the two previous inequalities we find $E_{V(p_0)} < E_{m_0+\nu_1}$ which implies, by the monotonicity of E_a, that

$$V(p_0) < m_0 + \nu_1.$$

Joining this information to the fact that $p_0 \in \Omega$ (and in particular $V(p_0) \ge m_0$) we have $p_0 \in \Omega[0, \nu_1)$, that is

$$\varepsilon \Upsilon(u_\varepsilon) \to p_0 \in \Omega[0, \nu_1) \quad \text{as } \varepsilon \to 0. \tag{5.3.51}$$

Exploiting again (5.3.50) (observe that $E_{m_0+\nu_1} < E_{m_0+\nu_0} < l_0$) together with $L'_{V(p_0)}(\tilde{U}) = 0$, and $\tilde{U} \neq 0$, we have that $\tilde{U}$ belongs to $S_{V(p_0)}$ up to translations, that is

$$U := \tilde{U}(\cdot - y_0) \in S_{V(p_0)} \subset \widehat{S}$$

for some suitable $y_0 \in \mathbb{R}^N$. So, set

$$p_\varepsilon := q_\varepsilon + y_0$$

we have

$$\|u_\varepsilon^{(1)} - U(\cdot - p_\varepsilon)\|_{H^s(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.3.52}$$

For a later use observe also that

$$\varepsilon p_\varepsilon \to p_0 \in \Omega[0, \nu_1) \quad \text{as } \varepsilon \to 0. \tag{5.3.53}$$

Step 10. *Conclusions.*
By (5.3.51) we have that

$$\varepsilon \Upsilon(u_\varepsilon) \in \Omega[0, \nu_1)$$

definitely for ε small. This is the first part of the claim. Moreover, by (5.3.52) and (5.3.42) we gain

$$\|u_\varepsilon - U(\cdot - p_\varepsilon)\|_{H^s(\mathbb{R}^N)} \to 0 \quad \text{as } \varepsilon \to 0 \tag{5.3.54}$$

and thus, since $\widehat{\rho}(u_\varepsilon) \le \|u_\varepsilon - U(\cdot - p_\varepsilon)\|_{H^s(\mathbb{R}^N)}$ by definition, also $\widehat{\rho}(u_\varepsilon) \to 0$ and hence

$$\widehat{\rho}(u_\varepsilon) \in [0, \rho_1]$$

definitely for ε small. This concludes the proof. ∎

In the next proposition we see that solutions of $J'_\varepsilon(u) = 0$ are, under suitable assumptions, also solutions of $I'_\varepsilon(u) = 0$.

Corollary 5.3.9. *Let $(u_\varepsilon)_\varepsilon$ be a sequence of critical points of J_ε, that is $J'_\varepsilon(u_\varepsilon) = 0$, satisfying*

$$u_\varepsilon \in S(r_2'), \quad J_\varepsilon(u_\varepsilon) \le l_0' \quad \textit{and} \quad \varepsilon \Upsilon(u_\varepsilon) \in \Omega[0, \nu_0]$$

for any $\varepsilon > 0$. Then, for ε sufficiently small, we have

$$Q_\varepsilon(u_\varepsilon) = 0, \quad \textit{and} \quad Q'_\varepsilon(u_\varepsilon) = 0.$$

In particular $I'_\varepsilon(u_\varepsilon) = 0$, which means that u_ε is a solution of (5.3.21).

Proof. By the proof of Lemma 5.3.6, we notice, since $1-\varphi^\varepsilon_{i_\varepsilon} \equiv 1$ outside $\Omega^\varepsilon_{i_\varepsilon+1}$, and thus outside Ω_{h_0}/ε, that

$$\|u_\varepsilon\|_{L^2(\mathbb{R}^N\setminus(\Omega_{h_0}/\varepsilon))} = \|u_\varepsilon^{(2)}\|_{L^2(\mathbb{R}^N\setminus(\Omega_{h_0}/\varepsilon))} \leq \|u_\varepsilon^{(2)}\|_{H^s(\mathbb{R}^N)} \tag{5.3.55}$$

and hence $\|u_\varepsilon\|_{L^2(\mathbb{R}^N\setminus(\Omega_{h_0}/\varepsilon))} \to 0$ by (5.3.42).

Through a careful analysis of the Steps 3–5 of the proof, that is by (5.3.41) and (5.3.55), we see, more precisely, that

$$\|u_\varepsilon\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{h_0}/\varepsilon))} \leq \frac{C}{n_\varepsilon} + \frac{C}{n_\varepsilon^{2s}} + o(1)$$

where $C = C(r_2')$ and $o(1)$ depends on the rate of convergence of $J'_\varepsilon(u_\varepsilon)$. Thus, since we assume $J'_\varepsilon(u_\varepsilon) \equiv 0$, we gain uniformity, i.e., called $\alpha^* := \min\{1, 2s\}$, we obtain

$$\|u_\varepsilon\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{h_0}/\varepsilon))} \leq \frac{C}{n_\varepsilon^{\alpha^*}} \sim \varepsilon^{\alpha^*/2} \tag{5.3.56}$$

As a consequence

$$\frac{1}{\varepsilon^\alpha}\|u_\varepsilon\|^2_{L^2(\mathbb{R}^N\setminus(\Omega_{h_0}/\varepsilon))} \to 0 \quad \text{as } \varepsilon \to 0$$

for $\alpha \in (0, \alpha^*/2)$, and hence $Q_\varepsilon(u_\varepsilon) \equiv Q'_\varepsilon(u_\varepsilon) \equiv 0$ for ε sufficiently small. ∎

We want to show now a (truncated) Palais-Smale-like condition.

Proposition 5.3.10. *There exists $r_2'' \in (0, \min\{r_0, r_1\})$ sufficiently small with the following property: let $\varepsilon > 0$ fixed and let $(u_j)_j \subset S(r_2'')$ be such that*

$$\|J'_\varepsilon(u_j)\|_{(H^s(\mathbb{R}^N))^*} \to 0 \quad \text{as } j \to +\infty \tag{5.3.57}$$

with the additional assumption

$$(\varepsilon\Upsilon(u_j))_j \subset \Omega[0, \nu_0].$$

Then $(u_j)_j$ admits a strongly convergent subsequence in $H^s(\mathbb{R}^N)$.

Proof. Let r_2'' to be fixed. Since $S(r_2'')$ is bounded, up to a subsequence we can assume $u_j \rightharpoonup u_0$ in $H^s(\mathbb{R}^N)$. We want to show that

$$\lim_{R\to+\infty}\lim_{j\to+\infty}\|u_j\|_{L^q(\mathbb{R}^N\setminus B_R)} = 0$$

for $q = 2$ and $q = p+1$ and conclude by Lemma 5.3.4.

Arguing similarly to Step 1 of the proof of Lemma 5.3.6, i.e. exploiting Remark 5.3.8, we obtain for $L \gg 0$, uniformly in $j \in \mathbb{N}$,

$$|||u_j|||_{\mathbb{R}^N\setminus B_L} \leq C r_2'';$$

indeed we work with the set $B_L - \Upsilon(u_j)$ which expands to $\mathbb{R}^N$ as $L, j \to +\infty$, since $\Upsilon(u_j) \in \Omega/\varepsilon$, a fixed bounded set. Moreover, for any $n \in \mathbb{N}$, we have

$$\sum_{i=1}^n \|u_j\|^2_{L^2(B_{L+ni}\setminus B_{L+n(i-1)})} \leq (C r_2'')^2$$

and similarly for the Gagliardo seminorm and the $(p+1)$-norm, thus for some $i_{j,n} \in \{1, \dots, n\}$

$$\|u_j\|^2_{A^{j,n}} + \|u_j\|^{p+1}_{L^{p+1}(A^{j,n})} \leq \frac{C}{n}$$

where $A^{j,n} := B_{L+ni_{j,n}} \setminus B_{L+n(i_{j,n}-1)}$. Again similarly to Step 2 of the proof of Lemma 5.3.6, we introduce $\psi_{j,n}$ such that

$$B_{L+n(i_{j,n}-1)} \prec \psi_{j,n} \prec B_{L+ni_{j,n}}$$

and $\|\nabla\psi_{j,n}\|_\infty = o(1)$ as $n \to +\infty$; moreover we set

$$\tilde{u}_{j,n} := (1-\psi_{j,n})u_j.$$

Observe that $\chi_{B_L} \leq \psi_{j,n}$, thus $\mathrm{supp}(\tilde{u}_{j,n}) \subset \complement(B_L)$. Arguing as in Step 5 and 3 of the proof of Lemma 5.3.6 we obtain

$$\int_{\mathbb{R}^{2N}} \frac{|\tilde{u}_{j,n}(x)-\tilde{u}_{j,n}(y)|^2}{|x-y|^{N+2s}} \leq 4\int_{\complement(B_L)\times\mathbb{R}^N} \frac{|\psi_{j,n}(x)-\psi_{j,n}(y)|^2|u_j(x)|^2}{|x-y|^{N+2s}}dx\,dy+$$
$$+4\int_{\complement(B_L)\times\mathbb{R}^N} \frac{|u_j(x)-u_j(y)|^2}{|x-y|^{N+2s}}dx\,dy \leq o(1)\|u_j\|^2_{L^2(\complement(B_L))} + C[u_j]^2_{\complement(B_L),\mathbb{R}^N}$$

thus $\|\tilde{u}_{j,n}\|_{H^s(\mathbb{R}^N)} \leq Cr_2''$ and hence, choosing r_2'' sufficiently small, we have

$$\|\tilde{u}_{j,n}\|_{H^s(\mathbb{R}^N)} \leq r_1;$$

by Lemma 5.3.2, for $q \in \{2, p+1\}$, we obtain

$$\|u_j\|^2_{L^q(\mathbb{R}^N\setminus B_{L+ni_{j,n}})} = \|\tilde{u}_{j,n}\|^2_{L^q(\mathbb{R}^N\setminus B_{L+ni_{j,n}})} \leq C\|\tilde{u}_{j,n}\|^2_{H^s(\mathbb{R}^N)} \leq CI'_\varepsilon(\tilde{u}_{j,n})\tilde{u}_{j,n}.$$

Thus the claim comes if we show that

$$I'_\varepsilon(\tilde{u}_{j,n})\tilde{u}_{j,n} \to 0 \quad \text{ as } j,n \to +\infty.$$

Indeed we have

$$I'_\varepsilon(\tilde{u}_{j,n})\tilde{u}_{j,n} = I'_\varepsilon(u_j)\tilde{u}_{j,n} - \int_{\mathbb{R}^{2N}} (-\Delta)^{s/2}(\psi_{j,n}u_j)(-\Delta)^{s/2}((1-\psi_{j,n})u_j)dx -$$
$$- \int_{A^{j,n}} V(\varepsilon x)\psi_{j,n}(1-\psi_{j,n})u_j^2 dx - \int_{A^{j,n}} (f((1-\psi_{j,n})u_j) - f(u_j))(1-\psi_{j,n})u_j dx$$
$$=: J'_\varepsilon(u_j)\tilde{u}_{j,n} - Q'_\varepsilon(u_j)\tilde{u}_{j,n} + (I) \leq o(1) + (I)$$

where we have used that $J'_\varepsilon(u_j) \to 0$ (as $j \to +\infty$, uniformly in $n \in \mathbb{N}$), the boundedness of $\|\tilde{u}_{j,n}\|_{H^s(\mathbb{R}^N)}$ and the positivity of $Q'_\varepsilon(u_j)\tilde{u}_{j,n}$. The term (I) can be estimated in the same way as done in Steps 3-4 of the proof of Lemma 5.3.6 (fixed $j \in \mathbb{N}$, and $n \to +\infty$), and hence we reach the claim. ∎

5.3.3 Deformation lemma on a neighborhood of expected solutions

We want to define now a neighborhood of expected solutions (see [119]), which will be invariant under a suitable deformation flow. Consider $r_3 := \min\{r', r_0', r_2', r_2''\}$ (see Lemma 5.2.6, Lemma 5.3.1, Theorem 5.3.7 and Proposition 5.3.10), and let us define

$$R(\delta, u) := \delta - \frac{\delta_2}{2}(\widehat{\rho}(u) - \rho_1)_+ \leq \delta$$

and

$$\mathcal{X}_{\varepsilon,\delta} := \Big\{u \in S(\rho_0) \mid \varepsilon\Upsilon(u) \in \Omega[0,\nu_0),\ J_\varepsilon(u) < E_{m_0} + R(\delta,u)\Big\}$$

where

$$0 < \rho_1 < \rho_0 < r_3,$$

ε is sufficiently small and

$$\delta \in \Big(0, \min\Big\{\tfrac{\delta_2}{4}(\rho_0-\rho_1),\ \delta_1,\ l_0' - E_{m_0}\Big\}\Big); \tag{5.3.58}$$

here δ_1 and δ_2 are the ones that appear in Lemma 5.3.1 and Theorem 5.3.7. Notice that the *height* of the sublevel in $\mathcal{X}_{\varepsilon,\delta}$ depends on u itself; this will be used to gain a deformation which preserves $\mathcal{X}_{\varepsilon,\delta}$.

We begin by pointing out some geometrical features of the neighborhood $\mathcal{X}_{\varepsilon,\delta}$.

- $\mathcal{X}_{\varepsilon,\delta}$ is open. Indeed, $S(\rho)$ and $\{J_\varepsilon(u) < E_{m_0} + R(\delta, u)\}$ are open, and $\Omega[0,\nu_0) = \Omega(-\gamma,\nu_0)$ for a whatever $\gamma > 0$ (since V cannot go under m_0 in Ω) and thus open. Moreover it is nonempty (see e.g. Section 5.3.4).
- If $v \in \mathcal{X}_{\varepsilon,\delta} \subset S(\rho_0)$, then by (5.2.17) we have $\widehat{\rho}(v) < \rho_0$.
- If $v \in \mathcal{X}_{\varepsilon,\delta} \subset \{\varepsilon\Upsilon(v) \in \Omega[0,\nu_0]\}$, then

$$\varepsilon\Upsilon(v) \in \Omega[0,\nu_1). \tag{5.3.59}$$

Indeed, if not, i.e. $\varepsilon\Upsilon(v) \in \Omega[\nu_1,\nu_0]$, then by Lemma 5.3.1 we have

$$J_\varepsilon(v) \geq I_\varepsilon(v) \geq E_{m_0} + \delta_1 > E_{m_0} + \delta \geq E_{m_0} + R(\delta, v)$$

which is an absurd.

- If $R(\delta, v) \geq -\delta$ then

$$v \in S(\rho_0). \tag{5.3.60}$$

Indeed $\frac{\delta_2}{4}(\widehat{\rho}(u) - \rho_1)_+ \leq \delta$ implies, by the restriction on δ,

$$(\widehat{\rho}(u) - \rho_1)_+ < \rho_0 - \rho_1.$$

If $\widehat{\rho}(u) < \rho_1$ then clearly $u \in S(\rho_1) \subset S(\rho_0)$. If instead $\widehat{\rho}(u) \geq \rho_1$, then $\widehat{\rho}(u) < \rho_0$, which again implies $u \in S(\rho_0)$.

We further define the set of critical points of J_ε lying in the neighborhood of expected solutions

$$K_c := \{u \in \mathcal{X}_{\varepsilon,\delta} \mid J'_\varepsilon(u) = 0,\ J_\varepsilon(u) = c\},$$

the sublevel

$$\mathcal{X}^c_{\varepsilon,\delta} := \mathcal{X}_{\varepsilon,\delta} \cap J^c_\varepsilon$$

and the strip level

$$(\mathcal{X}_{\varepsilon,\delta})^c_d := \{u \in \mathcal{X}_{\varepsilon,\delta} \mid d \leq J_\varepsilon(u) \leq c\},$$

for every $c, d \in \mathbb{R}$. We present now a deformation lemma with respect to K_c, for c sufficiently close to E_{m_0}.

Lemma 5.3.11. *Let $c \in (E_{m_0} - \delta, E_{m_0} + \delta)$. Then there exists a deformation at level c, which leaves the set $\mathcal{X}_{\varepsilon,\delta}$ invariant. That is, for every U neighborhood of K_c ($U = \emptyset$ if $K_c = \emptyset$), there exist a small $\omega > 0$ and a continuous deformation $\eta : [0,1] \times \mathcal{X}_{\varepsilon,\delta} \to \mathcal{X}_{\varepsilon,\delta}$ such that*

(i) $\eta(0,\cdot) = id$;

(ii) $J_\varepsilon(\eta(\cdot,u))$ *is non-increasing;*

(iii) $\eta(t,u) = u$ *for every* $t \in [0,1]$, *if* $J_\varepsilon(u) \notin (E_{m_0} - \delta, E_{m_0} + \delta)$;

(iv) $\eta(1, \mathcal{X}^{c+\omega}_{\varepsilon,\delta} \setminus U) \subset \mathcal{X}^{c-\omega}_{\varepsilon,\delta}$;

(v) $\eta(\cdot,u)$ *is a semigroup.*

Proof. Let $\mathcal{V} : \{u \in H^s(\mathbb{R}^N) \mid J'_\varepsilon(u) \neq 0\} \to H^s(\mathbb{R}^N)$ be a locally Lipschitz pseudo-gradient vector field associated to J_ε, and let $\phi \in Lip_{loc}(H^s(\mathbb{R}^N), \mathbb{R})$ be a cutoff function such that $\operatorname{supp}(\phi) \subset (\mathcal{X}_{\varepsilon,\delta})^{E_{m_0}+\delta}_{E_{m_0}-\delta}$ and $\phi = 1$ in a small neighborhood of c. We consider the Cauchy problem

$$\begin{cases} \dot{\eta} = -\phi(\eta)\dfrac{\mathcal{V}(\eta)}{\|\mathcal{V}(\eta)\|_{H^s(\mathbb{R}^N)}}, \\ \eta(0,u) = u. \end{cases} \tag{5.3.61}$$

The proof keeps on classically, obtaining a deformation $\eta : [0,1] \times \mathcal{X}_{\varepsilon,\delta} \to H^s(\mathbb{R}^N)$. We want to prove now that η goes into $\mathcal{X}_{\varepsilon,\delta}$.

Let $u \in \mathcal{X}_{\varepsilon,\delta}$. We need to show that $\eta(t,u) \in \mathcal{X}_{\varepsilon,\delta}$ for every $t > 0$. Since $\mathcal{X}_{\varepsilon,\delta}$ is open, $\eta(s,u)$ continues staying in $\mathcal{X}_{\varepsilon,\delta}$ for s small. Thus assume that

$$\eta(s,u) \in \mathcal{X}_{\varepsilon,\delta}, \qquad \text{for every } 0 \le s < t_0$$

for some $t_0 > 0$, and we want to show that $\eta(t_0,u) \in \mathcal{X}_{\varepsilon,\delta}$. Notice first that, by using (iii), (v), (ii) and the continuity of η and J_ε we can assume that

$$J_\varepsilon(\eta(t_0,u)) \in [E_{m_0} - \delta, E_{m_0} + \delta). \tag{5.3.62}$$

Step 1: $\varepsilon\Upsilon(\eta(t_0,u)) \in \Omega[0,\nu_0)$.
By (5.3.59) we have

$$\varepsilon\Upsilon(\eta(s,u)) \in \Omega[0,\nu_1), \qquad \text{for every } 0 \le s < t_0,$$

and thus by continuity $\varepsilon\Upsilon(\eta(t_0,u)) \in \overline{\Omega[0,\nu_1)} \subset \Omega[0,\nu_0]$.

Step 2: $J_\varepsilon(\eta(t_0,u)) < E_{m_0} + R(\delta,\eta(t_0,u))$.
If $\widehat{\rho}(\eta(t_0,u)) \le \rho_1$ then $R(\delta,\eta(t_0,u)) = \delta$ and we directly have the claim, recalled that $J_\varepsilon(\eta(t_0,u)) < E_{m_0} + \delta$ by (5.3.62). Assume instead $\widehat{\rho}(\eta(t_0,u)) > \rho_1$. By continuity, there exists $t_1 \in (0,t_0)$ such that we have

$$\widehat{\rho}(\eta(s,u)) > \rho_1, \qquad \text{for every } s \in [t_1,t_0].$$

In particular

$$\begin{cases} \eta(s,u) \in S(\rho_0) \subset S(r_3) \subset S(r_2'), \\ J_\varepsilon(\eta(s,u)) < E_{m_0} + \delta < E_{m_0+\nu_1}, \\ \widehat{\rho}(\eta(s,u)) \in (\rho_1,\rho_0], \\ \varepsilon\Upsilon(\eta(s,u)) \in \Omega[0,\nu_0] \end{cases}$$

for $s \in [t_1,t_0]$. Then by Theorem 5.3.7 we have

$$\|J_\varepsilon'(\eta(s,u))\|_{(H^s(\mathbb{R}^N))^*} \ge \delta_2, \qquad \text{for every } s \in [t_1,t_0].$$

We can thus compute with standard argument, by using (5.3.61), the properties of the pseudo-gradient and (5.2.18),

$$\begin{aligned} J_\varepsilon(\eta(t_0,u)) &\le J_\varepsilon(\eta(t_1,u)) - \frac{\delta_2}{2}\big(\widehat{\rho}(\eta(t_0,u)) - \widehat{\rho}(\eta(t_1,u))\big) \\ &< E_{m_0} + \delta - \frac{\delta_2}{2}\big(\widehat{\rho}(\eta(t_1,u)) - \rho_1\big) - \frac{\delta_2}{2}\big(\widehat{\rho}(\eta(t_0,u)) - \widehat{\rho}(\eta(t_1,u))\big) \\ &= E_{m_0} + R(\delta,\eta(t_0,u)), \end{aligned}$$

that is the claim.

Step 3: $\eta(t_0,u) \in S(\rho_0)$.
By the previous point we have $J_\varepsilon(\eta(t_0,u)) \le E_{m_0} + R(\delta,\eta(t_0,u))$. Since (5.3.62) implies $J_\varepsilon(\eta(t_0,u)) \ge E_{m_0} - \delta$, then by (5.3.60) we have $\eta(t_0,u) \in S(\rho_0)$, and thus the claim. ∎

5.3.4 Maps homotopic to the embedding

We search now for two maps Φ_ε, Ψ_ε such that, for a sufficiently small $\sigma_0 \in (0,1)$ and a sufficiently small $\hat{\delta} = \hat{\delta}(\sigma_0) \in (0,\delta)$ (see (5.3.58)), defined

$$I := [1-\sigma_0, 1+\sigma_0],$$

we have, for small ε,

$$I \times K \overset{\Phi_\varepsilon}{\to} \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}} \overset{\Psi_\varepsilon}{\to} I \times K_d$$

with the additional condition

$$\partial I \times K \overset{\Phi_\varepsilon}{\to} \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}-\hat{\delta}} \overset{\Psi_\varepsilon}{\to} (I \setminus \{1\}) \times K_d;$$

then we will prove that $\Psi_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the identity. While the first property is useful for category arguments to gain multiplicity of solutions, the second additional condition will be essential for developing *relative* category (and cup-length) arguments and controlling the sublevels of the functional *below* the expected critical level.

Definition of Φ_ε

Let us fix a ground state $U_0 \in S_{m_0} \subset \widehat{S}$, i.e. $L_{m_0}(U_0) = E_{m_0}$ (see Theorem 5.2.1 and (5.2.13)). Define, for $p \in K$ and $t \in I$ (σ_0 to be fixed)

$$\Phi_\varepsilon(t,p) := U_0\left(\tfrac{\cdot - p/\varepsilon}{t}\right) \in H^s(\mathbb{R}^N).$$

We show now that, for ε small, $\Phi_\varepsilon(t,p) \in \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}}$.

- $\Phi_\varepsilon(t,p) \in S(\rho_1) \subset S(\rho_0)$:

  indeed, recalled that the dilation $t \in \mathbb{R} \mapsto U_0(\cdot/t) \in H^s(\mathbb{R}^N)$ is continuous, we have

  $$\|U_0\left(\tfrac{\cdot - p/\varepsilon}{t}\right) - U_0(\cdot - p/\varepsilon)\|_{H^s(\mathbb{R}^N)} = \|U_0\left(\cdot/t\right) - U_0\|_{H^s(\mathbb{R}^N)} < \rho_1$$

  for $t \in I$ and sufficiently small $\sigma_0 = \sigma_0(U_0)$ (not depending on ε). Thus, setting $\varphi_t := U_0\left(\frac{\cdot - p/\varepsilon}{t}\right) - U_0(\cdot - p/\varepsilon)$ we have

  $$\Phi_\varepsilon(t,p) = U_0(\cdot - p/\varepsilon) + \varphi_t$$

  with $U_0 \in \widehat{S}$, $p/\varepsilon \in \mathbb{R}^N$ and $\|\varphi_t\|_{H^s(\mathbb{R}^N)} < \rho_1$, which is the claim.

- $\varepsilon\Upsilon(\Phi_\varepsilon(t,p)) \in \Omega[0,\nu_0)$:

  indeed, by the previous point and Lemma 5.2.8, we have

  $$|\Upsilon(\Phi_\varepsilon(t,p)) - p/\varepsilon| < 2R_0$$

  hence $|\varepsilon\Upsilon(\Phi_\varepsilon(t,p)) - p| < 2\varepsilon R_0$, and since $p \in K$

  $$d(\varepsilon\Upsilon(\Phi_\varepsilon(t,p)), K) < 2\varepsilon R_0.$$

  For sufficiently small ε, we have $K_{2\varepsilon R_0} \subset \Omega[0,\nu_0)$, and thus the claim. In particular, for a later use observe that

  $$\varepsilon\Upsilon(\Phi_\varepsilon(t,p)) = p + o(1). \tag{5.3.63}$$

- $J_\varepsilon(\Phi_\varepsilon(t,p)) < E_{m_0} + R(\delta, \Phi_\varepsilon(t,p))$:

  indeed $\Phi_\varepsilon(t,p) \in S(\rho_1)$, thus $\widehat{\rho}(\Phi_\varepsilon(t,p)) < \rho_1$, which implies $R(\delta, \Phi_\varepsilon(t,p)) = \delta$ and the claim comes from the following point, since $\hat{\delta} < \delta$.

- $J_\varepsilon(\Phi_\varepsilon(t,p)) < E_{m_0} + \hat{\delta}$:

  indeed we have by Lemma 5.2.3 (b)

  $$J_\varepsilon(\Phi_\varepsilon(t,p)) = L_{m_0}\left(\Phi_\varepsilon(t,p)\right) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - m_0)\Phi_\varepsilon^2(t,p)dx + Q_\varepsilon\left(\Phi_\varepsilon(t,p)\right)$$

$$=: L_{m_0}\left(U_0\left(\tfrac{\cdot - p/\varepsilon}{t}\right)\right) + (I) + (II) = g(t)E_{m_0} + o(1) \tag{5.3.64}$$
$$\leq E_{m_0} + o(1)$$

where we used $g(t) \leq 1$. Indeed, as regards (I) we have

$$(I) = \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x + p) - m_0)U_0^2(x/t)dx \to 0 \quad \text{as } \varepsilon \to 0$$

by exploiting that $p \in K$ and the dominated convergence theorem, together with the boundedness of V. Focusing on (II) instead, we have

$$(II) = \left(\frac{1}{\varepsilon^\alpha}\|U_0(\cdot/t)\|^2_{L^2(\mathbb{R}^N \setminus ((\Omega_{2h_0}-p)/\varepsilon)} - 1\right)_+^{\frac{p+1}{2}};$$

since $p \in K \subset \Omega \subset \Omega_{2h_0}$, we have $0 \in \Omega_{2h_0} - p$ and moreover $B_r \subset \Omega_{2h_0} - p$ for some ball B_r; notice that B_r/ε covers the whole $\mathbb{R}^N$ as $\varepsilon \to 0$. Therefore, by the polynomial estimate we have

$$\|U_0(\cdot/t)\|^2_{L^2(\mathbb{R}^N \setminus ((\Omega_{2h_0}-p)/\varepsilon)} \leq C\|\tfrac{1}{1+|x|^{N+2s}}\|^2_{L^2(\mathbb{R}^N \setminus (B_r/\varepsilon))} \leq C\varepsilon^{N+4s},$$

and hence $(II) \to 0$ as $\varepsilon \to 0$, since $\alpha < N + 4s$. Therefore, by choosing a sufficiently small ε, we obtain

$$J_\varepsilon(\Phi_\varepsilon(t,p)) \leq E_{m_0} + \tfrac{1}{2}\hat{\delta} < E_{m_0} + \hat{\delta}.$$

Finally, we show the additional condition.

- $J_\varepsilon(\Phi_\varepsilon(1 \pm \sigma_0, p)) < E_{m_0} - \hat{\delta}$:
  indeed, looking at (5.3.64) we see that, for small ε,

  $$J_\varepsilon(\Phi_\varepsilon(1 \pm \sigma_0, p)) < g(1 \pm \sigma_0)E_{m_0} + \hat{\delta};$$

  since $g(1 \pm \sigma_0) < 1$, we can find a small $\hat{\delta} < \frac{1-g(1\pm\sigma_0)}{2}E_{m_0}$ (not depending on ε) such that

  $$J_\varepsilon(\Phi_\varepsilon(1 \pm \sigma_0, p)) < g(1 \pm \sigma_0)E_{m_0} + \hat{\delta} < E_{m_0} - \hat{\delta} \tag{5.3.65}$$

  and thus the claim.

Definition of Ψ_ε

Define a truncation

$$T(t) := \begin{cases} 1 - \sigma_0 & \text{if } t \leq 1 - \sigma_0, \\ t & \text{if } t \in (1 - \sigma_0, 1 + \sigma_0), \\ 1 + \sigma_0 & \text{if } t \geq 1 + \sigma_0 \end{cases}$$

for $t \in \mathbb{R}$, and

$$\Psi_\varepsilon(u) := (T(P_{m_0}(u)), \varepsilon\Upsilon(u))$$

for every $u \in \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}}$. By the definition of T and property (5.3.59), we have directly

$$\Psi_\varepsilon(u) \in I \times \Omega[0, \nu_1] \subset I \times \Omega[0, \nu_0] \subset I \times K_d.$$

Assume now $u \in \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}-\hat{\delta}}$. We have, by using Lemma 5.3.5 and Lemma 5.2.6,

$$E_{m_0} - \hat{\delta} \geq J_\varepsilon(u) \geq L_{m_0}(u) - C_{min}\varepsilon^\alpha \geq g(P_{m_0}(u))E_{m_0} - C_{min}\varepsilon^\alpha$$

and hence

$$E_{m_0} \geq g(P_{m_0}(u))E_{m_0} + \hat{\delta} - C_{min}\varepsilon^\alpha > g(P_{m_0}(u))E_{m_0}$$

where the last inequality holds for ε small, not depending on u. Thus

$$g(P_{m_0}(u)) < 1$$

and this must imply, by the properties of g, that $P_{m_0}(u) \neq 1$, and in particular

$$T(P_{m_0}(u)) \neq 1.$$

This reaches the goal.

An homotopy to the identity

Introduce the notation of *topological pair* from the algebraic topology: we write, for $B \subset A$ and $B' \subset A'$,

$$f : (A, B) \to (A', B')$$

whenever

$$f \in C(A, A') \quad \text{and} \quad f(B) \subset B'.$$

Observed that Φ_ε and Ψ_ε are continuous, we can rewrite the stated properties as

$$\Phi_\varepsilon : \Big(I \times K,\, \partial I \times K\Big) \to \Big(\mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}},\, \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}-\hat{\delta}}\Big),$$

$$\Psi_\varepsilon : \Big(\mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}},\, \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}-\hat{\delta}}\Big) \to \Big(I \times K_d,\, (I \setminus \{1\}) \times K_d\Big)$$

and

$$\Psi_\varepsilon \circ \Phi_\varepsilon : \Big(I \times K,\, \partial I \times K\Big) \to \Big(I \times K_d,\, (I \setminus \{1\}) \times K_d\Big),$$

where a straightforward computation shows

$$(\Psi_\varepsilon \circ \Phi_\varepsilon)(t, p) = \left(t,\, \varepsilon \Upsilon\left(U_0\left(\tfrac{\cdot - p/\varepsilon}{t}\right)\right)\right),$$

thus actually $\Psi_\varepsilon \circ \Phi_\varepsilon : \Big(I \times K,\, \partial I \times K\Big) \to \Big(I \times K_d,\, \partial I \times K_d\Big)$. Clearly, we notice that the inclusion map has the same property, that is set $j(t,p) := (t,p)$ we have

$$j : \Big(I \times K,\, \partial I \times K\Big) \to \Big(I \times K_d,\, \partial I \times K_d\Big) \subset \Big(I \times K_d,\, (I \setminus \{1\}) \times K_d\Big).$$

We want to show that these maps are homotopic, information useful in the theory of relative cup-length.

Proposition 5.3.12. *For sufficiently small ε, the maps $\Psi_\varepsilon \circ \Phi_\varepsilon$ and j are homotopic, that is there exists a continuous map $H : [0,1] \times I \times K \to I \times K_d$ such that*

$$H(\theta, \cdot, \cdot) : \Big(I \times K,\, \partial I \times K\Big) \to \Big(I \times K_d,\, \partial I \times K_d\Big) \subset \Big(I \times K_d,\, (I \setminus \{1\}) \times K_d\Big)$$

for each $\theta \in [0,1]$, with $H(0, \cdot, \cdot) = \Psi_\varepsilon \circ \Phi_\varepsilon$ and $H(1, \cdot, \cdot) = j$.

Proof. Noticed that also $\Psi_\varepsilon \circ \Phi_\varepsilon$ fixes the first variable, it is sufficient to link the second variables through a segment, that is

$$H(\theta, t, p) := \left(t,\, (1-\theta)\varepsilon \Upsilon\left(U_0\left(\tfrac{\cdot - p/\varepsilon}{t}\right)\right) + \theta p\right),$$

with $\theta \in [0,1]$. We must check that H is well defined, since K_d is not a convex set, generally. Indeed we have, by (5.3.63)

$$(1-\theta)\varepsilon \Upsilon\left(U_0\left(\tfrac{\cdot - p/\varepsilon}{t}\right)\right) + \theta p = (1-\theta)p + o(1) + \theta p = p + o(1).$$

Since $p \in K$, for sufficiently small ε we have that $p + o(1) \in K_d$, and thus the claim. ∎

Before coming up to multiplicity results, we highlight that existence of a single solution could be obtained without any use of algebraic tools. Notice that we need only the map Φ_ε and the first component of Ψ_ε.

Proof (existence). Let $p \in K$. First observe that we can slightly change the map Φ_ε such that

$$\Phi_\varepsilon(1 \pm \sigma_0, p) \in \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}-\delta}; \tag{5.3.66}$$

indeed (see (5.3.64) and (5.3.65)), it is sufficient to take a smaller $\varepsilon > 0$ and $\hat{\delta} < \delta < \frac{1-g(1\pm\sigma_0)}{2} E_{m_0} < E_{m_0}$, where we point out that σ_0 depends only on U_0 and ρ_1 (and thus not on δ).

Let $c = E_{m_0}$; by contradiction, assume $K_c \neq 0$. Thus, by the Lemma 5.3.11, there exists a deformation η related to the regular value c. By Lemma 5.3.5 we have, for each $\sigma \in I$,

$$L_{m_0}(\eta(1, \Phi_\varepsilon(\sigma, p))) \leq J_\varepsilon(\eta(1, \Phi_\varepsilon(\sigma, p))) + C_{min}\varepsilon^\alpha \leq E_{m_0} - \delta + C_{min}\varepsilon^\alpha$$

where in the last inequality we have used that $\Phi_\varepsilon(\sigma, p) \in \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}} \subset \mathcal{X}_{\varepsilon,\delta}^{c+\delta}$. Thus, for ε small, we have

$$L_{m_0}(\eta(1, \Phi_\varepsilon(\sigma, p))) < E_{m_0} \quad \text{for each } \sigma \in I.$$

To conclude, we need to find a $\tilde{\sigma} \in I$ such that

$$P_{m_0}(\eta(1, \Phi_\varepsilon(\tilde{\sigma}, p))) = 1$$

since this implies $L_{m_0}(\eta(1, \Phi_\varepsilon(\tilde{\sigma}, p))) \geq C_{po,m_0} = E_{m_0}$ and thus an absurd.

Indeed, by (5.3.66) we have

$$P_{m_0}(\eta(1, \Phi_\varepsilon(1 \pm \sigma_0, p))) = P_{m_0}(\Phi_\varepsilon(1 \pm \sigma_0, p)) = 1 \pm \sigma_0$$

and the claim follows by the intermediate value theorem. ∎

5.4 Existence of multiple solutions

We finally come up to the existence of multiple solutions. Here the algebraic notions of *relative category* and *relative cup-length* (built on the Alexander-Spanier cohomology with coefficients in some field $\mathbb{F}$) are of key importance. We refer to the Appendix A for definitions, comments and properties of these algebraic tools.

Proof of Theorem 5.1.2. By construction of the neighborhood $\mathcal{X}_{\varepsilon,\delta}$ and Corollary 5.3.9 (recall that $\rho_0 < r_3 \leq r_2'$ and that $J_\varepsilon(u) < E_{m_0} + R(\hat{\delta}, u) \leq E_{m_0} + \hat{\delta} \leq l_0'$ for $u \in \mathcal{X}_{\varepsilon,\delta}$), we have

$$\left\{u \in (\mathcal{X}_{\varepsilon,\delta})_{E_{m_0}-\hat{\delta}}^{E_{m_0}+\hat{\delta}} \mid J_\varepsilon'(u) = 0\right\} \subset \left\{u \in H^s(\mathbb{R}^N) \mid I_\varepsilon'(u) = 0\right\}.$$

Thus we obtain

$$\begin{aligned}
\#\{u \text{ solutions of } (5.3.21)\} &\geq \#\left\{u \in (\mathcal{X}_{\varepsilon,\delta})_{E_{m_0}-\hat{\delta}}^{E_{m_0}+\hat{\delta}} \mid J_\varepsilon'(u) = 0\right\} \\
&\overset{(i)}{\geq} \operatorname{cat}\left(\mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}-\hat{\delta}}\right) \overset{(ii)}{\geq} \operatorname{cupl}\left(\mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}-\hat{\delta}}\right) + 1 \\
&\overset{(iii)}{\geq} \operatorname{cupl}(K) + 1
\end{aligned}$$

that is the claim, up to the proof of (i)–(iii). Indeed, (i) is obtained classically from the Deformation Lemma 5.3.11 as in Section A.5. Inequality (ii) is given by the algebraic-topological

Lemma A.10. Point (iii) is instead due to the existence of the homotopy gained in Proposition 5.3.12 and properties of the cup-length: indeed, by (A.3) in Lemma A.4 (a), we have

$$\mathrm{cupl}\Big(\mathcal{X}_{\varepsilon,\delta}^{E_{m_0}+\hat{\delta}}, \mathcal{X}_{\varepsilon,\delta}^{E_{m_0}-\hat{\delta}}\Big) \geq \mathrm{cupl}(\Psi_\varepsilon \circ \Phi_\varepsilon);$$

moreover, since $\Psi_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the immersion j thanks to Proposition 5.3.12, we have by Lemma A.4 (b)

$$\mathrm{cupl}(\Psi_\varepsilon \circ \Phi_\varepsilon) = \mathrm{cupl}(j),$$

which leads to the conclusion thanks to Lemma A.5. See Remark 5.4.1 for the proof of regularity. ∎

5.4.1 Concentration in the potential well

We prove now the polynomial decay and the concentration of the found solutions in K. To deal with uniform bound, we will make use of the fractional De Giorgi class recalled in Section 1.2.5.

Proof of Theorem 5.1.4. For ε sufficiently small, let u_ε be one of the $\mathrm{cupl}(K)+1$ critical points of J_ε built in Theorem 5.1.4, which by Corollary 5.3.9 is also a solution of (5.3.21), positive by (f2). In particular, since it satisfies the assumptions of Lemma 5.3.6, looking at the proof (see (5.3.54) and (5.3.53)) we obtain that

$$\|u_\varepsilon - U(\cdot - p_\varepsilon)\|_{H^s(\mathbb{R}^N)} \to 0$$

with $U \in S_{V(p_0)}$, $p_\varepsilon \in \mathbb{R}^N$ and

$$\varepsilon p_\varepsilon \to p_0 \in \Omega[0,\nu_1).$$

Step 1. Notice that we have found these solutions by fixing ν_0, l_0 and l_0'. Let them move, throughout three sequences $\nu_0^n \searrow 0$, $l_0^n \searrow E_{m_0}$, and $(l_0')^n \searrow E_{m_0}$, and find the corresponding (sufficiently small) $\varepsilon_n > 0$ such that $\mathrm{cupl}(K)+1$ solutions exist; let u_{ε_n} be one of those and p_{ε_n} as before. It is not reductive to assume $\varepsilon_n \to 0$ as $n \to +\infty$; by a diagonalization-like argument we obtain

$$u_{\varepsilon_n}(\cdot + p_{\varepsilon_n}) \to U \quad \text{in } H^s(\mathbb{R}^N), \quad \text{for some } U \text{ least energy solution of (5.1.7)}, \tag{5.4.67}$$

$$\varepsilon_n p_{\varepsilon_n} \to p_0 \in K,$$

as $n \to +\infty$.

Step 2. From now on we write $\varepsilon \equiv \varepsilon_n$ to avoid cumbersome notation. By $I_\varepsilon'(u_\varepsilon) = 0$ we obtain

$$(-\Delta)^s u_\varepsilon + V(\varepsilon x) u_\varepsilon = f(u_\varepsilon), \quad x \in \mathbb{R}^N, \tag{5.4.68}$$

thus (recall that u_ε is positive), by choosing $\beta < \underline{V}$ in (5.1.10),

$$(-\Delta)^s u_\varepsilon \leq -\underline{V} u_\varepsilon + f(u_\varepsilon) \leq (\beta - \underline{V}) u_\varepsilon + C_\beta u_\varepsilon^p \leq C_\beta u_\varepsilon^p, \quad x \in \mathbb{R}^N.$$

Therefore by Theorem 1.2.28 we have, choosing $q = p+1$, $d_1 = 0$ and $d_2 = C_\beta$,

$$u_\varepsilon \in \mathrm{DG}_+^{s,2}\Big(B_{R_0}(x_0), 0, H, 0, 1 - \frac{p+1}{2_s^*}, 2s, R_0\Big),$$

with $H = H(N,s,p,\beta)$ and R_0 depending on N, s, p, C_β and a uniform upper bound of the H^s-norms of u_ε.

We can thus use now [134, Proposition 6.1]: observing that $d(x_0, \partial B_{R_0}(x_0)) = R_0$, and that $\mu = 1 - \frac{p+1}{2_s^*}$, we obtain, for any $\omega \in (0,1]$ and $R \in (0, \frac{R_0}{2})$,

$$\sup_{B_R(x_0)} u_\varepsilon \leq \frac{C}{(N-2s)^{\frac{1}{2\mu}}} \frac{1}{\omega^{\frac{1}{2\mu}}} \frac{1}{(2R)^{N/2}} \|u_\varepsilon\|_{L^2(B_{2R}(x_0))} + \omega \mathrm{Tail}(u_\varepsilon; x_0, R)$$

that is, rewriting the constant $C = C(N, s, p, \beta)$,

$$\sup_{B_R(x_0)} u_\varepsilon \leq C \frac{1}{\omega^{\frac{1}{2\mu}}} \frac{1}{R^{N/2}} \|u_\varepsilon\|_{L^2(B_{2R}(x_0))} + \omega \mathrm{Tail}(u_\varepsilon; x_0, R).$$

Step 3. We have

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} &= \sup_{x_0 \in \mathbb{R}^N} \sup_{B_R(x_0)} u_\varepsilon \\ &\leq \sup_{x_0 \in \mathbb{R}^N} \left(C \frac{1}{\omega^{\frac{1}{2\mu}}} \frac{1}{R^{N/2}} \|u_\varepsilon\|_{L^2(B_{2R}(x_0))} + \omega \mathrm{Tail}(u_\varepsilon; x_0, R) \right). \end{aligned}$$

Observe that, by definition of Tail function (1.2.28) and Hölder inequality,

$$\begin{aligned} \mathrm{Tail}(u_\varepsilon; x_0, R) &\leq (1-s) R^{2s} \|u_\varepsilon\|_{L^2(\mathbb{R}^N \setminus B_R(x_0))} \|\tfrac{1}{|x-x_0|^{N+2s}}\|_{L^2(\mathbb{R}^N \setminus B_R(x_0))} \\ &\leq \frac{C}{R^{N/2}} \|u_\varepsilon\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Thus

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} &\leq \frac{C}{R^{N/2}} \sup_{x_0 \in \mathbb{R}^N} \left(\omega^{-\frac{1}{2\mu}} \|u_\varepsilon\|_{L^2(B_{2R}(x_0))} + \omega \|u_\varepsilon\|_{L^2(\mathbb{R}^N \setminus B_R(x_0))} \right) \\ &\leq \frac{C}{R^{N/2}} \left(\omega^{-\frac{1}{2\mu}} + \omega \right) \|u_\varepsilon\|_{L^2(\mathbb{R}^N)} \end{aligned}$$

which is uniformly bounded by the properties on u_ε. Hence u_ε are uniformly bounded in $L^\infty(\mathbb{R}^N)$.

In addition, by the estimates on V, f and u_ε, we have

$$g_\varepsilon(x) := -V(\varepsilon x) u_\varepsilon(x) + f(u_\varepsilon(x)) \in L^\infty(\mathbb{R}^N)$$

with bound uniform in ε; since

$$(-\Delta)^s u_\varepsilon = g_\varepsilon(x), \quad x \in \mathbb{R}^N,$$

by [134, Theorem 8.2] there exists $\sigma \in (0, 1)$, not depending on u_ε, and $C = C(N, s)$, such that, for each $R > 1$ and $x_0 \in \mathbb{R}^N$,

$$\begin{aligned} [u_\varepsilon]_{C^{0,\sigma}(B_R(x_0))} &\leq \frac{C}{R^\sigma} \left(\|u_\varepsilon\|_{L^\infty(B_{4R}(x_0))} + \mathrm{Tail}(u_\varepsilon; x_0, 4R) + R^{2s} \|g_\varepsilon\|_{L^\infty(B_{8R}(x_0))} \right) \\ &\leq C'. \end{aligned} \tag{5.4.69}$$

We highlight that, since the constant is uniform in $R > 1$, we obtain $u_\varepsilon \in C^{0,\sigma}(\mathbb{R}^N)$.

Step 4. By the local uniform estimate on u_ε we could gain $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N \setminus (\Omega_{h_0}/\varepsilon)_{2R_0})} \to 0$, but this lack of uniformity on the domain can be improved. Thus we exploit the tightness of $\tilde{u}_\varepsilon$ to reach the claim, where

$$\tilde{u}_\varepsilon := u_\varepsilon(\cdot + p_\varepsilon).$$

Indeed, by Step 2, and (5.4.67) we have

$$\begin{cases} (-\Delta)^s \tilde{u}_\varepsilon + V(\varepsilon x + \varepsilon p_\varepsilon) \tilde{u}_\varepsilon = f(\tilde{u}_\varepsilon), \quad x \in \mathbb{R}^N, \\ \|\tilde{u}_\varepsilon\|_\infty \leq C, \\ \tilde{u}_\varepsilon \to U \quad \text{in } H^s(\mathbb{R}^N) \text{ as } \varepsilon \to 0, \qquad U \text{ least energy solution of (5.1.7)}. \end{cases}$$

In particular, it is standard to show that $f(\tilde{u}_\varepsilon) \to f(U)$ in $L^2(\mathbb{R}^N)$, $\|f(\tilde{u}_\varepsilon)\|_\infty \leq C$ and $U, f(U) \in L^\infty(\mathbb{R}^N)$. By interpolation we thus obtain

$$\chi_\varepsilon := \tilde{u}_\varepsilon + f(\tilde{u}_\varepsilon) \to \chi := U + f(U) \quad \text{in } L^{q'}(\mathbb{R}^N)$$

for every $q' \in [2, +\infty)$, and $\|\chi_\varepsilon\|_\infty \leq C$. Proceeding as in the proof of Lemma 4.6.3 we gain

$$\tilde{u}_\varepsilon(x) \to 0 \quad \text{as } |x| \to +\infty, \quad \text{uniformly in } \varepsilon. \tag{5.4.70}$$

For the reader's convenience, we give some details. Indeed, being $\tilde{u}_\varepsilon$ solution of

$$(-\Delta)^s \tilde{u}_\varepsilon + \tilde{u}_\varepsilon = \chi_\varepsilon - V(\varepsilon x + \varepsilon p_\varepsilon)\tilde{u}_\varepsilon, \quad x \in \mathbb{R}^N,$$

we have the representation formula

$$\tilde{u}_\varepsilon = \mathcal{K}_{2s} * (\chi_\varepsilon - V(\varepsilon x + \varepsilon p_\varepsilon)\tilde{u}_\varepsilon)$$

where $\mathcal{K}_{2s}$ is the Bessel kernel. Let us fix $\eta > 0$; since V, $\tilde{u}_\varepsilon$ and $\mathcal{K}_{2s}$ are positive, we have, for $x \in \mathbb{R}^N$,

$$\begin{aligned}\tilde{u}_\varepsilon(x) &= \int_{\mathbb{R}^N} \mathcal{K}_{2s}(x-y)\big(\chi_\varepsilon(y) - V(\varepsilon x + \varepsilon p_\varepsilon)\tilde{u}_\varepsilon(y)\big)dy \\ &\leq \int_{|x-y|\geq 1/\eta} \mathcal{K}_{2s}(x-y)\chi_\varepsilon(y)dy + \int_{|x-y|<1/\eta} \mathcal{K}_{2s}(x-y)\chi_\varepsilon(y)dy.\end{aligned}$$

As regards the first piece

$$\int_{|x-y|\geq 1/\eta} \mathcal{K}_{2s}(x-y)\chi_\varepsilon(y)dy \leq \|\chi_\varepsilon\|_\infty \int_{|x-y|\geq 1/\eta} \frac{C}{|x-y|^{N+2s}}dy \leq C\eta^{2s}$$

while for the second piece, fixed a whatever $q \in (1, \min\{2, \frac{N}{N-2s}\})$ and its conjugate exponent $q' \in (\max\{2, \frac{N}{2s}\}, +\infty)$, we have by Hölder inequality

$$\begin{aligned}\int_{|x-y|<1/\eta} \mathcal{K}_{2s}(x-y)\chi_\varepsilon(y)dy &\leq \|\mathcal{K}_{2s}\|_q \|\chi_\varepsilon\|_{L^{q'}(B_{1/\eta}(x))} \\ &\leq \|\mathcal{K}_{2s}\|_q \left(\|\chi_\varepsilon - \chi\|_{q'} + \|\chi\|_{L^{q'}(B_{1/\eta}(x))}\right)\end{aligned}$$

where the first norm can be made small for $\varepsilon < \varepsilon_0 = \varepsilon_0(\eta)$, while the second for $|x| \gg 0$ (uniformly in ε). On the other hand, for $\varepsilon \geq \varepsilon_0$ (and thus for a finite number of elements, since we recall we are working with $\varepsilon \equiv \varepsilon_n$ small) the quantity $\|\chi_\varepsilon\|_{L^{q'}(B_{1/\eta}(x))}$ can be made small for $|x| \gg 0$, uniformly in ε. Joining the pieces, we have (5.4.70).

Step 5. Let now $y_\varepsilon \in \mathbb{R}^N$ be a maximum point for u_ε, which exists by the boundedness of u_ε and its continuity (see (5.4.69)). Therefore $z_\varepsilon := y_\varepsilon - p_\varepsilon$ is a maximum point for $\tilde{u}_\varepsilon$. In particular

$$\tilde{u}_\varepsilon(z_\varepsilon) = \max_{\mathbb{R}^N} \tilde{u}_\varepsilon = \|\tilde{u}_\varepsilon\|_\infty \not\to 0 \quad \text{as } \varepsilon \to 0$$

since on the contrary we would have $\tilde{u}_\varepsilon \to 0$ almost everywhere, which is in contradiction with the fact that $\tilde{u}_\varepsilon \to U \not\equiv 0$ almost everywhere (up to a subsequence). As a consequence, thanks to (5.4.70), we have that z_ε is bounded (up to a subsequence). That is, again up to a subsequence,

$$z_\varepsilon \to \overline{p}$$

for some $\overline{p} \in \mathbb{R}^N$. In particular

$$\varepsilon y_\varepsilon = \varepsilon z_\varepsilon + \varepsilon p_\varepsilon \to p_0 \in K$$

and, by the fact that

$$U(\cdot + z_\varepsilon) \to U(\cdot + \overline{p}) =: \overline{U} \quad \text{in } H^s(\mathbb{R}^N)$$

we have $u_\varepsilon(\cdot + y_\varepsilon) \to \overline{U}$ in $H^s(\mathbb{R}^N)$, $\overline{U}$ least energy solution of (5.1.7). We set

$$\overline{u}_\varepsilon := u_\varepsilon(\cdot + y_\varepsilon),$$

$$\overline{u}_\varepsilon \to \overline{U} \quad \text{in } H^s(\mathbb{R}^N), \qquad \overline{U} \text{ least energy solution of (5.1.7)};$$

in addition, $\overline{u}_\varepsilon$ is positive by (f2), and in the same way we obtained (5.4.70) we obtain also

$$\overline{u}_\varepsilon(x) \to 0 \quad \text{as } |x| \to +\infty, \quad \text{uniformly in } \varepsilon. \tag{5.4.71}$$

Moreover, by exploiting the uniform estimates in $L^\infty(\mathbb{R}^N)$ and $C^{0,\sigma}_{loc}(\mathbb{R}^N)$ we obtain by Ascoli-Arzellà theorem also that $\overline{u}_\varepsilon \to \overline{U} > 0$ in $L^\infty_{loc}(\mathbb{R}^N)$, with $\overline{U}$ continuous; this easily implies, for every $r > 0$, that

$$\min_{B_r} \overline{u}_\varepsilon \geq \frac{1}{2} \min_{B_r} \overline{U} > 0 \tag{5.4.72}$$

for ε small, depending on $\overline{U}$ and r.

Step 6. By (5.4.71) we have, for R' large (uniform in ε), that

$$\overline{u}_\varepsilon(x) \leq \eta', \qquad \text{for } |x| > R'$$

for every $\varepsilon > 0$, where $\eta' > 0$ is preliminary fixed. As a consequence, by (f1.2), we gain

$$-\frac{1}{2}\overline{V}\overline{u}_\varepsilon(x) \leq f(\overline{u}_\varepsilon(x)) \leq \frac{1}{2}\underline{V}\overline{u}_\varepsilon(x), \qquad \text{for } |x| > R',$$

where $\overline{V} := \|V\|_\infty$. We obtain by (5.4.68)

$$(-\Delta)^s \overline{u}_\varepsilon + \tfrac{1}{2}\underline{V}\overline{u}_\varepsilon \leq f(\overline{u}_\varepsilon) - \tfrac{1}{2}\underline{V}\overline{u}_\varepsilon \leq 0, \quad x \in \mathbb{R}^N \setminus B_{R'},$$

$$(-\Delta)^s \overline{u}_\varepsilon + \tfrac{3}{2}\overline{V}\overline{u}_\varepsilon \geq f(\overline{u}_\varepsilon) + \tfrac{1}{2}\overline{V}\overline{u}_\varepsilon \geq 0, \quad x \in \mathbb{R}^N \setminus B_{R'}.$$

Notice that we always intend differential inequalities in the weak sense. In addition, by Lemma 1.2.30 we have that there exist two positive functions $\underline{W}'$, $\overline{W}'$ and three positive constants R'', C' and C'' depending only on V, such that

$$\begin{cases} (-\Delta)^s \underline{W}' + \dfrac{3}{2}\overline{V}\,\underline{W}' = 0, & x \in \mathbb{R}^N \setminus B_{R''}, \\ \dfrac{C'}{|x|^{N+2s}} < \underline{W}'(x), & \text{for } |x| > 2R'', \end{cases}$$

and

$$\begin{cases} (-\Delta)^s \overline{W}' + \dfrac{1}{2}\underline{V}\,\overline{W}' = 0, & x \in \mathbb{R}^N \setminus B_{R''}, \\ \overline{W}'(x) < \dfrac{C''}{|x|^{N+2s}}, & \text{for } |x| > 2R'', \end{cases}$$

Set $R := \max\{R', 2R''\}$. Let $\underline{C}_1$ and $\overline{C}_1$ be some uniform lower and upper bounds for $\overline{u}_\varepsilon$ on B_R, $\underline{C}_2 := \min_{B_R} \overline{W}'$ and $\overline{C}_2 := \max_{B_R} \underline{W}'$, all strictly positive. Define

$$\underline{W} := \underline{C}_1 \overline{C}_2^{-1} \underline{W}', \quad \overline{W} := \overline{C}_1 \underline{C}_2^{-1} \overline{W}'$$

so that

$$\underline{W} \leq \overline{u}_\varepsilon \leq \overline{W}, \qquad \text{for } |x| \leq R.$$

Through a Comparison Principle (see Lemma 1.2.34), and redefining C' and C'', we obtain

$$\frac{C'}{|x|^{N+2s}} < \underline{W}(x) \leq \overline{u}_\varepsilon(x) \leq \overline{W}(x) < \frac{C''}{|x|^{N+2s}}, \qquad \text{for } |x| > R.$$

By the uniform boundedness of $\overline{u}_\varepsilon$ and (5.4.72) we also obtain

$$\frac{C'}{1 + |x|^{N+2s}} < \overline{u}_\varepsilon(x) < \frac{C''}{1 + |x|^{N+2s}}, \qquad \text{for } x \in \mathbb{R}^N.$$

Recalling the definition of $\overline{u}_\varepsilon$, we have finally obtained a sequence of solutions such that

$$\begin{cases} u_{\varepsilon_n}(y_{\varepsilon_n}) = \max_{\mathbb{R}^N} u_{\varepsilon_n}, \\ d(\varepsilon_n y_{\varepsilon_n}, K) \to 0, \\ \dfrac{C'}{1+|x-y_{\varepsilon_n}|^{N+2s}} \leq u_{\varepsilon_n}(x) \leq \dfrac{C''}{1+|x-y_{\varepsilon_n}|^{N+2s}}, \quad \text{for } x \in \mathbb{R}^N, \\ \|u_{\varepsilon_n}(\cdot + y_{\varepsilon_n}) - \overline{U}\|_{H^s(\mathbb{R}^N)} \to 0, \quad \text{for some } \overline{U} \text{ least energy solution of (5.1.7)}, \end{cases}$$

where the limits are given by $n \to +\infty$. Furthermore, by the uniform estimates in $L^\infty(\mathbb{R}^N)$ and the local uniform estimates in $C^{0,\sigma}_{loc}(\mathbb{R}^N)$ of u_{ε_n}, together with the locally-compact version of Ascoli-Arzelà theorem, we have that the last convergence is indeed uniform on compacts. Thus, recalled that $v_{\varepsilon_n} = u_{\varepsilon_n}(\cdot/\varepsilon_n)$ are solutions of the original problem (5.1.3), defined $x_{\varepsilon_n} := \varepsilon_n y_{\varepsilon_n}$ we obtain, as $n \to +\infty$,

$$\begin{cases} v_{\varepsilon_n}(x_{\varepsilon_n}) = \max_{\mathbb{R}^N} v_{\varepsilon_n}, \\ d(x_{\varepsilon_n}, K) \to 0, \\ \dfrac{C'}{1+|\frac{x-x_{\varepsilon_n}}{\varepsilon_n}|^{N+2s}} \leq v_{\varepsilon_n}(x) \leq \dfrac{C''}{1+|\frac{x-x_{\varepsilon_n}}{\varepsilon_n}|^{N+2s}}, \quad \text{for } x \in \mathbb{R}^N, \\ \|v_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n}) - \overline{U}\|_X \to 0, \quad X = H^s(\mathbb{R}^N) \text{ and } X = L^\infty_{loc}(\mathbb{R}^N), \end{cases}$$

for some $\overline{U}$ least energy solution of (5.1.7). This concludes the proof. ∎

Remark 5.4.1. *We observe that Steps 2 and 3 apply to a whatever family of equations $(u_\varepsilon)_{\varepsilon>0}$, that is why the regularity statement in Theorem 5.1.2 holds true. Moreover, the uniform concentration in K and the uniform polynomial decay are obtained by a contradiction argument.*

Proof of Lemma 5.2.4: polynomial decay of $\widehat{S}$

By adapting some argument of the proof of Theorem 5.1.4 we can now complete the proof of Lemma 5.2.4.

Proposition 5.4.2 (Polynomial decay)**.** *Assume (f1)–(f3). Let $a > 0$ and let U be a weak solution of*

$$(-\Delta)^s U + aU = f(U), \quad x \in \mathbb{R}^N.$$

Then there exist positive constants C'_a, C''_a such that

$$\frac{C'_a}{1+|x|^{N+2s}} \leq U(x) \leq \frac{C''_a}{1+|x|^{N+2s}}, \quad \textit{for } x \in \mathbb{R}^N.$$

These constants can be chosen uniform for $U \in \widehat{S}$.

Proof. The proof is similar to the one carried out in Theorem 5.1.4.

Indeed, as in Step 2 and Step 3, we obtain the uniform boundedness in $L^\infty(\mathbb{R}^N)$. We point out that the values C_δ, H, C and R_0 depend on $a \in [m_0, m_0+\nu_0]$, since they depend on δ and we must have $\delta < a$; on the other hand, it is sufficient to take $\delta < m_0$ to gain uniformity. The same can be said on the uniform boundedness in $C^{0,\sigma}(\mathbb{R}^N)$ and for the constants R'', C', C'' related to the comparison functions $\underline{W'}, \overline{W}'$, thanks to Lemma 1.2.30. As we will show, this allows us to gain that

$$\lim_{|x|\to+\infty} U(x) = 0 \quad \text{uniformly for } U \in \widehat{S}, \tag{5.4.73}$$

which leads, as in Step 6 of the proof, to

$$|f(U(x))| \leq \frac{1}{2} a U(x), \quad \text{for } |x| > R'$$

where R' does not depend on $a \in [m_0, m_0 + \nu_0]$. In addition, compactness of $\widehat{S}$ and a simple contradiction argument lead to $\min_{B_R} U \geq C > 0$ uniformly for $U \in \widehat{S}$. If we prove (5.4.73), we conclude as in Step 6.

Let us prove (5.4.73). By contradiction, there exist $(x_k)_k \subset \mathbb{R}^N$, $|x_k| \to +\infty$, $(U_k)_k \subset \widehat{S}$ and $\theta > 0$ such that $U_k(x_k) > \theta > 0$. Define

$$V_k := U_k(\cdot + x_k).$$

Since both are bounded sequences in $H^s(\mathbb{R}^N)$, we have $U_k \rightharpoonup U$ and $V_k \rightharpoonup V$ in $H^s(\mathbb{R}^N)$; moreover, by the uniform $L^\infty(\mathbb{R}^N)$ and $C^{0,\sigma}_{loc}(\mathbb{R}^N)$ estimates and Ascoli-Arzelà theorem, we have also that the convergences are pointwise. In particular by

$$U_k(0) \geq U_k(x_k) > \theta, \quad V_k(0) = U_k(x_k) > \theta$$

we obtain

$$U(0) \geq \theta > 0, \quad V(0) \geq \theta > 0.$$

As a consequence, U and V are not trivial. Let now $(a_k)_k \subset \mathbb{R}$ be such that $U_k \in S_{a_k}$; up to a subsequence we have $a_k \to a \in [m_0, m_0 + \nu_0]$. Observed that also V_k are solutions of $L'_{a_k}(V_k) = 0$, we obtain, as in Step 3 of Lemma 5.2.4 (see also Step 7 of the proof of Lemma 5.3.6), that U and V are (nontrivial) solutions of $L'_a(U) = 0$. Hence

$$E_{m_0} \leq E_a \leq L_a(U), \quad E_{m_0} \leq E_a \leq L_a(V).$$

By the Pohozaev identity (applied to U_k) we have the following chain of inequalities, once fixed $R > 0$ and $k \gg 0$ such that $|x_k| \geq 2R$,

$$\begin{aligned} l_0 &\geq \liminf_{k\to+\infty} L_{a_k}(U_k) = \frac{s}{N} \liminf_{k\to+\infty} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} U_k|^2 dx \\ &\geq \frac{s}{N} \liminf_{k\to+\infty} \left(\int_{B_R} |(-\Delta)^{s/2} U_k|^2 dx + \int_{B_R} |(-\Delta)^{s/2} V_k|^2 dy \right) \\ &\geq \frac{s}{N} \left(\int_{B_R} |(-\Delta)^{s/2} U|^2 dx + \int_{B_R} |(-\Delta)^{s/2} V|^2 dy \right) \end{aligned}$$

where in the last passage we have used that $U_k \rightharpoonup U$ in $H^s(\mathbb{R}^N)$, thus $(-\Delta)^{s/2} U_k \rightharpoonup (-\Delta)^{s/2} U$ in $L^2(\mathbb{R}^N)$, hence (by restriction) in $L^2(B_R)$, and the weak lower semicontinuity of the norm. Thus, by choosing R sufficiently large, we have, again by the Pohozaev identity (applied to U and V, we use (f3))

$$\begin{aligned} l_0 &\geq \frac{s}{N} \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} V|^2 dy \right) - \eta \\ &= L_a(U) + L_a(V) - \eta \geq 2E_{m_0} - \eta \end{aligned}$$

which leads to a contradiction if we choose $\eta \in (0, 2E_{m_0} - l_0)$, possible thanks to (5.2.14). ∎

Remark 5.4.3. *Actually, (f3) can be dropped, and we highlight here some modifications to the previous proofs.*

- *Define* $S_a := \{U \in H^s_r(\mathbb{R}^N) \setminus \{0\} \mid L'_a(U) = 0,\ L_a(U) \leq l_0,\ P_a(U) = 1\}$.

  We comment the proof of the compactness (Lemma 5.2.4). The nonemptiness si given by the existence of a ground state with $L_a(U) = E_a \leq l_0$*, which is automatically radially symmetric. The boundedness of* $\widehat{S}$ *is given by the extra condition on the Pohozaev; compactness is now enduced by the radial symmetry (and the fact that* $\int_{\mathbb{R}^N} g(u_n) u_n \to \int_{\mathbb{R}^N} g(u) u$*, see Proposition 1.5.5), and the strong convergence implies that the Pohozaev identity is preserved.*

Finally, the proof of uniform asymptotic decay (Proposition 5.4.2) is modified in the following way: after having shown that, for $U_k \in \widehat{S}$ and $|x_k| \to +\infty$, $V_k = U_k(\cdot + x_k)$ satisfies $V_k \rightharpoonup V \not\equiv 0$ (in $H^s(\mathbb{R}^N)$, thus in $L^p(\mathbb{R}^N)$, $p \in (2, 2_s^*)$), while $U_k \rightharpoonup U$ in $H^s_r(\mathbb{R}^N)$, by the compactness we have $U_k \to U$ in $L^p(\mathbb{R}^N)$, $p \in (2, 2_s^*)$. Thus, for every $\varphi \in L^{p'}(\mathbb{R}^N)$ we have $\varphi(\cdot - x_k) \rightharpoonup 0 \in L^{p'}(\mathbb{R}^N)$ and hence

$$\int_{\mathbb{R}^N} V_k \varphi = \int_{\mathbb{R}^N} U_k \varphi(\cdot - x_k) \to 0$$

i.e. $V_k \rightharpoonup 0$ in $L^p(\mathbb{R}^N)$, thus $V \equiv 0$, impossible.

- *Lemma 5.2.6, Lemma 5.2.3 (and whenever the Pohozaev identity is used for $\widehat{S}$), can be proved thanks to the extra condition in S_a.*

Her we kept the original definition of S_a (i.e. $U \in H^s(\mathbb{R}^N)$ such that $\max U = U(0)$), since this approach can be adapted also to frameworks where radial symmetry is not a feature of the limiting problem.

5.5 The critical case

Goal of this Section is to study equation (5.1.3), that is

$$\varepsilon^{2s}(-\Delta)^s v + V(x)v = f(v), \quad x \in \mathbb{R}^N,$$

where now f is assumed critical and satisfying general Berestycki-Lions type conditions. When $\varepsilon > 0$ is small, we obtain again existence and multiplicity of semiclassical solutions, relating the number of solutions to the cup-length of the set of local minima of V; these solutions are proved to concentrate in the potential well, exhibiting a polynomial decay. In particular, we improve the result in [221]. Finally, we prove the previous results also in the limiting local setting $s = 1$ and $N \geq 3$, with an exponential decay of the solutions.

Here, thus, we assume (V1)-(V2) where we recall

$$m_0 = \inf_{\Omega} V$$

with

$$K = \{x \in \Omega \mid V(x) = m_0\}, \tag{5.5.74}$$

and (f1)-(f3), where now (f1.3) is substituted with a critical (not pure) growth, i.e.

(f1') Berestycki-Lions type assumptions with respect to $m_0 > 0$, that is

- (f1.1) $f \in C(\mathbb{R}, \mathbb{R})$;
- (f1.2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
- (f1.3') $\lim_{t\to+\infty} \frac{f(t)}{t^{2_s^*-1}} = a > 0$, where $2_s^* = \frac{2N}{N-2s}$, and moreover for some $C > 0$ and $\max\{2_s^* - 2, 2\} < p < 2_s^*$, i.e. satisfying

$$p \in \begin{cases} \Big(\dfrac{4s}{N-2s}, \dfrac{2N}{N-2s}\Big) & N \in (2s, 4s), \\ \Big(2, \dfrac{2N}{N-2s}\Big) & N \geq 4s, \end{cases} \tag{5.5.75}$$

  it results that

$$f(t) \geq a t^{2_s^*-1} + C t^{p-1} \quad \text{for } t \geq 0;$$

- (f1.4) $F(t_0) > \frac{1}{2} m_0 t_0^2$ for some $t_0 > 0$.

See also Remark 5.5.2 for some weakening and comments on the assumptions (V1), (f1.3') and (f3). Notice that the stronger condition on p in the first line of (5.5.75) is verified, whenever $N \geq 2$, only if $N = 2$ and $s \in (\frac{1}{2}, 1]$, or $N = 3$ and $s \in (\frac{3}{4}, 1]$. We point out that the condition $C > 0$ in (f1.3') is of key importance: indeed, for pure critical nonlinearities of the type $f(t) = |t|^{2_s^*-2}t$, the limiting problem (5.1.5), that is

$$(-\Delta)^s u + m_0 u = |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^N$$

does not admit any variational solution [138].

The existence of a solution in a critical, fractional setting, in the case of local minima (V1)-(V2) and general Berestycki-Lions assumptions (f1')-(f2)-(f3), has been faced in [238] by assuming $V \in C^1(\mathbb{R}^N)$, and moreover in [220] by means of penalization methods.

Inspired by [326], multiplicity of solutions of (5.1.3) in the case of global minima of V was studied in [341] for power-type nonlinearities. Moreover, in [263] the authors consider functions of the type

$$f(t) = g(t) + |t|^{2_s^*-2}t, \tag{5.5.76}$$

where g is subcritical and satisfies a monotonicity condition which allows to implement the Nehari manifold tool, and they relate the number of solutions to the Lusternik-Schnirelmann category of the set of global minima.

Existence of multiple solutions for local minima of V has been investigated, in the spirit of [148], by [221] with sources of the type (5.5.76), where now g satisfies also an Ambrosetti-Rabinowitz condition: this assumption enables to employ Mountain Pass and Palais-Smale arguments, combined with a penalization scheme. Again, the authors are able to find $\mathrm{cat}(K)$ solutions, where K is the set of local minima of V and $\mathrm{cat}(K)$ denotes its Lusternik-Schnirelmann category.

In the present Section we prove a multiplicity result for equation (5.1.3) under almost optimal assumptions of f, showing the concentration of the solutions around local minima of V.

In particular, we prove the following result.

Theorem 5.5.1. *Assume $s \in (0,1)$, $N \geq 2$ and that (V1)-(V2), (f1')-(f2)-(f3) hold. Let K be defined by (5.5.74). Then, for small $\varepsilon > 0$ equation (5.1.3) has at least $\mathrm{cupl}(K) + 1$ positive solutions, which belong to $C^{0,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $\sigma \in (0,1)$. Moreover, each of these sequences v_ε concentrates in K as $\varepsilon \to 0$: namely, there exist $x_\varepsilon \in \mathbb{R}^N$ global maximum points of v_ε, such that*

$$\lim_{\varepsilon \to 0} d(x_\varepsilon, K) = 0$$

and

$$\frac{C'}{1 + |\frac{x - x_\varepsilon}{\varepsilon}|^{N+2s}} \leq v_\varepsilon(x) \leq \frac{C''}{1 + |\frac{x - x_\varepsilon}{\varepsilon}|^{N+2s}} \quad \text{for } x \in \mathbb{R}^N$$

where $C', C'' > 0$ are uniform in $\varepsilon > 0$. Finally, for every sequence $\varepsilon_n \to 0^+$ there exist a ground state solution U of (5.5.77) and a point $x_0 \in K$ such that, up to a subsequence,

$$x_{\varepsilon_n} \to x_0 \in K$$

and

$$v_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n}) \to U \quad \text{as } n \to +\infty$$

in $H^s(\mathbb{R}^N)$ and locally on compact sets.

We highlight that Theorem 5.5.1 extends the existence results in [220, 263] to a multiplicity result, and it improves the multiplicity theorem in [221], since we do not assume monotonicity nor Ambrosetti-Rabinowitz conditions on the nonlinearity. Moreover, no nondegeneracy and global conditions on V are considered.

Remark 5.5.2. *As observed in Remark 5.1.3, assumption (V1) in Theorem 5.5.1 can be relaxed without assuming the boundedness of V (see also [78, 81]). Moreover, the condition*

$$p > \max\{2_s^* - 2, 2\}$$

in (f1.3') can be relaxed in $p > 2$ by paying the cost of considering a sufficiently large $C \gg 0$; see for instance [220, 340]. Finally, we remark that (f3), instead of the mere continuity of f, is needed only to get a Pohozaev identity by means of the regularity of solutions (see Proposition 2.2.2). See also Remark 5.5.8 for further comments.

The idea of the present Section is the following: first, we gain compactness and uniform L^∞-bounds on the set of ground states of the critical limiting problem (5.1.5); to this aim we employ a Moser's iteration argument adapted to the fractional framework, without the use of the s-harmonic extension, and appropriate for weak solutions (see Proposition 1.2.24). The criticality of the problem, as well as the absence of a chain rule, make the argument more delicate. The gained uniformity allows then the introduction of a suitable truncation on the nonlinearity f; the new truncated function reveals thus to be subcritical.

Therefore, we can apply to the truncated problem the approach of the previous Sections: we employ a penalization argument on a neighborhood of expected solutions, perturbation of the ground states of a limiting problem, and this neighborhood results to be invariant under the action of a deformation flow. Compactness is restored also by the use of the new fractional center of mass, which engages the new strong seminorm; the topological machinery between two level sets of the associated indefinite energy functional is then built also through the use of the Pohozaev functional. The number of solutions is thus related to the cup-length of K and these solutions are proved to exhibit a polynomial decay and to converge to a ground state of the limiting equation. This last convergence allows finally to prove that these solutions solve the original critical problem (5.1.3).

We point out that the techniques employed in the previous Sections cannot be applied directly to the critical framework: indeed, the embedding of $H^s(\mathbb{R}^N)$ in $L^{2_s^*}(\mathbb{R}^N)$ is not compact, even if we reduce to radially symmetric functions or to bounded domains; in particular, the criticality obstructs the convergence of truncated Palais-Smale sequences related to the penalized functional, which is a key point in the proof. Moreover, the regularity results given by [134], exploited in the concentration and in the decay of the solutions, do not apply; in particular, L^∞-bounds and compactness of the set of ground states of the limiting problem have to be specifically investigated.

We highlight that the conclusions of Theorem 5.5.1 hold also for $s = 1$ and $N \geq 3$, as we state in Theorem 5.5.9. Regarding this local framework, Theorem 5.5.9 is the critical counterpart of the result in [119]: again, we point out that the arguments exploited in the subcritical setting of [119] cannot be directly implemented in our framework, because of the lack of compactness. In the critical case, previous results were given by [11, 21, 393]: in particular we extend here the existence result in [390] to a multiplicity result, and we improve the multiplicity theorem in [369] in the sense that we do not need to work with global minima of V nor we need monotonicity on f. In this setting, the solutions decay exponentially and enjoy more regularity. Notice that in such a case (f3) is no more needed.

This last part of the Chapter is organized as follows. In Section 5.5.1 we obtain compactness of the set of ground states and a crucial L^∞-bound on the critical limiting problem. In Section 5.5.2 we use this uniform estimate to introduce a truncation which brings the problem back to the subcritical case, and we prove Theorem 5.5.1. Finally, in Section 5.5.3 we deal with the local case.

5.5.1 Uniform L^∞-bound

Let us recall some crucial results on the limiting critical problem (5.1.5), that is

$$(-\Delta)^s U + m_0 U = f(U), \quad x \in \mathbb{R}^N. \tag{5.5.77}$$

We recall the energy $\mathcal{L} : H^s(\mathbb{R}^N) \to \mathbb{R}$

$$\mathcal{L}(U) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2\, dx + \frac{m_0}{2}\int_{\mathbb{R}^N} U^2\, dx - \int_{\mathbb{R}^N} F(U)\, dx, \quad U \in H^s(\mathbb{R}^N),$$

the related least energy

$$E_m := \inf\left\{\mathcal{L}(U) \mid U \in H^s(\mathbb{R}^N) \setminus \{0\},\ \mathcal{L}'(U) = 0\right\},$$

and the Mountain Pass level

$$C_{mp} := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{L}(\gamma(t))$$

with

$$\Gamma := \left\{\gamma \in C([0,1], H^s(\mathbb{R}^N)) \mid \gamma(0) = 0,\ \mathcal{L}(\gamma(1)) < 0\right\}.$$

We introduce also the following *minimization problem*

$$C_{min} := \inf\left\{\mathcal{T}(U) \mid U \in H^s(\mathbb{R}^N),\ \mathcal{V}(U) = 1\right\} \tag{5.5.78}$$

where

$$\mathcal{T}(U) := \int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2\, dx, \quad \mathcal{V}(U) := \int_{\mathbb{R}^N} \left(F(U) - \frac{m_0}{2} U^2\right) dx.$$

Notice that $\mathcal{L} = \frac{1}{2}\mathcal{T} - \mathcal{V}$. The following collection of results states the equivalence of the previous problems and the existence of a solution.

Proposition 5.5.3. *Assume (f1')-(f2)-(f3). Then there exists a ground state solution for the problem (5.5.77), that is a function U which solves the equation and such that*

$$\mathcal{L}(U) = E_m.$$

Moreover, every ground state is also a Mountain Pass solution and (up to scaling) also a solution for the minimization problem (5.5.78), and viceversa; in addition the following relations hold

$$E_m = C_{mp},$$

$$E_m = \frac{s}{N}(2_s^*)^{-\frac{N}{2_s^* s}} (C_{min})^{\frac{N}{2s}}, \tag{5.5.79}$$

and every ground state is positive. Finally, recalled that $\mathcal{S}$ is the best Sobolev constant for the embedding (1.2.7), we have that the following upper bound holds

$$C_{min} < \left(\frac{2_s^*}{a}\right)^{\frac{2}{2_s^*}} \mathcal{S} \tag{5.5.80}$$

where $a > 0$ appears in assumption (f1.3').

Proof. The positivity is a straightforward consequence of assumption (f2). Existence of a ground state solution can be achieved through the use of (5.5.80) and minimization of C_{min} as classically made by [50] (see also [80, Lemma 1]). The equivalence with the Mountain Pass formulation is instead discussed as in [237]. We refer to [238, Proposition 2.4 and Remark 1.3] for the precise statement and to [391, Section 4.1 and Remark 1.2], [255, Section 2] for details.

Moreover, as observed in Remark 5.5.2, to get the existence of a ground state, the restriction on the range of p in assumption (f1.3') can be substituted, by arguing as in [341, Lemma 3.3], with the request that C is sufficiently large (see also [220, Proposition 2.8] and references therein). We refer also to [22, Theorem 3.1.3, Theorem 3.1.5]. ∎

Thanks to Proposition 5.5.3 we can define

$$\widehat{S} := \{U \in H^s(\mathbb{R}^N) \setminus \{0\} \mid U \text{ ground state solution of (5.5.77)}, U(0) = \max_{\mathbb{R}^N} U\}.$$

We observe that, by the fractional version of the Pólya-Szegő inequality [311], every minimizer of C_{min} (i.e. every ground states of (5.5.77)) is actually radially symmetric decreasing up to a translation (see also Remark 2.2.4 and [79, Proposition B.3]). Thus, the request in $\widehat{S}$ for U to have a maximum in zero is equivalent to the radial symmetry of U; that is

$$\widehat{S} = \{U \in H^s(\mathbb{R}^N) \setminus \{0\} \mid U \text{ radially symmetric ground state solution of (5.5.77)}\}. \tag{5.5.81}$$

Proposition 5.5.4. *Every $U \in \widehat{S}$ satisfies the Pohozaev identity, i.e.*

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2 \, dx - 2_s^* \int_{\mathbb{R}^N} \left(F(U) - \frac{m_0}{2} U^2 \right) dx = 0. \tag{5.5.82}$$

Moreover, the set $\widehat{S}$ is compact.

Proof. Once one observes that $U \in L^\infty(\mathbb{R}^N)$, which follows from Proposition 1.2.24, the proof of (5.5.82) is gained by means of regularity results and explicit computations on the s-harmonic extension problem; the arguments can be easily adapted from [79, Proposition 1.1] to the critical case.

Let us show the boundedness of $\widehat{S}$. For any $U \in \widehat{S}$, the embedding (1.2.7) and the Pohozaev identity (5.5.82) lead to

$$\|U\|_{2_s^*} \leq \mathcal{S}^{-\frac{1}{2}} \|(-\Delta)^{s/2} U\|_2 = \mathcal{S}^{-\frac{1}{2}} \frac{N}{s} \mathcal{L}(U) = \mathcal{S}^{-\frac{1}{2}} \frac{N}{s} E_m;$$

moreover equation (5.5.77) and assumption (f1') imply

$$\|(-\Delta)^{s/2} U\|_2^2 + m_0 \|U\|_2^2 = \int_{\mathbb{R}^N} f(U) U \, dx \leq \delta \|U\|_2^2 + C_\delta \|U\|_{2_s^*}^{2_s^*}$$

for $\delta < m_0$ and some $C_\delta > 0$. The combination of the two bounds leads to the claim.

Let thus focus on compactness; we use some ideas from [392]. Let U_n be a sequence in $\widehat{S}$; by (5.5.81) we assume $(U_n)_n \subset H^s_r(\mathbb{R}^N)$, where $H^s_r(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ for $q \in (2, 2_s^*)$. By the boundedness of $\widehat{S}$ we can assume $U_n \rightharpoonup U$ in $H^s_r(\mathbb{R}^N)$. Set

$$\sigma := \left(\frac{1}{2_s^*} C_{min} \right)^{\frac{1}{2s}}$$

and

$$V_n := U_n(\sigma \cdot), \quad V := U(\sigma \cdot)$$

we have, by exploiting the Pohozaev identity, that V_n are solutions of the minimization problem (5.5.78), that is

$$\mathcal{T}(V_n) = C_{min}, \quad \mathcal{V}(V_n) = 1.$$

Thus we have $V_n \rightharpoonup V$ in $H^s_r(\mathbb{R}^N)$, and hence $V_n \to V$ in $L^q(\mathbb{R}^N)$, $q \in (2, 2_s^*)$, and $V_n \to V$ almost everywhere. By the lower semicontinuity of the norm we obtain

$$\mathcal{T}(V) \leq C_{min}; \tag{5.5.83}$$

hence, to conclude the proof, it is sufficient to show that $\mathcal{V}(V)=1$, since this implies also that $U=V(\sigma^{-1}\cdot)$ lies in $\widehat{S}$.

Set

$$W_n := V_n - V$$

we have by the Brezis-Lieb Lemma (since $(-\Delta)^{s/2}V_n \rightharpoonup (-\Delta)^{s/2}V$ in the Hilbert space $L^2(\mathbb{R}^N)$)

$$\begin{aligned} \mathcal{T}(W_n) &= \mathcal{T}(V_n) - \mathcal{T}(V) + o(1) \\ &= C_{min} - \mathcal{T}(V) + o(1) \qquad (5.5.84)\\ &\leq C_{min} + o(1). \qquad (5.5.85)\end{aligned}$$

Moreover, rewrite $\mathcal{V}(W_n)$ as

$$\mathcal{V}(W_n) = \int_{\mathbb{R}^N} \left(F(W_n) - \frac{a}{2_s^*} W_n^2 \right) dx + \frac{a}{2_s^*} \|W_n\|_{2_s^*}^{2_s^*} - \frac{m_0}{2}\|W_n\|_2^2. \tag{5.5.86}$$

Again by the Brezis-Lieb Lemma (since $V_n \rightharpoonup V$ in $L^q(\mathbb{R}^N)$, $q=2,2_s^*$ and $V_n \to V$ almost everywhere) we have

$$\|W_n\|_q^q = \|V_n\|_q^q - \|V\|_q^q + o(1), \quad q = 2, 2_s^*. \tag{5.5.87}$$

Set

$$g(t) := f(t) - a t^{2_s^*-1}$$

we have that g is subcritical at infinity by (f1.3'), and superlinear in zero by (f1.2); thus, set $G(t) := \int_0^t g(\tau)d\tau$, by Proposition 1.5.5 we have

$$\int_{\mathbb{R}^N} G(W_n)\,dx = o(1), \quad \int_{\mathbb{R}^N} G(V_n)\,dx = \int_{\mathbb{R}^N} G(V)\,dx + o(1). \tag{5.5.88}$$

Therefore by (5.5.86)–(5.5.88) we obtain

$$\begin{aligned} \mathcal{V}(W_n) &= \mathcal{V}(V_n) - \mathcal{V}(V) + o(1) \\ &= 1 - \mathcal{V}(V) + o(1). \qquad (5.5.89)\end{aligned}$$

Finally, through a simple scaling argument, we observe that

$$\mathcal{T}(u) \geq C_{min} (\mathcal{V}(u))^{\frac{2}{2_s^*}} \quad \text{for every } \mathcal{V}(u) \geq 0. \tag{5.5.90}$$

We pass to prove that $\mathcal{V}(V)=1$ by contradiction.
Case $\mathcal{V}(V) > 1$. In this case, by (5.5.90) we have

$$\mathcal{T}(V) \geq C_{min}(\mathcal{V}(V))^{\frac{2}{2_s^*}} > C_{min}$$

which contradicts (5.5.83).
Case $\mathcal{V}(V) < 0$. Then, by (5.5.89) we have that

$$\mathcal{V}(W_n) \geq 1 - \frac{1}{2}\mathcal{V}(V) > 1 \quad \text{for } n \gg 0.$$

Thus, by (5.5.90) we obtain

$$\mathcal{T}(W_n) \geq C_{min}(\mathcal{V}(W_n))^{\frac{2}{2_s^*}} \geq C_{min}\left(1 - \frac{1}{2}\mathcal{V}(V)\right)^{\frac{2}{2_s^*}}$$

which contradicts (5.5.85).
Case $\mathcal{V}(V) \in (0,1)$. Again by (5.5.89) we have that

$$\mathcal{V}(W_n) \geq \frac{1}{2}(1 - \mathcal{V}(V)) > 0 \quad \text{for } n \gg 0.$$

Thus by (5.5.84), (5.5.90) and (5.5.89) we gain

$$\begin{aligned} C_{min} &= \lim_n \left(\mathcal{T}(W_n) + \mathcal{T}(V)\right) \geq C_{min} \lim_n \left((\mathcal{V}(W_n))^{\frac{2}{2_s^*}} + (\mathcal{V}(V))^{\frac{2}{2_s^*}} \right) \\ &= C_{min} \left((1-\mathcal{V}(V))^{\frac{2}{2_s^*}} + (\mathcal{V}(V))^{\frac{2}{2_s^*}} \right) \\ &> C_{min}((1-\mathcal{V}(V)) + \mathcal{V}(V)) = C_{min} \end{aligned}$$

which is an absurd.
Case $\mathcal{V}(V) = 0$. By (5.5.89) we have

$$\mathcal{V}(W_n) = 1 + o(1), \tag{5.5.91}$$

and thus by (5.5.90) $\mathcal{T}(W_n) \geq C_{min}(1+o(1))^{\frac{2}{2_s^*}}$. This, combined with (5.5.85), gives

$$\mathcal{T}(W_n) = C_{min} + o(1). \tag{5.5.92}$$

Combining (5.5.91), (5.5.86) and (5.5.88) we obtain

$$1 + o(1) = \mathcal{V}(W_n) = \frac{a}{2_s^*} \|W_n\|_{2_s^*}^{2_s^*} - \frac{m_0}{2} \|W_n\|_2^2$$

that is

$$\begin{aligned} \|W_n\|_{2_s^*}^{2_s^*} &= \frac{2_s^*}{a} + \frac{2_s^* m_0}{2a} \|W_n\|_2^2 + o(1) \\ &\geq \frac{2_s^*}{a} + o(1). \end{aligned} \tag{5.5.93}$$

By (5.5.92), the Sobolev embedding (1.2.7) and (5.5.93) we gain

$$C_{min} + o(1) = \mathcal{T}(W_n) = \|(-\Delta)^{s/2} W_n\|_2^2 \geq \mathcal{S} \|W_n\|_{2_s^*}^2 \geq \mathcal{S} \left(\frac{2_s^*}{a} + o(1) \right)^{\frac{2}{2_s^*}}.$$

Letting $n \to +\infty$ we finally have

$$C_{min} \geq \left(\frac{2_s^*}{a} \right)^{\frac{2}{2_s^*}} \mathcal{S}$$

which is in contradiction with (5.5.80). This concludes the proof. ∎

As a key property to employ the truncation argument, and to detect a handy neighborhood of approximating solutions, we have the following result.

Proposition 5.5.5. *The following bound holds*

$$\sup_{U \in \widehat{S}} \|U\|_\infty < \infty.$$

Proof. Assume by contradiction that there exists $(U_n)_n \subset \widehat{S}$ such that $\|U_n\|_\infty \to +\infty$ as $n \to +\infty$. By the compactness of $\widehat{S}$ in Proposition 5.5.4 we may assume that U_n is positive and convergent in $H^s(\mathbb{R}^N)$; in particular U_n converges in $L^{2_s^*}(\mathbb{R}^N)$ and is equibounded a. e. pointwise by a function in $L^{2_s^*}(\mathbb{R}^N)$. If we prove that

$$\sup_n \|U_n\|_\infty < +\infty$$

we get a contradiction and conclude the proof. In order to do this, we argue as in the proof of Proposition 1.2.24, uniformly in n for $U_n = U_n^+$; the idea is a Moser's iteration argument in a critical, fractional framework, appropriate for weak solutions. We refer to [197] for details. ∎

5.5.2 The truncated problem

In virtue of Proposition 5.5.5, let

$$M := \sup_{U\in\widehat{S}} \|U\|_\infty + 1.$$

We preliminary observe that we can find a $t_0 \in [0, M]$ such that

$$F(t_0) > \frac{1}{2} m_0 t_0^2. \tag{5.5.94}$$

Indeed fixed a whatever $U \in \widehat{S}$, by the Pohozaev identity (5.5.82) we have (notice that $(-\Delta)^{s/2}U$ cannot identically vanish)

$$\int_{\mathbb{R}^N} \left(F(U) - \frac{m_0}{2}U^2\right) dx = \frac{1}{2_s^*}\|(-\Delta)^{s/2}U\|_2^2 > 0$$

and thus there exists an $x_0 \in \mathbb{R}^N$ such that

$$F(U(x_0)) > \frac{m_0}{2}U(x_0)^2;$$

setting $t_0 := U(x_0) \in [0, M]$ we have the claim.

We thus set

$$k := \sup_{t\in[0,M]} f(t) \in (0, +\infty),$$

where we observe that the strict positivity is due to the fact that $F(t_0) > 0$. Moreover we define the *truncated* nonlinearity $f_k : \mathbb{R} \to \mathbb{R}$

$$f_k(t) := \min\{f(t), k\}, \quad t \in \mathbb{R}.$$

We have the following properties on $f_k : \mathbb{R} \to \mathbb{R}$:

- $f_k(t) \leq f(t)$ for each $t \in \mathbb{R}$,
- $f_k(t) = f(t)$ whenever $|t| \leq M$,
- $f_k(U) = f(U)$ for every $U \in \widehat{S}$.

Notice that the same relations hold also for F and

$$F_k(t) := \int_0^t f_k(\tau) d\tau.$$

We have that f_k is subcritical, i.e. f_k satisfies assumptions (f1)–(f3); here $p \in (1, 2_s^* - 1)$ is however fixed and $t_0 \in [0, M]$ is the one appearing in (5.5.94); notice that t_0 does not depend on k.

Consider now the truncated problem

$$\varepsilon^{2s}(-\Delta)^s v + V(x)v = f_k(v), \quad x \in \mathbb{R}^N \tag{5.5.95}$$

and the corresponding limiting truncated problem

$$(-\Delta)^s U + m_0 U = f_k(U), \quad x \in \mathbb{R}^N. \tag{5.5.96}$$

Notice again that, since f_k satisfies (f2), all the ground states of (5.5.96) are positive. Thus define

$$\widehat{S}_k := \{U \in H^s(\mathbb{R}^N) \setminus \{0\} \mid U \text{ ground state solution of (5.5.96)}, U(0) = \max_{\mathbb{R}^N} U\}.$$

We have that the following key relation holds.

Proposition 5.5.6. *It results that $\widehat{S} = \widehat{S}_k$. Moreover, the least energy levels coincide.*

Proof. Let us denote by $\mathcal{L}_k$, Γ_k, $\mathcal{V}_k$, $E^k_m = C^k_{mp}$, C^k_{min} the quantities of the truncation problem analogous to the ones introduced in Section 5.5.1 for the critical problem.

First observe that, by $\mathcal{L}_k \geq \mathcal{L}$, we have $\Gamma_k \subset \Gamma$ and

$$C^k_{mp} \geq C_{mp}; \tag{5.5.97}$$

moreover for any $V \in \widehat{S}$ we have also $\mathcal{L}'_k(V) = 0$, and hence

$$\min_{V\in\widehat{S}} \mathcal{L}_k(V) \geq \min_{\mathcal{L}'_k(V)=0} \mathcal{L}_k(V) = E^k_m. \tag{5.5.98}$$

Let now $U \in \widehat{S}$. We have by (5.5.97) and (5.5.98)

$$C^k_{mp} \geq C_{mp} = \mathcal{L}(U) = E_m = \min_{V\in\widehat{S}} \mathcal{L}(V) = \min_{V\in\widehat{S}} \mathcal{L}_k(V) \geq E^k_m.$$

Therefore

$$\mathcal{L}_k(U) = \mathcal{L}(U) = C^k_{mp} = E^k_m$$

which, together with $\mathcal{L}'_k(U) = \mathcal{L}'(U) = 0$, gives that $U \in \widehat{S}_k$. Hence $\widehat{S} \subset \widehat{S}_k$. As a further consequence we gain

$$E^k_m = E_m. \tag{5.5.99}$$

We show now that $\widehat{S}_k \subset \widehat{S}$. By (5.5.99), (5.5.79) and the analogous relation on the subcritical problem, we have

$$C^k_{min} = C_{min},$$

thus, by rescaling, it is sufficient to prove that every minimizer of C^k_{min} is also a minimizer of C_{min}. Let thus U be a minimizer for C^k_{min}, i.e. $\mathcal{T}(U) = C^k_{min}$ and $\mathcal{V}_k(U) = 1$. Since $\mathcal{T}(U) = C_{min}$, it suffices to prove that $\mathcal{V}(U) = 1$. By definition, we have

$$\mathcal{V}(U) \geq \mathcal{V}_k(U) = 1.$$

On the other hand, set $\theta := (\mathcal{V}(U))^{\frac{1}{N}}$ we obtain, by scaling, that $\mathcal{V}(U(\theta\cdot)) = 1$ and thus

$$\mathcal{T}(U) = C_{min} \leq \mathcal{T}(U(\theta\cdot)) = \theta^{-\frac{N+2s}{N}} \mathcal{T}(U)$$

from which we achieve

$$\mathcal{V}(U) \leq 1.$$

This concludes the proof. ∎

We are now ready to prove Theorem 5.5.1.

Proof of Theorem 5.5.1.
Step 1. We first look at the truncated problem (5.5.95). Indeed, by Theorems 5.1.2 and 5.1.4 we obtain the existence of $\mathrm{cupl}(K) + 1$ sequences of solutions of (5.5.95) satisfying the properties of Theorem 5.5.1 for $\varepsilon > 0$ small. For each of these sequences v_ε of solutions of (5.5.95), called $x_\varepsilon \in \mathbb{R}^N$ a global maximum point of v_ε, we obtain

$$\lim_{\varepsilon\to 0} d(x_\varepsilon, K) = 0$$

and

$$\frac{C'}{1 + |\frac{x-x_\varepsilon}{\varepsilon}|^{N+2s}} \leq v_\varepsilon(x) \leq \frac{C''}{1 + |\frac{x-x_\varepsilon}{\varepsilon}|^{N+2s}} \quad \text{for } x \in \mathbb{R}^N$$

where $C', C'' > 0$ are uniform in $\varepsilon > 0$.

Moreover, for every sequence $\varepsilon_n \to 0^+$ there exist $U \in \widehat{S}_k$ and an $x_0 \in \mathbb{R}^N$ such that, up to subsequences,

$$v_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n}) \to U(\cdot + x_0), \quad \text{as } n \to +\infty \tag{5.5.100}$$

in $H^s(\mathbb{R}^N)$ and locally on compact sets.

Step 2. Notice that by Proposition 5.5.6 we have $U \in \widehat{S}$, thus $U(\cdot + x_0)$ is a ground state of (5.5.77). We prove now that v_ε are solutions of the original equation, which is given by

$$\|v_\varepsilon\|_\infty < M \quad \text{definitely for } \varepsilon \text{ small.} \tag{5.5.101}$$

Assume by contradiction that (5.5.101) does not hold: thus there exists a sequence $\varepsilon_n \to 0$ such that

$$\|v_{\varepsilon_n}\|_\infty \geq M \quad \text{for each } n \in \mathbb{N}.$$

By the previous Step, there exists an $U \in \widehat{S}_k$ and an $x_0 \in \mathbb{R}^N$ such that, up to subsequence, (5.5.100) holds. In particular, by the pointwise convergence we obtain

$$\|v_{\varepsilon_n}\|_\infty = v(x_{\varepsilon_n}) \to U(x_0) \leq \|U\|_\infty < M$$

which implies

$$\|v_{\varepsilon_n}\|_\infty < M$$

definitely for $n \gg 0$, which is an absurd. Thus (5.5.101) holds. As a consequence

$$f_k(v_\varepsilon) = f(v_\varepsilon)$$

and hence v_ε are solutions of the original problem (5.1.3), satisfying the desired properties. ∎

Remark 5.5.7. *We point out that the found solutions are perturbations of ground states of the truncated limiting problem* (5.5.96) *which are, on the other hand, coinciding with the ground states of the critical limiting problem* (5.5.77) *thanks to Proposition 5.5.6. One may think to search directly the solutions as perturbation of functions in the compact set $\widehat{S}$, but actually the direct approach in a critical setting reveals several problems, such as the convergence of the Palais-Smale sequences. A different and direct approach is given in [21] by means of Concentration-Compactness techniques, but in the assumptions that f satisfies monotonicity and Ambrosetti-Rabinowitz conditions.*

Remark 5.5.8. *We see that actually the ideas of this Section adapts to study the case of f negatively critical $a < 0$, or subcritical $a = 0$ (but not in the strict sense of* (f1.3) *treated in the previous Sections), that is*

$$\lim_{t \to +\infty} \frac{f(t)}{|t|^{2_s^*-1}} = a \in (-\infty, 0],$$

instead of (f1.3'), *filling the gap between the papers [111] and [197]. This case covers functions of the type $f(t) = |t|^{p-2}t - |t|^{2_s^*-2}t$ and $f(t) = \frac{|t|^{2_s^*-2}t}{\log(t^2+2)}$.*

In order to achieve this result, we sketch the steps:

- *We substitute the existence result Proposition 5.5.3 with the one by [95], observing that $f \in C^1$ is needed only to get the Pohozaev identity, thus our assumptions* (f3) *is enough (see also Remark 2.2.5).*
- *The uniform L^∞-bound of Proposition 5.5.5 can be easily adapted.*
- *The uniform $C^{0,\sigma}$-bound can be obtained as in the Step 3 of the proof of Theorem 5.1.4.*

- *The compactness result Proposition 5.5.4 can be obtained as in the strict-subcritical case Lemma 5.2.4. When $a = 0$, the proof follows verbatim, otherwise we adapt Step 4 in the following way.*
  Let $f = f^+ - f^-$. If $a < 0$ it means that $f^+(t) = 0$ for $t \gg 0$; in particular f^+ is subcritical. By knowing $U_k \rightharpoonup U$ in $H^s(\mathbb{R}^N)$, $a_k \to a$ and $L'_{a_k}(U_k)U_k = 0 = L'_a(U)U$ we want to show that $\|(-\Delta)^{s/2}U_k\|_2^2 + a_k\|U_k\|_2^2 \to \|(-\Delta)^{s/2}U\|_2^2 + a\|U\|_2^2$. Observe that, by (5.2.15), Proposition 1.5.5 and Fatou's Lemma, for any $\eta > 0$ there exists $R \gg 0$ such that

$$\left|\int_{B_R^c} f(U_k)U_k\right|, \left|\int_{B_R^c} f(U)U\right| < \eta \quad \textit{for each } k \in \mathbb{N},$$

$$\int_{B_R} f^+(U_k)U_k \to \int_{B_R} f^+(U)U,$$

$$\liminf_{k\to+\infty} \int_{B_R} f^-(U_k)U_k \geq \int_{B_R} f^-(U)U.$$

Thus

$$\begin{aligned}
&\limsup_k \left(\|(-\Delta)^{s/2}U_k\|_2^2 + a_k\|U_k\|_2^2\right) \leq \limsup_k \int_{\mathbb{R}^N} f(U_k)U_k \\
\leq\ & \limsup_k \int_{B_R^c} f(U_k)U_k + \limsup_k \int_{B_R} f^+(U_k)U_k - \liminf_k \int_{B_R} f^-(U_k)U_k \\
\leq\ & \eta + \int_{B_R} f^+(U)U - \int_{B_R} f^-(U)U = \eta + \int_{\mathbb{R}^N} f(U)U - \int_{B_R^c} f(U)U \\
\leq\ & 2\eta + \int_{\mathbb{R}^N} f(U)U = 2\eta + \|(-\Delta)^{s/2}U\|_2^2 + a\|U\|_2^2.
\end{aligned}$$

Letting $\eta \to 0$ we obtain

$$\limsup_k \left(\|(-\Delta)^{s/2}U_k\|_2^2 + a_k\|U_k\|_2^2\right) \leq \|(-\Delta)^{s/2}U\|_2^2 + a\|U\|_2^2$$

which, together with the semicontinuity of the norm, gives the claim. ∎

The remaining part of the proof follows the lines of the critical case treated in this Section.

5.5.3 The local case

The arguments presented in Theorem 5.5.1 apply, with suitable modifications, also to local nonlinear Schrödinger equations. We give here some details. Condition (f1') rewrites in the local case $s = 1$ as

(f1') Berestycki-Lions type assumptions with respect to $m_0 > 0$, that is

- (f1.1) $f \in C(\mathbb{R}, \mathbb{R})$;
- (f1.2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
- (f1.3') $\lim_{t\to+\infty} \frac{f(t)}{t^{2^*-1}} = a > 0$, where $2^* = \frac{2N}{N-2}$, and moreover for some $C > 0$ and $\max\{2^* - 2, 2\} < p < 2^*$, i.e. satisfying

$$p \in \begin{cases} (4,6) & N = 3, \\ \left(2, \dfrac{2N}{N-2}\right) & N \geq 4, \end{cases}$$

  it results that

$$f(t) \geq at^{2^*-1} + Ct^{p-1} \quad \text{for } t \geq 0;$$

(f1.4) $F(t_0) > \frac{1}{2}m_0 t_0^2$ for some $t_0 > 0$.

See also Remark 5.5.2 for some weakening and comments on the assumption (f1.3').

Theorem 5.5.9. *Suppose $s = 1$, $N \geq 3$ and that (V1)-(V2), (f1')-(f2) hold. Let K be defined by (5.5.74). Then, for small $\varepsilon > 0$ the equation*

$$-\varepsilon^2 \Delta v + V(x)v = f(v), \quad x \in \mathbb{R}^N$$

has at least $\mathrm{cupl}(K)+1$ *positive solutions, which belong to $C^{1,\sigma}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for some $\sigma \in (0,1)$. Moreover, each of these sequences v_ε concentrates in K as $\varepsilon \to 0$: namely, there exist $x_\varepsilon \in \mathbb{R}^N$ global maximum points of v_ε, such that*

$$\lim_{\varepsilon \to 0} d(x_\varepsilon, K) = 0$$

and

$$v_\varepsilon(x) \leq C' \exp\Big(- C'' \Big| \frac{x - x_\varepsilon}{\varepsilon} \Big| \Big) \qquad \textit{for } x \in \mathbb{R}^N \tag{5.5.102}$$

where $C', C'' > 0$ are uniform in $\varepsilon > 0$. Finally, for every sequence $\varepsilon_n \to 0^+$ there exists a ground state solution U of

$$-\Delta U + m_0 U = f(U), \quad x \in \mathbb{R}^N$$

and a point $x_0 \in K$ such that, up to a subsequence, $x_{\varepsilon_n} \to x_0$ and

$$v_{\varepsilon_n}(\varepsilon_n \cdot + x_{\varepsilon_n}) \to U \quad \textit{as } n \to +\infty$$

in $H^s(\mathbb{R}^N)$ and locally on compact sets.

Proof. The arguments of the previous Sections apply mutatis mutandis. Indeed, we define in the same way the set of ground states $\widehat{S}$, which turns to be nonempty [392] and compact. Moreover to get the uniform $L^\infty(\mathbb{R}^N)$ bound, one can easily adapt the proof of Proposition 5.5.5 after observing that by the chain rule it holds

$$|\nabla h(U)|^2 = \nabla U \cdot \nabla \tilde{h}(U), \quad U \in H^1(\mathbb{R}^N),$$

where we recall that $\tilde{h}' = (h')^2$. Then the truncation machinery can be set in motion, and one can prove $\widehat{S}_k = \widehat{S}$. Existence, multiplicity and decay of solutions of the truncated problem are given by [119, Theorem 1.1 and Remark 1.3]; the regularity is instead a consequence of standard elliptic estimates [354, Appendix B]. ∎

APPENDIX A

A Some algebraic topology: the relative cup-length

In order to estimate the number of critical points of certain functionals not bounded from below and above, it is useful to implement the algebraic-topological tool of the *relative cup-length*, together with the more used *relative category*. In this Appendix we briefly recall the basic notion of algebraic topology needed to define this object; afterwards we will highlight how it relates to the category and how they are exploited in order to gain multiple solutions of PDEs. Finally we will briefly recall also the definition of the genus.

A.1 The singular cohomology

We start by defining the *singular cohomology*. Here we essentially follow the self-contained description due to [94], but we refer also to [69, 160, 168, 210, 210, 217, 287, 288, 350, 368].

Fix X a topological space (in our case it will be a subset of some Hilbert space, such as $\mathbb{R}^N$ or $H^s(\mathbb{R}^N)$, see Section 5.4), and fix an abelian group G: actually the choice of G does not heavily influence the main properties of cohomology, and usually G is chosen as a generic field $\mathbb{F}$ [119], or some specific ones like the real field $\mathbb{R}$ [185, 187] or the $\mathbb{Z}_2$ field [358].

Let $q \in \mathbb{N}$, and let Δ_q be the *q-simplex* defined by

$$\Delta_q := \left\{ \sum_{j=0}^q \lambda_j e_j \mid \lambda_j \geq 0, \ \sum_{j=0}^q \lambda_j = 1 \right\} \equiv \left\{ (\lambda_0, \lambda_1, \dots, \lambda_q, 0, \dots) \mid \lambda_j \geq 0, \ \sum_{j=0}^q \lambda_j = 1 \right\}$$

where $e_0 := (0, 0, \dots)$, $e_1 := (0, 1, \dots)$ and so on, are vectors in $\mathbb{R}^\infty$. We define the set of *singular q-simplexes* by

$$\Sigma_q(X) := \{\sigma : \Delta_q \to X \ \text{ continuous}\}.$$

Starting from $\Sigma_q(X)$ and G we can build the free abelian group $C_q(X, G)$ with bases $\Sigma_q(X)$, that is

$$C_q(X, G) := \left\{ \sum_{i, \text{ finite}} g_i \sigma_i \mid g_i \in G, \ \sigma_i \in \Sigma_q(X) \right\}$$

where the linear combination has to be intended in the formal sense[1]. We call $C_q(X, G)$ the set

[1]For example, if $G = R$ is a ring with unit 1_R, we define the free abelian group in this way [368, page 4]: start by identifying the elements $\sigma \in \sigma_q$ with the functions $f_\sigma : \Sigma_q(X) \to R$, $f_\sigma(\tau) := \begin{cases} 1_R \text{ if } \tau = \sigma \\ 0_R \text{ if } \tau \neq \sigma \end{cases}$. Then set

$$C_q(X, G) := \{f : \Sigma_q(X) \to \mathbb{Z} \mid f(\sigma) \neq 0 \text{ for a finite number of } \sigma \in \Sigma_q(X)\}$$

of *singular q-chains*; here an inner summation and an external product (through G) can be easily defined.

We define now a boundary operator on $C_q(X,G)$, introducing it first on $\Sigma_q(X)$ and then extending it by linearity. Indeed, for any $q \geq 1$ and $\sigma \in \Sigma_q(X)$ and $j = 0 \dots q$ we define $\sigma^{(j)} \in \Sigma_{q-1}$ by

$$\sigma^{(j)}(x_0, x_1, \dots, x_{q-1}) := \sigma(x_0, x_1, \dots x_{j-1}, 0, x_j, \dots x_{q-1})$$

where 0 is in the j-th position. Thus the *boundary operator* is defined as

$$\partial\sigma := \sum_{j=0}^{q} (-1)^j \sigma^{(j)}$$

and hence

$$\partial : C_q(X,G) \to C_{q-1}(X,G).$$

If $q = 0$, the boundary operator $\partial : C_0(X,G) \to G$ is defined as $\partial(\sum g_i\sigma_i) := \sum g_i$ (we are formally setting $C_{-1}(X,G) := G$). We have that ∂ is a homomorphism. Set

$$C_*(X,G) := \bigoplus_{q\geq 0} C_q(X,G)$$

we have $\partial : C_*(X,G) \to C_*(X,G)$. It is a straightforward computation showing that

$$\partial^2 = 0$$

which is of key importance in the theory of homologies and cohomologies. With these ingredients it is possible to define a homology $H_*(X,G)$; we are anyway interested in *cohomologies*, and thus we need first to pass on homomorphisms and dualities. Thus we define the set of *singular q-cochains*

$$C^q(X,G) := Hom(C_q(X,G), G);$$

by using the bracket notation

$$[\sigma, c] := c(\sigma)$$

for every $c \in C^q(X,G)$ and $\sigma \in C_q(X,G)$, the definition of $C^q(X,G)$ rewrites as

$$[\sigma_1 + \sigma_2, c] = [\sigma_1, c] + [\sigma_2, c] \quad \text{and} \qquad [g\sigma, c] = g[\sigma, c]$$

for every $c \in C^q(X,G)$, $\sigma, \sigma_1, \sigma_2 \in C_q(X,G)$ and $g \in G$. We can hence define the dual operator of ∂, named the *coboundary operator*, by

$$[\sigma, \delta c] := [\partial\sigma, c]$$

for every $c \in C^{q-1}(X,G)$, $\sigma \in C_q(X,G)$; thus

$$\delta : C^{q-1}(X,G) \to C^q(X,G),$$

which is a homomorphism. Set

$$C^*(X,G) := \bigoplus_{q\geq 0} C^q(X,G)$$

we have $\delta : C^*(X,G) \to C^*(X,G)$, and we obtain

$$\delta^2 = 0.$$

and observe that $\Sigma_q(X) \equiv \{f_\sigma\}_{\sigma\in\Sigma_q(X)}$ is a basis for $C_q(X,G)$, that is, elements of $C_q(X,G)$ are of the form

$$f = \sum_{i,\text{ finite}} g_i f_{\sigma_i}.$$

In particular the last property easily implies that $\mathrm{Im}(\delta) \lhd \mathrm{Ker}(\delta)$, thus we are allowed to define the *singular q-cohomology group*

$$H^q_\Delta(X,G) := \mathrm{Ker}(\delta_{|C^q(X,G)})/\mathrm{Im}(\delta_{|C^{q-1}(X,G)})$$

the sets $\mathrm{Ker}(\delta)$ and $\mathrm{Im}(\delta)$ are said, respectively, the sets of the *cocycles* and of the *coboundaries*. We highlight that $H^0(X,G)$ may be interpreted as the set of functions $X \to G$ constant on path-components of X [217, pages 198-199] (see also [368, Proposition 3.11], [210, page 183], [350, page 244], [287, Lemma 1.2]), while $H^q(\emptyset, G)$ is the trivial cohomology [287, page 192].

Moreover we define the *singular cohomology group* on X with coefficients in G

$$H^*_\Delta(X,G) := \bigoplus_{q\geq 0} H^q_\Delta(X,G).$$

Assume from now on $G = R$ to be a commutative ring with unit. On the cohomology $H^*_\Delta(X,R)$ (also called *cohomology ring* of X [160, Remark 8.17]) we can define a *cup product*: instead of introducing it in terms of *cross product*, we give here a direct construction. We start by defining it on $C^*(X,R)$, then by quotient we obtain it also on $H^*_\Delta(X,R)$. Indeed, fixed $p,q \geq 0$, we define

$$\phi_p : \Delta_p \to \Delta_{p+q}, \quad \beta_q : \Delta_q \to \Delta_{p+q}$$

the immersions in the first p components and in the last q components respectively, i.e.

$$\phi_p(\lambda_1,\dots,\lambda_p,0,\dots) := (\lambda_1,\dots,\lambda_p,0,\dots,0,0,\dots),$$

$$\beta_q(\lambda_1,\dots,\lambda_q,0,\dots) := (0,\dots,0,\lambda_1,\dots,\lambda_q,0,\dots),$$

so that, if $\sigma \in C_{p+q}(X,R)$, then $\sigma\phi_p \in C_p(X,R)$ and $\sigma\beta_q \in C_q(X,R)$. Thus we define, through the product in R, the cup product

$$[\sigma, c \smile d] := [\sigma\phi_p, c][\sigma\beta_q, d]$$

for any $c \in C^p(X,R)$, $d \in C^q(X,R)$ and $\sigma \in C_{p+q}(X,R)$, which implies

$$\smile : C^p(X,R) \times C^q(X,R) \to C^{p+q}(X,R)$$

and more generally, $\smile : C^*(X,R) \times C^*(X,R) \to C^*(X,R)$. Notice that multiplying $c \in C^p(X,R)$ with $d \in C^0(X,R)$ means multiplying by constant elements of the form $\sum_{i,\,\text{finite}} g_i\sigma_i(e_0)$, with $g_i \in G$ and $\sigma_i \in \Sigma_p(X)$. The cup product results bilinear, associative and unitary. Moreover, it satisfies $c \smile d = (-1)^{pq} d \smile c$ (since R is commutative [217, Theorem 3.11]), which implies that it is *skew*-commutative: even if not properly commutative, it nevertheless satisfies

$$c \smile d = 0 \iff d \smile c = 0. \tag{A.1}$$

Moreover, it holds

$$\delta(c \smile d) = \delta c \smile d + (-1)^p \smile \delta d,$$

for $c \in C^p(X,R)$ and $d \in C^q(X,R)$; in particular, this easily implies that $\mathrm{Ker}(\delta)$ is a subalgebra of $C^*(X,R)$ and $\mathrm{Im}(\delta)$ is an ideal of $\mathrm{Ker}(\delta)$. Thus, $\smile$ can be passed to the quotient and hence defined on

$$\smile : H^*_\Delta(X,R) \times H^*_\Delta(X,R) \to H^*_\Delta(X,R).$$

Starting from a whatever cohomology $H^*(X,R) = H^*_\Delta(X,R)$ (see Section A.2), we can define the *cup-length* as the length of the longest nontrivial chain of cup products in $H^*(X,R)$ (up to the constants in $H^0(X,R)$).

Definition A.1. *Let X be topological space and R be a commutative ring with unit. We define the* cup-length of X *as*

$$\operatorname{cupl}(X,R) := \max\{l \in \mathbb{N}^* \mid \exists \alpha_i \in H^{q_i}(X,R), q_i \geq 1 \text{ for } i = 1 \dots l, \\ \text{s.t. } \alpha_1 \smile \cdots \smile \alpha_l \neq 0\};$$

if such l does not exists, but $H^0(X,R)$ is not trivial, we set $\operatorname{cupl}(X,R) := 0$*, otherwise (if $H^0(X,R) = \{0\}$) we set* $\operatorname{cupl}(X,R) := -1$.

We notice that by (A.1), the order in the cup product is of no importance. In the case X is not connected, a slightly different definition (which makes the cup-length additive) can be found in [33].

For explicit computations of the cup-length we refer to [185, Example 3.4 and page 19] and to [133]: for instance if $B \subset \mathbb{R}^N$ is the closed unit ball, then $\operatorname{cupl}(\partial B) = 1$ for $N \geq 2$; if T^N is the N-dimensional torus, then $\operatorname{cupl}(T^N) = N$.

Singular relative cohomology and cup-length

We define now the cohomology and the cup-length relative to a *topological pair* (X,Y), that is $Y \subset X$ topological spaces. Observed that

$$C_q(Y,G) \triangleleft C_q(X,G)$$

and that ∂ conserves $C_q(Y,G)$, we can define the *singular q-relative chain module*

$$C_q(X,Y,G) := C_q(X,G)/C_q(Y,G).$$

Notice that $C_q(X,\emptyset,G) \equiv C_q(X,G)$. Considered the canonical projection $\pi_q : C_q(X,G) \to C_q(X,Y,G)$, we introduce

$$\tilde{\partial} : C_q(X,Y,G) \to C_{q-1}(X,Y,G)$$

the well defined function such that the canonical diagram commutes

$$\tilde{\partial} \circ \pi_q = \pi_{q-1} \circ \partial.$$

The other definitions follows in the same way as before:

$$C^q(X,Y,G) := Hom(C_q(X,Y,G),G),$$
$$\tilde{\delta} : C^{q-1}(X,Y,G) \to C^q(X,Y,G),$$
$$H^q_\Delta(X,Y,G) := \operatorname{Ker}(\tilde{\delta})/\operatorname{Im}(\tilde{\delta}),$$

and also $C_*(X,Y,G)$, $C^*(X,Y,G)$, $H^*_\Delta(X,Y,G)$ and $\smile$ (see also [217, page 209]). Notice that, if X is path-connected and $Y \neq \emptyset$, then $H^0(X,Y,R)$ is trivial [210, page 183].

When $G = R$, we can define the *relative cup-length* as the length of the longest chain of cup products in $H^*(X,R)$ multiplied with an element of $H^*(X,Y,R)$; see also [185, 187, 358].

Definition A.2. *Let (X,Y) be a topological pair and R be a commutative ring with unit. We define the* cup-length of X, relative to Y *as*

$$\operatorname{cupl}(X,Y,R) := \\ \max\{l \in \mathbb{N}^* \mid \exists \alpha_i \in H^{q_i}(X,R), q_i \geq 1 \text{ for } i = 1 \dots l, \exists \alpha_0 \in H^*(X,Y,R) \\ \text{s.t. } \alpha_0 \smile \alpha_1 \smile \cdots \smile \alpha_l \neq 0\}.$$

if such $l \in \mathbb{N}$ does not exists, but $H^*(X,Y,R)$ is not trivial, we set $\operatorname{cupl}(X,Y,R) := 0$*, otherwise (if $H^*(X,Y,R) = \{0\}$) we set* $\operatorname{cupl}(X,Y,R) := -1$.

Notice that

$$\operatorname{cupl}(X,R) = \operatorname{cupl}(X,\emptyset,R);$$

this is the same as taking $\alpha_0 \in H^0(X,R)$, since $H^*(X,\emptyset,R)$ is essentially $H^*(X,G)$ (see also [287, page 256] and [69, Proposition 12.3]). Again, for explicit examples we refer to [185, Example 3.4]: for instance, if $B \subset \mathbb{R}^N$ is the closed unit ball, then $\operatorname{cupl}(B,\partial B) = 0$.

Cup-length relative to a function

Let us consider two topological pairs (X, Y) and (X', Y') and a continuous map $f : (X, Y) \to (X', Y')$, that is $f : X \to X'$ with $f(Y) \subseteq Y'$. It is possible to prove[2] that f induces and homomorphism of groups

$$f^* : H^*(X', Y', G) \to H^*(X, Y, G)$$

which is *suitable functorial*, namely

$$(id)^* = id, \quad (gf)^* = f^* g^*, \quad f^* = g^* \text{ whenever } f, g \text{ homotopic.}$$

Moreover one can show that

$$\tilde{\partial} \circ f^* = f^* \circ \tilde{\partial}, \quad f^*(\alpha_1 \smile \alpha_2) = f^*(\alpha) \smile f^*(\alpha_2);$$

the second is said the *naturality* of the cup product (see [160, Section 7.8.6]). With this tool, when $G = R$, we can define the *cup-length relative to* f, as the length of the longest chain of cup products in $f^*(H^*(X', R)) \subset H^*(X, R)$ multiplied with an element of $f^*(H^*(X', Y', R)) \subset H^*(X, Y, R)$; see also [38].

Definition A.3. *Let (X, Y), (X', Y') be two topological pairs, R be a commutative ring with unit and $f : (X, Y) \to (X', Y')$ be continuous, with f^* the induced homomorphism on the relative cohomolgies. We define the* cup-length relative to f *as*

$$\begin{aligned} &\operatorname{cupl}(f, R) := \\ &\quad \max\{l \in \mathbb{N}^* \mid \quad \exists \alpha_i \in H^{q_i}(X', R), q_i \geq 1 \textit{ for } i = 1 \dots l, \exists \alpha_0 \in H^*(X', Y', R) \\ &\qquad\qquad\qquad\quad \textit{s.t. } f^*(\alpha_0) \smile f^*(\alpha_1) \smile \dots \smile f^*(\alpha_l) \neq 0\}; \end{aligned}$$

if such $l \in \mathbb{N}$ does not exist, but $f^* \not\equiv 0$, it results $\operatorname{cupl}(f) := 0$*, otherwise (if $f^* \equiv 0$) we define* $\operatorname{cupl}(f) := -1$.

Notice that

$$\operatorname{cupl}(X, Y, R) = \operatorname{cupl}(id_{(X,Y)}, R),$$

and in particular $\operatorname{cupl}(X, R) = \operatorname{cupl}(id_{(X,\emptyset)}, R)$.

A.2 Other cohomologies

We highlight that other cohomologies could be used to define the cup-length: for instance, the Alexander-Spanier cohomology [119] and the Čech cohomology [358]. We sketch here how they are built, and then we point out how they are closely related to the singular cohomology.

[2] We show here the standard construction in the non-relative case [288, Section VII.3]. Consider the induced function

$$f^\# : C^q(X') \to C^q(X)$$

such that

$$(f^\#(c))(\sigma) = c(f \circ \sigma) \quad \text{(i.e. } [\sigma, f^\#(c)] = [f \circ \sigma, c])$$

for every $c \in C^q(X')$ and $\sigma \in C_q(X)$. A straightforward computation shows that $f^\# \circ \delta' = \delta \circ f^\#$, where $\delta' : C^q(X') \to C^{q+1}(X')$; this easily implies that $f^\#(\mathrm{Ker}(\delta')) \subset \mathrm{Ker}(\delta)$ and $f^\#(\mathrm{Im}(\delta')) \subset \mathrm{Im}(\delta)$. This allows to pass to the quotient and define

$$f^* : \mathrm{Ker}(\delta')/\mathrm{Im}(\delta') \to \mathrm{Ker}(\delta)/\mathrm{Im}(\delta).$$

Alexander-Spanier cohomology. We refer to [286,287] (see also [350]). Let X be a topological space and G be a group. We define the *abelian group of all the q-functions*

$$\Phi^q(X,G) := \{\varphi : X^{q+1} \to G\}$$

and its normal subgroup

$$\Phi^q_0(X,G) := \{\varphi \in \Phi^q(X,G) \mid \varphi = 0 \text{ in a neighborhood of the } \textit{diagonal}\}.$$

On $\Phi^q(X,G)$ we can define the *coboundary operator* $\bar{\delta} : \Phi^q(X,G) \to \Phi^{q+1}(X,G)$ by

$$(\bar{\delta}\varphi)(x_0 \dots x_{q+1}) := \sum_{j=1}^{q+1} (-1)^j \varphi(x_0, \dots \hat{x}_j, \dots x_{q+1})$$

where $\hat{x}_j$ means that the variable is omitted; we have $\bar{\delta}^2 = 0$. Moreover, we can define the *q-cochain*

$$\bar{C}^q(X,G) := \Phi^q(X,G)/\Phi^q_0(X,G)$$

and then the *Alexander-Spanier cochain* $\bar{C}^*(X,G)$, on which we can define $\bar{\delta}$ through quotients. Thanks to the property $\bar{\delta}^2 = 0$ we can pass to the quotient of $\mathrm{Ker}(\bar{\delta})$ over $\mathrm{Im}(\bar{\delta})$ and obtain the *Alexander-Spanier cohomology* $\bar{H}^*(X,G)$. Slightly different definitions, which focus on *locally finitely valued* q-functions or which define Φ^q_0 through *supports*, can be found in [286, 287].

Once defined a *relative Alexander-Spanier cohomology* $\bar{H}^*(X,Y,G)$, by exploiting [288, Theorem 14.6.1 and Proposition 14.6.2] one can show that actually, for Y closed subset of X paracompact Hausdorff space (for example a manifold, such as $\mathbb{R}^N$ or a more general Hilbert space, see Section 5.4) and $G = R$ ring, it results that

$$\bar{H}^*(X,Y,R) \approx H^*_\Delta(X,Y,R) \tag{A.2}$$

that is, the Alexander-Spanier cohomology and the singular cohomology are isomorphic.

Čech cohomology. We refer to [67, 377] (see also [160, 168, 217, 350]). Let X be a topological space and G be a group (notice that we focus only on the case of a *constant presheaf* with *identical restrictions*). Let $\mathfrak{U}$ be a open covering of X and define

$$\sigma := (U_0, \dots U_q)$$

to be a *q-simplex* if $U_i \in \mathfrak{U}$ and $|\sigma| := \bigcap_{i=1}^q U_i \neq \emptyset$; $|\sigma| \subset X$ is called *support* of σ. We thus define $\check{\Sigma}_q$ as the set of all the q-simplexes, and

$$\check{C}^q(\mathfrak{U},G) := \{\varphi : \check{\Sigma}_q \to G\}$$

the set of all the *q-cochains*. On $\check{C}^q(\mathfrak{U},G)$ we can define the *coboundary operator* $\check{\delta} : \check{C}^q(\mathfrak{U},G) \to \check{C}^{q+1}(\mathfrak{U},G)$ as

$$(\check{\delta}\varphi)(U_0 \dots U_{q+1}) := \sum_{j=1}^{q+1} (-1)^j \varphi(U_0, \dots \hat{U}_j, \dots U_{q+1})$$

satisfying $\check{\delta}^2 = 0$. Thanks to this property we can define $\check{H}^*(\mathfrak{U},G)$ by passing to the quotient the kernel and the image of $\check{\delta}$. Finally, considering the coverings of X ordered by inclusion, we can define the *Čech cohomology* as

$$\check{H}(X,G) := \varinjlim \check{H}^*(\mathfrak{U},G)$$

in the sense of the *direct limits*. Notice that, if X is an n-dimensional manifold and $\mathfrak{U}$ is a *good cover*, i.e. every finite intersection of its elements is diffeomorphic to $\mathbb{R}^N$ (and there always exists such a good cover [67, Theorem 5.1]), then there is no need of passing to the direct limit, since it results that

$$\check{H}^*(X, G) \approx \check{H}^*(\mathfrak{U}, G);$$

in particular, the right-hand side does not depend on the particular good cover $\mathfrak{U}$.

By [67, Proposition 15.8] (see also [217, page 257]) we have that

$$\check{H}^*(X, \mathbb{Z}) \approx H^*_\Delta(X, \mathbb{Z})$$

whenever X is a manifold. Moreover [350, Corollary 6.9.9]

$$\check{H}^*(X, G) \approx \bar{H}^*(X, G)$$

whenever X is a closed subset of a manifold (or more generally, X is a Hausdorff space with coefficients in a module G [350, Corollary 6.8.8]).

Once defined also the *relative* Čech cohomology, one can prove [350, pages 342 and 359]

$$\check{H}^*(X, Y, G) \approx \bar{H}^*(X, Y, G)$$

whenever X, Y are closed subset of a manifold. Thus, by combining this result with (A.2), whenever Y and X are closed subsets of a manifold (such as $\mathbb{R}^N$ or a more general Hilbert space, see Section 5.4) and $G = R$ ring, we have

$$\check{H}^*(X, Y, G) \approx H^*_\Delta(X, Y, G);$$

see also [160, Proposition 8.6.12] for a direct proof in the case of a pair of *ENR* (*Euclidean Neighborhood Retracts*, which is the case for example of $Y \subset X \subset \mathbb{R}^N$ with X retractible).

See also [69] for further relations on these three cohomologies.

A.3 Properties of the cup-length

Here we focus on the case $G := \mathbb{F}$ for some field $\mathbb{F}$, and we drop the dependence on G in the notations. We collect some properties of the cup-length, see e.g. [38, Lemma 2.6].

Lemma A.4. *We have the following properties.*

(a) For any $f : (A, B) \to (A', B')$ and $f' : (A', B') \to (A'', B'')$ it results that

$$\mathrm{cupl}(f' \circ f) \leq \min\{\mathrm{cupl}(f'), \mathrm{cupl}(f)\}.$$

As a consequence,

$$\mathrm{cupl}(f' \circ f) \leq \mathrm{cupl}(A', B'). \tag{A.3}$$

(b) For any $f, g : (A, B) \to (A', B')$ homotopic, we have

$$\mathrm{cupl}(f) = \mathrm{cupl}(g).$$

Finally, we cite the following key result [34] which can be found in [119, Lemma 5.5].

Lemma A.5. *Consider the inclusion*

$$j : (I \times K, \partial I \times K) \to (I \times K_d, \partial I \times K_d)$$

for a whatever $K \subset \mathbb{R}^N$ compact, $K_d := \{x \in \mathbb{R}^N \mid d(x, K) \leq d\}$, and $I = [a, b]$. Then, for $d > 0$ sufficiently small, we have

$$\mathrm{cupl}(j) \geq \mathrm{cupl}(K).$$

A.4 Relation with the Ljusternik-Schnirelmann category

We recall here the definition of *relative category*, by following [185, 358] and references therein (see also [33]).

Definition A.6. *Let X be a topological space and let A, B be two closed subsets of X. We call the* category of A in X, relative to B, *and write*

$$k = \mathrm{cat}_{X,B}(A),$$

the least integer $k \in \mathbb{N}$ such that there exist $A_0, A_1, \ldots, A_k$ closed subsets of X which verify

- *$(A_i)_{i=0\ldots k}$ cover A;*
- *$(A_i)_{i=1\ldots k}$ are contractible in X, i.e. $id : A_i \to X$ is homotopic to a constant;*
- *A_0 is deformable in B, i.e. there exists a continuous $h_0 : [0,1] \times (A_0 \cup B) \to X$ such that $h_0(0,\cdot) = id$, $h_0(1, A_0) \subset B$ and $h_0(t, B) \subset B$ for each $t \in [0,1]$.*

If such k does not exists, we set $\mathrm{cat}_{X,B}(A) := +\infty$.

Examples of computations can be found in [185, Examples 2.2 and 3.7] and [186, Remark 3.2]. For example, if B is the unit ball in $\mathbb{R}^N$, then $\mathrm{cat}_{B,\partial B}(B) = 1$ (while it is equal zero if B is the unit ball in $H^s(\mathbb{R}^N)$); if A is the annulus in $\mathbb{R}^N$ with $N \geq 2$, then $\mathrm{cat}_{A,\partial A}(A) = 2$; moreover $\mathrm{cat}_{\mathbb{R}^2,\mathbb{R}}(\mathbb{R}^2) = \mathrm{cat}_{\mathbb{R}^2,(0,0)}(\mathbb{R}^2) = 0$.

Remark A.7.

- *If we drop the condition on B, A_0 and h_0, we have the classical definition of category, and simply write $\mathrm{cat}_X(A)$; more precisely*

$$\mathrm{cat}_X(A) = \mathrm{cat}_{X,\emptyset}(A).$$

  This definition can be given for a whatever A (even not closed), and a posteriori one has $\mathrm{cat}_X(A) = \mathrm{cat}_X(\overline{A})$ (see [133, Remark 1.12]).
- *We required the covering to be closed, but equivalently one can ask $A_0 \ldots A_k$ to be open (see [133, Proposition 1.10]).*
- *We do not require that $B \subset A_0 \subset A$, even if equivalent definitions could be given in this way (see e.g. [185]).*
- *Some authors require the stronger condition $h_0(t, \cdot_{|B}) = id_B$ (see e.g. [119, 186, 187] and Remark 2.2 in [358]), and this modification would bring no differences in what follows.*
- *Considered a continuous map $f : (A,B) \to (A', B')$ one can define the category of f by substituting, in the definition (with $A = X$), "$id : A_i \to A$" with "$f_{|A_i} : A_i \to A$", "$h_0 : [0,1] \times (A_0 \cup B) \to A$" with "$h_0 : [0,1] \times (A_0 \cup B) \to A$" and "$h_0(0,\cdot) = id$" with "$h_0(0,\cdot) = f$"; in this case $\mathrm{cat}_{A,B}(A) = \mathrm{cat}(id_{(A,B)})$. See [38]. Anyway, we will not use this tool.*

The following classical properties on category can be found, e.g., in [38, Lemma 2.2] and [133, Lemma 1.13] (see also [185, Proposition 2.9]).

Lemma A.8. *Let A be a closed subset of X.*

- *$\#A \geq \mathrm{cat}_X(A)$;*

- *If A is compact, and every point in A has an open neighborhood in X contractible in X, then there exists an open neighborhood $N \subset X$ of A such that $\mathrm{cat}_X(N) = \mathrm{cat}_X(A)$. In particular, if $A \subset X \subset X' \subset H$, with A compact and X open subset of the Hilbert space H, then the claim holds true for $\mathrm{cat}_{X'}(A)$.*

Next proposition deals with some properties on relative category, and can be found, for instance, in [358, Propositions 2.5 and 2.8] or [185, Propositions 2.4 and 2.9] (see also [186, Remark 3.2 and Propositions 3.4 and 3.5]).

Lemma A.9. *Let A, A', B, V be closed subsets of X.*

- *Then $\mathrm{cat}_{X,B}(A) = 0$ if and only if A can be deformed in B, i.e. there exists $h : (A \cup B) \times [0, 1] \to X$ such that $h(0, \cdot) = id$, $h(t, B) \subset B$ for each $t \in [0, 1]$ and $h(1, A) \subset B$. As a consequence, if $A \subset B$, then $\mathrm{cat}_{X,B}(A)$. In particular, $\mathrm{cat}_{X,A}(A) = 0$.*
- *If $A \subset A'$, then $\mathrm{cat}_{X,B}(A) \leq \mathrm{cat}_{X,B}(A')$.*
- *If $A \cup B \subset X \subset X'$, then $\mathrm{cat}_{X,B}(A) \geq \mathrm{cat}_{X',B}(A)$. In particular, if $B \subset A \subset X$, then $\mathrm{cat}_{A,B}(A) \geq \mathrm{cat}_{X,B}(A)$.*
- *If $\mathrm{cat}_X(V) < \infty$, then $\mathrm{cat}_{X,B}(\overline{A \setminus V}) \geq \mathrm{cat}_{X,B}(A) - \mathrm{cat}_X(V)$.*
- *If there exists $\eta : [0, 1] \times (A \cup B) \to X$ such that $\eta(1, A) \subset A'$ and $\eta([0, 1], B) \subset B$, then $\mathrm{cat}_{X,B}(A) \leq \mathrm{cat}_{X,B}(A')$.*

The following lemma links the concepts of category (when $A = X$) and cup-length, and it can be found in [358, Proposition 2.6 and Remark 2.7] (see also [187, Theorem 1] and [185, Theorem 3.6]).

Lemma A.10. *Let B be a closed subset of a metric space A. Then*

$$\mathrm{cat}_{A,B}(A) \geq \mathrm{cupl}(A, B) + 1.$$

In particular, $\mathrm{cat}_A(A) \geq \mathrm{cupl}(A) + 1$.

To avoid cumbersome notation we will write

$$\mathrm{cat}(A) := \mathrm{cat}_A(A), \quad \text{and} \quad \mathrm{cat}(A, B) := \mathrm{cat}_{A,B}(A).$$

Notice that, if $A \subset X$, then $\mathrm{cat}(A, B) \geq \mathrm{cat}_{X,B}(A)$ (and in particular $\mathrm{cat}(A) \geq \mathrm{cat}_X(A)$).

Remark A.11. *We notice that in standard examples the inequality in Lemma A.10 is actually an equality. Indeed, if K is a contractible set or it is finite (e.g. a single point), then*

$$\mathrm{cupl}(K) + 1 = \mathrm{cat}_K(K) = 1;$$

if $K = S^{N-1}$ is the $N - 1$ dimensional sphere in $\mathbb{R}^N$, then

$$\mathrm{cupl}(K) + 1 = \mathrm{cat}_K(K) = 2;$$

if $K = T^N$ is the N-dimensional torus, then

$$\mathrm{cupl}(K) + 1 = \mathrm{cat}_K(K) = N + 1.$$

However in general the strict inequality may hold, see [133, Sections 2.8 and 9.23] for some examples.

Remark A.12. *When one deals with a functional which is not bounded from below, the tool of the relative category is needed. On the other hand, for any interval $I \subset \mathbb{R}$ and any neighborhood K_d of K, considered the inclusion*

$$j : (I \times K, \partial I \times K) \to (I \times K_d, \partial I \times K_d)$$

the key relation

$$\mathrm{cat}(j) \geq \mathrm{cat}_K(K),$$

essential in the estimation of the relative category of two sublevels of the indefinite functional (see [119, Remark 4.3]) does not generally hold [133, Remark 7.47]. Nevertheless, the same relation for the cup-length

$$\mathrm{cupl}(j) \geq \mathrm{cupl}(K)$$

holds true, as proved in [119, Lemma 5.5] (see also [185, Proposition 3.5]). That is why we take advantage of the relative cup-length in order to get a bound on the number of solutions.

A.5 Application to multiplicity of solutions

We sketch now how to obtain multiple solutions from the information on the category of a set.

Let indeed $J : X \to \mathbb{R}$ to be a C^1-functional on a function space X, and denote, for every $c \in \mathbb{R}$, $J^c := \{J \leq c\}$ the sublevel at c and $K_c := \{J = c,\ J' = 0\}$ the set of critical points at c. Assume the following:

- there exist $\bar{c} \in \mathbb{R}$ and $\delta > 0$ such that K_c is compact for every $c \in [\bar{c} - \delta, \bar{c} + \delta]$ (for example, a Palais-Smale type condition holds at level c) and a Deformation Lemma holds around K_c;
- $\bar{c} + \delta$ is a regular value; this is not restrictive, up to choosing properly δ (small), since otherwise we would have a sequence of critical values at levels $c + \delta_n$ with $\delta_1 > ... > \delta_n \to 0$.
- there exist a compact K and two continuous maps ϕ_1, ϕ_2 such that $(I \times K, \partial I \times K) \overset{\phi_1}{\to} (J^{\bar{c}+\delta}, J^{\bar{c}-\delta}) \overset{\phi_2}{\to} (I \times K_d, \partial I \times K_d)$ is homotopic to the inclusion $j : (I \times K, \partial I \times K) \to (I \times K_d, \partial I \times K_d)$, where $K_d = \{x \in \mathbb{R}^N \mid d(x, K) \leq d\}$ and $I = [a, b]$ for some $a, b \in \mathbb{R}$.

We want to show

$$\begin{aligned} \#\{u \in X \mid J(u) \in [\bar{c} - \delta, \bar{c} + \delta],\ J'(u) = 0\} &\overset{(i)}{\geq} \mathrm{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(J^{\bar{c}+\delta}) \\ &\overset{(ii)}{\geq} \mathrm{cupl}\left(J^{\bar{c}+\delta},\ J^{\bar{c}-\delta}\right) + 1 \\ &\overset{(iii)}{\geq} \mathrm{cupl}(K) + 1. \end{aligned}$$

which is an estimate from below on the number of critical points of J.

Proof of (i). This is a consequence of the Deformation Lemma and of the compactness of critical level sets, as done in [358, Proposition 3.2] and [186, Theorem 4.2] (see also [187, Theorem 3] and [185, Theorem 6.1]). Let thus define

$$k := \mathrm{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(J^{\bar{c}+\delta}) \in \mathbb{N} \cup \{+\infty\};$$

if $k = 0$ the claim is trivial, thus we can assume $k \geq 1$. For each $j = 1 \ldots k$ define

$$\Gamma_j := \{A \subset X \mid A \text{ closed},\ \mathrm{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(A) \geq j\},$$

$$c_j := \inf_{A \in \Gamma_j} \sup_A J.$$

Notice that, since $j \geq 1$, then each $A \in \Gamma_j$ cannot be included in $J^{\bar{c}-\delta}$, that is $c_j \geq \bar{c} - \delta$; moreover, since $j \leq k$, then $J^{\bar{c}+\delta} \in \Gamma_j$, which implies $c_j \leq \bar{c} + \delta$. Therefore

$$\bar{c} - \delta \leq c_1 \leq c_2 \leq \cdots \leq \bar{c} + \delta.$$

Fix $j \in \{1 \ldots k\}$ and let $p \in \mathbb{N}$ be such that

$$c_j = \cdots = c_{j+p} =: c \in [\bar{c} - \delta, \bar{c} + \delta];$$

to reach the claim, it is sufficient to show that

$$\operatorname{cat}_{J^{\bar{c}+\delta}}(K_c) \geq p + 1 \tag{A.4}$$

since $\# K_c \geq \operatorname{cat}_{J^{\bar{c}+\delta}}(K_c)$ and by combining the estimates for different values of c_j (if $c_i \neq c_j$ we clearly have different critical points at the two levels).

We do some preliminary work. We first exploit that $\bar{c} + \delta$ is a regular point to show that $c < \bar{c} + \delta$. Indeed, since $K_{\bar{c}+\delta} = \emptyset$, by the Deformation Lemma there exist $\eta : [0,1] \times X \to X$ and an $\omega > 0$ such that

- $J(\eta(t, \cdot)) \leq J(\cdot)$ for each $t \in [0,1]$, and thus $\eta : [0,1] \times J^{\bar{c}+\delta} \to J^{\bar{c}+\delta}$,
- $\eta(1, J^{\bar{c}+\delta+\omega}) \subset J^{\bar{c}+\delta-\omega}$, and thus $\eta(1, J^{\bar{c}+\delta}) \subset J^{\bar{c}+\delta-\omega}$,
- $J(t, J^{\bar{c}-\delta}) \subset J^{\bar{c}-\delta}$ for each $t \in [0,1]$;

by Lemma A.9 we have

$$\operatorname{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(J^{\bar{c}+\delta-\omega}) \geq \operatorname{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(J^{\bar{c}+\delta}) = k \geq j;$$

thus $J^{\bar{c}+\delta-\omega} \in \Gamma_j$, which implies $c_j \leq \bar{c} + \delta - \omega < \bar{c} + \delta$ for each j, which is the claim.

Since $c < \bar{c} + \delta$, we have $K_c \subset \{J < \bar{c} + \delta\} \subset J^{\bar{c}+\delta}$; moreover K_c is compact; thus by Lemma A.8 we have that there exists an open neighborhood N of K_c such that

$$\operatorname{cat}_{J^{\bar{c}+\delta}}(N) = \operatorname{cat}_{J^{\bar{c}+\delta}}(K_c).$$

Corresponding to N, again by the Deformation Lemma there exist an $\omega \in (0, \bar{c} + \delta - c)$ and an $\eta : [0,1] \times J^{\bar{c}+\delta} \to J^{\bar{c}+\delta}$ (notice that $J^{c+\omega} \cup J^{\bar{c}-\delta} \subset J^{\bar{c}+\delta}$) such that $\eta(1, J^{c+\omega} \setminus N) \subset J^{c-\omega}$ and $\eta(t, J^{\bar{c}-\delta}) \subset J^{\bar{c}-\delta}$ for each $t \in [0,1]$. By Lemma A.9 we have

$$\operatorname{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(J^{c+\omega} \setminus N) \leq \operatorname{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(J^{c-\omega}). \tag{A.5}$$

Corresponding to ω, by definition of $c = c_{j+p}$ there exists an $A \in \Gamma_{j+p}$ such that $\sup_A J < c + \omega$, which means that $A \subset J^{c+\omega}$ and thus

$$A \setminus N \subset J^{c+\omega} \setminus N. \tag{A.6}$$

We prove now (A.4) by contradiction. Assume $\operatorname{cat}(K_c) \leq p < \infty$. Thus, by (A.5), (A.6) and Lemma A.9 (notice that $A \setminus N$ is closed) we have

$$\begin{aligned}\operatorname{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(J^{c-\omega}) &\geq \operatorname{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(J^{c+\omega} \setminus N) \geq \operatorname{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(A \setminus N) \\ &\geq \operatorname{cat}_{J^{\bar{c}+\delta}, J^{\bar{c}-\delta}}(A) - \operatorname{cat}_{J^{\bar{c}+\delta}}(N) \geq (j+p) - p = j.\end{aligned}$$

This means that $J^{c-\omega} \in \Gamma_j$, and thus

$$c_j \leq \sup_{J^{c-\omega}} J \leq c - \omega = c_j - \omega$$

which is an absurd.

Proof of (ii). This is a consequence of the property of algebraic topology given in Lemma A.10.

Proof of (iii). This is due to the existence of the homotopy and properties of the cup-length. Indeed, by (A.3) in Lemma A.4 (a), we have

$$\mathrm{cupl}\left(J^{\bar{c}+\delta}, J^{\bar{c}-\delta}\right) \geq \mathrm{cupl}(\phi_2 \circ \phi_1);$$

we highlight that the left-hand side deals with subsets of the function space X, while the right-hand side deals with subsets of $\mathbb{R}^N$. Since $\phi_2 \circ \phi_1$ is homotopic to the immersion j, we have by Lemma A.4 (b)

$$\mathrm{cupl}(\phi_2 \circ \phi_1) = \mathrm{cupl}(j).$$

Finally, we conclude thanks to Lemma A.5. ∎

A.6 The Krasnoselskii genus: a particular category

In order to obtain existence of multiple solutions in the entire space $\mathbb{R}^N$, without any topology related to some potential V, it is useful to exploit some symmetry of the functionals, and some tool related to them.

In particular, we introduce the well known Krasnoselskii genus.

Definition A.13. *For any A closed subset of $\mathbb{R}^N \setminus \{0\}$, symmetric with respect to the origin (i.e. $A = -A$), the* Krasnoselskii genus *is defined by*

$$\mathrm{genus}(A) := \max\left\{n \in \mathbb{N} \mid \exists \beta : A \to \mathbb{R}^n \setminus \{0\} \text{ continuous and odd}\right\};$$

if such n does not exists, $\gamma(A) := +\infty$; moreover $\gamma(A) = 0$ if (and only if) $A = \emptyset$.

The genus enjoys several standard properties [324, Section 3].

Proposition A.14. *Let $A, B \subset \mathbb{R}^N \setminus \{0\}$ be closed and symmetric.*

- *if A is finite, then* $\mathrm{genus}(A) = 1$;
- $\mathrm{genus}(A \cup B) \leq \mathrm{genus}(A) + \mathrm{genus}(B)$;
- *if* $\mathrm{genus}(B) < \infty$, *then* $\mathrm{genus}(\overline{A \setminus B}) \leq \mathrm{genus}(A) - \mathrm{genus}(B)$;
- *if $h : \mathbb{R}^N \to \mathbb{R}^N$ is continuous and odd, then* $\mathrm{genus}(A) \leq \mathrm{genus}(\overline{h(A)})$;
- *if A is compact, then there exists a closed, symmetric neighborhood $U \not\ni 0$ of A such that* $\mathrm{genus}(U) = \mathrm{genus}(A) < \infty$;
- *if U is a symmetric neighborhood of the origin, then* $\mathrm{genus}(\partial U) = N$.

Example A.15. *The genus describes, roughly, how a set is wrapped near the origin. Let A be a closed subset of $\mathbb{R}^N \setminus \{0\}$, such that $A = -A$. If $A = B \cup (-B)$, with $B \cap (-B) = \emptyset$, then* $\mathrm{genus}(A) = 1$. *If A is connected, then* $\mathrm{genus}(A) \geq 2$. *Moreover,* $\mathrm{genus}(S^N) = N + 1$.

Actually, this tool reveals to be a subcase of the already introduced category. Indeed, considered the action of $\mathbb{Z}_2$ over $\mathbb{R}^N$ (which identifies x with $-x$) we have the following relation [324, Theorem 3.7] (see also [171])

$$\mathrm{genus}(A) = \mathrm{cat}_{(\mathbb{R}^N \setminus \{0\})/\mathbb{Z}_2}(A/\mathbb{Z}_2).$$

This relation highlights the fact that the genus tool exploits not the topology of a particular subset of $\mathbb{R}^N$, but the topology induced by a symmetry relation.

Acknowledgments. The number of people to thank for this three years journey is quite long, starting from my advisors, passing through collaborators and colleagues, up to family and friends. I have written a truly marvelous praise about this, which this margin unfortunately is too narrow to contain.

So long, and thanks for all the fish!

Bibliography

[1] N. Abatangelo, *Higher-order fractional Laplacians: an overview*, Bruno Pini Math. Anal. Semin. **12** (2021), no. 1, 53–80. 5, 6

[2] N. Abatangelo, S. Jarohs, A. Saldaña, *Positive powers of the Laplacian: from hypersingular integrals to boundary value problems*, Commun. Pure Appl. Anal. **17** (2018), no. 3, 899–922. 6, 10, 30

[3] N. Abatangelo, S. Jarohs, A. Saldaña, *On the loss of maximum principles for higher-order fractional Laplacians*, Proc. Amer. Math. Soc. **146** (2018), no. 12, 4823–4835. 6, 10

[4] N. Abatangelo, S. Jarohs, A. Saldaña, *On the maximum principle for higher-order fractional Laplacians*, arXiv.1607.00929 (2018), pp. 38. 6, 30, 31

[5] N. Abatangelo, S. Jarohs, A. Saldaña, *Green function and Martin kernel for higher-order fractional Laplacians in balls*, Nonlinear Anal. **175** (2018), no. 11, 173–190. 6, 23

[6] N. Abatangelo, E. Valdinoci, *Getting acquainted with the fractional Laplacian*, in "Contemporary research in elliptic PDEs and related topics" (ed. S. Dipierro), Springer INdAM Series **33**, Springer Nature Switzerland AG (2019), 1–105. 3, 5

[7] M. Abramowitz, I. Stegun, "Handbook of mathematical functions with formulas, graphs, and mathematical tables", Dover Publications Inc., 1964. 12, 13

[8] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z. **248** (2004), no. 2, 423–443. 81

[9] D. R. Adams, L. I. Hedberg, "Function spaces and potential theory", Springer-Verlag, Berlin Heidelberg, 1996. 9, 30

[10] C. O. Alves, V. Ambrosio, *A multiplicity result for a nonlinear fractional Schrödinger equation in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz condition*, J. Math. Anal. Appl. **466** (2018), no. 1, 498–522. v, 170

[11] C. O. Alves, J. M. do Ó, M. A. S. Souto, *Local mountain-pass for a class of elliptic problems in $\mathbb{R}^N$ involving critical growth*, Nonlinear Anal. **46** (2001), no. 4, 495–510. 209

[12] C. O. Alves, O. H. Miyagaki, *Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, article ID 47, pp. 19. 170

[13] A. Ambrosetti, M. Badiale, S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Ration. Mech. Anal. **140** (1997), no. 3, 285–300. 169, 177

[14] A. Ambrosetti, A. Malchiodi, "Nonlinear analysis and semilinear elliptic problems", Cambridge University Press, New York, 2010. 67

[15] A. Ambrosetti, A. Malchiodi, W.-M. Ni, *Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres: part I*, Comm. Math. Phys. **235** (2003), no. 3, 427–466.

[16] A. Ambrosetti, A. Malchiodi, W.-M. Ni, *Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres: part II*, Indiana Univ. Math. J. **53** (2004), no. 2, 297–330.

[17] A. Ambrosetti, A. Malchiodi, S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Ration. Mech. Anal. **159** (2001), no. 3, 253–271. 169, 170

[18] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), no. 4, 349–381. iv, 46

[19] V. Ambrosio, *Boundedness and decay of solutions for some fractional magnetic Schrödinger equations in* $\mathbb{R}^N$, Milan J. Math. **86** (2018), 125–136. 145, 148

[20] V. Ambrosio, *Concentrating solutions for a class of nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, Rev. Mat. Iberoam. **35** (2019), no. 5, 1367–1414. 170

[21] V. Ambrosio, *Existence and concentration results for some fractional Schrödinger equations in* $\mathbb{R}^N$ *with magnetic fields*, Comm. Partial Differential Equations **44** (2019), no. 8, 637–680. 209, 216

[22] V. Ambrosio, *Nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, Frontiers in Elliptic and Parabolic Problems, Birkhäuser, Springer Nature Switzerland AG, 2021. 3, 24, 41, 44, 210

[23] G. E. Andrews, R. Askey, R. Roy, "Special Functions", Cambridge University Press, 1999. 12

[24] C. Argaez, M. Melgaard, *Solutions to quasi-relativistic multi-configurative Hartree-Fock equations in quantum chemistry*, Nonlinear Anal. **75** (2012), no. 1, 384–404. 107

[25] A. Atangana, "Fractional operators with constant and variable order with application to geo-hydrology", Academic Press, London, 2018. 44

[26] M. Badiale, T. D'Aprile, *Concentration around a sphere for a singularly perturbed Schrödinger equation*, Nonlinear Anal. **49** (2002), no. 7, 974–985.

[27] M. Badiale, L. Pisani, S. Rolando, *Sum of weighted Lebesgue spaces and nonlinear elliptic equations*, Nonlinear Differ. Equ. Appl. **18** (2011), no. 4, 369–405. 37

[28] A. Bahri, J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), no. 3, 253–294. 169

[29] D. Baleanu, S. Etemad, S. Rezapour, *A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions*, Bound. Value Probl. **2020** (2020), no. 64, pp. 16. 44

[30] D. Baleanu, A. Jajarmi, H. Mohammadi, S. Rezapour, *A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative*, Chaos Solitons Fractals **134** (2020), article ID 109705, pp. 7. 44

[31] G. Barles, E. Chasseigne, C. Imbert, *On the Dirichlet Problem for Second-Order Elliptic Integro-Differential Equations*, Indiana Univ. Math. J. **57** (2008), no. 1, 213–246. 26

[32] B. Barrios. L Montoro, I. Peral, *Neumann conditions for the higher order s-fractional Laplacian* $(-\Delta)^s$ *with* $s > 1$, Nonlinear Anal. **193** (2020), article ID 111368, pp. 34. 6, 10

[33] T. Bartsch, "Topological methods for variational problems with symmetries", Lectures Notes in Math. **1560**, Springer-Verlag Berlin Heidelber, 1993. 222, 226

[34] T. Bartsch, *Note on category and cup-length*, private communication (2012). 225

[35] T. Bartsch, S. de Valeriola, *Normalized solutions of nonlinear Schrödinger equations*, Arch. Math. (Basel) **100** (2013), no. 1, 75–83. 45

[36] T. Bartsch, L. Jeanjean, N. Soave, *Normalized solutions for a system of coupled cubic Schrödinger equations on* $\mathbb{R}^3$, J. Math. Pures Appl. **106** (2016), no. 4, 583–614. 45

[37] T. Bartsch, Y. Liu, Z. Liu, *Normalized solutions for a class of nonlinear Choquard equations*, SN Partial Differ. Equ. Appl. **1** (2020), no. 5(34), pp. 25. 41, 81, 82

[38] T. Bartsch, T. Weth, *The effect of the domain's configuration space on the number of nodal solutions of singularly perturbed elliptic equations* Topol. Methods Nonlinear Anal. **26** (2005), no. 1, 109–133. 223, 225, 226

[39] L. Battaglia, J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations in the plane*, Adv. Nonlinear Stud. **17** (2017), no. 3, 581–594.

[40] W. Beckner, *Sobolev inequalities, the Poisson semigroup, and analysis on the sphere* S^n, Proc. Natl. Acad. Sci. USA **89** (1992), no. 11, 4816–4819. 35

[41] P. Belchior, H. Bueno, O. H. Miyagaki, G. A. Pereira, *Remarks about a fractional Choquard equation: ground state, regularity and polynomial decay*, Nonlinear Anal. **164** (2017), 38–53. 108, 142

[42] J. Bellazzini, M. Ghimenti, C. Mercuri, V. Moroz, J. Van Schaftingen, *Sharp Gagliardo-Nirenberg inequalities in fractional Coulomb-Sobolev spaces*, Trans. Amer. Math. Soc. **370** (2018), no. 11, 8285–8310. 8

[43] V. Benci, G. Cerami, *Existence of positive solutions of the equation* $-\Delta u + a(x)u = u^{\frac{N+2}{N-2}}$ *in* $\mathbb{R}^N$, J. Funct. Anal. **88** (1990), no. 1, 90–117. 179

[44] V. Benci, G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Ration. Mech. Anal. **114** (1991), no. 1, 79–93. 169

[45] V. Benci, T. D'Aprile, *The semiclassical limit of the nonlinear Schrödinger equation in a radial potential*, J. Differential Equations **184** (2002), no. 1, 109–138.

[46] V. Benci, M. Ghimenti, A.M. Micheletti, *On the dynamics of solitons in the nonlinear Schrödinger equation*, Arch. Ration. Mech. Anal. **205** (2012), no. 2, 467–492. 169

[47] J. Benedikt, V. Bobkov, R. N. Dhara, P. Girg, *Nonradiality of second eigenfunctions of the fractional Laplacian in a ball*, Proc. Amer. Math. Soc. **150** (2022), 5335–5348. 35

[48] H. Berestycki, L. Caffarelli, L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **25** (1997), no. 1-2, 69–94. 44

[49] H. Berestycki, T. Gallouet, O. Kavian, *Équations de champs scalaires euclidiens non linéaires dans le plan*, C. R. Acad. Sci. Paris Ser. I Math. **297** (1983), no. 5, 307–310. iv

[50] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations I: existence of a ground state*, Arch. Ration. Mech. Anal. **82** (1983), no. 4, 313–345. iii, iv, 35, 44, 48, 49, 82, 109, 112, 113, 139, 210

[51] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations II: existence of infinitely many solutions*, Arch. Ration. Mech. Anal. **82** (1983), no. 4, 347–375. iii, iv, v, 35, 44, 71, 79, 82, 84, 86, 93

[52] F. Bethuel, H. Brezis, F. Hélein, "Ginzburg-Landau vortices", Progr. Nonlinear Differential Equations Appl. **13**, Birkhäuser Boston, Springer International Publishing AG, 2017. 169

[53] S. Bhattara, *On fractional Schrödinger systems of Choquard type*, J. Differential Equations **263** (2017), no. 6, 3197–3229. 112

[54] U. Biccari, M. Warma, E. Zuazua, *Local elliptic regularity for the Dirichlet fractional Laplacian*, Adv. Nonlinear Stud. **17** (2017), no. 2, 378–409. 5, 171

[55] B. Bieganowski, J. Mederski, *Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth*, J. Funct. Anal. **280** (2021), no. 11, article ID 108989, pp. 26. 45

[56] G. M. Bisci, V. D. Rădulescu, R. Servadei, "Variational methods for nonlocal fractional problems", Encyclopedia Math. Appl. **162**, Cambridge University Press, 2016. 3

[57] C. Bonanno, P. d'Avenia, M. Ghimenti, M. Squassina, *Soliton dynamics for the generalized Choquard equation*, J. Math.Anal.Appl. **417** (2014), no. 1, 180–199. 169

[58] D. Bonheure, J.-B. Casteras, E. M. Dos Santos, R. Nascimento, *Orbitally stable standing waves of a mixed dispersion nonlinear Schrödinger equation*, SIAM J. Math. Anal. **50** (2018), no. 5, 5027–5071. 45

[59] D. Bonheure, J.-B. Casteras, T. Gou, L. Jeanjean, *Normalized solutions to the mixed dispersion nonlinear Schrödinger equation in the mass critical and supercritical regime*, Trans. Amer. Math. Soc. **372** (2019), no. 3, 2167–2212. 45

[60] D. Bonheure, J. Földes, E. M. Dos Santos, A. Saldaña, H. Tavares, *Paths to uniqueness of critical points and applications to partial differential equations*, Trans. Amer. Math. Soc. **370** (2018), no. 10, 7081–7127. 45

[61] D. Bonheure, S. Cingolani, M. Nys, *Nonlinear Schrödinger equation: concentration on circles driven by an external magnetic field*, Calc. Var. Partial Differential Equations **55** (2016), n. 82, pp. 33.

[62] D. Bonheure, S. Cingolani, S. Secchi, *Concentration phenomena for the Schrödinger-Poisson system in* $\mathbb{R}^2$, Discrete Contin. Dyn. Syst. Ser. S **14** (2021), n. 5, 1631–1648.

[63] D. Bonheure, S. Cingolani, J. Van Schaftingen, *The logarithmic Choquard equation: sharp asymptotics and nondegeneracy of the groundstate*, J. Funct. Anal. **272** (2017), no. 12, 5255–5281.

[64] D. Bonheure, J. Di Cosmo, C. Mercuri, *Concentration on circles for nonlinear Schrödinger-Poisson systems with unbounded potentials vanishing at infinity*, Commun. Contemp. Math. **14** (2012), no. 2, article ID 1250009, pp. 31.

[65] D. Bonheure, J. Di Cosmo, J. Van Schaftingen, *Nonlinear Schrödinger equation with unbounded or vanishing potentials: solutions concentrating on lower dimensional spheres*, J. Differential Equations **252** (2012), no. 2, 941–968.

[66] D. Bonheure, J. Van Schaftingen, *Bound state solutions for a class of nonlinear Schrödinger equations*, Rev. Mat. Iberoamericana **24** (2008), no. 1, 297–351. 169

[67] R. Bott, L. W. Tu, "Differential forms in algebraic topology", Grad. Texts in Math. **82**, Springer-Verlag New York, 1982. 224, 225

[68] L. Brasco, S. Mosconi, M. Squassina, *Optimal decay of extremals for the fractional Sobolev inequality*, Calc. Var. Partial Differential Equations **55**(2) (2016), no. 23, pp. 32. 14

[69] G. E. Bredon, "Sheaf theory", Grad. Texts in Math. **170**, Springer-Verlag New York, 1997. 219, 222, 225

[70] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477. 44

[71] H. Brezis, T. Kato, *Remarks on the Schrödinger operator with singular complex potential*, J. Math. Pure. Appl. **58** (1979), no. 2, 137–151. 125

[72] F. Brock, A. Y. Solynin, *An approach to symmetrization via polarization*, Trans. Amer. Math. Soc. **352** (2000), no. 4, 1759–1796. 35

[73] J. C. Bronski, R. L. Jerrard, *Soliton dynamics in a potential*, Math. Res. Lett. **7** (2002), no. 2-3, 329–342. v, 169

[74] K. J. Brown, Y. Zhang, *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Differential Equations **193** (2003), no. 2, 481–499. iv

[75] C. Bucur, *Some observation on the Green function for the ball in the fractional Laplace framework*, Comm. Pure Appl. Math. **15** (2016), no. 2, 657–699. 29, 30

[76] C. Bucur, E. Valdinoci, "Nonlocal Diffusion and Applications", Lect. Notes Unione Mat. Ital. **20**, Springer, Switzerland, 2016. 3, 44

[77] A. Burchard, *A short course on rearrangement inequalities*, course notes (2009). 35

[78] J. Byeon, L. Jeanjean, *Standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Arch. Ration. Mech. Anal. **185** (2007), no. 2, 185–200, and *Erratum*: Arch. Ration. Mech. Anal. **190** (2008), no. 13, 549–551. 169, 170, 171, 175, 182, 209

[79] J. Byeon, O. Kwon, J. Seok, *Nonlinear scalar field equations involving the fractional Laplacian*, Nonlinearity **30** (2017), no. 4, 1659–1681. iv, 3, 5, 24, 25, 35, 43, 44, 48, 49, 50, 107, 108, 121, 123, 173, 211

[80] J. Byeon, L. Jeanjean, M. Maris, *Symmetry and monotonicity of least energy solutions*, Calc. Var. Partial Differential Equations **36** (2009), no. 4, 481–492. 48, 49, 210

[81] J. Byeon, K. Tanaka, *Semi-classical standing waves for nonlinear Schrödinger equations at structurally stable critical points of the potential*, J. Eur. Math. Soc. **15** (2013), no. 5, 1859–1899. v, 169, 171, 179, 209

[82] J. Byeon, K. Tanaka, "Semi-classical standing waves with clustering peaks for nonlinear Schrödinger equations", Mem. Amer. Math. Soc. **229** (2014), no. 1076, 1–89. 169

[83] J. Byeon, Z.-Q. Wang, *Standing waves with a critical frequency for nonlinear Schrödinger equations*, Arch. Rational Mech. Anal. **165** (2002), 295–316. 182

[84] X. Cabré, Y. Sire, *Nonlinear equations for fractional Laplacians, I: regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), no. 1, 23–53. iii, 3, 24, 44

[85] L. A. Caffarelli, S. Dipierro, E. Valdinoci, *A logistic equation with nonlocal interactions*, Kinet. Relat. Models **10** (2017), no. 1, 141–170. 44, 107

[86] L. A. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 8, 1245–1260. iii, 6, 44

[87] L. A. Caffarelli, L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. **62** (2009), no. 5, 597–638. iii, 16, 26

[88] L. A. Caffarelli, Y. Sire, *On some pointwise inequalities involving nonlocal operators*, in "Harmonic Analysis, Partial Differential Equations and Applications" (eds. S. Chanillo, B. Franchi, G. Lu, C. Perez, E. T. Sawyer), Springer, 2017. iii, 17

[89] D. Cao, W. Dai, G. Qin, *Super poly-harmonic properties, Liouville theorems and classification of nonnegative solutions to equations involving higher-order fractional Laplacians*, Trans. Amer. Math. Soc. **374** (2021), no. 7, 4781–4813. 6, 10

[90] P. C. Carrião, O. H. Miyagaki, J. C. Pádua, *Radial solutions of elliptic equations with critical exponents in* $\mathbb{R}^N$, Differential Integral Equations **11** (1998), no. 1, 61–68. 36

[91] T. Cazenave, "Semilinear Schrödinger equations", Courant Lecture Notes **10**, AMS, 2003. 11

[92] T. Cazenave, P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), no. 4, 549–561. 36, 44, 45

[93] S. Chandrasekhar, *The maximum mass of ideal white dwarfs*, Astrophys. J. **74** (1931), 81–82 and J. Astrophys. Astron. **15** (1994), no. 2, 115–116. iii, 109

[94] K.-C. Chang, "Infinite dimensional Morse theory and multiple solution problems", Progr. Nonlinear Differential Equations Appl. **6**, Birkhäuser, Boston, 1993. 219

[95] X. Chang, Z.-Q. Wang, *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearities*, Nonlinearity **26** (2013), no. 2, 479–494. iv, 35, 38, 39, 43, 44, 48, 50, 216

[96] G. Chen, *Multiple semiclassical standing waves for fractional nonlinear Schrödinger equations*, Nonlinearity **28** (2015), no. 4, 927–949. v, 169, 170

[97] H. Chen, P. Felmer, A. Quaas, *Large solutions to elliptic equations involving fractional Laplacian*, Ann. Inst. H. Poincaré Anal. Non Linéaire C **32** (2015), no. 6, 1199–1128. 16, 25

[98] S. Chen, R. L. Frank, T. Weth, *Remainder terms in the fractional Sobolev inequality*, Indiana Univ. Math. J. **62** (2013), no. 4, 1381–1397. 6, 8, 12, 31

[99] W. Chen, Y. Li, P. Ma, "The Fractional Laplacian", World Scientific, 2020. 3

[100] W. Chen, L. D'Ambrosio, Y. Li, *Some Liouville theorems for the fractional Laplacian*, Nonlinear Anal. **121** (2015), 370–381. 31

[101] W. Chen, C. Li, B. Ou, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math. **59** (2006), no. 3, 330–343. 142

[102] Y.-H. Chen, C. Liu, *Ground state solutions for non-autonomous fractional Choquard equations*, Nonlinearity **29** (2016), no. 6, 1827–1842. 108

[103] Y. Cho, M. M. Fall, H. Hajaiej, P. A. Markowich, S. Trabelsi, *Orbital stability of standing waves of a class of fractional Schrödinger equations with Hartree-type nonlinearity*, Anal. Appl. (Singap.) **15** (2017), no. 5, 699–729. 107, 108, 113

[104] Y. Cho, G. Hwang, H. Hajaiej, T. Ozawa, *On the Cauchy problem of fractional Schrödinger equation with Hartree type nonlinearity*, Funkcial. Ekvac. **56** (2013), no. 2, 193–224. 108

[105] Y. Cho, T. Ozawa, *Sobolev inequalities with symmetry*, Commun. Contemp. Math. **11** (2009), no. 3, 355–365. 11

[106] P. Choquard, Symposium on Coulomb Systems, Lausanne (1976), *unpublished.* iii, 79

[107] P. Choquard, J. Stubbe, *The one-dimensional Schrödinger-Newton Equations*, Lett. Math. Phys. **81** (2007), no. 2, 177–184.

[108] P. Choquard, J. Stubbe, M. Vuffray, *Stationary solutions of the Schrödinger-Newton model - an ODE approach*, Differential Integral Equations **21** (2008), no. 7-8, 665–679. 80

[109] S. Cingolani, M. Clapp, S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys. **63** (2012), no. 2, 233–248. iii, 80, 142

[110] S. Cingolani, M. Clapp, S. Secchi, *Intertwining semiclassical solutions to a Schrödinger-Newton system*, Discrete Contin. Dyn. Syst. Ser. S **6** (2013), no. 4, 891–908.

[111] S. Cingolani, M. Gallo, *On the fractional NLS equation and the effects of the potential well's topology*, Adv. Nonlinear Stud. **21** (2021), no. 1, 1–40. vi, 12, 24, 26, 168, 216, 248

[112] S. Cingolani, M. Gallo, *On some qualitative aspects for doubly nonlocal equations*, Discrete Contin. Dyn. Syst. Ser. S **15** (2022), no. 12, 3603–3620. vi, 107, 248

[113] S. Cingolani, M. Gallo, K. Tanaka, *Normalized solutions for fractional nonlinear scalar field equation via Lagrangian formulation*, Nonlinearity **34** (2021), no. 6, 4017–4056. vi, 43, 107, 112, 248

[114] S. Cingolani, M. Gallo, K. Tanaka, *Symmetric ground states for doubly nonlocal equations with mass constraint*, Symmetry **13** (2021), no. 7, article ID 1199, pp. 17. vi, 43, 63, 99, 107, 112, 114, 248

[115] S. Cingolani, M. Gallo, K. Tanaka, *On fractional Schrödinger equations with Hartree type nonlinearities*, Math. Eng. **4** (2022), no. 6, 1–33. vi, 1, 19, 28, 107, 112, 143, 248

[116] S. Cingolani, M. Gallo, K. Tanaka, *Multiple solutions for the nonlinear Choquard equation with even or odd nonlinearities*, Calc. Var. Partial Differential Equations, **61** (2022), no. 68(2), pp. 34. vi, 79, 248

[117] S. Cingolani, M. Gallo, K. Tanaka, *Infinitely many free or prescribed mass solutions for fractional Hartree equations and Pohozaev identities*, to appear on Adv. Nonlinear Stud. (2024), arXiv:2305.14003, pp. 33. vi, 107, 112, 162, 248

[118] S. Cingolani, L. Jeanjean, *Stationary solutions with prescribed L^2-norm for the planar Schrödinger-Poisson system*, SIAM J. Math. Anal. **51** (2019), no. 4, 3533–3568. 81

[119] S. Cingolani, L. Jeanjean, K. Tanaka, *Multiplicity of positive solutions of nonlinear Schrödinger equations concentrating at a potential well*, Calc. Var. Partial Differential Equations **53** (2015), no. 1-2, 413–439. v, 169, 171, 175, 179, 182, 194, 209, 218, 219, 223, 225, 226, 228

[120] S. Cingolani, M. Lazzo, *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations*, Topol. Methods Nonlinear Anal. **10** (1997), no. 1, 1–13. 169, 170

[121] S. Cingolani, M. Lazzo, *Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions*, J. Differential Equations **160** (2000), no. 1, 118–138. 169

[122] S. Cingolani, S. Secchi, M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), no. 5, 973–1009. 175

[123] S. Cingolani, K. Tanaka, *Semi-classical states for the nonlinear Choquard equations: existence, multiplicity and concentration at a potential well*, Rev. Mat. Iberoam. **35** (2019), no. 6, 1885–1924. v, 170, 171

[124] S. Cingolani, K. Tanaka, *Ground state solutions for the nonlinear Choquard equation with prescribed mass*, in: V. Ferone, T. Kawakami, P. Salani, F. Takahashi (eds) "Geometric Properties for Parabolic and Elliptic PDE's" (2021), Springer INdAM Series **47**, Cortona 2019, 23–41. 81, 82, 87, 96

[125] S. Cingolani, K. Tanaka, *Deformation argument under PSP condition and applications*, Anal. Theory Appl. **37** (2021), no. 2, 191–208. 100, 106

[126] S. Cingolani, T. Weth, *On the planar Schrödinger-Poisson system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 1, 169–197.

[127] S. Cingolani, T. Weth, *Trudinger-Moser-type inequality with logarithmic convolution potentials*, J. London Math. Soc. **105** (2022), no. 3, 1897–1935.

[128] M. Clapp, D. Salazar, *Positive and sign changing solutions to a nonlinear Choquard equation*, J. Math. Anal. Appl. **407** (2013), no. 1, 1–15. iii, 80, 81, 142

[129] R. Clemente , J. C. de Albuquerque, E. Barboza, *Existence of solutions for a fractional Choquard-type equation in $\mathbb{R}$ with critical exponential growth*, Z. Angew. Math. Phys. **72** (2021), no. 16, 1–13. 108

[130] G. M. Coclite, S. Dipierro, F. Maddalena, E. Valdinoci, *Wellposedness of a nonlinear peridynamic model*, Nonlinearity **32** (2019), no. 1, 1–21. 12

[131] S. Coleman, V. Glaser, A. Martin, *Action minima among solutions to a class of Euclidean scalar field equations*, Commun. Math. Phys. **58** (1978), 211–221. 109, 112

[132] G. E. Comi, G. Strefani, *A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up*, J. Func. Anal. **277** (2019), no. 10, 3373–3435. 167

[133] O. Cornea, G. Lupton, J. Oprea, D. Tanré, "Lusternik-Schnirelmann category", Math. Surveys Monogr. **103**, AMS, USA, 2003. 222, 226, 227, 228

[134] M. Cozzi, *Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: a unified approach via fractional De Giorgi classes*, J. Funct. Anal. **272** (2017), no. 11, 4762–4837. 23, 24, 201, 202, 209

[135] L. D'Ambrosio, M. Ghergu, *Representation formulae for nonhomogeneous differential operators and applications to PDEs*, J. Differential Equations **317** (2022), 706–753. 142

[136] T. D'Aprile, J. Wei, *On bound states concentrating on spheres for the Maxwell-Schrödinger equation*, SIAM J. Math. Anal. **37** (2005), no. 1, 321–342.

[137] P. D'Avenia, J. Mederski, A. Pomponio, *Nonlinear scalar field equation with competing nonlocal terms*, Nonlinearity **34** (2021), no. 8, 5687–5707. 82

[138] P. D'Avenia, G. Siciliano, M. Squassina, *On the fractional Choquard equations*, Math. Models Methods Appl. Sci. **25** (2015), no. 8, 1447–1476. v, 35, 107, 108, 109, 112, 121, 123, 131, 138, 139, 142, 147, 162, 208

[139] P. D'Avenia, G. Siciliano, M. Squassina, *Existence results for a doubly nonlocal equation*, São Paulo J. Math. Sci. **9** (2015), no. 2, 311–324. 108, 142

[140] P. d'Avenia, A. Pomponio, D. Ruiz, *Semiclassical states for the nonlinear Schrödinger equation on saddle points of the potential via variational methods*, J. Funct. Anal. **262** (2012), no. 10, 4600–4633. *Erratum*: J. Funct. Anal. **284** (2023), no. 7, Paper No. 109833. 169

[141] W. Dai, Y. Fang, G. Quin, *Classification of positive solutions to fractional order Hartree equations via a direct method of moving planes*, J. Differential Equations **265** (2018), no. 5, 2044–2063. 108

[142] A. Dall'Acqua, T. Østergaard Sørensen, E. Stockmeyer, *Hartree-Fock theory for pseudorelativistic atoms*, Ann. Henri Poincaré **9** (2008), no. 4, 711–742. 107

[143] E. N. Dancer, K. Y. Lam, S. Yan, *The effect of the graph topology on the existence of multipeak solutions for nonlinear Schrödinger equations*, Abstr. Appl. Anal. **6** (2001), no. 2, 71–99. 169

[144] E. N. Dancer, J. Wei, *On the effect of domain topology in a singular perturbation problem*, Topol. Methods Nonlinear Anal. **11** (1998), no. 2, 227–248. 169

[145] J. Dávila, M. del Pino, I. Guerra, *Non-uniqueness of positive ground states of non-linear Schrödinger equations*, Proc. London Math. Soc. **106** (2012), no. 2, 318–344. 177

[146] J. Dávila, M. del Pino, J. Wei, *Concentrating standing waves for the fractional nonlinear Schrödinger equation*, J. Differential Equations **256** (2014), no. 2, 858–892. 170, 177

[147] M. Degiovanni, A. Musesti, M. Squassina, *On the regularity of solutions in the Pucci–Serrin identity*, Calc. Var. Partial Differential Equations **18** (2003), no. 3, 317–334.

[148] M. del Pino, P. L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations **4** (1996), no. 2, 121–137. 169, 170, 208

[149] M. del Pino, P. L. Felmer, *Semi-classical states for nonlinear Schrödinger equations*, J. Funct. Anal. **149** (1997), no. 1, 245–265. 169, 179

[150] F. Demengel, G. Demengel, "Functional spaces for the theory of elliptic partial differential equations", Universitext, Springer-Verlag London, 2012. 3, 7

[151] A. Di Castro, T. Kuusi, G. Palatucci, *Nonlocal Harnack inequalities*, J. Funct. Anal. **267** (2014), no. 6, 1807–1836. 23

[152] A. Di Castro, T. Kuusi, G. Palatucci, *Local behaviour of fractional p-minimizers*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 5, 1279–1299. 23

[153] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573. 3, 4, 5, 6, 7, 8, 10, 25, 44

[154] S. M. Djitte, *Fractional Hadamard formulas, Pohozaev type identities and applications*, Ph.D. thesis (2022). 163, 167

[155] S. M. Djitte, M. M. Fall, T. Weth, *A fractional Hadamard formula and applications*, Calc. Var. Partial Differential Equations **60** (2021), no. 6, article ID 231, pp. 31. 163

[156] V. D. Dinh, *Existence, non-existence and blow-up behaviour of minimizers for the mass-critical fractional non-linear Schrödinger equations with periodic potentials*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 6, 3252–3292. 45

[157] S. Dipierro, M. Medina, E. Valdinoci, "Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^n$", Lecture Notes, Edizioni della Normale (2017), pp. 162. 130

[158] S. Dipierro, G. Palatucci, E. Valdinoci, *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian*, Matematiche **68** (2013), no. 1, 201–216 iii

[159] S. Dipierro, O. Savin, E. Valdinoci, *Definition of fractional Laplacian for functions with polynomial growth*, Rev. Mat. Iberoam. **35** (2019), no. 4, 1079–1122. 31

[160] A. Dold, "Lectures on algebraic topology", Springer-Verlag Berlin Heidelberg, 1995. 219, 221, 223, 224, 225

[161] L. Dong, D. Liu, W. Qi, L. Wang, H. Zhou, P. Peng, C. Huang, *Necklace beams carrying fractional angular momentum in fractional systems with a saturable nonlinearity*, Commun. Nonlinear Sci. Numer. Simul. **99** (2021), article ID 105840, pp. 8. 37

[162] S. Dovetta, E. Serra, P. Tilli, *Action versus energy ground states in nonlinear Schrödinger equations*, Math. Ann. (2022), pp. 32. 78

[163] M. Du, T. Weth, *Ground states and high energy solutions of the planar Schrödinger-Poisson system*, Nonlinearity **30** (2017), no. 9, 3492–3515.

[164] N. Du Plessis, *Some theorems about the Riesz fractional integral*, Trans. Am. Math. Soc. **80** (1955), 124–134. 2, 32

[165] S. Duo, Y. Zhang, *Computing the ground and first excited states of the fractional Schrödinger equation in an infinite potential well*, Commun. Comput. Phys. **18** (2015), no. 2, 321–350. 44, 47

[166] B. Dyda, A. Kuznetsov, M. Kwaśnicki, *Fractional Laplace operator and Meijer G-function*, Constr. Approx. **45** (2017), no. 3, 427–448. 12

[167] Y. Ebihara, T. P. Schonbek, *On the (non)compactness of the radial sobolev spaces*, Hiroshima Math. J. **16** (1986), no. 3, 665–669. 11

[168] S. Eilenberg, N. Steenrod, "Foundations of algebraic topology", Princeton Legacy Library **2193**, Princeton University Press, New Jersey, 1952. 219, 224

[169] A. Elgart, B. Schlein, *Mean field dynamics of boson stars*, Comm. Pure Appl. Math. **60** (2007), no. 4, 500–545. 108, 109

[170] L. C. Evans, R. F. Gariepy, "Measure theory and fine properties of functions", Textbooks in Mathematics (eds. A. Boggess, K. Rosen), CRC Press, 2015. 163

[171] E. Fadell, *The relationship between Lusternik-Schnirelmann category and the concept of genus*, Pacific J. of Math. **89** (1980), 33–42. 230

[172] M. M. Fall, *Entire s-harmonic functions are affine*, Proc. Amer. Math. Soc. **144** (2016), no. 6, 2587–2592. 31

[173] M. M. Fall, *Semilinear elliptic equations for the fractional Laplacian with Hardy potential*, Nonlinear Anal. **193** (2020), article ID 111311. 14

[174] M. M. Fall, F. Mahmoudi, E. Valdinoci, *Ground states and concentration phenomena for the fractional Schrödinger equation*, Nonlinearity **28** (2015), no. 6, 1937–1961. v, 169

[175] M. M. Fall, T. Weth, *Liouville theorems for a general class of nonlocal operators*, Potential Anal. **45** (2016), no. 1, 187–200. 31

[176] M. Fazly, Y. Sire, *Symmetry properties for solutions of nonlocal equations involving nonlinear operators*, Ann. Inst. H. Poincaré Anal. Non Linéaire **36** (2019), no. 2, 523–543. 6

[177] P. Felmer, A. Quaas, J. Tan, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 6, 1237–1262. iii, 3, 8, 10, 24, 25, 44, 173

[178] M. Felsinger, M. Kassmann, P. Voigt, *The Dirichlet problem for nonlocal operators*, Math. Z. **279** (2015), 779–809. 12

[179] B. Feng, *Ground states for the fractional Schrödinger equation*, Electron. J. Differential Equations **2013** (2013), no. 127, 1–11. 45

[180] B. Feng, H. Zhang, *Stability of standing waves for the fractional Schrödinger-Hartree equation*, J. Math. Anal. Appl. **460** (2018), no. 1, 352–364. 113

[181] F. Ferrari, I.. E. Verbitsky, *Radial fractional Laplace operators and Hessian inequalities*, J. Differential Equations **253** (2012), no. 1, 244–272. 10, 12

[182] R. P. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. **20** (1948), no. 2, 367–386. ii, 43

[183] G. M. Figueiredo, G. Siciliano, *A multiplicity result via Ljusternick-Schnirelmann category and Morse theory for a fractional Schrödinger equation in $\mathbb{R}^N$*, NoDEA Nonlinear Differential Equations Appl. **23** (2016), no. 2, article ID 12, pp. 22. v, 170

[184] A. Floer, A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. **69** (1986), no. 3, 397–408. 169, 170, 177

[185] G. Fournier, D. Lupo, M. Ramos, M. Willem, *Limit relative category and critical point theory*, in "Dynamics reported. Expositions in dynamical systems" (eds. C. K. R. T. Jones, U. Kirchgraber, H. O. Walther), Springer-Verlag New Series **3**, Berlin (1994), 1–24. 219, 222, 226, 227, 228

[186] G. Fournier, M. Willem, *Multiple solutions of the forced double pendulum equation*, Ann. Inst. H. Poincaré Anal. Non Lineaire **6** (1989), 259–281. 226, 227, 228

[187] G. Fournier, M. Willem, *Relative Category and The Calculus of Variations*, in "Variational Problems" (eds. H. Berestycki, J. M. Coron, I. Ekeland), Birkhäuser, Basel (1990), 95–104. 219, 222, 226, 227, 228

[188] L. E. Fraenkel, *On steady vortex rings of small cross-section in an ideal fluid*, Proc. Royal Soc. A **316** (1970), no. 1524, 29–62.

[189] R. L. Frank, E. Lenzmann, *On ground states for the L^2-critical boson star equation*, arXiv:0910.2721 (2010), pp. 16. 108, 109

[190] R. L. Frank, E. Lenzmann, L. Silvestre, *Uniqueness of radial solutions for the fractional Laplacian*, Comm. Pure Appl. Math. **69** (2016), no. 9, 1671–1726. iii, 24, 30, 44, 145, 170, 177

[191] J. Fröhlich, S. Gustafson, B. L. G. Jonsson, I. M. Sigal, *Solitary wave dynamics in an external potential*, Comm. Math. Phys. **250** (2004), no. 3, 613–642. 169

[192] J. Fröhlich, B. L. G. Jonsson, E. Lenzmann, *Effective dynamics of boson stars*, Nonlinearity **20** (2007), no. 5, 1031–1075. 108, 109

[193] J. Fröhlich, B. L. G. Jonsson, E. Lenzmann, *Boson stars as solitary waves*, Comm. Math. Phys. **274** (2007), no. 1, 1–30. 108, 109

[194] J. Fröhlich, E. Lenzmann, *Mean-field limit of quantum Bose gases and nonlinear Hartree equation*, in Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau (2004), talk no. 18, pp. 26. 80, 108

[195] J. Fröhlich, E. Lenzmann, *Blowup for nonlinear wave equations describing boson stars*, Comm. Pure Appl. Math. **60** (2007), no. 11, 1691–1705. 108

[196] J. Fröhlich, T.-P. Tsai, H.-T. Yau, *On the point-particle (Newtonian) limit of the non-linear Hartree equation*, Comm. Math. Phys. **225** (2002), no. 2, 223–274. 80, 109

[197] M. Gallo, *Multiplicity and concentration results for local and fractional NLS equations with critical growth*, Adv. Differential Equ., **26** (2021), no. 9-10, 397–424. vi, 1, 19, 107, 168, 213, 216, 248

[198] M. Gallo, *Asymptotic decay of solutions for sublinear fractional Choquard equations*, Nonlinear Anal. **242** (2024), article ID 113515, DOI: 10.1016/j.na.2024.113515, pp. 21. vi, 1, 143, 248

[199] Z. Gao, X. Tang, S. Chen, *Ground state solutions of fractional Choquard equations with general potentials and nonlinearities*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 1, 2037–2057. 112

[200] M. Á. García-Ferrero, A. Rüland, *Strong unique continuation for the higher order fractional Laplacian*, Math. Eng. **1** (2019), no. 4, 715–774. 6

[201] N. Garofalo, *Fractional thoughts*, in “New Developments in the Analysis of Nonlocal Operators” **723** (eds. D. Danielli, A. Petrosyan, C. A. Pop), AMS, 2019. iii, 3, 4, 5, 6, 10, 11, 15, 17, 30

[202] V. Georgiev, *Evolution flow and ground states for fractional Schrödinger-Hartree equations*, Mathematics and Education in Mathematics, Proc. of 50th Spring Conference of the Union of Bulgarian Mathematicians **50** (2021), 45–54. 108

[203] V. Georgiev, A. Stefanov, *On the classification of the spectrally stable standing waves of the Hartree problem*, Phys. D **370** (2018), 29–39. 108

[204] M. Ghergu, Y. Miyamoto, V. Moroz, *Polyharmonic inequalities with nonlocal terms*, J. Differential Equations **296** (2021), 799–821. 29

[205] M. Ghimenti, V. Moroz, J. Van Schaftingen, *Least action nodal solutions for the quadratic Choquard equation*, Proc. Amer. Math. Soc. **145** (2017), no. 2, 737–747. iii, 80

[206] M. Ghimenti, J. Van Schaftingen, *Nodal solutions for the Choquard equation*, J. Funct. Anal. **271** (2016), no. 1, 107–135. iii, 80

[207] J. Giacomoni, D. Goel, K. Sreenadh, *Regularity results on a class of doubly nonlocal problems*, J. Differential Equations **268** (2020), 5301–5328. 108, 125

[208] L. Grafakos, S. Oh, *The Kato-Ponce Inequality*, Comm. Partial Differential Equations **39** (2014), no. 6, 1128–1157. 5

[209] D. Greco, *Optimal decay and regularity for a Thomas-Fermi type variational problem*, arXiv:2302.12586, (2023), pp. 56. 146

[210] M. J. Greenberg, J. R. Harper, “Algebraic topology: a first course”, Mathematics Lecture Note Series **58**, The Benjamin/Cummings Publishing Company, Canada, 1981. 219, 221, 222

[211] G. Grillo, M. Muratori, *On the asymptotic behaviour of solutions to the fractional porous medium equation with variable density*, Discrete Contin. Dyn. Syst. **35** (2015), no. 12, 5927–5962. 29, 147

[212] Q. Guo, S. Zhu, *Sharp threshold of blow-up and scattering for the fractional Hartree equation*, J. Differential Equations **264** (2018), no. 4, 2802–2832. 108

[213] C. Hainzl, B. Schlein, *Stellar collapse in the time dependent Hartree-Fock approximation*, Commun. Math. Phys. **287** (2009), no. 2, 705–717. 109

[214] C. Hainzl, E. Lenzmann, M. Lewin, B. Schlein, *On blowup for time-dependent generalized Hartree-Fock equations*, Ann. Henri Poincaré **11** (2010), no.6, 1023–1052. 108, 109

[215] H. Hajaiej, P. A. Markowich, S. Trabelsi, *Multiconfiguration Hartree-Fock Theory for pseudorelativistic systems: the time-dependent case*, Math. Models Methods Appl. Sci. **24** (2014), no. 3, 599–626. 107

[216] D. Hartree, *The wave mechanics of an atom with a non-Coulomb central field. Part I. Theory and methods*, Math. Proc. Cambridge Philos. Soc. **24** (1928), no. 1, 89–110. 80

[217] A. Hatcher, “Algebraic Topology”, Cambridge University Press (2002). 219, 221, 222, 224, 225

[218] X. He, V. D. Rădulescu, *Small linear perturbations of fractional Choquard equations with critical exponent*, J. Differential Equations **282** (2021), no. 1, 481–540. 108

[219] X. He, V. D. Rădulescu, W. Zou, *Normalized ground states for the critical fractional Choquard equation with a local perturbation*, J. Geom. Anal. **32** (2022), no. 10, pp. 51. 108

[220] Y. He, *Singularly perturbed fractional Schrödinger equations with critical growth*, Adv. Nonlinear Stud. **18** (2018), no. 3, pp. 25. 208, 209, 210

[221] Y. He, W. Zou, *Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities*, Calc. Var. Partial Differential Equations **55** (2016), no. 4, 55–91. 207, 208

[222] S. Herr, E. Lenzmann, *The Boson star equation with initial data of low regularity*, Nonlinear Anal. **97** (2014), 125–137. 108

[223] J. Hirata, N. Ikoma, K. Tanaka, *Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches*, Topol. Methods Nonlinear Anal. **35** (2010), no. 2, 253–276. 59

[224] J. Hirata, K. Tanaka, *Nonlinear scalar field equations with L^2 constraint: mountain pass and symmetric mountain pass approaches*, Adv. Nonlinear Stud. **19** (2019), no. 2, 263–290. iv, 45, 46, 56, 59, 61, 82, 83, 84, 100, 113

[225] R. R. Huilgol, *On Liouville's theorem for biharmonic functions*, SIAM J. Appl. Math. **20** (1971), no. 1, 37–39. 31

[226] D. Hundertmark, Y.-R. Lee, T. Ried, V. Zharnitsky, *Dispersion managed solitons in the presence of saturated nonlinearity*, Phys. D **365-357** (2017), 65–69. 37

[227] I. Ianni, *Solutions of the Schrödinger-Poisson problem concentrating on spheres II: existence*, Math. Models Methods Appl. Sci. **19** (2009), no. 6, 877–910.

[228] I. Ianni, G. Vaira, *Solutions of the Schrödinger-Poisson problem concentrating on spheres I: necessary conditions*, Math. Models Methods Appl. Sci. **19** (2009), no. 5, 707–720.

[229] N. Ikoma, *Existence of solutions of scalar field equations with fractional operator*, J. Fixed Point Theory Appl. **19** (2017), no. 2, 649–690. *Erratum*: J. Fixed Point Theory Appl. **19** (2017), no. 2, 1649–1652. iv, 44

[230] N. Ikoma, *Multiplicity of radial and nonradial solutions to equations with fractional operators*, Commun. Pure Appl. Anal. **19** (2020), no. 7, 3501–3530. iv, 44, 50

[231] N. Ikoma, K. Tanaka, *A note on deformation argument for L^2-normalized solutions of nonlinear Schrödinger equations and systems*, Adv. Differential Equations **24** (2019), no. 11-12, 609–646. 46, 56, 61, 83, 100, 113

[232] A. D. Ionescu, F. Pusateri, *Nonlinear fractional Schrödinger equations in one dimension*, J. Funct. Anal. **266** (2014), no. 1, 139–176. 44

[233] S. Jarohs, *Strong comparison principle for the fractional p-Laplacian and applications to starshaped rings*, Adv. Nonlinear Stud. **18** (2018), no. 4, 691–704. 26

[234] S. Jarohs, T. Weth, *On the strong maximum principle for nonlocal operators*, Adv. Nonlinear Stud. **18** (2018), no. 4, 691–704. 12

[235] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal. **28** (1997), no. 10, 1633–1659. 45, 59

[236] L. Jeanjean, S.-S. Lu, *On global minimizers for a mass constrained problem*, Calc. Var. Partial Differential Equations **61** (2022), no. 6, article ID 214, pp. 18. 78

[237] L. Jeanjean, K. Tanaka, *A remark on least energy solutions in $\mathbb{R}^N$*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2399–2408. iv, 35, 44, 48, 112, 113, 210

[238] H. Jin, W. Liu, J. Zhang, *Singularly perturbed fractional Schrödinger equation involving a general critical nonlinearity*, Adv. Nonlinear Stud. **18** (2018), no. 3, pp. 13. 208, 210

[239] S. Jin, *Multi-bump standing waves for nonlinear Schrödinger equations with a general nonlinearity: the topological effect of potential wells*, Adv. Nonlinear Stud. **21** (2021), no. 2, 369–396. 169

[240] T. Jin, Y. Y. Li, J. Xiong, *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, J. Eur. Math. Soc. **16**, (2014), 1111–1171. 24, 25

[241] B. L. G. Jonsson, J. Fröhlich, S. Gustafson, *Long time motion of NLS solitary waves in a confining potential*, Ann. Henri Poincaré **7** (2006), no. 4, 621–660. 169

[242] K. Kirkpatrick, E. Lenzmann, G. Staffilani, *On the continuum limit for discrete NLS with long-range lattice interactions*, Comm. Math. Phys. **317** (2013), no. 3, 563–591. 44

[243] C. Klein, C. Sparber, P. Markowich, *Numerical study of fractional nonlinear Schrödinger equations*, Proc. Royal Soc. A **470** (2014), no. 470, article ID 20140364, pp. 26. 44

[244] I. Kuzin, S. I. Pohozaev, "Entire solution of semilinear elliptic equations", Progr. Nonlinear Differential Equations Appl. **33**, ed: H. Brezis, Birkhäuser Verlag, Basel Switzerland, 1997. iv

[245] M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal. **20** (2017), no. 1, 7–51. 6, 31

[246] M. Kwaśnicki, *Fractional Laplace operator and its properties*, in: A. Kochubei, Y. Luchko, "Handbook of Fractional Calculus with Applications, Vol. 1: Basic Theory", De Gruyter, Berlin, 2019. 12, 14

[247] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$, Arch. Ration. Mech. Anal. **105** (1989), no. 3, 243–266. 177

[248] N. S. Landkof, "Foundations of modern potential theory", Springer-Verlag, Berlin Heidelberg, 1972. 28, 29, 30, 33

[249] N. Laskin, *Fractional quantum mechanics*, Phys. Rev. E **62** (2000), no. 3, 3125–3145. ii, 43

[250] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A **268** (2000), no. 4–6, 298–305. ii, 43

[251] N. Laskin, *Fractional Schrödinger equation*, Phys. Rev. E **66** (2002), no. 5, article ID 056108, pp. 7. ii, 43

[252] N. Laskin, "Fractional quantum mechanics", World Sci., Singapore, 2018. ii, 43

[253] P. Le, *Liouville theorem and classification of positive solutions for a fractional Choquard type equation*, Nonlinear Anal. **185** (2019), 123–141. 12, 147

[254] P. Le, *Symmetry of positive solutions to Choquard type equations involving the fractional p-Laplacian*, Acta Appl. Math. **170** (2020), no. 1, 387–398. 148

[255] R. Lehrer, L. A. Maia, M. Squassina, *Asymptotically linear fractional Schrödinger equations*, Complex Var. Elliptic Equ. **60** (2007), no. 4, 529–558. 49, 210

[256] E. Lenzmann, *Well-posedness for semi-relativistic Hartree equations of critical type*, Math. Phys. Anal. Geom. **2** (2007), no. 1, 43–64. 108, 109

[257] E. Lenzmann, *Uniqueness of ground states for pseudorelativistic Hartree equations*, Anal. PDE **2** (2009), no. 1, 1–27. iii, 80, 108, 109

[258] E. Lenzmann, M. Lewin, *On singularity formation for the* L^2*-critical Boson star equation*, Nonlinearity **24** (2011), no. 12, 3515–3540. 108, 109

[259] E. Lenzmann, J. Sok, *A sharp rearrangement principle in Fourier space and symmetry results for PDEs with arbitrary order*, Int. Math. Res. Not. IMRN **2021** (2021), no. 19, 15040–15081. 35

[260] M. Lewin, N. Rougerie, *Derivation of Pekar's polarons from a microscopic model of quantum crystal*, SIAM J. Math. Anal. **45** (2013), no. 3, 1267–1301. 79

[261] G.-B. Li, H.-Y. Ye, *The existence of positive solutions with prescribed* L^2*-norm for nonlinear Choquard equations*, J. Math. Phys. **55** (2014), no. 12, article ID 12501, 1–19. 81

[262] P. Li, B. A. Malomed, D. Mihalache, *Vortex solitons in fractional nonlinear Schrödinger equation with the cubic-quintic nonlinearity*, Chaos, Solitons and Fractals **137** (2020), article ID 109783, pp. 10. 44

[263] Q. Li, K. Teng, J. Zhang, W. Wang, *Concentration behaviour of solutions for fractional Schrödinger equations involving critical exponent*, J. Math. Phys. **61** (2020), no. 7, article ID 071513, pp. 5. 208

[264] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57** (1977), no. 2, 93–105. iii, 79

[265] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. **118** (1983), no. 2, 349–374. iii, 12, 80

[266] E. H. Lieb, M. Loss, "Analysis", Grad. Texts in Math. **14**, Amer. Math. Soc., USA, 2001. 28, 29, 31

[267] E. H. Lieb, R. Seiringer, J. P. Solovej, J. Yngvason, "The mathematics of the Bose gas and its condensation", in: Oberwolfach Seminars **34**, Birkäuser Basel, 2005. 47

[268] E. H. Lieb, W. Thirring, *Gravitational collapse in quantum mechanics with relativistic kinetic energy*, Ann. Phys. **155** (1984), no. 2, 494–512. iii, 109

[269] E. H. Lieb, H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112** (1987), no. 1, 147–174. iii, 108, 109

[270] E. H. Lieb, H.-T. Yau, *A rigorous examination of the Chandrasekhar theory of stellar collapse*, Astrophys. J. **323** (1987), 140–144. iii, 109

[271] P.-L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), no. 6, 1063–1073. iii, iv, 81

[272] P.-L. Lions, *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal. **49** (1982), no. 3, 315–334. 11

[273] Z. Liu, V. D. Rădulescu, C. Tang, J. Zhang, *Another look at planar Schrödinger-Newton systems*, J. Differential Equations **328** (2022), no. 4, 65–104.

[274] S. Longhi, *Fractional Schrödinger equation in optics*, Optics Lett. **40** (2015), no. 6, 1117–1120. 44

[275] O. Lopes, M. Maris, *Symmetry of minimizers for some nonlocal variational problems*, J. Func. Anal. **254** (2008), no. 2, 535–592 . 28, 29, 30, 49, 78, 140

[276] J. Lu, V. Moroz, C. B. Muratov, *Orbital-free density functional theory of out-of-plane charge screening in graphene*, J. Nonlinear. Sci. **25** (2015), no. 6, 1391–1430. 107

[277] H. Luo, *Ground state solutions of Pohozaev type for fractional Choquard equations with general nonlinearities*, Comput. Math. Appl. **77** (2019), no. 3, 877–887. 108, 112, 123

[278] H. Luo, Z. Zhang, *Normalized solutions to the fractional Schrödinger equations with combined nonlinearities*, Calc. Var. Partial Differential Equations **59** (2020), no. 143, pp. 35. iv, 45

[279] L. Ma, L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal. **195** (2010), no. 2, 455–467. iii, 80

[280] P. Ma, J. Zhang, *Existence and multiplicity of solutions for fractional Choquard equations*, Nonlinear Anal. **164** (2017), pp. 100–117. 142

[281] L. A. Maia, E. Montefusco, B. Pellacci, *Weakly coupled nonlinear Schrödinger systems: the saturation effect*, Calc. Var. Partial Differential Equations **46** (2013), no. 1-2, 325–351. 37

[282] L. A. Maia, B. Pellacci, D. Schiera, *Positive bound states to nonlinear Choquard equations in the presence of nonsymmetric potentials*, arXiv:2106.04519 (2021), pp. 16. 145

[283] L. A. Maia, B. Pellacci, D. Schiera, *Symmetric positive solutions to nonlinear Choquard equations with potentials*, Calc. Var. Partial Differential Equations **61** (2022), no. 61, pp. 34. 145

[284] S. Masaki, *Local existence and WKB approximation of solutions to Schrödinger-Poisson system in the two-dimensional whole space*, Comm. Partial Differential Equations **35** (2010), no. 12, 2253–2278.

[285] S. Masaki, *Energy solution to a Schrödinger-Poisson system in the two-dimensional whole space,*, SIAM J. Math. Anal. **43** (2011), no. 6, 2719–2731.

[286] W. S. Massey, *How to give an exposition of the Čech-Alexander-Spanier type homology theory*, The Amer. Math. Monthly **85** (1978), no. 2, 75–83. 224

[287] W. S. Massey, "Homology and cohomology theory: an approach based on Alexander-Spanier cochains", Pure and Applied Mathematics **46**, Marcel Dekker Inc., New York, 1978. 219, 221, 222, 224

[288] W. S. Massey, "A basic course in algebraic topology", Grad. Texts in Math. **127**, Springer-Verlag, New York Inc., 1991. 219, 223, 224

[289] W. McLean, "Strongly elliptic systems and boundary integral equations", Cambridge University Press, USA, 2000. 7

[290] J. Mederski, *Nonradial solutions of nonlinear scalar field equations*, Nonlinearity **33** (2020), no. 12, article ID 6349, pp. 33. iv, 44

[291] J. Mederski, *General class of optimal Sobolev inequalities and nonlinear scalar field equations*, J. Differential Equations **281** (2021), no. 2, 411–441. 36

[292] J. Mederski, J. Schino, *Least energy solutions to a cooperative system of Schrödinger equations with prescribed L^2-bounds: at least L^2-critical growth*, Calc. Var. Partial Differential Equations **61** (2022), no. 10, pp. 31. 45

[293] P. Menzala, *On regular solutions of a nonlinear equation of Choquard's type*, Proc. Roy. Soc. Edinburgh Sect. A **86** (1980), no. 3-4, 291–301. 80

[294] C. Mercuri, V. Moroz, J. Van Schaftingen, *Groundstates and radial solutions to nonlinear Schrödinger-Poisson-Slater equations at the critical frequency*, Calc. Var. Partial Differential Equations **55**(146) (2016), no. 6, pp. 58. 87

[295] R. Metzler, J. Klafter, *The random walks guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 1–77. 44

[296] A. Michelangeli, B. Schlein, *Dynamical collapse of boson stars*, Commun. Math. Phys. **311** (2012), no. 3, 645–687. 109

[297] Y. Mizuta, "Potential theory in Euclidean spaces", Gakuto International Series, Mathematical Sciences and Applications **6**, Gakkōtosho, Tokyo (1996). 28, 29, 30, 31, 32

[298] D. R. Moreira, E. V. O. Teixeira, *Weak convergence under nonlinearities*, An. Acad. Brasil. Ciênc. **75** (2003), no. 1, 9–19. 38, 42

[299] I. M. Moroz, R. Penrose, P. Tod, *Spherically-symmetric solutions of the Schrödinger-Newton equations*, Topology of the Universe Conference (Cleveland, OH, 1997), Classical Quantum Gravity **15** (1998), no. 9, 2733–2742. iii, 80

[300] V. Moroz, J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), no. 2, 153–184. iii, 30, 35, 80, 109, 112, 139, 142, 143, 145

[301] V. Moroz, J. Van Schaftingen, *Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains*, J. Differential Equations **254** (2013), no. 8, 3089–3145. 29, 98, 142

[302] V. Moroz, J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), no. 9, 6557–6579. iv, v, 35, 41, 79, 80, 81, 82, 84, 86, 87, 97, 98, 107, 108, 109, 112, 113, 123, 124, 131, 138, 139, 141, 149

[303] V. Moroz, J. Van Schaftingen, *Semi-classical states for the Choquard equation*, Calc. Var. Partial Differential Equations **52** (2015), no. 1-2, 199–235.

[304] V. Moroz, J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. **19** (2017), no. 1, 773–813. 27, 28, 80, 142, 145

[305] T. Mukherjee, K. Sreenadh, *Fractional Choquard equation with critical nonlinearities*, Nonlinear Differ. Equ. Appl. **24** (2017), no. 6, pp. 34. 108

[306] Z. Nehari, *On a class of nonlinear second-order differential equations*, 7rans. Amer. Math. Soc. **95** (1960), no. 1, 101–123. iv

[307] D.-T. Nguyen, *On blow-up profile of ground states of boson stars with external potential*, J. Stat. Phys. **169** (2017), no. 2, 395–422. 109

[308] D.-T. Nguyen, *Blow-up analysis for focusing many-body quantum systems*, Ph.D. thesis (2020). 109

[309] Y.-G. Oh, *Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class* $(V)_a$, Comm. Math. Phys. **13** (1988), no. 12, 1499–1519. 169, 170

[310] R. S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), no. 1, 19–30. 11, 82, 97

[311] Y. J. Park, *Fractional Polya-Szegö inequality*, J. Chungcheong Math. Soc. **24** (2011), no. 2, 267–271. 211

[312] Y. J. Park, *Fractional Gagliardo-Nirenberg inequality*, J. Chungcheong Math. Soc. **24** (2011), no. 3, 583–586. 8

[313] S. Pekar, "Untersuchung über die elektronentheorie der kristalle", Akademie Verlag, Berlin, 1954. iii, 79

[314] R. Penrose, *On gravity's role in quantum state reduction*, Gen. Relativity Gravitation **28** (1996), no. 5, 581–600. iii, 80

[315] R. Penrose, *Quantum computation, entanglement and state reduction*, Philos. Trans. Roy. Soc. A **356** (1998), no. 1743, 1927–1939. iii, 80

[316] R. Penrose, "The road to reality. A complete guide to the laws of the universe", Alfred A. Knopf Inc., New York, 2005. iii, 80

[317] M. M. Petroski, M. S. Petrović, M. R. Belić, *Quasi-stable propagation of vortices and soliton clusters in previous saturable Kerr media with square-root nonlinearity*, Opt. Commun. **279** (2007), no. 11, 196–202. 37

[318] S. I. Pohozaev, *On the eigenfunctions of the equation* $\Delta + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR **165** (1965), 36–39. iv, 45

[319] S. I. Pohozaev, *On the solvability of nonlinear equations with odd operators*, Funktsional. Anal. i Prilozhen. **1** (1967), no. 3, 66–73. iv

[320] L. Pitaevskii, S. Stringari, "Bose-Einstein condensation", Oxford University Press, USA, 2003. 47

[321] F. Punzo, G. Terrone, *On a fractional sublinear elliptic equation with a variable coefficient*, Appl. Anal. **94** (2015), no. 4, 800–818. 147

[322] A. Quaas, A. Xia, *Liouville type theorems for nonlinear elliptic equations and systems involving fractional Laplacian in the half space*, Calc. Var. Partial Differential Equations **52** (2015), 641–659. 26

[323] D. Qin, V. D. Rădulescu, X. Tang, *Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations*, J. Differential Equations **275** (2021), no. 901, 652–683. 81

[324] P. H. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math. **3** (1973), no. 2, 161–202. 230

[325] P. H. Rabinowitz, "Minimax methods in critical point theory with applications to differential equations", CBMS Reg. Conf. Ser. Math. **65**, Amer. Math. Soc., Providence, 1986. 72, 106

[326] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), no. 2, 270–291. 169, 208

[327] S. T. R. Rizvi, I. Ali, K. Ali, M. Younis, *Saturation of the nonlinear refractive index for optical solitons in two-core fibers*, Optik **127** (2016), no. 13, 5328–5333. 37

[328] X. Ros-Oton, J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. **101** (2014), no. 3, 275–302. iii, 16, 24, 32

[329] X. Ros-Oton, J. Serra, *The extremal solution for the fractional Laplacian*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 723–750. iii

[330] X. Ros-Oton, J. Serra, *Local integration by parts and Pohozaev identities for higher order fractional Laplacians*, Discrete Contin. Dyn. Syst. **35** (2015), no. 5, 2131–2150. 6, 23

[331] D. Ruiz, *Semiclassical states for coupled Schrödinger-Maxwell equations: concentration around a sphere*, Math. Models Methods Appl. Sci. **15** (2005), no. 1, 141–164.

[332] D. Ruiz, J. Van Schaftingen, *Odd symmetry of least energy nodal solutions for the Choquard equation*, J. Differential Equations **264** (2018), no. 2, 1231–1262. iii, 80

[333] T. Runst, W. Sickel, "Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations", de Gruyter Ser. Nonlinear Anal. Appl. **3**, Berlin, 1996. 10

[334] S. G. Samko, "Hypersingular integrals and their applications", Anal. Methods Spec. Funct. **5**, Taylor & Francis, USA and Canada, 2002. 6, 29, 30, 31

[335] J. Schino, *Normalized ground states to a cooperative system of Schrödinger equations with generic* L^2*-subcritical or* L^2*-critical nonlinearity*, Adv. Differential Equations **27** (2022), no. 7-8, 467–496. 45

[336] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, J. Math. Phys. **54** (2013), no. 3, article ID 031501, pp. 17. 5, 170, 171

[337] S. Secchi, M. Squassina, *Soliton dynamics for fractional Schrödinger equations*, Appl. Anal. **93** (2014), no. 8, 102–1729. v, 169

[338] S. Seok, *Spike-layer solutions to nonlinear fractional Schrödinger equations with almost optimal nonlinearities*, Electron. J. Differential Equations **2015** (2015), no. 196, 1–19. v, 170

[339] R. Servadei, E. Valdinoci, *Weak and viscosity solutions of the fractional Laplace equation*, Publ. Mat. **58** (2014), no. 1, 133–154. 3, 16, 26

[340] X. Shang, J. Zhang, *Existence and concentration of positive solutions for fractional nonlinear Schrödinger equation with critical growth*, J. Math. Phys. **58** (2017), no. 8, article ID 081502, pp. 18. 209

[341] X. Shang, J. Zhang, Y. Yang, *On fractional Schrödinger equation in $\mathbb{R}^N$ with critical growth*, J. Math. Phys. **54** (2013), no. 12, article ID 121502, pp. 20. 208, 210

[342] Z. Shen, F. Gao, M. Yin, *Ground state for nonlinear fractional Choquard equations with general nonlinearities*, Math. Methods Appl. Sci. **39** (2016), no. 14, 4082–4098. 108, 112, 121, 123, 138, 162

[343] M. Shibata, *Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term*, Manuscripta Math. **143** (2014), no. 1-2, 221–237. iv, 45, 46

[344] T.-T. Shieh, D. E. Spector, *On a new class of fractional partial differential equations*, Adv. Calc. Var. **8** (2014), no. 4, pp. 16. 25

[345] M. Šilhavý, *Fractional vector analysis based on invariance requirements (critique of coordinate approaches)*, Continuum Mech. Thermodyn. **32** (2020), no. 1, 207–228. 167

[346] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2006), 67–112. iii, 3, 5, 6, 22, 30

[347] Y. Sire, E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), no. 6, 1842–1864. iii

[348] V. Skarka, V. I. Berezhiani, V. Boucher, *Self-trapping of electromagnetic pulses in narrow-gap semiconductors*, Phys. Rev. B **61** (2000), no. 15, article ID 10201 , pp. 5. 37

[349] N. Soave, E. Valdinoci, *Overdetermined problems for the fractional Laplacian in exterior and annular sets*, J. Anal. Math. **137** (2019), no. 1, 101–134. 25

[350] E. H. Spanier, “Algebraic topology”, Springer-Verlag New York, 1989. 219, 221, 224, 225

[351] E. M. Stein, “Singular Integrals and Differentiability Properties of Functions”, Princeton Mathematical Series **30**, Princeton University Press, Princeton, New Jersey (1970). 2, 29, 30

[352] P. R. Stinga, *User’s guide to the fractional Laplacian and the method of semigroups*, in: A. Kochubei, Y. Luchko, “Handbook of Fractional Calculus with Applications, Vol. 2: Basic Theor”, De Gruyter, Berlin, 2019. 2, 6, 22, 30

[353] W. A. Strauss, *Existence of solitary waves in higher dimensions*, Commun. Math. Phys. **55** (1977), no. 2, 149–162. 11, 36

[354] M. Struwe, “Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems”, A Series of Modern Surveys in Mathematics **34**, Springer-Verlag Berlin Heidelberg, 2008. 218

[355] C. Stuart, *Existence theory for the Hartree equation*, Arch. Ration. Mech. Anal. **51** (1973), 60–69. 80

[356] C. Stuart, *Bifurcation from the continuous spectrum in the L^2-theory of elliptic equations on $\mathbb{R}^n$*, in: Recent Methods in Nonlinear Analysis and Applications, Liguori, Napoli (1980), 231–300. 45

[357] J. Stubbe, M. Vuffray, *Bound states of the Schrödinger-Newton model in low dimensions*, Nonlinear Anal. **73** (2010), no. 10, 3171–3178.

[358] A. Szulkin, *A relative category and applications to critical point theory for strongly indefinite functionals*, Nonlinear Anal. **15** (1990), no. 8, 725–739. 219, 222, 223, 226, 227, 228

[359] J. Thim, *Asymptotics and inversion of Riesz potentials through decomposition in radial and spherical parts*, Ann. Mat. **195** (2015), no. 2, 232–241. 84, 87, 88, 93

[360] W. Thirring, *Bosonic black holes*, Physics Letters B **127** (1983), no. 1-2, 27–29. iii, 109

[361] P. Tod, *The ground state energy of the Schrödinger-Newton equation*, Phys. Lett. A **280** (2001), no. 4, 173–176. 80

[362] P. Tod, I. M. Moroz, *An analytical approach to the Schrödinger-Newton equations*, Nonlinearity **12** (1999), no. 2, 201–216. 80

[363] H. Triebel, "Interpolation theory, function spaces, differential operators", North-Holland **18**, 1978. 7, 10

[364] J. Van Schaftingen, *Explicit approximation of the symmetric rearrangement by polarizations*, Arch. Math. (Basel) **93** (2009), no. 2, 181–190. 35

[365] J. Van Schaftingen, M. Willem, *Symmetry of solutions of semilinear elliptic problems*, J. Eur. Math. Soc. **10** (2008), no. 2, 439–456. 35

[366] J. L. Vázquez, *Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type*, J. Eur. Math. Soc. **16** (2014), no. 4, 769–803. 14

[367] J. L. Vázquez, *The mathematical theories of diffusion: nonlinear and fractional diffusion*, in "Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions" (eds. M. Bonforte, G. Grillo), Springer (2017), 205–278. 44

[368] J. W. Vick, "Homology theory: an introduction to algebraic topology", Springer -Verlag New York (1994). 219, 221

[369] J. Wang, D. Lu, J. Xu, F. Zhang, *Multiple positive solutions for semilinear Schrödinger equations with critical growth in* $\mathbb{R}^N$, J. Math. Phys. **56** (2015), no. 4, article ID 041502, pp. 27. 209

[370] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Commun. Math. Phys. **153** (1993), no. 2, 229–244. v, 169

[371] X. Wang, Z. Yang, *Symmetry and monotonicity of positive solutions for a Choquard equation with the fractional Laplacian*, Complex Var. Elliptic Equ. **67** (2021), no. 5, 1211-1228. 140, 148

[372] Z.-Q. Wang, J. Xia, *Saddle solutions for the Choquard equation II*, Nonlinear Anal. **201** (2020), article ID 112053, pp. 25. iii, 80

[373] Z.-Q. Wang, C. Zhang, *Convergence from power-law to logarithm-law in nonlinear scalar field equations*, Arch. Rational Mech. Anal. **231** (2019), no. 6, 45–61.

[374] Z. X. Wang, D. R. Guo, "Special functions", World Sci., Singapore, 2010. 12

[375] J. Wei, M. Winter, *Strongly interacting bumps for the Schrödinger-Newton equation*, J. Math. Phys. **50** (2009), no. 1, article ID 012905, pp. 23. 80

[376] J. Wei, Y. Wu, *On some nonlinear Schrödinger equations in* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A (2022), 1–26, DOI: 10.1017/prm.2022.56. 177

[377] R. O. Wells, O. Garcia-Prada, "Differential analysis on complex manifolds", Grad. Texts in Math. **65**, Springer Science + Business Media, 2008. 224

[378] T. Weth, *Spectral and variational characterizations of solutions to semilinear eigenvalue problems*, Dissertation, Johannes Gutenberg-Universität, Mainz, 2001. iii, 80

[379] M. Willem, "Minimax theorems", Progr. Nonlinear Differential Equations Appl. **24**, Birkäuser, 1996. iv, 122

[380] M. Willem, "Functional analysis: fundamentals and applications", Cornerstones, Springer Science+Business Media, Birkäuser, 1996. 35

[381] J. Wloka, "Partial differential equations", Cambridge University Press, Cambridge, 1987. 23

[382] D. Wu, *Existence and stability of standing waves for nonlinear fractional Schrödinger equations with Hartree type nonlinearity*, J. Math. Anal. Appl. **411** (2014), no. 2, 530–542. 108

[383] Z. Wu, S. Cao, W. Che, F. Yang, X. Zhu, Y. He, *Solitons supported by parity-time-symmetric optical lattices with saturable nonlinearity in fractional Schrödinger equation*, Results Phys. **19** (2020), no. 6, article ID 103381, pp. 6. 37, 44

[384] J. Xia, Z.-Q. Wang, *Saddle solutions for the Choquard equation*, Calc. Var. Partial Differential Equations **58**(3) (2019), no. 85, pp. 30. iii, 80

[385] J. Yang, *The existence of normalized Solutions for a nonlocal problem in* $\mathbb{R}^3$, Adv. Math. Phys. **2020** (2020), article ID 3186135, 1–8. 45

[386] Z. Yang, F. Zhao, *Multiplicity and concentration behaviour of solutions for a fractional Choquard equation with critical growth*, Adv. Nonlinear Anal. **10** (2021), no. 1, 732–774. 108

[387] X. Yao, X. Liu, *Solitons in the fractional Schrödinger equation with parity-time-symmetric lattice potential*, Photonics Res. **6** (2018), no. 9, 875–879. 44

[388] H. Ye, *Mass minimizers and concentration for nonlinear Choquard equations in $\mathbb{R}^N$*, Topol. Methods Nonlinear Anal. **48** (2016), no. 2, 393–417. 81

[389] A. C. Zaanen, *Some examples in weak sequential convergence*, Amer. Math. Monthly **69** (1962), no. 2, 85–93. 34

[390] J. Zhang, Z. Chen, W. Zou, *Standing waves for nonlinear Schrödinger equations involving critical growth*, J. London Math. Soc. **90** (2014), no. 2, 827–844. 209

[391] J. Zhang, J. M. do Ó, M. Squassina, *Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity*, Adv. Nonlinear Stud. **15** (2016), no. 1, 15–30. 210

[392] J. Zhang, W. Zou, *A Berestycki-Lions theorem revisited*, Commun. Contemp. Math. **14** (2012), no. 5, article ID 1250033, pp. 14. 36, 211, 218

[393] J. Zhang, W. Zou, *Solutions concentrating around the saddle points of the potential for critical Schrödinger equations*, Calc. Var. Partial Differential Equations **54** (2015), no. 4, 4119–4142. 209

[394] R. Zhang, Z. Han, Y. Shao, Z. Wang, Y. Wang, *The numerical study for the ground and excited states of fractional Bose-Einstein condensates*, Comput. Math. Appl. **78** (2019), 1548–1561. 44

[395] S. Zhao, Y. Yu, *Sign-changing solutions for a fractional Choquard equation with power nonlinearity*, Nonlinear Anal. **221** (2022), article ID 112917, pp. 18. 142

Fundings: The research in this thesis has been supported by INdAM-GNAMPA and by PRIN 2017JPCAPN "Qualitative and quantitative aspects of nonlinear PDEs".

Mathematics Subject Classification:
30H40, 33C05, 35A01, 35A09, 35A15, 35A21, 35B06, 35B09, 35B25, 35B33, 35B38, 35B40, 35B50, 35B65, 35D30, 35D35, 35D40, 35G20, 35G50, 35J05, 35J10, 35J15, 35J20, 35J47, 35J50, 35J60, 35J61, 35J91, 35Q40, 35Q55, 35Q60, 35Q70, 35Q75, 35Q85, 35Q92, 35R09, 35R11, 44A10, 45G15, 45K05, 45M05, 45M20, 46M20, 47F05, 47F10, 47G10, 47H10, 47J30, 49J35, 55M30, 55N05, 55N10, 58E05, 58J05.

Key words:
Asymptotic behaviour; Center of mass; Choquard-Pekar equation; Concentration phenomena; Critical exponent; Double nonlocality; Even and odd nonlinearities, Existence and multiplicity; Fractional Laplacian; Ground states; Hartree-type term; L^2-constraint; Lagrange multiplier; Mountain Pass paths; Nonlocal sources; Nonlinear PDEs; Normalized solutions; Pohozaev identity; Polynomial decay; Positivity and sign; Prescribed mass; Qualitative properties; Radial symmetry; Regularity; Relative cup-length; Riesz potential; Schrödinger equation; Singular perturbation; Spike solutions; Sublinear nonlinearities.

Contacts: marco.gallo@uniba.it, Dipartimento di Matematica, UNIBA (Bari).
Edit: marco.gallo1@unicatt.it, Università Cattolica del Sacro Cuore (Brescia).

This thesis has been elaborated in the three years 2019–2022, submitted first on 28th December 2022, and defended on 21st March 2023. See also arXiv:2402.08338.

The contents are mainly based on the papers [111–117, 197, 198], and elaborated in collaboration with Prof. Silvia Cingolani (Università degli Studi di Bari Aldo Moro), Prof. Denis Bonheure (Université libre de Bruxelles) and Prof. Kazunaga Tanaka (Waseda University, Tokyo).